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ON S-DECOMPOSABLE OPERATORS

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1. INTRODUCTION

Residually decomposable operators, introduced by F.-H. Vasilescu [14], [15], as well as bounded and unbounded S-decomposable operators, studied by I. Bacalu [2], [3] and the author [8], are linear operators that show a "good spectral behaviour" only outside a certain part of the spectrum. There are more related definitions of this good behaviour, which are all connected with the concept of decomposability in the sense of C. Foiaş (see, e.g., [7]). It was proved in [9] that for any closed operator there is a unique minimal closed subset of the spectrum, called the strong spectral residuum, outside which the operator has a spectral behaviour of this kind (strong decomposability).

The main result of this paper is that for any bounded operator there is a unique minimal closed subset of the spectrum, called the spectral residuum, outside which the operator shows a similar spectral behaviour (decomposability). The spectral residuum is contained (maybe properly) in the strong spectral residuum and is, in general, different from the spectral residuum in Vasilescu's sense [15], p. 385 (which was proved to exist only for a certain class of operators). As a preparation we extend a recent result by M. Radjabalipour [12] and prove the equivalence of bounded $(S, 1)$- and S-decomposable operators. In Section 2 we shall give the necessary definitions for closed operators, but we shall restrict most of the discussion in Section 3 to the bounded case, leaving open the question whether these results are valid if the operator is unbounded.

2. PRELIMINARIES

Let X be a complex Banach space and let $\mathscr{C}(X)$ and $\mathscr{B}(X)$ denote the class of closed and bounded linear operators on X, respectively. $\mathbf{C}$ and $\bar{\mathbf{C}}$ will stand for the complex plane and its compactification, respectively. If $F \subset \bar{\mathbf{C}}$, then F^c denotes $\bar{\mathbf{C}} \setminus F$ and $\bar{F}$ denotes the closure of F in $\bar{\mathbf{C}}$. For $T \in \mathscr{C}(X)$, $\mathscr{D}(T)$ is its domain and $\sigma(T)$ denotes its extended spectrum, which is its spectrum $s(T)$ if $T \in \mathscr{B}(X)$ and is

$s(T) \cup \{\infty\}$ otherwise. We put $\rho(T) = \sigma(T)^c$. If Y is a closed subspace of X and $T(Y \cap \mathscr{D}(T)) \subset Y$, then we write $Y \in \mathscr{I}(T)$, and $T|Y$ or T_Y denote the restriction of T to $Y \cap \mathscr{D}(T)$. Let $\bar{x}$ denote the coset $x + Y$ for any x in X and let X/Y denote the quotient Banach space. Put

$$\mathscr{D}(T^Y) = \{\bar{x} \in X/Y;\ \bar{x} \cap \mathscr{D}(T) \neq \emptyset\}$$

and $T^Y\bar{x} = \overline{Tx}$ for any $\bar{x} \in \mathscr{D}(T^Y)$ and $x \in \bar{x} \cap \mathscr{D}(T)$. T^Y is the operator induced in X/Y by T. If $\sigma(T) \cup \sigma(T_Y) \neq \bar{\mathbf{C}}$, then $T^Y \in \mathscr{C}(X/Y)$ ([10]).

We recall some definitions and facts from [14]. Let $T \in \mathscr{C}(X)$. For $x \in X$, $z \in \bar{\mathbf{C}}$ we write $z \in \delta_T(x)$ if in a neighborhood U of z there is a holomorphic function f such that $(u - T)f(u) = x$ for $u \in U \cap \mathbf{C}$. There is a unique maximal open set O_T in $\bar{\mathbf{C}}$ with the following property: if $G \subset O_T$ is an open set and $f\colon G \to \mathscr{D}(T)$ is a holomorphic function such that $(u - T)f(u) = 0$ for $u \in G \cap \mathbf{C}$, then $f(u) = 0$ for $u \in G$. Set $S_T = O_T^c$ and for any x in X set $\gamma_T(x) = \delta_T(x)^c$, $\sigma_T(x) = \gamma_T(x) \cup S_T$ and $\rho_T(x) = \sigma_T(x)^c$. If $S_T = \emptyset$, we say that T has the single-valued extension property. For any $T \in \mathscr{C}(X)$, $H \subset \bar{\mathbf{C}}$ set

$$X_T(H) = \{x \in X;\ \sigma_T(x) \subset H\}.$$

$X_T(H)$ is a linear manifold in X. A subspace Y in $\mathscr{I}(T)$ belongs to the class $\mathscr{I}_T$ if $T_Y \in B(Y)$.

Let F be a closed set in $\bar{\mathbf{C}}$ and define

$$\mathscr{I}(T, F) = \{Y \in \mathscr{I}(T);\ \sigma(T_Y) \subset F\}.$$

If $\mathscr{I}(T, F)$ has an upper bound (with respect to the relation $\subset$), which belongs to $\mathscr{I}(T, F)$, then the upper bound is denoted by $X(T, F)$. If $X(T, F) \in \mathscr{I}_T$, then it is denoted also by $X_{T,F}$. Subspaces of the form $X(T, F)$ (with F closed in $\bar{\mathbf{C}}$) are called spectral maximal spaces of T.

Let S be closed in $\bar{\mathbf{C}}$. A finite family of open sets $(G_1, \ldots, G_n; G_0)$ is called an open S-covering of the closed set $H \subset \bar{\mathbf{C}}$ if

$$\bigcup_{i=0}^{n} G_i \supset H \cup S \text{ and } \bar{G}_i \cap S = \emptyset \text{ for } i = 1, \ldots, n.$$

An open S-covering of $\emptyset$ is simply called an S-system.

Let n be a positive integer. $T \in \mathscr{C}(X)$ is called strongly (S, n)-decomposable if for any open S-covering $(G_1, \ldots, G_n; G_0)$ of $\sigma(T)$ there are spectral maximal spaces of T, $X_i \subset \mathscr{D}(T)$ $(i = 1, \ldots, n)$ and $X_0 \subset X$ such that

1° $\sigma(T|X_i) \subset \bar{G}_i$ for $i = 0, 1, \ldots, n$,

2° for any spectral maximal space Y of T

$$Y = \sum_{i=0}^{n} (Y \cap X_i).$$

T is called (S, n)-decomposable if we postulate 2° only for $Y = X$. T is said to be (strongly) S-decomposable if it is (strongly) (S, n)-decomposable for every positive integer n. (Strongly) $\emptyset$-decomposable operators are called (strongly) decomposable.

We note that E. Albrecht [1] has shown that not every decomposable operator is strongly decomposable. Further, an operator T is (S, n)-decomposable in our terminology if and only if T is (S, n)-decomposable and has property (γ) ([16], pp. 1574—1575) in Vasilescu's sense (cf. [8]). Moreover, many properties of S-decomposable operators in [2], [3], [4] and [8] are, in fact, consequences of $(S, 1)$-decomposability and will be used accordingly without further comment. Finally, T is (S, n)-decomposable if and only if it is $(S \cap \sigma(T), n)$-decomposable, therefore we may assume that $S \subset \sigma(T)$.

We also note that (S, n)-decomposable operators differ from decomposable operators in two important respects: they need not to have the single-valued extension property, and there is no description of their spectral maximal spaces in terms of the manifolds $X_T(F)$.

3. THE RESULTS

Until we state otherwise, assume that the operator T belongs to $\mathscr{B}(X)$. We shall say that a closed subset F of $\sigma(T)$ is a set-spectrum of T if $X(T, F)$ exists and $\sigma(T|X(T, F)) = F$ (cf. [6]). The interior of any subset A of $\sigma(T)$ in the topology of $\sigma(T)$ will be denoted by A^i.

1. LEMMA. *Let F be a closed subset of $\sigma(T)$.*

a) *If $X(T, F)$ exists, then there is a largest set-spectrum F_m contained in F.*

b) *If T is $(S, 1)$-decomposable and $F \cap S = \emptyset$, then $\bar{F^i} \subset F_m$.*

Proof. a) Put $F_m = \sigma(T|X(T, F))$; then $F_m \subset F$. If $Y \in \mathscr{I}(T, F_m)$, then $\sigma(T_Y) \subset$ $\subset F_m \subset F$, hence $Y \subset X(T, F)$. Thus $X(T, F_m)$ exists and is identical with $X(T, F)$. If F_0 is any set-spectrum in F, then $X(T, F_0) \subset X(T, F) = X(T, F_m)$. By [7], 1.3.4 we obtain that

$$F_0 = \sigma(T|X(T, F_0)) \subset \sigma(T|X(T, F_m)) = F_m.$$

Hence F_m is the largest set-spectrum in F.

b) $X(T, F)$ exists (cf. [16], p. 1574), hence F_m exists by a). There is an open set G_1 such that $\bar{G_1} \cap S = \emptyset$ and $F^i = G_1 \cap \sigma(T)$. Let $z \in F^i$ and let G_0 be an open set such that $z \notin G_0$ and $(G_1; G_0)$ is an S-covering of $\sigma(T)$. Since T is $(S, 1)$-decomposable, there are spectral maximal spaces X_1, X_0 such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ $(i = 1, 0)$. According to [5], Lemma 1, we have

$$\sigma(T) = \sigma(T|X_1) \cup \sigma(T|X_0) \cup S_T.$$

Hence $z \in \sigma(T|X_1)$. Since $\sigma(T|X_1) \subset F_m$, we obtain that $\overline{F^i} \subset F_m$.

REMARK . The relation $\overline{F^i} \subset F_m$ follows also from [5], Lemma 2.

2. LEMMA. *Let T be $(S, 1)$-decomposable, let G be an open subset of $\mathbf{C}$ such that $\bar{G} \cap S = \emptyset$, and put $Y = X(T, G)$. Then $\sigma(T^Y) \cap G = \emptyset$.*

Proof. We clearly have

$$Y = X(T, G \cap \sigma(T)) \supset X(T, \overline{G \cap \sigma(T)}).$$

By Lemma 1 and [7], 1.3.4, we obtain

$$\overline{G \cap \sigma(T)} \subset \sigma(T_Y) \subset \bar{G} \cap \sigma(T).$$

An application of Proposition 5 in [5] yields

$$\sigma(T^Y) = \overline{\sigma(T) \setminus \sigma(T_Y)} \subset G^c.$$

The following lemma generalizes a result due to M. Radjabalipour [12], and will provide us with an important tool for proving Theorem 6.

3. LEMMA. *Let T be $(S, 1)$-decomposable and let $F \subset \mathbf{C}$ be closed and such that $X(T, F)$ exists. If $(G_1; G_0)$ is an open S-covering of F, then*

$$X(T, F) \subset X(T, \bar{G}_1) \dotplus X_T(\bar{G}_0).$$

Proof. We may and will assume that $F \subset \sigma(T)$, that G_1, G_0 are not disjoint from $\sigma(T)$ and $\sigma(T)$ is not contained in $G_1 \cup G_0$. Set $K = \overline{G_1 \cap G_0}$. Then $X(T, K)$ exists and will be denoted by Y.

Since the resolvent $(z - T|X(T, F))^{-1}$ exists for $z \in F^c$, for any x in $X(T, F)$ there is a holomorphic function f such that

$$(z - T)f(z) = x \qquad \text{for } z \in F^c.$$

Let $\bar{x}$ denote the coset $x + Y$; then

$$(z - T^Y)\overline{f(z)} = \bar{x} \qquad \text{for } z \in F^c,$$

and the function $\bar{f}$ defined by $\bar{f}(z) = \overline{f(z)}$ is holomorphic. By Lemma 2, we have $G_1 \cap G_0 \subset \rho(T^Y)$, hence the function $\bar{f}$ defined by

$$\bar{f}(z) = \begin{cases} \overline{f(z)} & \text{for} \quad z \in F^c \\ (z - T^Y)^{-1}\bar{x} & \phantom{\text{for}} \quad z \in G_1 \cap G_0 \end{cases}$$

is well-defined, holomorphic and satisfies

$$(z - T^Y)\bar{f}(z) = \bar{x} \text{ for } z \in F^c \cup (G_1 \cap G_0).$$

Put $H_1 = F \setminus G_0$ and $H_0 = F \setminus G_1$. Then H_1 and H_0 are disjoint compact sets and $(H_1 \cup H_0)^c = F^c \cup (G_1 \cap G_0)$. Let D_j be bounded Cauchy domains ([13], p. 288) such that $H_j \subset D_j$ $(j = 1, 0)$ and $\bar{D}_1 \cap \bar{D}_0 = \emptyset$. Let B_j denote the positively oriented boundary of D_j $(j = 1, 0)$ and put $c = (2\pi i)^{-1}$. Then

$$\bar{x} = c \int_{|z| = |T^Y| + 1} (z - T^Y)^{-1}\, \bar{x}\, dz = c \int_{B_1} \bar{f}(z) dz + c \int_{B_0} \bar{f}(z) dz.$$

Setting

$$\bar{x}_j = c \int_{B_j} \bar{f}(z) dz \quad (j = 1, 0),$$

we shall show that $H_j^c \subset \delta_{T^Y}(\bar{x}_j)$. Indeed, if $v_0 \in H_j^c$, then we can find a smaller Cauchy domain D_j' (with boundary B_j') such that $D_j' \supset H_j$ and $v_0 \notin \bar{D}_j'$. Then for some neighborhood N of v_0 we have $\bar{N} \cap \bar{D}_j' = \emptyset$, and for $v \in N$

$$(v - T^Y) c \int_{B_j'} \frac{\bar{f}(z)}{v - z}\, dz = c \int_{B_j'} (v - z + z - T^Y) \frac{\bar{f}(z)}{v - z}\, dz = \bar{x}_j.$$

Since the function

$$g_j(v) = c \int_{B_j'} \frac{\bar{f}(z)}{v - z}\, dz$$

is holomorphic for $v \in N$, we have proved that $v_0 \in \delta_{T^Y}(\bar{x}_j)$.

According to [15], Lemma 2.1, in a neighborhood $N' \subset N$ of v_0 there is a holomorphic X-valued function h_j such that $\overline{h_j(v)} = g_j(v)$, hence

$$\overline{(v - T)\, h_j(v)} = \bar{x}_j \qquad \text{for } v \in N'.$$

Let $x_j \in \bar{x}_j$ and define the holomorphic function r_j by $r_j(v) = (v - T)\, h_j(v) - x_j$ $(v \in N')$. Assume that, in addition to our hypothesis, $v_0 \in H_j^c \cap K^c$, then $v_0 \in \rho(T_Y)$. Since $r_j(v) \in Y$, for some neighborhood $N'' \subset N'$ of v_0 we have

$$(v - T)(h_j(v) - (v - T_Y)^{-1} r_j(v)) = x_j \quad (v \in N'').$$

Since the function $h_j(v) - (v - T_Y)^{-1} r_j(v)$ is holomorphic, we obtain that $\gamma_T(x_j) \subset H_j \cup K$ and $x = x_1 + x_0 + y$ for some $y \in Y$. Here $\sigma_T(y) \subset K \cup S \subset \bar{G}_0$ and $\sigma_T(x_0) \subset H_0 \cup K \cup S \subset \bar{G}_0$, since $S_T \subset S$. Further, the relation $\gamma_T(x_1) \subset \bar{G}_1$ and Theorem 4.2 in [14] imply that $x_1 \in X(T, \bar{G}_1) + X_T(S_T)$. Hence $x \in X(T, \bar{G}_1) + X_T(\bar{G}_0)$. The proof is complete.

4. THEOREM. *Let T be $(S, 1)$-decomposable. Then T is S-decomposable; further for any closed F in $\mathbf{C}$ such that $X(T, F)$ exists and for any open S-covering $(G_1, \ldots, G_n; G_0)$ of F*

$$X(T, F) \subset \sum_{j=1}^{n} X(T, \bar{G}_j) + X_T(\bar{G}_0). \tag{1}$$

Proof. The relation (1) for the value $n = 1$ was proved in Lemma 3. Assume that (1) is valid for n covering sets, and the covering $(G_1, \ldots, G_n; G_0)$ is given. Then there is an open S-covering $(D_1, \ldots, D_n; D_0)$ of F such that $\bar{D}_j \subset G_j$ for $j = 0, 1, \ldots, n$. By assumption,

$$X(T, F) \subset \sum_{j=1}^{n-1} X(T, \bar{D}_j) + X_T(\overline{D_0 \cup D_n}).$$

Applying Lemma 3, we obtain

$$X_T(\overline{D_0 \cup D_n}) \subset X(T, \bar{G}_n) + X_T(\bar{G}_0).$$

Since $X(T, \bar{D}_j) \subset X(T, \bar{G}_j)$, we have (1) for $n + 1$ covering sets. Setting $F = \sigma(T)$, we obtain that T is S-decomposable.

5. DEFINITION. Let $T \in \mathscr{C}(X)$ and let $\mathscr{Q} = \mathscr{Q}(T)$ denote the family of all closed sets S such that $S_T \subset S \subset \sigma(T)$ and T is S-decomposable. If there exists $S^* \in \mathscr{Q}$ such that $S^* \subset S$ for any $S \in \mathscr{Q}$, then S^* is called the *spectral residuum* of T.

Note that this notion of the spectral residuum is different from that given by F.-H. Vasilescu in [15].

6. THEOREM. *The spectral residuum exists for each operator $T \in \mathscr{B}(X)$.*

Proof. Since $\sigma(T)$ belongs to $\mathscr{Q}$, the family $\mathscr{Q}$ is nonempty. Let $\{S_a; a \in A\}$ be a totally ordered (with respect to the relation $\subset$) subfamily of $\mathscr{Q}$ and let $S_0 = \cap \{S_a; a \in A\}$. If $F \subset \mathbf{C}$ is a closed set disjoint from S_0 then, since $\sigma(T)$ is compact, $F \cap S_a$ is void for some $a \in A$. Hence an S_0-covering of $\sigma(T)$ is an S_a-covering of $\sigma(T)$ for some $a \in A$. Since T is S_a-decomposable, it is also S_0-decomposable. By Zorn's lemma, there is a minimal element in $\mathscr{Q}$. We shall show that if $S_1, S_2 \in \mathscr{Q}$ and $S = S_1 \cap S_2$, then $S \in \mathscr{Q}$, which will complete the proof.

Let $(G; G_S)$ be an open S-covering of $\sigma(T)$. Then there exist open sets G_k, G_{S_k} $(k = 1, 2)$ with the following properties (cf. [9]):

$$G_{S_k} \supset S_k \cup G_S \ (k = 1, 2) \quad \text{and} \quad \bar{G}_{S_1} \cap \bar{G}_{S_2} = \bar{G}_S;$$

$G_k \subset G$, $\bar{G}_k \cap S_k = \emptyset$ and $G_k \cup G_{S_k} \supset G$ $(k = 1, 2)$. Thus $(G_k; G_{S_k})$ is an open S_k-covering of $\sigma(T)$. Hence there is an open set G'_{S_2} such that $\bar{G}'_{S_2} \subset G_{S_2}$ and $(G_2; G'_{S_2})$ is an S_2-covering of $\sigma(T)$. Since T is S_2-decomposable, we have

$$X = X(T, \bar{G}_2) + X_T(\bar{G}'_{S_2}).$$

Since T is S_1- and S_2-decomposable and $\bar{G} \cap S = \emptyset$, a part of the proof of [9], Theorem 1 shows that the spectral maximal space $X(T, \bar{G})$ exists. Since $G_2 \subset G$, we obtain

$$X = X(T, \bar{G}) + X_T(\bar{G}'_{S_2}). \tag{2}$$

Now put $F = \bar{G}'_{S_2} \cap \sigma(T)$ and $G_0 = G_{S_1} \cap G_{S_2}$. Then $X(T, F)$ exists (see [2], Theorem 1.5), and

$$S_1 \cap F \subset G_{S_1} \cap G_{S_2} = G_0 \subset \bar{G}_S;$$

furthermore

$$G_1 \cup G_0 \supset (G_1 \cup G_{S_1}) \cap G_{S_2} \supset \sigma(T) \cap F = F.$$

Hence $(G_1; G_0)$ is an open $S_1 \cap F$-covering of F such that $\bar{G}_1 \cap S_1 = \emptyset$. Since T is S_1-decomposable and the set K defined by $K = \overline{G_1 \cap G_0}$ is disjoint from S_1, we can apply almost all of the proof of Lemma 3 to the present situation and obtain the following statement: If $x \in X(T, F)$, then

$$x = x_1 + x_0 + y,$$

where $y \in X(T, K)$ and $\gamma_T(x_j) \subset H_j \cup K$ with $H_j = F \setminus G_k$ $(j, k = 1,0;\ j \neq k)$

Since $S_1, S_2 \in \mathcal{Q}$, we have $S_T \subset S_1 \cap S_2$. Hence $\sigma_T(y) \subset K \cup S_T \subset K \cup (S_1 \cap F) \subset \bar{G}_0$. Since $(G_1; G_0)$ covers F, we obtain $\sigma_T(x_0) \subset H_0 \cup K \cup S_T \subset \bar{G}_0$, and $\gamma_T(x_1) \subset H_1 \cup K \subset \bar{G}_1$. Thus [14], Theorem 4.2 implies that $x_1 \in X(T, \bar{G}_1) + X_T(S_T)$. Hence

$$X_T(\bar{G}'_{S_2}) = X(T, F) \subset X(T, \bar{G}_1) + X_T(\bar{G}_0) \subset X(T, \bar{G}) + X_T(\bar{G}_S).$$

By (2) we obtain

$$X = X(T, \overline{G}) + X_T(\overline{G}_S),$$

thus T is $(S, 1)$-decomposable. An application of Theorem 4 ends the proof.

7. REMARKS. The author does not know whether the above results generalize to the case when $T \in \mathscr{C}(X)$. Since an operator T in $\mathscr{C}(X)$ is strongly (S, n)-decomposable if and only if for any spectral maximal space Y of T the restriction T_Y is (S, n)-decomposable (cf. [4]), the following Theorem 8 needs an independent proof only when T is not in $\mathscr{B}(X)$; otherwise it follows from Theorem 4.

Moreover, in the case $T \in \mathscr{B}(X)$ we can deduce the assertion of [9], Theorem 1 from Theorem 6, i.e. we can prove that there is a smallest one (called the strong spectral residuum) among the sets S for which T is strongly S-decomposable. Indeed, let the latter class of sets be denoted by $\mathscr{R} = \mathscr{R}(T)$. It is easily seen that the intersection of sets in a totally ordered subclass of $\mathscr{R}$ also belongs to $\mathscr{R}$. Further, if S_1, $S_2 \in \mathscr{R}$ and Y is a spectral maximal space of T, then $S_1, S_2 \in \mathscr{Q}(T_Y)$ and Theorem 6 implies $S_1 \cap S_2 \in \mathscr{Q}(T_Y)$. Hence $S_1 \cap S_2 \in \mathscr{R}(T)$, which completes the proof.

Let $O(T)$, $S(T)$ and $V(T)$ denote the spectral residuum, the strong spectral residuum, and the spectral residuum in Vasilescu's sense ([15], p. 385), respectively, of the operator T. If $T \in \mathscr{B}(X)$, then $O(T) \subset S(T)$. If T is the decomposable operator in [1], Theorem 2.1, then this inclusion is proper; further $S(T) = V(T)$, by [9], Theorem 2. If $T \in \mathscr{B}(X)$ then Theorem 4 and [16], Theorem 2.10 imply $O(T^*) \subset O(T)$, where T^* is the adjoint of T.

8. THEOREM. *If $T \in \mathscr{C}(X)$ is strongly $(S, 1)$-decomposable, then T is strongly S-decomposable.*

Proof. First we show that if $(G_1, \ldots, G_n; G_0)$ is an S-system of open sets, then

$$X_T\left(\bigcup_{k=0}^{n} G_k\right) \subset X_T(G_0) + \sum_{k=1}^{n} X_{T, \overline{G}_k}. \tag{3}$$

Assume that $n = 1$ and $x \in X_T(G_0 \cup G_1)$. Let $Y = X_T(\sigma_T(x) \cup S)$; then Y is a spectral maximal space of T, by [2], Theorem 1.5. Hence T_Y is $(S, 1)$-decomposable. Since $\sigma(T_Y) \subset \sigma_T(x) \cup S \subset G_0 \cup G_1$, the system $(G_1; G_0)$ is an open S-covering of $\sigma(T_Y)$. Thus there exist spectral maximal spaces $Y_1 \subset D(T_Y)$ and Y_0 of T_Y such that

$$Y = Y_0 + Y_1 \text{ and } \sigma(T|Y_i) \subset G_i \quad (i = 0, 1).$$

Hence $x \in Y \subset X_T(G_0) + X_{T, \overline{G}_1}$, which proves (3) for $n = 1$. If (3) is valid for a fixed n, and an S-system of $n+2$ sets is given, then we obtain

$$X_T\left(\bigcup_{k=0}^{n+1} G_k\right) = X_T\left(\left(\bigcup_{k=0}^{n} G_k\right) \cup G_{n+1}\right) \subset X_T\left(\bigcup_{k=0}^{n} G_k\right) + X_{T,\bar{G}_{n+1}} \subset$$

$$\subset X_T(G_0) + \sum_{k=1}^{n+1} X_{T,\bar{G}_k}.$$

Thus (3) is proved by induction.

Now let Z be a spectral maximal space of T and let $(G_1, \ldots, G_n; G_0)$ be an open S-covering of $\sigma(T_Z)$. By assumption, T_Z is strongly $(S, 1)$-decomposable (cf. [4], Proposition 4). Thus the preceding paragraph yields

$$Z = Z_{T|Z}\left(\bigcup_{k=0}^{n} G_k\right) \subset Z_{T|Z}(\bar{G}_0) + \sum_{k=1}^{n} Z_{T|Z, \bar{G}_k} \subset Z.$$

Hence T_Z is S-decomposable, so T is strongly S-decomposable.

REMARK. The above proof is an extension of a proof of S. Plafker [11] for the case $S = \emptyset$.

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