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ON THE DISTANCE TO THE SET OF COMPACT PERTURBATIONS OF NILPOTENT OPERATORS

NORBERTO SALINAS

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ be the algebra of all (bounded, linear) operators on $\mathscr{H}$. Also, we denote by $\mathscr{K}$ the ideal of compact operators on $\mathscr{H}$, and by $\pi : \mathscr{L}(\mathscr{H}) \to \mathscr{Q}(\mathscr{H})$ $(= \mathscr{L}(\mathscr{H})/\mathscr{K})$ the canonical quotient map onto the Calkin algebra $\mathscr{Q}(\mathscr{H})$.

In this paper, we deal with precise estimates of the distance from an arbitrary operator T in $\mathscr{L}(\mathscr{H})$ to the set $\mathscr{N}$ of all nilpotent operators on $\mathscr{H}$ (Theorem 2.3(a)). We also estimate the distance $\mathrm{d}(T, \mathscr{N} + \mathscr{K})$ (Theorem 2.3 (b)), and we establish a spectral formula for $\mathrm{d}(T, \mathscr{N} + \mathscr{K})$ for the class of essentially (G_1) operators (i.e. those operators whose essential resolvent function has first order growth, [12]), whose semi-Fredholm domain contains no infinite indices (Theorem 3.1). In particular we obtain a spectral characterization of $\mathrm{d}(T, \mathscr{N} + \mathscr{K})$ for the class of essentially normal operators (Remark 3.2(a)).

In the last section of the paper, we give an interesting example of an essentially (G_1) operator S (in fact, an essentially hyponormal operator S) for which $\mathrm{d}(S, \mathscr{N}+\mathscr{K})$ is not obviously computable. We also pose some other problems concerning this operator S, and we ask some questions related to the material presented in the previous sections.

In what follows $\sigma(T)$ denotes the spectrum of an operator T in $\mathscr{L}(\mathscr{H})$, and $E(T)$ denotes the essential spectrum of T (i.e. $E(T) = \sigma(\pi(T))$). Also, we shall denote by $E_l(T)$ and $E_r(T)$ the left and right essential spectra of T, and we let $E_{lr}(T) = E_l(T) \cap E_r(T)$. Finally, $\Omega(T)$ will denote the Weyl spectrum of T, i.e. $\Omega(T) = \{\lambda \in \mathbf{C} \mid T - \lambda$ is not Fredholm of index $0\}$ [16], and $\sigma_0(T)$ will denote the set of normal eigen-values of T, i.e. $\sigma_0(T) = \{\lambda \in \mathbf{C} \mid \lambda$ is an isolated point of $\sigma(T)$ and $T - \lambda$ is Fredholm$\}$ (recall that the Browder spectrum $B(T)$ is defined to be $B(T) = \sigma(T) \setminus \sigma_0(T)$ [17]).

2. A SPECTRAL ESTIMATE OF $\mathrm{d}(T, \mathscr{N} + \mathscr{K})$

In order to state the main result of this section we need to introduce some non-standard terminology.

DEFINITION 2.1. Given a bounded open subset U of $\mathbf{C}$, we denote by $m_t(U)$ (the modulus of thickness of U) the non-negative number: $\mathrm{m}_t(U) = \sup_{\lambda \in \mathbf{C}} \mathrm{d}(\lambda, \mathbf{C} \setminus U)$. We also define the modulus of disconnectedness $\mathrm{m}_d(K)$ of a nonempty compact subset K of $\mathbf{C}$, as follows: given $\varepsilon > 0$, we let $K_\varepsilon = \{\lambda \in \mathbf{C} \mid \mathrm{d}(\lambda, K) \leqslant \varepsilon\}$, and then we let $\mathrm{m}_d(K)$ be the infimum of those positive numbers ε such that K_ε is connected.

REMARK 2.2. (a) It is clear that $\mathrm{m}_t(U)$ is the largest radius of those open disks contained in U.

(b) If $\delta = \sup_{\lambda \in K} |\lambda|$, then K_δ is simply connected so that $\mathrm{m}_d(K) \leqslant \delta$.

(c) If K is disconnected, then $\mathrm{m}_d(K)$ is the supremum of those positive numbers ε such that K_ε is disconnected.

(d) If L is a compact subset of K such that $\partial(K) \subseteq L$ (here, ∂ denotes boundary), then $\mathrm{m}_d(K) \leqslant \mathrm{m}_d(L)$ and $K_{\mathrm{m}_d(K)} \subseteq L_{\mathrm{m}_d(L)}$.

(e) If $\mathrm{m}_d(K) > 0$, then for $0 < \delta < \mathrm{m}_d(K)$ there exist two disjoint nonempty closed subsets K_1 and K_2 of K, and two contours Γ_1 and Γ_2 consisting of the disjoint union of finitely many simple closed Jordan curves, such that

$$K = K_1 \cup K_2, \quad \mathrm{int}(\Gamma_1) \cap \mathrm{int}(\Gamma_2) = \emptyset, \; K_j \subseteq \mathrm{Int}(\Gamma_j),$$

and

$$\delta < \mathrm{d}(\Gamma_j, K_j) = \mathrm{d}(\Gamma_j, K) < \mathrm{m}_d(K), \quad j = 1, 2.$$

We are now in a position to state our main theorem.

THEOREM 2.3. *Let* $T \in \mathscr{L}(\mathscr{H})$.

(a) $\mathrm{d}(T, \mathscr{N}) \leqslant \max\{\mathrm{m}_t(\Omega(T) - E_{lr}(T)),\ \mathrm{m}_d(\Omega(T)),\ \sup_{\lambda \in \{0\} \cup \sigma_0(T)} \mathrm{d}(\lambda, \Omega(T))\}$.

(b) $\mathrm{d}(T, \mathscr{N} + \mathscr{K}) \leqslant \max\{\mathrm{m}_t(\Omega(T) - E_{lr}(T)),\ \mathrm{m}_d(\Omega(T)),\ \mathrm{d}(0, \Omega(T)\}$.

REMARK 2.4. In order to prove Theorem 2.3 we shall need to establish some auxiliary results. It may be worth mentioning that since the right hand side of the inequality in part (a) is less than or equal to the spectral radius of T, Theorem 2.3 (a), implies [5, Thm.3.5]. On the other hand, since

$$E_{lr}(T) \subseteq E(T) \subseteq \Omega(T) \subseteq B(T),$$

and

$$\partial(B(T)) \subseteq E_{lr}(T),$$

it follows that

$$\mathrm{d}(\lambda, E(T)) = \mathrm{d}(\lambda, \Omega(T)) \quad \text{for every } \lambda \notin \Omega(T),$$

$$\mathrm{d}(\lambda, \Omega(T)) = \mathrm{d}(\lambda, B(T)) \quad \text{for every } \lambda \notin B(T),$$

and

$$\mathrm{m_d}(B(T)) \leqslant \mathrm{m_d}(\Omega(T)) \leqslant \mathrm{m_d}(E(T)).$$

Also,

$$\mathrm{m_d}(\sigma(T)) \leqslant \max \left\{ \mathrm{m_d}\,(B(T), \frac{1}{2} \sup_{\lambda \in \sigma_0(T)} \mathrm{d}(\lambda, B(T)) \right\} \leqslant$$

$$\leqslant \max \left\{ \mathrm{m_d}(\Omega(T)), \frac{1}{2} \sup_{\lambda \in \sigma_0(T)} \mathrm{d}(\lambda, \Omega(T)) \right\},$$

$$\max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T)), \mathrm{d}(0, \Omega(T))\} = \max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T), \ \mathrm{d}(0, E(T))\} =$$

$$= \max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T)), \mathrm{d}(0, E_{\mathrm{lr}}(T))\},$$

and

$$\max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T)), \mathrm{m_d}(\Omega(T))\} =$$

$$= \max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T)), \mathrm{m_d}(E(T))\} =$$

$$= \max \{\mathrm{m_t}(\Omega(T) - E_{\mathrm{lr}}(T), \mathrm{m_d}(E_{\mathrm{lr}}(T))\}.$$

Therefore, the right hand side of the inequality in Theorem 2.3(a) is zero if and only if $\sigma(T)$ and $E(T)$ are connected, $0 \in E(T)$, and $\Omega(T) = E_{\mathrm{lr}}(T)$. This means that Theorem 2.3(a) is a generalization of [2, Thm.2.4] (see also [19]). We should also point out that in the proof of Theorem 2.3, we shall use [2, Prop. 1.6].

For the benefit of the reader, we begin by stating some known results from the literature that we need later.

LEMMA 2.5. ([2, Prop 1.6]). *Let T be in $\mathscr{L}(\mathscr{H})$, and suppose that $\sigma(T) = E_{\mathrm{lr}}(T)$, $0 \in \sigma(T)$, and that $\sigma(T)$ is connected. Then T is in the norm-closure of $\mathscr{N}$.*

LEMMA 2.6. *Let T be in $\mathscr{L}(\mathscr{H})$. Then $\lambda \in E_{\mathrm{l}}(T)$ if and only if*

$$\inf_{\substack{\mathscr{M} \subseteq \mathscr{H} \\ \mathrm{codim}\,(\mathscr{M}) < \infty}} \sup_{\substack{x \in \mathscr{M} \\ \|x\| = 1}} \|(T - \lambda)x\| = 0.$$

Proof. It is a direct consequence of [7, Theorem 1.1].

LEMMA 2.7. *Let T be in $\mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$. If $\sigma(T) \subseteq (B(T))_\varepsilon$, then there exists $J_\varepsilon \in \mathscr{K}$ with $\|J_\varepsilon\| \leqslant \varepsilon$, such that $\sigma(T + J_\varepsilon) = \Omega(T)$.*

Proof. It readily follows from [1, Theorem 4.5].

Some variations of the next result have already appeared in the literature (see the proof of [4, Prop. 1.4] and [5, Theorem 3.5]). However, since the following lemma is central to our purpose, and we have not found a reference to it in its explicit form, we present its proof here.

LEMMA 2.8. *Let T be in $\mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$. Then there exists $K_\varepsilon \in \mathscr{K}$, $\|K_\varepsilon\| \leqslant \varepsilon$ and a unitary transformation $U_\varepsilon : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ such that*

$$U_\varepsilon(T - K_\varepsilon)U_\varepsilon^* = \begin{bmatrix} M & A & B \\ 0 & C & D \\ 0 & 0 & N \end{bmatrix}$$

where $A, B, C, D \in \mathscr{L}(\mathscr{H})$, M and N are diagonalizable normal operators in $\mathscr{L}(\mathscr{H})$ of uniform infinite multiplicity (i.e. *they have systems of eigenvectors which constitute orthonormal bases for $\mathscr{H}$ and each of their eigenvalues has infinite multiplicity) and such that*

$$\sigma(M)(= E(M)) = E_1(T),$$

$$\sigma(N)(= E(N)) = E_r(T),$$

$$E_1(C) = E_1(T),$$

$$E_r(C) = E_r(T),$$

$$\Omega(C) = \Omega(T),$$

and

$$\operatorname{ind}(T - \lambda) = \operatorname{ind}(C - \lambda) \qquad \textit{for every } \lambda \notin E_{1r}(T).$$

Moreover, if $\sigma(T) \subseteq (B(T))_\varepsilon$ we can further require that $\sigma(C) = \Omega(C)$.

Proof. Let $\{\lambda_n\} \subseteq E_1(T)$; $\{\mu_n\} \subseteq E_r(T)$ be dense sequences in $E_1(T)$ and $E_r(T)$, respectively satisfying

$$\operatorname{card}\{\lambda_k | \lambda_k = \lambda_n\} = \operatorname{card}\{\mu_k | \mu_k = \mu_n\} = \aleph_0, \quad n = 1, 2, \ldots .$$

From Lemma 2.6 we can define inductively an orthonormal sequence x_n in $\mathscr{H}$ such that

$$\langle x_{2n-1}, (T^* - \mu_k)x_{2k}\rangle = \langle x_{2n}, (T - \lambda_k)x_{2k-1}\rangle = 0\ , \quad k, n = 1, 2, \ldots$$

and

$$\|(T - \lambda_n)x_{2n-1}\| < \frac{\varepsilon}{2^{2n-1}},$$

$$\|(T^* - \mu_n)x_{2n}\| < \frac{\varepsilon}{2^{2n}} .$$

By deleting a convenient subsequence of $\{x_n\}$ (if necessary), we may assume that $[\text{span } x_n]^\perp$ is infinite dimensional. Now, if we define K_ε in $\mathscr{L}(\mathscr{H})$ by

$$K_\varepsilon(x) = \sum_{n=1}^{\infty} \langle x, x_{2n-1}\rangle (T - \lambda_n)x_{2n-1} + \sum_{n=1}^{\infty} \langle x, x_{2n}\rangle (T - \mu_n)x_{2n},$$

$x \in \mathscr{H}$, it readily follows that $K_\varepsilon \in \mathscr{K}$ and $\|K_\varepsilon\| < \varepsilon$. Defining

$$\mathscr{H}_1 = \text{span } x_{2n-1}, \ \mathscr{H}_3 = \text{span } x_{2n}$$

and

$$\mathscr{H}_2 = \mathscr{H} \ominus (\mathscr{H}_1 \oplus \mathscr{H}_3)$$

we see that there exists a unitary transformation

$$U_\varepsilon : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$$

taking $\mathscr{H}_1$ onto $\mathscr{H} \oplus \{0\} \oplus \{0\}$, $\mathscr{H}_2$ onto $\{0\} \oplus \mathscr{H} \oplus \{0\}$ and $\mathscr{H}_3$ onto $\{0\} \oplus \oplus \{0\} \oplus \mathscr{H}$ and such that

$$U_\varepsilon(T - K_\varepsilon)U_\varepsilon^* = \begin{bmatrix} M & A & B \\ 0 & C & D \\ 0 & 0 & N \end{bmatrix},$$

where M and N are diagonalizable normal operators in $\mathscr{L}(\mathscr{H})$ with uniform infinite multiplicity such that $\sigma(M) = E_l(T)$ and $\sigma(N) = E_r(T)$. A short matricial calculation shows that $E_l(C) \subseteq E_l(T)$ and $E_r(C) \subseteq E_r(T)$. Moreover, since M and N are unitarily equivalent to $M \oplus M$ and $N \oplus N$ respectively, we can further assume (changing the matrix representation of $U_\varepsilon(T - K_\varepsilon)U_\varepsilon^*$ according to the above identification), that $E_l(C) = \sigma(M)$ and $E_r(C) = \sigma(N)$. Now, an easy exercise from Fredholm theory shows that

$$\Omega(C) \setminus E_{lr}(C) = \Omega(T) \setminus E_{lr}(T),$$

$$\text{ind}(T - \lambda) = \text{ind}(C - \lambda), \quad \lambda \notin E_{lr}(T).$$

The proof of the first part of the lemma is complete. In order to prove the last assertion we first choose J_ε in $\mathscr{K}$, $\|J_\varepsilon\| \leqslant \varepsilon$ such that

$$\sigma(T + J_\varepsilon) = \Omega(T + J_\varepsilon) = \Omega(T)$$

and we apply the first part of the proof to the operator $T + J_\varepsilon$. Consequently we obtain K_ε in $\mathscr{K}$, $\|K_\varepsilon\| \leqslant 2\varepsilon$ and a unitary operator

$$U_\varepsilon : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$$

satisfying the first statement of the lemma, but now $\sigma(C) \subset (\Omega(T))_\varepsilon$. Another application of Lemma 2.7 completes the proof of the lemma.

Proof of Theorem 2.3. We begin by proving (a). Let $\varepsilon > 0$ be given and let $\Lambda^\varepsilon = \sigma(T)\setminus(B(T))_\varepsilon$. If $\Lambda^\varepsilon \neq \emptyset$, then it is finite and the spectral idempotent P^ε associated with Λ^ε is finite dimensional. Let $\mathscr{L} = \text{Range } P^\varepsilon$ and $\mathscr{M} = \text{null } P^\varepsilon$. Then $\mathscr{H}$ is the (not necessarily orthogonal) direct sum of $\mathscr{L}$ and $\mathscr{M}$, $T\mathscr{L} \subseteq \mathscr{L}$, $T\mathscr{M} \subseteq \mathscr{M}$ and

$$\sigma(T|\mathscr{L}) = \Lambda^\varepsilon,$$

$$\sigma(T|\mathscr{M}) \subseteq (B(T))_\varepsilon = (B(T|\mathscr{M}))_\varepsilon,$$

$$E_l(T|\mathscr{M}) = E_l(T), \quad E_r(T|\mathscr{M}) = E_r(T),$$

and

$$\Omega(T|\mathscr{M}) = \Omega(T).$$

From Lemma 2.8, there exists a compact operator K_ε on $\mathscr{M}$, with $\|K_\varepsilon\| \leqslant \varepsilon$ and a unitary transformation

$$U_\varepsilon : \mathscr{M} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$$

such that

$$U_\varepsilon(T|\mathscr{M} - K_\varepsilon)U_\varepsilon^* = \begin{bmatrix} M & A & B \\ 0 & C & D \\ 0 & 0 & N \end{bmatrix},$$

where A, B, C, D, M and N are as in Lemma 2.8. We observe that

$$\Omega(C)\setminus E_{lr}(C) = \Omega(T)\setminus E_{lr}(T)$$

and

$$E_{lr}(T) = E_{lr}(C).$$

Therefore,

$$\max\{m_t(\Omega(T)\setminus E_{lr}(T)), m_d(\Omega(T))\} = \max\{m_t(\Omega(C)\setminus E_{lr}(C)), m_d(\Omega(C))\}.$$

Let M' and N' be diagonalizable normal operators in $\mathscr{L}(\mathscr{H})$ of uniform infinite multiplicity such that

$$\sigma(M') = (\sigma(M))_{\delta+\varepsilon},$$

$$\sigma(N') = (\sigma(N))_{\delta+\varepsilon},$$

$$\|M' - M\| \leqslant \delta + 2\varepsilon,$$

$$\|N' - N\| \leqslant \delta + 2\varepsilon,$$

where

$$\delta = \max\{m_t(\Omega(T)\setminus E_{lr}(T)),\ m_d(\Omega(T)),\ \sup_{\lambda\in\{0\}\cup\sigma_0(T)} d(\lambda, \Omega(T))\}$$

(cf. the proof of [5, Thm 3.5]). Now, let $T_\varepsilon \in \mathcal{L}(\mathcal{H})$ be such that $T_\varepsilon|\mathcal{M}^\perp = T|\mathcal{M}^\perp$, and define T_ε on $\mathcal{M}$ by

$$U_\varepsilon(T_\varepsilon|\mathcal{M})\,U_\varepsilon^* = \begin{bmatrix} M' & A & B \\ 0 & C & D \\ 0 & 0 & N' \end{bmatrix}.$$

It readily follows that

$$E_{lr}(T_\varepsilon) = \sigma(T_\varepsilon) = \sigma(M') \cup \sigma(N'),$$

$0 \in \sigma(T_\varepsilon)$ and $\sigma(T_\varepsilon)$ is connected. From Lemma 2.5, $T_\varepsilon \in \overline{\mathcal{N}}$. Then

$$d(T, \mathcal{N}) \leqslant \|T - T_\varepsilon\| = \|(T - T_\varepsilon)|\mathcal{M}\| \leqslant \varepsilon + \|(T - T_\varepsilon)|\mathcal{M} - K_\varepsilon\| \leqslant \varepsilon +$$

$$+ \max\{\|M' - M\|, \|N' - N\|\} \leqslant 3\varepsilon + \delta.$$

Since ε is arbitrary the proof of the first part of the theorem is complete. For the proof of (b) we let $K \in \mathcal{K}$ such that $\sigma(T + K) = \Omega(T)$ ([21, §3]), and then we apply part (a) to the operator $T + K$.

REMARK 2.9. From [13] it follows that the set $\mathcal{N}_e$ of nilpotent elements in $\mathcal{Q}(\mathcal{H})$ coincide with $\pi\mathcal{N}$. Therefore, $d(T, \mathcal{N} + \mathcal{K}) = d(\pi T, \pi\mathcal{N}) = d(\pi T, \mathcal{N}_e)$.

3. A SPECTRAL FORMULA FOR $d(T, \mathcal{N} + \mathcal{K})$ IN THE CASE OF ESSENTIALLY (G_1) OPERATORS

An operator T in $\mathcal{L}(\mathcal{H})$ is called essentially (G_1) ([12]) if $\|\pi(T - \lambda)^{-1}\| = \dfrac{1}{d(\lambda, E(T))}$ for every $\lambda \notin E(T)$ (see also [18 ,§3]).

THEOREM 3.1. *Let T be in $\mathcal{L}(\mathcal{H})$ and assume that T is essentially (G_1) and that $E_{lr}(T) = E(T)$. Then,* $d(T, \mathcal{N} + \mathcal{K}) = \max\{m_t(\Omega(T)\setminus E(T)), m_d(\Omega(T)), d(0, \Omega(T))\}$.

Proof. In view of Theorem 2.3 and Remark 2.4 we need only show that

$$\|\pi(T - Q)\| \geqslant \max\{m_t(\Omega(T)\setminus E(T)), m_d(E(T)), d(0, E(T))\}$$

for every $Q \in \mathcal{N}$. Arguing by contradiction, assume that there exists $Q \in \mathcal{N}$ such that

$$\|\pi(T - Q)\| < m_t(\Omega(T)\setminus E(T)).$$

Then, we can find $\mu \in \Omega(T) \setminus E(T)$ such that

$$d(\mu, E(T)) > \| \pi(T - Q) \|.$$

It follows that

$$\|\pi(T - \mu)^{-1}\pi(T - Q)\| \leqslant \frac{\|\pi(T - Q)\|}{d(\mu, E(T))} < 1,$$

and hence $1 + \alpha[\pi(T - \mu)]^{-1}\pi(Q - T)$ is invertible in $\mathscr{Q}(\mathscr{H})$, $0 \leqslant \alpha \leqslant 1$ and lies in the connected component of invertible elements of $\mathscr{Q}(\mathscr{H})$ containing the identity. Therefore, ind $((T - \mu) + \alpha(Q - T))$ is a constant, $0 \leqslant \alpha \leqslant 1$. This is a contradiction because ind $(T - \mu) \neq 0$ while $Q - \mu$ is either not semi-Fredholm or ind $(Q - \mu) = 0$. Next, assuming $0 \notin E(T)$ we prove that $d(0, E(T)) \leqslant \|\pi(T - Q)\|$, for every $Q \in \mathscr{N}$. To this end let $Q \in \mathscr{N}$, and using the fact that $0 \in E_1(Q)$, we can find an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that $\lim_{n \to \infty} \|Qx_n\| = 0$. Since T is essentially (G_1) we deduce that $d(0, E(T)) = \|\pi(T)^{-1}\|^{-1}$. On the other hand,

$$\|\pi(T)^{-1}\|^{-1} = \|[\pi(T^*T)^{\frac{1}{2}}]^{-1}\|^{-1} = \inf \varphi(\pi(T^*T)^{\frac{1}{2}})$$

where the infimum is taken over all the states φ of $\mathscr{Q}(\mathscr{H})$. Let ψ be the state on $\mathscr{L}(\mathscr{H})$ defined by

$$\psi(A) = \text{gen}\cdot\lim \langle Ax_n, x_n \rangle, \ A \in \mathscr{L}(\mathscr{H}).$$

Then $\mathscr{K} \subseteq \operatorname{Ker} \psi$ and ψ induces a state ρ on $\mathscr{Q}(\mathscr{H})$. It follows that for every $K \in \mathscr{K}$,

$$d(0, E(T)) = \|\pi(T)^{-1}\|^{-1} \leqslant \rho(\pi((T^*T)^{\frac{1}{2}})) = \psi((T^*T)^{\frac{1}{2}}) \leqslant$$

$$\leqslant \limsup_{n \to \infty} < (T^*T)^{\frac{1}{2}}(x_n), \ x_n > \leqslant \limsup_{n \to \infty} \| (T - K)(x_n) \| \leqslant$$

$$\leqslant \limsup_{n \to \infty} \|(T - K - Q)x_n\| + \lim_{n \to \infty} \|Q(x_n)\| \leqslant \|T - K - Q\|,$$

and hence

$$d(0, E(T)) \leqslant \|\pi(T - Q)\|.$$

Now assume that there exists $Q \in \mathscr{N}$ such that $\|\pi(T - Q)\| < m_d(E(T))$. In particular $E(T)$ is disconnected and can be written as $E(T) = E_1 \cup E_2$, where $E_1 \cap E_2 = \emptyset$, E_j is a nonempty compact set, $j = 1, 2$, and

$$\inf_{\substack{\lambda \in E_1 \\ \mu \in E_2}} |\lambda - \mu| = 2m_d(E(T)).$$

Let Γ_j be suitable contours so that

$$E_j \subseteq \operatorname{int}\Gamma_j, \quad \operatorname{int}\Gamma_1 \cap \operatorname{int}\Gamma_2 = \emptyset,$$

$$d(\lambda, E_j) = d(\lambda, E(T)) \quad \text{for every } \lambda \in \Gamma_j,$$

and

$$\inf_{\lambda \in \Gamma_j} d(\lambda, E_j) > \|\pi(T - Q)\|,$$

$j = 1, 2$ (see Remark 2.2(e)). Then, for every $\lambda \in \Gamma_j$

$$\|\pi(T-\lambda)^{-1}\pi(Q-T)\| \leqslant \|\pi(T-\lambda)^{-1}\| \; \|\pi(Q-T)\| = \|\pi(Q-T)\|/d(\lambda, E(T)) =$$

$$= \|\pi(Q-T)\|/d(\lambda, E_j) \leqslant \|\pi(Q-T)\|/\inf_{\lambda\in\Gamma_j} d(\lambda, E_j) < 1.$$

Hence for every $0 \leqslant \alpha \leqslant 1$, and $\lambda \in \Gamma_j$,

$$\pi(T-\lambda) + \alpha\pi(Q-T) = \pi(T-\lambda)\,(1 + \alpha\pi(T-\lambda)^{-1}\pi(Q-T))$$

is invertible. That is $\Gamma_j \subset \mathbf{C}\setminus E(T + \alpha(Q-T))$, $0 \leqslant \alpha \leqslant 1$. Now we define the idempotents

$$p_j(\alpha) = \frac{1}{2\pi i}\int_{\Gamma_j} (\pi[T + \alpha(Q-T)] - \lambda)^{-1}d\lambda, \quad 0 \leqslant \alpha \leqslant 1, \; j = 1, 2.$$

Since $p_j(0) \neq 0$, and $p_j(\alpha)$ is continuous for $0 \leqslant \alpha \leqslant 1$ we deduce $p_j(1) \neq 0, j = 1, 2$. This contradicts the fact that $E(Q)\,(=\{0\})$ is connected.

REMARK 3.2. (a) If T in $\mathscr{L}(\mathscr{H})$ is essentially normal (i.e. $\pi(T^*T) = \pi(TT^*)$) then it readily follows that T is essentially (G_1) and $E_{lr}(T) = E(T)$. Therefore, Theorem 3.1 applies and one may explicitly calculate $d(T, \mathscr{N} + \mathscr{K})$.

(b) If T is essentially (G_1), $E_{lr}(T) = E(T)$, and T is a (G_1) operator, i.e. $\|(T-\lambda)^{-1}\| = \dfrac{1}{d(\lambda, \sigma(T))}$ for every $\lambda \notin \sigma(T)$, then, an argument similar to that given in the proof of Theorem 3.1 shows that

$$d(T, \mathscr{N}) \geqslant \max\{m_t(\Omega(T)\setminus E(T), m_d(\Omega(T)), d(0, \Omega(T)), m_d(\sigma(T))\}.$$

(Notice that if T is essentially (G_1), then there exists a compact perturbation of T which is (G_1) ([10])). On the other hand, it will follow from the next two results that if T is a (G_1) operator then

$$d(T, \mathscr{N}) \leqslant \max\left\{m_t(\Omega(T)\setminus E_{lr}(T)),\; m_d(\Omega(T)),\; d(0, \Omega(T)), \frac{1}{2}\sup_{\lambda\in\sigma_0(T)} d(\lambda, \Omega(T))\right\}.$$

(cf. Remark 2.4). Moreover, it follows from the present Remark and Theorem 3.1 that if T is (G_1) and essentially (G_1), $E_{1r}(T) = E(T)$, and $B(T) = \sigma(T)$, then

$$\mathrm{d}(T, \mathcal{N}) = \max\{\mathrm{m_t}(\Omega(T)\setminus E(T)), \mathrm{m_d}(\Omega(T)), \mathrm{d}(0, \Omega(T))\}.$$

The idea of the proof of the second inequality in the following lemma is based on [9], and was suggested to us by D. Herrero.

LEMMA 3.3. *Let S_n be the forward finite shift acting in the usual fashion on* $\mathbf{C}^n$, i.e. *if $e_1, e_2, \ldots, e_n$ are the vectors in the natural basis of $\mathbf{C}^n$, then*

$$S_n(e_j) = e_{j+1},\ 1 \leqslant j \leqslant n-1,\ S_n(e_n) = 0.$$

Also, let P_n be the projection from $\mathbf{C}^n$ onto the span of the vectors $f_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j$.

Then,

$$\frac{1}{2} \leqslant \left\| P_n - \frac{1}{2} S_n \right\| \leqslant \frac{1}{2} + \frac{1+\sqrt{n-1}}{2n}.$$

Proof. The proof of the first inequality follows the same pattern of the proof of Theorem 3.1 (see Remark 3.2). To prove the second inequality, we first notice that $P_nS_nP_n - 2P_n = -\left(1 + \frac{1}{n}\right)P_n$. Indeed, let $f \in \mathbf{C}^n$ and write $f = \alpha f_n + g_n$, where $\alpha \in \mathbf{C}$ and $g_n \in \{f_n\}^{\perp}$. Then,

$$P_nS_nP_nf = P_nS_n\alpha f_n = \alpha P_n \frac{1}{\sqrt{n}} \sum_{j=2}^{n} e_j = \alpha P_n\left(f_n - \frac{e_1}{\sqrt{n}}\right) =$$

$$= \alpha f_n - \frac{\alpha}{\sqrt{n}} \langle e_1, f_n \rangle f_n = \alpha\left(1 - \frac{1}{n}\right) f_n = \left(1 - \frac{1}{n}\right) P_nf.$$

Now, since Range P_n is one dimensional, it follows that

$$\|(1-P_n)S_nP_n\|^2 = \|(1-P_n)S_nf_n\|^2 = \|S_nf_n\|^2 - \|P_nS_nf_n\|^2 =$$

$$= \left\| \frac{1}{\sqrt{n}} \sum_{j=2}^{n} e_j \right\|^2 - \|P_nS_nP_n\|^2 = \frac{(n-1)}{n} - \left(1 - \frac{1}{n}\right)^2 = \frac{n-1}{n^2}.$$

Analogously

$$\|P_nS_n(1-P_n)\|^2 = \|(1-P_n)S_n^*P_n\|^2 =$$

$$= \|(1-P_n)S_n^*f_n\|^2 = \|S_n^*f_n\|^2 - \|P_nS_n^*f_n\|^2 =$$

$$= \left\| \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} e_j \right\|^2 - \|P_nS_n^*P_n\|^2 =$$

$$= \frac{n-1}{n} - \left(1 - \frac{1}{n}\right)^2 = \frac{n-1}{n^2}.$$

We conclude that

$$\|S_n - 2P_n\| \leqslant \|P_n(S_n - 2P_n)P_n + (1 - P_n)S_n(1 - P_n)\| +$$

$$+ \|(1 - P_n)S_nP_n + P_nS_n(1 - P_n)\| \leqslant$$

$$\leqslant \max\{\|P_nS_nP_n - 2P_n\|, \|(1 - P_n)S_n(1 - P_n)\|\} +$$

$$+ \max\{\|(1 - P_n)S_nP_n\|, \|P_nS_n(1 - P_n)\|\} =$$

$$= 1 + \frac{1}{n} + \frac{\sqrt{n-1}}{n}.$$

This completes the proof of the lemma.

THEOREM 3.4. *Let T be in $\mathscr{L}(\mathscr{H})$, and suppose that every eingenvalue λ in $\sigma_0(T)$ satisfies the condition that the spectral idempotent associated with $\{\lambda\}$ is selfadjoint (or, equivalently,* $\mathrm{null}\,(T - \lambda) = \mathrm{null}\,(T^* - \bar{\lambda})$). *Then,*

$$d(T, \mathscr{N}) \leqslant \max\{m_t(\Omega(T) \setminus E_{lr}(T)), m_d(\Omega(T)), d(0, \Omega(T)), \frac{1}{2} \sup_{\lambda \in \sigma_0(T)} d(\lambda, \Omega(T))\}.$$

Proof. Under the assumption of the theorem, T can be written as $T = T_1 \oplus T_2$, where T_2 is a diagonalizable normal operator all of whose eigenvalues are finite dimensional, and they cluster (if there are infinitly many) on the boundary of $B(T)$, $\sigma_0(T) = \sigma_0(T_2)$, and $\sigma(T_1) = B(T_1) = B(T)$. Since

$$d(T, \mathscr{N}) \leqslant \max\{d(T_1, \mathscr{N}), d(T_2, \mathscr{N})\},$$

from the proof of Theorem 2.3 we see that it suffices to show the present theorem in the case that T is a diagonalizable normal operator. Moreover, perusual of the proof of Theorem 2.3 shows that we can further assume that $E(T)$ is connected, $0 \in E(T)$ and $\sigma(T) \setminus E(T)$ is finite say $\sigma(T) \setminus E(T) = \{\lambda_1, \ldots, \lambda_m\}$. (Notice that in this case $E_{lr}(T) = \Omega(T)$). Let $\mu_j \in \partial(E(T))$ such that

$$d(\lambda_j, E(T)) = |\lambda_j - \mu_j|, \quad 1 \leqslant j \leqslant m.$$

Then, up to a unitary equivalence and a small norm compact perturbation (see for instance [9, Lemma 4], [18, §4], [21, §2]) we know that we can write T as

$$S \oplus \left(\bigoplus_{j=1}^{m} (\mu_j + (\lambda_j - \mu_j)Q_j)\right),$$

where S is a diagonalizable normal operator in $\mathscr{L}(\mathscr{H})$ such that $E(S) = \sigma(S) = E(T)$ and Q_j is a finite dimensional projection in $\mathscr{L}(\mathscr{H})$, $1 \leqslant j \leqslant m$. Given $\varepsilon > 0$ we see from Lemma 3.3 that there exists $N_j \in \mathscr{N}$

such that $\|Q_j - N_j\| < \varepsilon + \frac{1}{2}$. On the other hand, we deduce from Lemma 2.5 that the operator

$$S \oplus \left(\bigoplus_{j=1}^{m} (\mu_j + (\lambda_j - \mu_j)N_j) \right)$$

is in $\overline{\mathscr{N}}$. Thus,

$$\mathrm{d}(T, \mathscr{N}) \leqslant \left\| \left(S \oplus \left(\bigoplus_{j=1}^{m} (\mu_j + (\lambda_j - \mu_j)Q_j) \right) \right) - \left(S \oplus \left(\bigoplus_{j=1}^{m} (\mu_j + (\lambda_j - \mu_j)N_j) \right) \right) \right\| \leqslant$$

$$\leqslant \max_{1 \leqslant j \leqslant m} \{ |\lambda_j - \mu_j| \, \|Q_j - N_j\| \} \leqslant \left(\varepsilon + \frac{1}{2} \right) \sup_{\lambda \in \sigma_0(T)} \mathrm{d}(\lambda, E(T)).$$

Since ε is arbitrary the proof of the theorem is complete.

REMARK 3.5. Let $\mathscr{G}$ be the set of all invertible operators in $\mathscr{L}(\mathscr{H})$. Since the right hand sides of the inequalities in Theorem 2.3(a), (b) are invariant under similarity it follows that

$$\sup_{S \in \mathscr{G}} \mathrm{d}(STS^{-1}, \mathscr{N}) \leqslant \max \{ \mathrm{m_t}(\Omega(T) \setminus E_{1r}(T)), \mathrm{m_d}(\Omega(T)), \sup_{\lambda \in \{0\} \cup \sigma_0(T)} \mathrm{d}(\lambda, \Omega(T)) \},$$

$$\sup_{S \in \mathscr{G}} \mathrm{d}(STS^{-1}, \mathscr{N} + \mathscr{K}) \leqslant \max \{ \mathrm{m_t}(\Omega(T) \setminus E_{1r}(T)), \mathrm{m_d}(\Omega(T)), \mathrm{d}(0, \Omega(T)) \}.$$

We shall see, in the remainder of this section, that the last inequality becomes an equality when T satisfies the condition $E(T) = E_{1r}(T)$.

We recall that an operator T in $\mathscr{L}(\mathscr{H})$ is called quasitriangular if there exists an increasing sequence $\{P_n\} \subset \mathscr{L}(\mathscr{H})$ of finite rank projections, tending strongly to the identity, and such that $\lim_{n \to \infty} \|(1 - P_n)T P_n\| = 0$. T is called bi-quasitriangular if both T and T^* are quasitriangular. The set of all bi-quasitriangular operators in $\mathscr{L}(\mathscr{H})$ will be denoted by BQT.

THEOREM 3.6. *Let T be in $\mathscr{L}(\mathscr{H})$. Then*

$$\mathrm{m_t}(\Omega(T) \setminus E(T)) \leqslant \sup_{S \in \mathscr{G}} \mathrm{d}(STS^{-1}, \mathrm{BQT}) \leqslant \mathrm{m_t}(\Omega(T) \setminus E_{1r}(T)).$$

Proof. Since the proof follows the same pattern of the proofs of Theorem 4.3 and Corollary 4.4 of [3], the details will be omitted.

THEOREM 3.7. *Let T be in $\mathscr{L}(\mathscr{H})$. Then*

$$\sup_{S \in \mathscr{G}} \mathrm{d}(STS^{-1}, \mathscr{N} + \mathscr{K}) \geqslant \max \{ \mathrm{m_t}(\Omega(T) \setminus E(T)), \mathrm{m_d}(\Omega(T)), \mathrm{d}(0, \Omega(T)) \}.$$

Proof. We first observe that, since $\mathcal{N} + \mathcal{K} \subseteq$ BQT [3], Theorem 3.6 implies

$$\sup_{S \in \mathcal{G}} \mathrm{d}(STS^{-1}, \mathcal{N} + \mathcal{K}) \geqslant \mathrm{m}_{\mathrm{t}}(\Omega(T) \setminus E(T)).$$

On the other hand, by [21, §3] there exists a T' in $\mathcal{L}(\mathcal{H})$ such that

$$T - T' \in \mathcal{K}, \text{ and } \sigma(T') = \Omega(T') = \Omega(T).$$

Furthermore, from [11, Thm. 3] we deduce that there exists a normal operator A in the norm closure of $\{ST'S^{-1} | S \in \mathcal{G}\}$ such that $E(A) = \Omega(T')$. Therefore, from Theorem 3.1 we deduce that

$$\sup_{S \in \mathcal{G}} \mathrm{d}(STS^{-1}, \mathcal{N} + \mathcal{K}) = \sup_{S \in \mathcal{G}} \mathrm{d}(ST'S^{-1}, \mathcal{N} + \mathcal{K}) \geqslant \mathrm{d}(A, \mathcal{N} + \mathcal{K}) =$$

$$= \max\{\mathrm{m}_{\mathrm{d}}(E(A)), \mathrm{d}(0, E(A))\} = \max\{\mathrm{m}_{\mathrm{d}}(\Omega(T)), \mathrm{d}(0, \Omega(T))\}.$$

The proof of the theorem is complete.

COROLLARY 3.8. *Let T in $\mathcal{L}(\mathcal{H})$ be such that $E_{\mathrm{l}}(T) = E_{\mathrm{r}}(T)$. Then*

$$\sup_{S \in \mathcal{G}} \mathrm{d}(STS^{-1}, \mathcal{N} + \mathcal{K}) = \max\{\mathrm{m}_{\mathrm{t}}(\Omega(T) \setminus E(T)), \mathrm{m}_{\mathrm{d}}(\Omega(T)), \mathrm{d}(0, \Omega(T))\}.$$

Proof. It is an immediate consequence of Remark 3.5 and Theorem 3.7.

4. OPEN QUESTIONS AND CONCLUDING REMARKS

In view of Theorem 2.3 and Remark 3.2 it is natural to expect at least in the case when $E_{\mathrm{l}}(T) = E_{\mathrm{r}}(T)$ that

$$\sup_{S \in \mathcal{G}} \mathrm{d}(STS^{-1}, \mathcal{N}) = \max\{\mathrm{m}_{\mathrm{t}}(\Omega(T) \setminus E(T)), \mathrm{m}_{\mathrm{d}}(\Omega(T)), \mathrm{d}(0, \Omega(T)), \mathrm{m}_{\mathrm{d}}(\sigma(T))\}.$$

This formula might be easier to prove if one can first prove it for the case of normal operators (the case in which $\sigma(T) = E(T)$ has already been taken care of in Remark 3.2).

Now we present an example of an essentially hyponormal operator S which is quasidiagonal (and hence it is essentially (G_1), and $E_{\mathrm{lr}}(S) = \Omega(S)$), but for which Theorem 3.1 does not give much information on the precise value of $\mathrm{d}(T, \mathcal{N} + \mathcal{K})$. Let A_n be the n by n Hermitian matrix all of whose diagonal terms are zero and whose $(j, k)^{\text{th}}$ entry is $i/(j-k), j \neq k$, and let B_n be the diagonal n by n matrix whose j^{th} diagonal term is j/n. It follows that $\|B_n\| = 1$ while $\|A_n\| \leqslant \pi$. Indeed, in order to prove the last assertion we simply observe that A_n is the n^{th} section of the Toeplitz matrix A all of whose diagonal terms are zero and whose $(j, k)^{\text{th}}$ entry is $i/(j-k)$, $j \neq k$, $j, k = 1, 2, \ldots$. To show that $\|A\| = \pi$ one argues as follows: Let φ in $L^\infty(S^1)$ be given by

$$\varphi(\lambda) = \arg(\lambda) \quad \text{for } |\arg(\lambda)| < \pi,$$

$$\varphi(\lambda) = 0 \quad \text{for } |\arg(\lambda)| = \pi, \ \lambda \in S^1.$$

Then the Fourier coefficients of φ are given by

$$\hat{\varphi}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \, \mathrm{e}^{-\mathrm{i}n\theta} \, \mathrm{d}\theta = \frac{\mathrm{i}(-1)^n}{n} \quad \text{for } n \neq 0 \text{ and } \hat{\varphi}_0 = 0.$$

If T_φ is the Toeplitz operator associated with φ on $H^2(S^1)$, then $\sigma(T_\varphi) = [-\pi, \pi]$ and hence $\|T_\varphi\| = \pi$ (see [6]). Since T_φ is easily seen to be unitarily equivalent to the Toeplitz matrix A our claim is established. Let $C_n = A_n + \mathrm{i}\, B_n$, $n = 2, 3, \ldots$. Then a straightforward matricial calculation shows that $C_n^* C_n - C_n C_n^*$ $(=2\mathrm{i}(A_n B_n - B_n A_n))$ is twice the n by n matrix whose diagonal terms are zero and all of whose nondiagonal terms are $\frac{1}{n}\cdot$ Hence, $C_n^* C_n - C_n C_n^* = 2\left(P_n - \frac{1}{n}\right)$, where P_n is a one dimensional projection. In particular, the eigenvalues of $C_n^* C_n - C_n C_n^*$ are $2\left(1 - \frac{1}{n}\right)$ (with multiplicity 1), and $-\frac{2}{n}$ (with multiplicity $n-1$). It follows that the operator $S = \bigoplus_{n=2}^{\infty} C_n$ is quasidiagonal and essentially hyponormal (i.e. $\pi(S^* S - SS^*) \geqslant 0$). From [15] page 46 we deduce that

$$E(S) \subseteq [-\pi, \pi] \times [0, 1] \quad \text{and hence} \quad \|\pi(S)\| \leqslant \sqrt{1 + \pi^2}.$$

We suspect that $S \in (\mathcal{N} + \mathcal{K})^-$. The above operator S serves as an interesting testing example for the following question:

Is a quasidiagonal essentially hyponormal operator a compact perturbation of a (quasidiagonal) hyponormal operator? If such a hyponormal operator exists in the case of our operator S, it cannot be block diagonal, because block diagonal hyponormal operators are normal, and the operator S is not essentially normal. The first example of a quasidiagonal hyponormal operator which is not essentially normal was given in [14].

We point out that there is another question (whose answer is unknown) connected with the above lifting problem, namely:

Is the set of compact perturbations of quasidiagonal hyponormal operators norm closed in $\mathcal{L}(\mathcal{H})$? Observe that the set of essentially hyponormal operators is norm closed, as is the set of quasidiagonal operators ([8]). Therefore, if the lifting problem for quasidiagonal essentially hyponormal operators is answered in the affirmative, this last question also has an affirmative answer. As far as we know, the question of whether or not the set of compact perturbations of hyponormal operators is norm closed ([20]) remains unanswered, and solving the above lifting problem may be an important step towards the solution of this question.

Added in proof. Employing reasonings analogous to the one given in the proof of Theorem 3.4 of the present paper and in the proof of Proposition 6.6 of the recent paper "Quasidiagonality, similarity and approximation by nilpotent operators" by D. Herrero to be published in the *Indiana University Mathematics Journal* one can easily show that the conclusion of Theorem 3.4 holds for every operator T on $\mathscr{H}$.

REFERENCES

1. Apostol, C., The correction by compact perturbations of the singular behaviour of operators, *Rev. Roumaine Math. Pures Appl.*, **21** (1976), 155–175.
2. Apostol, C.; Foiaş, C.; Voiculescu, D., On the norm closure of nilpotents II, *Rev. Roumaine Math. Pures Appl.*, **19** (1974), 549–577.
3. Apostol, C.; Foiaş, C.; Voiculescu, D., Some results on non-quasitriangular operators VI, *Rev. Roumaine Math. Pures Appl.*, **18** (1973), 1473–1494.
4. Apostol, C.; Morrel, B., On uniform approximation of operators by simple models, *Indiana Univ. Math. J.*, **26** (1977), 427–442.
5. Apostol, C.; Salinas, N., Nilpotent approximation and quasinilpotent operators, *Pacific J. Math.*, **61** (1975), 327–337.
6. Douglas, R. G., *Banach algebra techniques in the theory of Toeplitz operators*, Lectures given at the CBMS Regional Conference at the University of Georgia, 1972.
7. Fillmore, P. A.; Stampfli, J. G.; Williams, J. P., On the essential numerical range, the essential spectrum, and a problem of Halmos, *Acta Sci. Math. (Szeged)*, **33** (1972), 179–192.
8. Halmos, P. R., Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887–933.
9. Halmos, P. R., Limits of shifts, *Acta. Sci. Math. (Szeged)*, **34** (1973), 131–139.
10. Herrero, D. A., Lifting essentially (G_1) operators, preprint.
11. Herrero, D. A., Closure of similarity orbits of Hilbert space operators III, *Math.Ann.*, **232** (1978), 195–204.
12. Luecke, G. R., Essentially (G_1) operators and essentially convexoid operators on Hilbert space, *Illinois J. Math.*, **19** (1975), 389–399.
13. Olsen, C. L., A structure theorem for polynomially compact operators, *Amer. J. Math.*, **93**(1971), 686–698.
14. Pearcy, C.; Salinas, N., Operators with compact self-commutator, *Canad. J. Math.*, **26**(1974), 115–120.
15. Putnam, C. R., *Commutation properties of Hilbert space operators and related topics*, Ergebrisse der. math. und ihrer Grenzgebiete, Band 36, Springer-Verlag, New York, 1967.

16. SALINAS, N., Operators with essentially disconnected spectrum, *Acta. Sci. Math. (Szeged)*, **33**(1972), 193–205.

17. SALINAS, N., A characterization of the Browder spectrum, *Proc. Amer. Math. Soc.*, **38** (1973), 369–373.

18. SALINAS, N., Reducing essential eigenvalues, *Duke Math. J.*, **40** (1973), 561–580.

19. SALINAS, N., Subnormal limits of nilpotent operators, *Acta. Sci. Math. (Szeged)*, **37** (1975) 117–124.

20. STAMPFLI, J. G., On hyponormal and Toeplitz operators, *Math. Ann.*, **183** (1969), 328–336.

21. STAMPFLI, J. G., Compact perturbations, normal eigenvalues, and a problem of Salinas, *J. London Math. Soc.*, **2**:9 (1974), 165–175.

NORBERTO SALINAS
University of Kansas
Mathematics Department
Lawrence, Kansas 66045
U.S.A.

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