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SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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1. We shall consider Schrödinger Hamiltonians, $H = -\Delta + V$ defined on $\mathbf{R}^n$, where V is a complex valued singular potential. Several recent papers [2], [7] have considered questions of closure and maximal accretiveness for such operators. In this paper we shall also consider such questions and obtain results which include recent results of R. Jensen [6].

We shall use the method of semi-groups and in particular those semi-groups associated with a Wiener process. In this connection I learned much from a paper of R. Carmona [3]. But also see the papers of N. Portenko [8] and Berthier and Gaveau [1].

2. We shall suppose that $V = U + \mathrm{i}W$ belongs to $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and shall be concerned with the maximal realization of $H = -\Delta + V$ whose domain is given by

$$D(H) = \{u \in L^2 : Vu \in L^1_{\mathrm{loc}} \text{ and } Hu \in L^2\}.$$

If we take $U^+ = \max(U, 0)$, $U^- = \max(-U, 0)$, then we shall suppose that

$$\lim_{\alpha\to\infty} \operatorname*{esssup}_x \frac{1}{4\pi^{n/2}} \int_{\mathrm{R}^n} \frac{U^-(x-y)}{|y|^{n-2}} \left(\int_{\alpha|y|^2}^{\infty} \tau^{\frac{n}{2}-2} \mathrm{e}^{-\tau}\, \mathrm{d}\tau \right) \mathrm{d}y = \mu_0 < \infty. \tag{2.1}$$

We can now state the main result of this paper:

THEOREM. *If $\mu_0 < 1$, then there exists a real λ so that $H + \lambda$ has an m-accretive closure. If $\mu_0 < 1/2$ or if $W \in L^2_{\mathrm{loc}}$, then H is closed and $H + \lambda$ is m-accretive for some real λ.*

In [6] R. Jensen assumed that V is real, belongs to L^2_{loc} and that there exists an $r_0 > 0$ so that (for $n \geqslant 3$)

$$\sup_x \frac{\Gamma(n/2)}{2(n-2)\,\pi^{n/2}} \int_{|y|\leqslant r_0} \frac{V^-(x-y)}{|y|^{n-2}}\, \mathrm{d}y \leqslant \mu_0 < 1. \tag{2.2}$$

Under these conditions he showed that H is selfadjoint. Actually Jensen proved somewhat more. He assumed that $V = V_1 - V_2$, where $V_2 \geqslant 0$ satisfies (2.2) and

$$V_1(x) \geqslant -a|x|^2 - b, \quad a, b > 0. \tag{2.3}$$

However, this extension represents no essential difficulty once the theorem has been proved for $V_1 \geqslant 0$ (see [4], Theorem 2). It is not difficult to show that the class of functions which satisfy (2.1) with $\mu_0 < 1$, contains as a proper subclass the class of functions which satisfy (2.2).

The proof of our theorem will be constructed with the help of a number of lemmas. Toward this end let $w(t)$ be an n-dimensional normalized Brownian motion, i.e. a Wiener process. Instead of $w(t)$ we shall use the process $\xi(t) = \sqrt{2}w(t)$, since the differential generator of the latter process is Δ, whereas the differential generator of the Wiener process is $\Delta/2$.

LEMMA 1. *If $\mu_0 < 1$, then for every $t \in \mathbf{R}^+$,*

$$\operatorname*{esssup}_x E_x e^{\int_0^t U^-(\xi(s))\, ds} < \infty. \tag{2.4}$$

Proof. For the genesis of this proof see [1], [3], [8]. We shall expand $\exp \int_0^t U^-(\xi(s)) ds$ into a power series and get an estimate on the $(k+1)$-st term. For simplicity of notation let us drop the superscript on U^- and replace it by U, but keeping in mind that it is non-negative. We may write

$$E_x \left[\int_0^t U(\xi(s))\, ds \right]^k = k! \int_0^t \int_{s_1}^t \cdots \int_{s_{k-1}}^t E_x \prod_{j=1}^k U(\xi(s_j))\, ds_k \ldots ds_1. \tag{2.5}$$

For $0 < s_1 < s_2 < \ldots < s_k \leqslant t$ we have the standard formula

$$E_x \prod_{j=1}^k U(\xi(s_j)) = \int_{\mathbf{R}^{kn}} \prod_{j=1}^k \frac{U(y_j) \exp\left\{-\dfrac{|y_j - y_{j-1}|^2}{4(s_j - s_{j-1})}\right\}}{[4\pi(s_j - s_{j-1})]^{n/2}}\, dy_j \tag{2.6}$$

where $y_0 = x$, and dy_j is Lebesgue measure on $\mathbf{R}^n$.

Integrate both sides of (2.6) with respect to s_k over the interval $[s_{k-1}, t]$. After a suitable change of variables we have

$$\int_{s_{k-1}}^t \frac{\exp\left\{-\dfrac{|y_k - y_{k-1}|^2}{4(s_k - s_{k-1})}\right\}}{[4\pi(s_k - s_{k-1})]^{n/2}}\, ds_k = \frac{1}{4\pi^{n/2} |y_k - y_{k-1}|^{n-2}} \int_{\frac{|y_k - y_{k-1}|^2}{4(t - s_{k-1})}}^{\infty} \tau^{\frac{n}{2} - 2} e^{-\tau}\, d\tau. \tag{2.7}$$

Now multiply (2.7) by $U(y_k)$ and integrate over $\mathbf{R}^n$ with respect to dy_k. The resulting integral is

$$\frac{1}{4\pi^{n/2}}\int_{\mathbf{R}^n}\frac{U(y_{k-1}-y_k)}{|y_k|^{n-2}}\left(\int_{\frac{|y_k|^2}{4(t-s_{k-1})}}^{\infty}\tau^{\frac{n}{2}-2}e^{-\tau}\,d\tau\right)dy_k. \tag{2.8}$$

By hypothesis there exists a $t_0 > 0$ so that $0 < t \leqslant t_0$ implies that for almost all y_{k-1}, (2.8) is $\leqslant \mu_0 < 1$. If we repeat this process k times we see from (2.5) and (2.6) that

$$\operatorname*{esssup}_x E_x\left[\int_0^t U(\xi(s))\,ds\right]^k \leqslant k!\,\mu_0^k.$$

Thus

$$\operatorname*{esssup}_x E_x e^{\int_0^t U(\xi(s))\,ds} \leqslant \frac{1}{1-\mu_0} < \infty,$$

so that (2.4) is true for $0 \leqslant t \leqslant t_0$.

By exactly the same reasoning as above it is clear that for every $s \in \mathbf{R}^+$

$$\operatorname*{esssup}_x E_x e^{\int_s^{t+s} U(\xi(\tau))\,d\tau} < \infty, \quad 0 \leqslant t \leqslant t_0.$$

For any bounded Borel measurable function u, and $t \leqslant t_0$ let us set

$$T(t)\,u(x) = E_x e^{\int_0^t U(\xi(\tau))\,d\tau}\,u(\xi(t)). \tag{2.9}$$

This is well defined and is again a bounded Borel measurable function. Thus for $s \leqslant t_0$, $t \leqslant t_0$ we have

$$T(s)\,T(t)\,u(x) = E_x e^{\int_0^s U(\xi(\tau))\,d\tau}\,T(t)u(\xi(s)).$$

But by the Markov property,

$$T(t)\,u(\xi(s)) = E_{x,s}\,[e^{\int_0^t U(\xi(\tau))\,d\tau}\,u(\xi(t))|\mathscr{F}_t^s] = E_x[e^{\int_s^{t+s} U(\xi(\tau))\,d\tau}\,u(\xi(t+s))|\mathscr{F}_t^s].$$

Let I_n be the indicator function of $\left\{\int_s^{t+s} U(\xi(\tau))\,\mathrm{d}\tau \leqslant n\right\}$. Since $U \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$, for almost all x, $I_n \to 1$ a.s. - P_x. Taking $u \geqslant 0$ and using the monotone convergence theorem we get

$$T(s)\,T(t)\,u(x) = E_x \mathrm{e}^{\int_0^s U(\xi(\tau))\,\mathrm{d}\tau} E_x[\mathrm{e}^{\int_s^{t+s} U(\xi(\tau))\,\mathrm{d}\tau} u(\xi(t+s))|\mathscr{F}_t^s] =$$

$$= \lim_{n\to\infty} E_x \mathrm{e}^{\int_0^s U(\xi(\tau))\,\mathrm{d}\tau} E_x[\mathrm{e}^{\int_s^{t+s} U(\xi(\tau))\,\mathrm{d}\tau} I_n u(\xi(t+s))|\mathscr{F}_t^s] =$$

$$= \lim_{n\to\infty} E_x \mathrm{e}^{\int_0^{t+s} U(\xi(\tau))\,\mathrm{d}\tau} I_n u(\xi(t+s)) = E_x \mathrm{e}^{\int_0^{t+s} U(\xi(\tau))\,\mathrm{d}\tau} u(\xi(t+s)).$$

Take $u = 1$ and we get (2.4) for $0 \leqslant t \leqslant 2t_0$. Proceeding by induction we get (2.4) for all $t \in \mathbf{R}^+$.

In the last paragraph we have, of course, just repeated the standard proof that $\{T(t)\}$ is a semi-group, making sure that all manipulations were valid by the introduction of the indicator function I_n.

LEMMA 2. *Under the hypothesis of Lemma* 1, *the semi-group defined by* (2.9) *may be extended so as to be a semi-group of bounded operators from* $L^p \to L^p$, $1 \leqslant p \leqslant \infty$. *If* $\mu_0 < 1/q$, $T(t)$ *may be extended to be a bounded map from* $L^p \to L^\infty$, $q' \leqslant p \leqslant \infty$, $1/q + 1/q' = 1$.

Proof. The proof of this may be found in [3] and [5]. We shall reproduce it here for the reader's convenience. Let

$$K(q, t)^q = \operatorname*{esssup}_x E_x \mathrm{e}^{q\int_0^t U(\xi(s))\,\mathrm{d}s}.$$

It is clear that $\|T(t)u\|_\infty \leqslant K(1, t)\|u\|_\infty$. Further, if $u \in L^1 \cap L^\infty$ we have

$$\|T(t)u\|_1 \leqslant \int_{\mathbf{R}^n} E_x |u(\xi(t))| \mathrm{e}^{\int_0^t U(\xi(s))\,\mathrm{d}s}\,\mathrm{d}x = \int_{\mathbf{R}^n} |u(x)| E_x \mathrm{e}^{\int_0^t U(\xi(s))\,\mathrm{d}s}\,\mathrm{d}x \leqslant K(1, t)\|u\|_1.$$

The Riesz-Thorin interpolation theorem can now be used for $1 < p < \infty$.

For the second part of the lemma we apply Hölder's inequality to get

$$|T(t)\,u(x)| \leqslant \{E_x|u(\xi(t))|^p\}^{1/p} \{E_x \mathrm{e}^{p'\int_0^t U(\xi(s))\,\mathrm{d}s}\}^{1/p'} \leqslant (4\pi t)^{-n/2p} K(p', t)\|u\|_p.$$

Since $q' \leqslant p$, the result follows.

For $u \in L^\infty$, let us now consider the semi-group

$$S(t)\,u(x) = E_x \mathrm{e}^{-\int_0^t V(\xi(s))\,\mathrm{d}s}\, u(\xi(t))\,. \tag{2.10}$$

LEMMA 3. *If $\mu_0 < 1$, the semi-group defined by (2.10) may be extended as a bounded semi-group from $L^p \to L^p$, $1 \leqslant p \leqslant \infty$. If $\mu_0 < 1/q$, then $S(t)$ may be extended as a bounded operator from $L^p \to L^\infty$, $q' \leqslant p \leqslant \infty$. The semi-group $\{S(t)\}$ is strongly continuous in t when considered from $L^p \to L^p$, $1 \leqslant p < \infty$, and indeed $S(t) \to I$ as $t \to 0$ in the strong topology.*

Proof. The boundedness properties of the operator $S(t)$ is an immediate consequence of Lemma 2. The semi-group property is also a consequence of Lemma 2 and a standard argument as given e.g. in the proof of Lemma 2.

To prove the strong continuity of $S(t)$ as a function of t, when $S(t)$ is acting from $L^p \to L^p$, $1 \leqslant p < \infty$, it is enough to prove it for C_0^∞ functions $u(x)$. From Ito's formula we have

$$S(t)\,u(x) - u(x) = E_x[\mathrm{e}^{-\int_0^t V(\xi(s))\,\mathrm{d}s}\, u(\xi(t))] - u(x) =$$

$$= -E_x \int_0^t \mathrm{e}^{-\int_0^s V(\xi(\tau))\,\mathrm{d}\tau}\, Hu(\xi(s))\,\mathrm{d}s = -\int_0^t S(s)\,Hu(x)\,\mathrm{d}s\,.$$

Since $Hu \in L^1(\mathbf{R}^n)$ and $\|T(t)\|_1 \leqslant M$ we see that

$$\|S(t)u - u\|_1 \leqslant M\int_0^t \|Hu\|_1\,\mathrm{d}s = Mt\|Hu\|_1 \to 0 \quad \text{as } t \to 0\,.$$

For $1 \leqslant p < \infty$, we may write $p = 1 + \delta$, $\delta \geqslant 0$. Thus

$$\|S(t)\,u - u\|_p^p = \int_{\mathbf{R}^n} |S(t)u(x) - u(x)|^p\,\mathrm{d}x \leqslant$$

$$\leqslant \|S(t)u - u\|_\infty^\delta \int_{\mathbf{R}^n} |S(t)\,u(x) - u(x)|\,\mathrm{d}x \leqslant$$

$$\leqslant K\|u\|_\infty^\delta\, \|S(t)\,u - u\|_1 \to 0 \qquad \text{as } t \to 0.$$

LEMMA 4. *Let U be a non-negative function which satisfies* (2.1). *Then for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ so that for every $u \in H^1(\mathbf{R}^n)$*

$$\int_{\mathbf{R}^n} U|u|^2 \leqslant (\mu_0 + \varepsilon)\,\|\nabla u\|^2 + C_\varepsilon\,\|u\|^2. \tag{2.11}$$

Proof. The function $U_1 = U/(\mu_0 + \varepsilon)$ satisfies (2.1) with some $\mu_1 < 1$ replacing μ_0. Let $\{T_1(t)\}$ be the semi-group given by (2.9) with U replaced by U_1. From Lemma 3 it follows that $\{T_1(t)\}$ has a selfadjoint infinitesimal generator in L^2, say A_1, which is bounded above. Let us assume at first that $U \in L^2_{\text{loc}}$. It then follows by Ito's formula that $C_0^\infty(\mathbf{R}^n) \subset D(A_1)$ and $u \in C_0^\infty$ implies $A_1 u = H_1 u = \Delta u + U_1 u$. Indeed, using Ito's formula,

$$\frac{T_1(t)\, u(x) - u(x)}{t} - H_1 u(x) = \int_0^t (T(s) - I)\, H_1\, u(x)\, \mathrm{d}s\,.$$

Since

$$\sup_{0 \leqslant s \leqslant t} \|(T(s) - I) H_1 u\|_1 \to 0 \qquad \text{as } t \to 0,$$

we have the assertion.

Let λ_ε be a real number so that $A_1 - \lambda_\varepsilon I \leqslant 0$. If $u \in C_0^\infty$ we have

$$(\Delta u, u) - \lambda_\varepsilon \|u\|^2 + \int_{\mathbf{R}^n} U_1 |u|^2 \leqslant 0\,.$$

After an integration by parts this yields

$$\int_{\mathbf{R}^n} U_1 |u|^2 \leqslant \|\nabla u\|^2 + \lambda_\varepsilon \|u\|^2,$$

or, if we use the definition of U_1, we get (2.11) in this case:

$$\text{(2.11')} \qquad \int_{\mathbf{R}^n} U |u|^2 \leqslant (\mu_0 + \varepsilon) \|\nabla u\|^2 + C_\varepsilon \|u\|^2\,.$$

In case $U \in L^1_{\text{loc}}(\mathbf{R}^n)$ let us set

$$U_n = \begin{cases} U & \text{if } U \leqslant n\,, \\ n & \text{otherwise}\,. \end{cases}$$

If $\{T_n(t)\}$ is the semi-group corresponding to U_k, and $\{T(t)\}$ the semi-group corresponding to U, it is immediate from (2.9) that $\|T_n(t)\|_p \leqslant \|T(t)\|_p$, where $1 \leqslant p \leqslant \infty$. Let A be the infinitesimal generator of $\{T(t)\}$ and A_n the infinitesimal generator of $\{T_n(t)\}$ in L^2. Let λ_ε be a real number so that $\lambda_\varepsilon I - A$ is positive definite. From the formula

$$(\lambda_\varepsilon - A_n)^{-1} = \int_0^\infty e^{-\lambda_\varepsilon t}\, T_n(t)\, \mathrm{d}t$$

it follows that $\lambda_\varepsilon - A_n$ is positive definite for all n. Thus from (2.11′) we get for every n,

$$\int_{\mathbf{R}^n} U_n|u|^2 \leqslant (\mu_0 + \varepsilon) \| \nabla u\|^2 + C_\varepsilon \|u\|^2 .$$

Applying Fatou's lemma we get (2.11).

The following lemma is essentially due to R. Jensen [6] (but see also [2]). The proof given here follows his basic idea but differs somewhat in the details.

LEMMA 5. *Let U be a non-negative function which satisfies* (2.1) *with* $\mu_0 < 1$. *There exists a* $\lambda_0 \geqslant 0$ *so that if* $\lambda > \lambda_0$, $u \in L^2(\mathbf{R}^n)$, $Uu \in L^1_{\text{loc}}(\mathbf{R}^n)$ *and (as a distribution)*

$$-\Delta u + \lambda u - Uu \leqslant 0 , \tag{2.12}$$

then $u \leqslant 0$.

Proof. Let $\{T(t)\}$ be the semi-group given by (2.9). From the proofs of Lemmas 1 and 2, it is clear that there exists a $\lambda_0 \geqslant 0$ so that $\|T(t)\|_p \leqslant \exp \lambda_0 t$, $1 \leqslant p \leqslant \infty$. Let us set

$$U_n = \begin{cases} U & \text{if } U \leqslant n , \\ n & \text{otherwise} . \end{cases}$$

If $\{T_n(t)\}$ is the corresponding semi-group then $\|T_n(t)\|_p \leqslant \|T(t)\|_p \leqslant \exp \lambda_0 t$, $1 \leqslant p \leqslant \infty$. If A_n is the infinitesimal generator of $\{T_n(t)\}$ in L^2, then A_n is selfadjoint, $D(A_n) = H^2(\mathbf{R}^n)$, and $A_n u = \Delta u + U_n u$.

Let $g \in L^2 \cap L^\infty$, $g \geqslant 0$, $\lambda > \lambda_0$, and let $u_n \in H^2$ be the unique solution in L^2 to the equation

$$-\Delta u_n + \lambda u_n - U_n u_n = g. \tag{2.13}$$

Since

$$u_n = (\lambda - A_n)^{-1} g = \int_0^\infty e^{-\lambda t}\, T_n(t)\, g \,dt$$

it follows that

$$\begin{gathered} 0 \leqslant u_n(x) \leqslant K\|g\|_\infty, \quad \text{a.e.}, \\ \|u_n\|_2 \leqslant K\|g\|_2. \end{gathered} \tag{2.14}$$

If we multiply (2.13) by u_n and integrate by parts we find

$$\| \nabla u_n\|_2^2 = (g, u_n) - \lambda\|u_n\|_2^2 + \int_{\mathbf{R}^n} U_n|u_n|^2 \leqslant K\|g\|_2^2 + \int_{\mathbf{R}^n} U\|u_n\|^2 \leqslant$$

$$\leqslant K\|g\|_2^2 + (\mu_0 + \varepsilon)\| \nabla u_n\|_2^2 + C_\varepsilon\|g\|_2^2 ,$$

where the last inequality follows from (2.11). Thus there exists an $M > 0$ so that for all n

$$\|\nabla u_n\|_2 \leqslant M . \tag{2.15}$$

Now let ψ be a non-negative element of C_0^∞. Since u_n can be approximated in H^2 by non-negative elements of C_0^∞, in the computations which follow there is no loss in generality if we suppose that u_n itself is such an element. Keeping this in mind we have from (2.13)

$$\begin{aligned}(\psi u, g) &= (\psi u\,, -\Delta u_n + \lambda u_n - U_n u_n) = \\ &= \langle -\Delta\psi u + \lambda\psi u - U\psi u\,, u_n > + ((U - U_n)\,\psi u\,, u_n),\end{aligned} \tag{2.16}$$

where $\langle,\rangle$ indicates the distribution pairing. Now,

$$\begin{aligned}\langle -\Delta\psi u + \lambda\psi u - U\psi u\,, u_n\rangle &= \langle -\Delta u + \lambda u - Uu, \psi u_n\rangle - 2\langle \nabla\psi . \nabla u\,, u_n\rangle - \\ &- \langle u\Delta\psi\,, u_n\rangle \leqslant -2\langle \nabla\psi\cdot\nabla u\,, u_n\rangle - \langle u\Delta\psi\,, u_n\rangle .\end{aligned}$$

we also have

$$-2\langle \nabla\psi\cdot\Delta u\,, u_n\rangle - \langle u\Delta\psi\,, u_n\rangle = 2(u\,, \nabla\psi\cdot\nabla u_n) + (u\,, u_n\Delta\psi) .$$

Let us now choose a sequence $\{\psi_m\} \subseteq C_0^\infty$ so that $\psi_m = 1$ on $B(0, m)$, $\psi_m = 0$ outside of $B(0, m + 1)$, $0 \leqslant \psi_m \leqslant 1$ and so that $|\Theta\psi_m| + |\nabla\psi_m| \leqslant K$. If we use ψ_m instead of ψ in the last equality we see from (2.14) and (2.15) that these terms go to zero as $m \to \infty$. Also, for fixed m we have, since $0 \leqslant u_n \leqslant K$,

$$\psi_m(U - U_n)\, uu_n \to 0 \qquad \text{as } n \to \infty\,, \text{a.e.}\,,$$

and

$$|\psi_m(U - U_n)\, uu_n| \leqslant K\psi_m U|u| \in L^1 .$$

Using Lebesgue's convergence theorem we see that

$$\int_{\mathbf{R}^n} \psi_m(U - U_n)\, uu_n \to 0 \qquad \text{as } n \to \infty .$$

If we use all of these facts in (2.16) we see that $(u, g) \leqslant 0$. From this the conclusion of the lemma is immediate.

LEMMA 6. *Let $S(t)$ be the semi-group given by (2.10), A its infinitesimal generator, and λ_0 a real number so that $\lambda > \lambda_0$ implies $(\lambda - A)^{-1}$ exists. If $C = (\lambda - A)^{-1} L^2 \cap L^\infty$, then*

$$-A|C \subseteq H. \tag{2.17}$$

If $\mu_0 < 1/2$ *or if* $\mathrm{Im} V \in L^2_{\mathrm{loc}}$, *then*

$$-A \subseteq H. \tag{2.18}$$

Proof. Recall we had set $V = U + iW$. Let

$$U_n^{\pm} = \begin{cases} U^{\pm} & \text{if } U^{\pm} \leqslant n\,, \\ n & \text{otherwise}\,, \end{cases}$$

$$W_n = \begin{cases} W & \text{if } |W| \leqslant n\,, \\ n & \text{if } W \geqslant n\,, \\ -n & \text{if } W \leqslant -n\,. \end{cases}$$

Let $V_n = U_n^+ - U_n^- + iW_n$, let $\{S_n(t)\}$ be the semi-group corresponding to the potential V_n given by (2.10), and A_n its infinitesimal generator in L^2. It is almost immediate that

$$-A_n = H_n = -\Delta + V_n\,, \quad D(A_n) = H^2(\mathbf{R}^n) = D(H_n)\,. \tag{2.19}$$

Indeed, the proof is as follows. Let $H_{n,0}$ be the closure in L^2 of $H_n|C_0^\infty$. Since V_n is bounded, $H_{n,0} = H_n$. By Ito's formula (see the proof of Lemma 4) it follows that $H_n = H_{n,0} \subseteq -A_n$. On the other hand let V_n^+ be the complex conjugate of V_n and $H_n^+ = -\Delta + V_n^+$. It is elementary that $H_n^* = H_n^+$ and if $\{S_n^+(t)\}$ is the semi-group corresponding to V_n^+, then $S_n(t)^* = S_n^+(t)$. Thus $H_n^+ \subseteq -A_n^+$ and taking adjoints $-A_n \subseteq H_n$.

From standard semi-group theory we know there exists a λ_0 so that $\lambda > \lambda_0$ implies $(\lambda - A)^{-1}$ exists. Further, we have already essentially noted in the proof of Lemma 4 that $\lambda > \lambda_0$ implies $(\lambda - A_n)^{-1}$ exists for all n. Let $g \in L^2 \cap L^\infty$ and $u_n = (\lambda - A_n)^{-1}g$. From (2.19) it follows that

$$\Delta u_n = \lambda u_n + V_n u_n - g\,. \tag{2.20}$$

From the representation (2.10) we see that $S_n(t)g(x) \to S(t)g(x)$ a.e. in x From the representation of $(\lambda - A_n)^{-1}$ in terms of the semi-group $\{S_n(t)\}$, it follows that $u_n \to u = (\lambda - A)^{-1}g$, a.e. and in L^2. Further, it follows from Lemma 2 that $|u_n(x)| \leqslant K\|g\|_\infty$. Thus we have $V_n u_n \to Vu$ a.e. and $|V_n u_n| \leqslant K\|g\|_\infty |V|$. By Lebesgue's dominated convergence theorem $V_n u_n \to Vu$ in L^1_{loc}. It follows from (2.20) that, as a distribution, we have $\Delta u = \lambda u + Vu - g$. This is the content of the inclusion (2.17).

To get (2.18) we first consider the case $\mu_0 < 1/2$. From Lemma 2 it follows that $(\lambda - A)^{-1}L^2 \subseteq L^\infty$. The proof then proceeds exactly as above. In case $W \in L^2_{\mathrm{loc}}$ and $\mu_0 < 1$ we take any $g \in L^2$ and set $u_n = (\lambda - A_n)^{-1}g$. It follows that $u_n \to u =$

$= (\lambda - A)^{-1}g$ in L^2. Further $\|u_n\|_2 \leqslant K\|g\|_2$ and from (2.19) that $u_n \in H^2$. If we multiply (2.20) by u_n, integrate by parts and use Lemma 4 we see that there is a constant K so that for all n

$$\|\nabla u_n\|_2^2 + \int_{\mathbf{R}^n} U_n^+ |u_n|^2 \leqslant K. \tag{2.21}$$

Applying Fatou's lemma (to a subsequence of $\{u_n\}$ if necessary) we see that

$$\int_{\mathbf{R}^n} U^+ |u|^2 \leqslant K.$$

Now, there exists a subsequence of $U_n^+ u_n$, again denoted by the same symbols, so that $U_n^+ u_n \to U^+ u$, a.e. To show that this convergence is in L^1_{loc} we adopt an argument from [2]. By Egoroff's theorem, it is enough to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ so that if E is any measurable set contained in a fixed bounded set with $|E| < \delta$, then $\int_E U_n^+ |u_n| < \varepsilon$ for all n. But for every $R > 0$

$$U_n^+ |u_n| < \frac{1}{2} R U_n^+ + \frac{1}{2R} U_n^+ |u_n|^2,$$

so that

$$\int_E U_n^+ |u_n| \leqslant \frac{1}{2} R \int_E U^+ + \frac{K}{2R}.$$

Fix. R sufficiently large so that $K/2R < \varepsilon$ and then δ sufficiently small so that $R\int_E U^+ < \varepsilon$.

Since by Lemma 4 we have

$$\int_{\mathbf{R}^n} U_n^- |u_n|^2 \leqslant (\mu_0 + \varepsilon) \|\nabla u_n\|_2^2 + C_\varepsilon \|u_n\|_2^2$$

it follows from (2.21) and the fact that $\|u_n\|_2 \leqslant K\|g\|_2$, that there is a constant K so that for all n

$$\int_{\mathbf{R}^n} U_n^- |u_n|^2 \leqslant K.$$

We can now use the same argument as before to show that $U_n^- u_n \to U^- u$ in L^1_{loc}. Finally, since $W \in L^2_{\text{loc}}$ and $W_n \to W$, $u_n \to u$ in L^2_{loc}, it follows that $W_n u_n \to Wu$ in L^1_{loc}. Using these facts in (2.20) we see that we have (2.18).

Proof of Theorem. Let $u \in D(H)$ and let $g = (\lambda + H)u$, $\lambda > \lambda_0$. There exists a $v \in D(A)$ so that $g = (\lambda - A)v$. Since the manifold C of (2.17) is a core for A, there exists a sequence $\{v_n\} \subseteq C$ so that $(\lambda - A)v_n = g_n \to g$ in L^2. From (2.17) and Kato's inequality we have

$$-\Delta|v_n - u| + \lambda|v_n - u| - U^-|v_n - u| \leqslant |g_n - g|.$$

Let w_n be the solution to

$$-\Delta w_n + \lambda w_n - U^- w_n = |g_n - g|,$$

where, of course, we have taken λ_0 large enough to assure us that this has a solution. From Lemma 5 it follows that $|v_n - u| \leqslant w_n$. Further $\|w_n\|_2 \leqslant K\|g_n - g\|_2 \to 0$ as $n \to \infty$. Since $v_n \to v$ in L^2 it follows that $v = u$. Thus $D(H) \subseteq D(A)$ so that H has a closure and from (2.17), $-A = \overline{H}$.

Suppose now that $\mu_0 < 1/2$ or that $W \in L^2_{\text{loc}}$. Let $u \in D(H)$ and $g = (\lambda + H)u$, $\lambda > \lambda_0$. There exists $v \in D(A)$ so that $g = (\lambda - A)v$. Using (2.18) and Kato's inequality we find

$$-\Delta|u - v| + \lambda|u - v| - U^-|u - v| \leqslant 0,$$

which by Lemma 5 implies $u = v$. Thus $H \subseteq -A$, so that by (2.18), $H = -A$. The proof is complete.

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