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FUNCTIONAL CALCULUS AND INVARIANT SUBSPACES

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INTRODUCTION

This paper is an attempt to generalize the results of J. Agler [1], S. Brown [4], S. Brown, B. Chevreau and C. Pearcy [5] and J. G. Stampfli [27] on invariant subspaces. We shall show that the techniques of S. Brown can be used to produce invariant subspaces for polynomially bounded m-tuples of operators acting in Banach spaces. We have to say that our results are not complete as in the above quoted papers (except for $m = 1$ in particular cases) because of the difficulties of spectral nature for $m > 1$ and of the imperfection of general Banach spaces.

The paper is divided in five sections. In §§ 1 and 2 we develop an $H^\infty(\mathbf{D}^m)$ — functional calculus for a polynomially bounded m-tuple $A \in \mathscr{L}(\mathscr{X})^m$, where $\mathscr{X}$ denotes a complex separable Banach space. If the approximate point spectrum of A is enough rich then the $H^\infty(\mathbf{D}^m)$ — functional calculus becomes a weak* homeomorphism between $H^\infty(\mathbf{D}^m)$ and a weak* closed subspace of $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ regarded as the dual of a tensor product space. Thus we can speak about the weak* closure of the algebra generated by A. In § 3 we produce hyperinvariant subspaces only for the reductions we shall need in the sequel. In § 4 we produce invariant subspaces for A in case it has rich approximate point spectrum and $\mathscr{X}$ has an unconditional basis or an unconditional finite-dimensional decomposition determined by some compact injective scalar operator. Theorem 4.13 is a direct correspondent of the main result of [5]. In § 5 we release the hypothesis on $\mathscr{X}$ and our results involve only restrictions to invariant subspaces of m-tuples having functional calculus with continuous functions on $\mathbf{D}^m$ (i.e. quasiscalar m-tuples). Sample results in this section are:

— If $\mathscr{X}$ is reflexive, $T \in \mathscr{L}(\mathscr{X})$ is subscalar and $\partial\mathbf{D} \subset \sigma(T) \subset \bar{\mathbf{D}}$ then T has a proper invariant subspace (Theorem 5.8).

— The restriction to an invariant subspace of the multiplication with the argument in $L^p(m)$, $1 < p < \infty$ (where m denotes a finite positive Borel measure in $\mathbf{C}$ with compact support) has a proper invariant subspace.

Both results generalize Scott Brown's Theorem [4].

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§ 1. FUNCTIONAL CALCULUS

Let $\mathscr{X}$ be a separable Banach space over the complex field $\mathbf{C}$ and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators acting in $\mathscr{X}$. The dual $\mathscr{X}^*$ of $\mathscr{X}$ is the Banach space of all bounded linear functionals defined in $\mathscr{X}$. The action of $x^* \in \mathscr{X}^*$ applied to $x \in \mathscr{X}$ will be denoted either $x^*(x)$ or $\langle x, x^* \rangle$. Recall that the $\mathscr{X}^*$-topology of $\mathscr{X}$ is called the "weak topology" and the $\mathscr{X}$-topology of $\mathscr{X}^*$ is called the "weak*-topology".

Correspondingly we shall use the symbols "w-lim", "w^*-lim". The symbol "s-lim" will denote the strong limit. If $\{T_\iota\}_{\iota \in I} \subset \mathscr{L}(\mathscr{X})$, is given, where I is a directed set, then "$w\text{-}\lim\limits_{\iota} T_\iota''$, $s\text{-}\lim\limits_{\iota} T_\iota$ are defined pointwise (in case the corresponding limits exist).

For any natural number $m \geqslant 1$, we denote by $\mathbf{C}^m$ the Carthesian product of m copies of $\mathbf{C}$, i.e. $\omega \in \mathbf{C}^m$ is an m-tuple of the form $\omega = (\omega_1, \ldots, \omega_m)$, $\omega_k \in \mathbf{C}$. The open unit polydisk in $\mathbf{C}^m$ will be denoted by $\mathbf{D}^m$. We shall denote by $H^\infty(\mathbf{D}^m)$ the algebra of all bounded complex analytic functions defined in $\mathbf{D}^m$, endowed with the supnorm topology determined by the supremum norm $\|\cdot\|_\infty$.

Let $A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m$ be a commutative m-tuple ,i.e. $A_j A_k = A_k A_j$. If p is a polynomial of the form $p(\omega) = p(\omega_1, \ldots, \omega_m)$, then $p(A)$ is defined by $p(A) = p(A_1, \ldots, A_m)$. We shall say that $A \in \mathscr{L}(\mathscr{X})^m$ is *polynomially bounded* if it is commutative and

$$c_A = \sup \{\|p(A)\| : p\text{-polynomial}, \ \|p\|_\infty \leqslant 1\} < \infty,$$

where $\|p\|_\infty$ is the norm of p as an element of $H^\infty(\mathbf{D}^m)$. Assume that A is polynomially bounded and denote by A^* the m-tuple $A^* = (A_1^*, \ldots, A_m^*)$. Then A^* is polynomially bounded. For any $h \in H^\infty(\mathbf{D}^m)$, $0 \leqslant r < 1$, the function $h_r \in H^\infty(\mathbf{D}^m)$ defined by the equation $h_r(\omega) = h(r\omega)$ is analytic in a neighbourhood of $\bar{\mathbf{D}}^m$, thus we can define $h_r(A)$ as a Cauchy integral. We shall say that A is *H^∞-bounded* if $\lim\limits_{r \to 1} < h_r(A)\, x, x^* > = \langle \Phi^A(h)\, x, x^* \rangle$ exists for any $h \in H^\infty(\mathbf{D}^m)$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$. $\Phi^A(h)$ is a bounded linear operator mapping $\mathscr{X}$ in $\mathscr{X}^{**}$. The restriction of $\Phi^A(h)^* \in \mathscr{L}(\mathscr{X}^{***}, \mathscr{X}^*)$ to $\mathscr{X}^*$ will be denoted by $\Phi^{*A}(h)$. We shall say that Φ^A is **-multiplicative* if Φ^{*A} is multiplicative as a map of $H^\infty(\mathbf{D}^m)$ into $\mathscr{L}(\mathscr{X}^*)$. Observe that we have

$$\Phi^A(h) = w^*\text{-}\lim_{r \to 1} h_r(A), \quad \Phi^{*A}(h) = w^*\text{-}\lim_{r \to 1} h_r(A^*),$$

where the first limit involves the weak* topology of $\mathscr{X}^{**}$ and the second one the weak* topology of $\mathscr{X}^*$, and

$$\|\Phi^A(h)\| = \|\Phi^{*A}(h)\| \leqslant c_A \|h\|_\infty.$$

If $\mathscr{X}$ is reflexive we identify $\mathscr{X}^{**}$ with $\mathscr{X}$, thus $\Phi^A(h) \in \mathscr{L}(\mathscr{X})$, $\Phi^{*A} = \Phi^{A^*}$. We shall call A *strongly H^∞-bounded* if

$$\Phi^A(h) = s\text{-}\lim_{r \to 1} h_r(A).$$

If A is strongly H^∞-bounded we have again $\Phi^A(h) \in \mathscr{L}(\mathscr{X})$ and Φ^A is multiplicative. For any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ define the rank-one operator $x \otimes x^* \in \mathscr{L}(\mathscr{X})$ by the equation

$$(x \otimes x^*)\, z = \langle z, x^* \rangle\, x, \quad z \in \mathscr{X},$$

and put

$$\|x \otimes^A x^*\|^* = \sup \{|\langle p(A)\, x, x^* \rangle| : p\text{-polynomial},\ \|p\|_\infty \leqslant 1\}.$$

Now consider the following possible situations for A:

1) $\quad s\text{-}\lim_{n \to \infty} A_k^n = 0, \quad 1 \leqslant k \leqslant m,$

2) $\quad s\text{-}\lim_{n \to \infty} A_k^{*n} = 0, \quad 1 \leqslant k \leqslant m,$

3) $\quad$ either $s\text{-}\lim_{n \to \infty} A_k^n = 0$ or $s\text{-}\lim_{n \to \infty} A_k^{*n} = 0, \quad 1 \leqslant k \leqslant m,$

4) $\quad \lim_{n \to \infty} \|A_k^n x \otimes^A x^*\|^* = 0, \quad (\forall)\ x \in \mathscr{X}, \quad x^* \in \mathscr{X}^*.$

The classes determined by the above situations will be denoted by $C_{0\cdot}$, $C_{\cdot 0}$, $C_{0 \vee 0}$, C^0.

In the sequel of this section $A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m$ will denote a fixed polynomially bounded m-tuple. We shall define the operators $\alpha_{k,n}, \beta_{k,r}, \gamma_{k,n} \in \mathscr{L}(H^\infty(\mathbf{D}^m))$, $1 \leqslant k \leqslant m$, $n \geqslant 1$, as follows: if $h \in H^\infty(\mathbf{D}^m)$ has a Taylor expansion with respect to ω_k, of the form $h(\omega) = \sum_{l=0}^{\infty} h^{(k,l)}(\omega)\, \omega_k^l$, where $h^{(k,l)} \in H^\infty(\mathbf{D}^m)$ and $\dfrac{\partial h^{(k,l)}}{\partial \omega_k} = 0$, we put

$$(\alpha_{k,n} h)(\omega) = \sum_{l=0}^{n^3-1} \frac{n^3 - l}{n^3} h^{(k,l)}(\omega)\, \omega_k^l,$$

$$(\beta_{k,n} h)(\omega) = \sum_{l=0}^{n-1} \frac{l}{n^3} h^{(k,l)}(\omega)\, \omega_k^l,$$

$$(\gamma_{k,n} h)(\omega) = (\omega_k)^{-n}[h(\omega) - (\alpha_{k,n} h)(\omega) - (\beta_{k,n} h)(\omega)].$$

1.1. Proposition. *The following relations hold*

$$C^0 \supset C_{0 \vee 0} \supset C_{0\cdot} \cup C_{\cdot 0}\,.$$

The proof is a simple verification which we omit.

1.2. LEMMA. *For any* $1 \leqslant k \leqslant m$, $n \geqslant 1$ *we have*

$$\|\alpha_{k,n}\| = 1, \quad \|\beta_{k,n}\| \leqslant \frac{1}{2n}, \quad \|\gamma_{k,n}\| \leqslant 2 + \frac{1}{2n}.$$

Proof. $\alpha_{k,n}h$ is the Cesaro mean of order n^3 of h as a function of ω_k and the relation $\|\alpha_{k,n}\| = 1$ becomes obvious in view of [18], p. 33. Since plainly we have $\|h^{(k,l)}\|_\infty \leqslant \|h\|_\infty$ we derive

$$\|\beta_{k,n}h\|_\infty \leqslant \frac{1}{n^3}\left(\sum_{l=0}^{n-1} l\right)\|h\|_\infty \leqslant \frac{\|h\|_\infty}{2n}.$$

We also have

$$\|\gamma_{k,n}h\|_\infty = \|h - (\alpha_{k,n}h) - (\beta_{k,n}h)\|_\infty \leqslant \left(2 + \frac{1}{2n}\right)\|h\|_\infty.$$

1.3. LEMMA. *If* $\lim_{n\to\infty} \|A_k^n x \otimes^A x^*\|^* = 0$, $(\forall)\, x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, *resp. if* $A_k \in C_0$, *then*

$$w^*\text{-}\lim_{n\to\infty} g(A)\,(h - \alpha_{k,n}h)_r(A) = 0, \quad \textit{resp.} \quad s\text{-}\lim_{n\to\infty} g(A)\,(h - \alpha_{k,n}h)_r(A) = 0,$$

uniformly with respect to $0 \leqslant r < 1$, $g, h \in H^\infty(\mathbf{D}^m)$, $\|g\|_\infty \leqslant 1$, $\|h\|_\infty \leqslant 1$, *where* g *is a polynomial.*

Proof. Using the relations

$$|\langle g(A)\,(h - \alpha_{k,n}h)_r(A)\,x, x^*\rangle| =$$

$$= |\langle g(A)\,(\beta_{k,n}h)_r(A)\,x, x^*\rangle + \langle g(A)\,(\gamma_{k,n}h)_r(A)\,A_k^n x, x^*\rangle| \leqslant$$

$$\leqslant c_A\left(\frac{\|x\|\cdot\|x^*\|}{2n} + \left(2 + \frac{1}{2n}\right)\|A_k^n x \otimes^A x^*\|^*\right) \leqslant$$

$$\leqslant c_A\|x^*\|\left(\frac{\|x\|}{2n} + \left(2 + \frac{1}{2n}\right)\|A_k^n x\|\right),$$

the assertions in our Lemma become obvious.

In the next three lemmas we shall assume $m > 1$ and we shall consider the $(m-1)$-tuple $A' = (A_1, \ldots, A_{m-1}) \in \mathscr{L}(\mathscr{X})^{m-1}$. We shall denote by $H_m^\infty(\mathbf{D}^m)$ the subalgebra in $H^\infty(\mathbf{D}^m)$ consisting of polynomials in ω_m whose coefficients do not depend on ω_m, but could depend on $\omega' = (\omega_1, \ldots, \omega_{m-1})$.

1.4. LEMMA. *Assume* A' *is* H^∞*-bounded, resp. strongly* H^∞*-bounded. Then for any* $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $g, g' \in H_m^\infty(\mathbf{D}^m)$ *the limits*

$$\lim_{r\to 1} \langle g_r(A)\,x, x^*\rangle, \text{ resp. } s\text{-}\lim_{r\to 1} g_r(A)$$

*exist. Moreover if $\Phi^{A'}$ is *-multiplicative we have*

$$(w^*\text{-}\lim_{r\to 1} g_r(A^*))\,(w^*\text{-}\lim_{r\to 1} g'_r(A^*)) = w^*\text{-}\lim_{r\to 1}\,(gg')_r(A^*).$$

Proof. Let us put

$$G = \left\{ f \in H^\infty(\mathbf{D}^m) : \frac{\partial f}{\partial \omega_m} \equiv 0 \right\}.$$

For any $f \in G$ we define $\hat{f} \in H^\infty(\mathbf{D}^{m-1})$ by the equation $\hat{f}(\omega') = f(\omega)$. It is obvious that the map $f \to \hat{f}$ is an algebraic isomorphism and $f_r(A) = \hat{f}_r(A')$. Thus the limits

$$\lim_{r\to 1} \langle f_r(A)\,x, x^*\rangle, \text{ resp. } s\text{-}\lim_{r\to 1} f_r(A)$$

exist. Since g is of the form $g(\omega) = \sum_{l=0}^{\infty} g^{(m,l)}(\omega)\,\omega_m^l$, where $g^{(m,l)} \in G$ and the series is in fact a finite sum, the rest of the proof will be a simple verification.

1.5. LEMMA. *If A' is H^∞-bounded and $\lim_{n\to\infty} \|A_m^n x \otimes^A x^*\|^* = 0$, $(\forall)\, x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, then A is H^∞-bounded. Moreover, if $\Phi^{A'}$ is *-multiplicative then Φ^A is *-multiplicative.*

Proof. Let $h, g \in H^\infty(\mathbf{D}^m)$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ be given. Since $\alpha_{m,n}h \in H_m^\infty(\mathbf{D}^m)$ and by Lemma 1.3 and Lemma 1.4 we have

$$\lim_{\substack{r\to 1\\ r'\to 1}} \langle h_r(A)\,x - h_{r'}(A)\,x,\ x^*\rangle =$$

$$= \lim_{n\to\infty} \lim_{\substack{r\to 1\\ r'\to 1}} \langle (\alpha_{m,n}h)_r(A)x - (\alpha_{m,n}h)_{r'}(A)\,x, x^*\rangle = 0,$$

it follows that A is H^∞-bounded. Using again Lemma 1.3 we derive

$$\langle x, \Phi^{*A}(hg)\,x^* - \Phi^{*A}(h)\Phi^{*A}(g)x^*\rangle =$$

$$= \lim_{r\to 1} \langle x, h_r(A^*)\,(g_r(A^*) - \Phi^{*A}(g))\,x^*\rangle =$$

$$= \lim_{n\to\infty} \lim_{r\to 1} \langle x, (\alpha_{m,n}h)_r(A^*)\,[(\alpha_{m,n}g)_r(A^*)\,x^* - \Phi^{*A}(\alpha_{m,n}g)\,x^*]\rangle =$$

$$= \lim_{n\to\infty} \langle x, \Phi^{*A}((\alpha_{m,n}h)\,(\alpha_{m,n}g))\,x^* - \Phi^{*A}(\alpha_{m,n}h)\,\Phi^{*A}(\alpha_{m,n}g)\,x^*\rangle,$$

thus if $\Phi^{A'}$ is *-multiplicative, Lemma 1.4 will imply that Φ^A is *-multiplicative.

1.6. LEMMA. *If A' is strongly H^∞-bounded and $A_m \in C_0$. then A is strongly H^∞-bounded.*

Proof. We imitate the proof of Lemma 1.5, using the second part of Lemma 1.3 and the second part of Lemma 1.4.

1.7. THEOREM. *If $A \in C^0$, resp. $A \in C_0$. then A is H^∞-bounded and Φ^A is *-multiplicative, resp. A is strongly H^∞-bounded.*

Proof. By Lemma 1.5 and Lemma 1.6 we may suppose $m = 1$, $A = A_1$. Let $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $h \in H^\infty(\mathbf{D})$. Since $\alpha_{1,r}h$ is a polynomial and by Lemma 1.3 we have

$$\lim_{\substack{r\to 1\\ r'\to 1}} \langle h_r(A)\,x - h_{r'}(A)\,x, x^*\rangle =$$

$$= \lim_{n\to\infty} \lim_{\substack{r\to 1\\ r'\to 1}} \langle(\alpha_{1,n}h)_r(A)\,x - (\alpha_{1,n}h)_{r'}(A)\,x, x^*\rangle = 0,$$

we derive that A is H^∞-bounded. The fact that Φ^A is *-multiplicative is actually proven in Lemma 1.5. The proof of the "strong" part is similar.

1.8. PROPOSITION. *If $A \in C^0$, resp. $A \in C_0$. and if $\{h_n\}_{n=1}^\infty \subset H^\infty(\mathbf{D}^m)$ is a bounded sequence pointwise convergent to 0 then for any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ we have*

$$w^*\text{-}\lim_{n\to\infty} \Phi^{*A}(h_n)\,x^* = 0, \quad resp.\ s\text{-}\lim_{n\to\infty} \Phi^A(h_n)\,x = 0.$$

Proof. Using Lemma 1.3 and proceeding as in Lemma 1.5 and Lemma 1.6 we can reduce to the case $m = 1$, $A = A_1$. Because $\{h_n\}_{n=1}^\infty$ tends uniformly to 0 on compact sets we may suppose

$$h_n(\omega_1) = \omega_1^{l_n} h_n'(\omega_1), \quad \lim_{n\to\infty} l_n = \infty,$$

consequently

$$|\langle\Phi^A(h_n)\,x, x^*\rangle| = |\langle A^{l_n}\Phi^A(h_n')\,x, x^*\rangle| \leqslant$$

$$\leqslant \|h_n'\|_\infty \|A^{l_n}x \otimes^A x^*\|^* \leqslant \|h_n'\|_\infty \|x^*\|\, \|A^{l_n}x\|\, c_A.$$

Since we have $\lim_{n\to\infty} \|A^n x \otimes^A x^*\|^* = 0$, resp. $\lim_{n\to\infty} \|A^n x\| = 0$, the proof is concluded.

A direct consequence of Proposition 1.8 is the following:

1.9. COROLLARY. *If $A \in C^0$, resp. $A \in C_0$. then for any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ the sets*

$$\{\Phi^{*A}(h)\,x^* : h \in H^\infty(\mathbf{D}^m),\ \|h\|_\infty \leqslant 1\},\quad \{\Phi^A(h)\,x : h \in H^\infty(\mathbf{D}^m),\ \|h\|_\infty \leqslant 1\}$$

are sequentially compact in the weak topology, resp. strong topology.*

Proof. If $\{h_n\}_{n=1}^{\infty} \subset H^{\infty}(\mathbf{D}^m)$ is a bounded sequence we can find a w^*-convergent subsequence $\{h_{k_n}\}_{n=1}^{\infty}$, $w^*\text{-}\lim\limits_{n\to\infty} h_{k_n} = h$. Since $\{h_{k_n}\}_{n=1}^{\infty}$ is pointwise convergent to h, applying Proposition 1.8 we derive

$$w^*\text{-}\lim_{n\to\infty} \Phi^{*A}(h_{k_n})\, x^* = \Phi^{*A}(h)\, x^*, \text{ resp. } s\text{-}\lim_{n\to\infty} \Phi^{A}(h_{k_n})\, x = \Phi^{A}(h)\, x.$$

1.10. Remark. The proofs of Theorem 1.7 and Proposition 1.8, in case $m = 1$, can be achieved using Taylor expansions in place of Cesaro means.

§ 2. THE WEAK* CLOSURE OF THE ALGEBRA GENERATED BY A POLYNOMIALLY BOUNDED m-TUPLE

Let $A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m$ be a polynomially bounded m-tuple. For any $x \in \mathscr{X}$ put $Ax = (A_1x, \ldots, A_mx) \in \mathscr{X}^m$, $|||Ax||| = \sum\limits_{k=1}^{m} \|A_k x\|$. Then the *left approximate point spectrum* of A is the set

$$\tau_l(A) = \{\omega \in \mathbf{C}^m : \inf_{\|x\|=1} |||(A-\omega)\, x||| = 0\}.$$

The right approximate point spectrum and the approximate point spectrum of A are the sets

$$\tau_r(A) = \tau_l(A^*),\ \tau(A) = \tau_l(A) \cup \tau_r(A).$$

Observe that we have $\tau_l(A) = \tau^{\text{left}}_{\mathscr{L}(\mathscr{X})}(A)$, where the right hand term is defined in [17], Definition 1.3. We also define the essential approximate point spectra of A by

$$\tau_{le}(A) = \{\omega \in \mathbf{C}^m : \inf_{\|x\|=1,\ x\in\mathscr{Y}} |||(A-\omega)\, x||| = 0 \text{ if } \dim \mathscr{X}/\mathscr{Y} < \infty\},$$

$$\tau_{re}(A) = \tau_{le}(A^*),\ \tau_e(A) = \tau_{le}(A) \cup \tau_{re}(A).$$

Let $\mathscr{X} \underline{\otimes} \mathscr{X}^*$ denote the projective tensor product of $\mathscr{X}$ with $\mathscr{X}^*$, i.e. the completion in the projective norm, denoted below by $\|\cdot\|_*$, of the algebraic tensor product (see [16] and [25], Ch. III, § 6). By [25], Ch. III, Theorem 6.2 and Theorem 6.4 we know that any $u \in \mathscr{X} \underline{\otimes} \mathscr{X}^*$ is of the form

$$u = \sum_{n=1}^{\infty} x_n \underline{\otimes}\, x_n^*,\ \sum_{n=1}^{\infty} \|x_n\|\, \|x_n^*\| < \infty$$

and the map

$$\mathscr{L}(\mathscr{X}, \mathscr{X}^{**}) \ni T \to \varphi_T \in (\mathscr{X} \underline{\otimes} \mathscr{X}^*)^*$$

where $\varphi_T(u) = \sum_{n=1}^{\infty} (Tx_n)(x_n^*)$, identifies $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ with $(\mathscr{X} \underline{\otimes} \mathscr{X}^*)^*$. The weak* topology of $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ will be determined by the above identification. Since $\mathscr{L}(\mathscr{X})$ is a subspace in $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ we define the ultraweak operator topology of $\mathscr{L}(\mathscr{X})$ by the restriction of the weak* topology to $\mathscr{L}(\mathscr{X})$. The ultraweak and weak operator topology of $\mathscr{L}(\mathscr{X})$ agree on bounded sets. If $\mathscr{X}$ is reflexive then we can speak about the weak* topology of $\mathscr{L}(\mathscr{X})$ ($= \mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$) or equivalently the ultraweak operator topology of $\mathscr{L}(\mathscr{X})$. Let $\mathscr{A}_A$ denote the weak* closure in $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ of the algebra generated in $\mathscr{L}(\mathscr{X})$ by $A_1, \ldots, A_m$. If we denote by $\mathscr{X} \overset{A}{\underline{\otimes}} \mathscr{X}^*$ the quotient space of $\mathscr{X} \underline{\otimes} \mathscr{X}^*$ by the subspace

$$\{u \in \mathscr{X} \underline{\otimes} \mathscr{X}^* : \varphi_T(u) = 0, \ (\forall)\ T \in \mathscr{A}_A\},$$

then $\mathscr{A}_A$ can be canonically identified with the dual of $\mathscr{X} \overset{A}{\underline{\otimes}} \mathscr{X}^*$ and this identification will determine a weak* topology in $\mathscr{A}_A$. The norm in $\mathscr{X} \overset{A}{\underline{\otimes}} \mathscr{X}^*$ will be denoted as in $\mathscr{X} \underline{\otimes} \mathscr{X}^*$, by $\|\cdot\|_*$. Let $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ be given. There exists a unique continuous linear operator, mapping $\mathscr{X} \underline{\otimes} \mathscr{X}^*$ onto nuclear operators in $\mathscr{X}$ such that

$$\mathscr{X} \underline{\otimes} \mathscr{X}^* \ni x \underline{\otimes} x^* \to x \otimes x^* \in \mathscr{L}(\mathscr{X})$$

(recall that we already defined in § 1, $x \otimes x^*$, by the equation $(x \otimes x^*)\, y = \langle y, x^* \rangle x$, $y \in \mathscr{X}$). The above map is injective if and only if $\mathscr{X}$ has the approximation property (see [19], Theorem 1. e.4) in which case $\mathscr{L}(\mathscr{X}, \mathscr{X}^{**})$ is the dual of nuclear operators in $\mathscr{X}$ (endowed with the nuclear norm). If $\mathscr{X}$ has an unconditional basis (or an unconditional finite-dimensional decomposition) then $\mathscr{X}$ has the approximation property. The image of $x \underline{\otimes} x^*$ in $\mathscr{X} \overset{A}{\underline{\otimes}} \mathscr{X}^*$ will be denoted by $x \overset{A}{\underline{\otimes}} x^*$.

Now let $\mathbf{T}^m$ denote the m-dimensional torus and let $\mathscr{L}^\infty(\mathbf{T}^m)$ be the algebra of all essentially bounded classes of complex functions of $\mathbf{T}^m$, with respect to the normalized Lebesgue measure. The weak* topology of $L^\infty(\mathbf{T}^m)$ is determined by the duality relation $\mathscr{L}^1(\mathbf{T}^m)^* = \mathscr{L}^\infty(\mathbf{T}^m)$. By [24], Chapter III, 3.4.4 (c) we identify $H^\infty(\mathbf{D}^m)$ with a weak* closed subspace in $L^\infty(\mathbf{T}^m)$ and this determines a weak* topology in $H^\infty(\mathbf{D}^m)$.

Further we assume that $A \in \mathscr{L}(\mathscr{X})^m$ is a C^0-class m-tuple. Since A is H^∞-bounded and Φ^A is well defined (see Theorem 1.7) we may consider the linear map

$$\Phi_*^A : (H^\infty(\mathbf{D}^m), w^*) \to (\mathscr{A}_A, w^*)$$

defined by the equation

$$\Phi_*^A(h) = \Phi^A(h), \ h \in H^\infty(\mathbf{D}^m).$$

In case Φ_*^A maps homeomorphic $H^\infty(\mathbf{D}^m)$ onto $\mathscr{A}_A$ and $\mathscr{E}_\omega$ denotes the evaluation at ω in $H^\infty(\mathbf{D}^m)$, then the equation

$$\mathscr{E}_\omega^A \circ \Phi_*^A = \mathscr{E}_\omega$$

determines a w^*-continuous linear functional $\mathscr{E}_\omega^A \in \mathscr{X} \underset{\overline{}}{\overset{A}{\otimes}} \mathscr{X}^*$. Finally recall that a subset $\sigma \subset \mathbf{D}^m$ is *dominating* if (see [6], [23])

$$\sup_{\omega \in \sigma} |h(\omega)| = \|h\|_\infty, \qquad (\forall)\, h \in H^\infty(\mathbf{D}^m).$$

2.1. THEOREM. *Let A be a C^0-class m-tuple. The map Φ_*^A is continuous. If $\omega^0 \in \mathbf{D}^m$ is given and $\{x_n\}_{n=1}^\infty \subset \mathscr{X}$, $\{x_n^*\}_{n=1}^\infty \subset \mathscr{X}^*$ are bounded sequences such that $\lim_{n\to\infty} |||(A - \omega^0)\, x_n||| = \lim_{n\to\infty} |||(A^* - \omega^0)\, x_n^*||| = 0$ then we have*

$$\lim_{n\to\infty} \|(\Phi^A(h) - h(\omega^0))\, x_n\| = \lim_{n\to\infty} \|(\Phi^{*A}(h) - h(\omega^0))\, x_n^*\| = 0,$$

uniformly with respect to $h \in H^\infty(\mathbf{D}^m)$, $\|h\|_\infty \leqslant 1$.

Proof. Let $\{h_n\}_{n=1}^\infty \subset H^\infty(\mathbf{D}^m)$ be weak* convergent to 0. Since in particular $\{h_n\}_{n=1}^\infty$ is a bounded sequence, pointwise convergent to 0 and by Proposition 1.8 we have

$$\lim_{n\to\infty} \langle \Phi_*^A(h_n)\, x, x^* \rangle = 0, \quad x \in \mathscr{X},\ x^* \in \mathscr{X}^*,$$

we derive easily

$$\lim_{n\to\infty} \varphi_{\Phi_*^A(h_n)}(u) = 0, \quad u \in \mathscr{X} \underline{\otimes} \mathscr{X}^*.$$

Because $H^\infty(\mathbf{D}^m)$ is separable, the continuity of Φ_*^A will follow by [5], Theorem 2.3. Now if $h \in H^\infty(\mathbf{D}^m)$ is given define $h' \in H^\infty(\mathbf{D}^m)$ by the equation

$$h'(\omega) = h(\omega_1, \ldots, \omega_{m-1}, \omega_m^0).$$

Then we have

$$h(\omega) - h'(\omega) = g(\omega)\,(\omega_m - \omega_m^0), \quad g \in H^\infty(\mathbf{D}^m),$$

$$\|(\Phi^A(h) - h(\omega^0))\, x_n\| \leqslant \|\Phi^A(g)\,(A_m - \omega_m^0)\, x_n\| + \|(\Phi^A(h') - h(\omega^0))\, x_n\|,$$

$$\|(\Phi^{*A}(h) - h(\omega^0))\, x_n^*\| \leqslant \|\Phi^{*A}(g)\,(A_m^* - \omega_m^0)\, x_m^*\| + \|(\Phi^{*A}(h') - h'(\omega^0))\, x_n^*\|.$$

Since h' is in fact an element of $H^\infty(\mathbf{D}^{m-1})$ (if $m > 1$) the rest of the proof can be done by induction and we omit it.

2.2. REMARK. Theorem 2.1 is analogous to a spectral mapping theorem. In fact it implies the spectral inclusion $h(\tau_r(A) \cap \mathbf{D}^m) \subset \tau_l(\Phi^{*A}(h))$, $h \in H^\infty(\mathbf{D}^m)$.

2.3. THEOREM. *Let A be a C^0-class m-tuple. If $\tau(A) \cap \mathbf{D}^m$ is dominating then Φ_*^A is a homeomorphism onto $\mathscr{A}_A$ and for any $h \in H^\infty(\mathbf{D}^m)$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ we have*

$$\|h\|_\infty \leqslant \|\Phi_*^A(h)\| \leqslant c_A \|h\|_\infty\,, \quad \|x \overset{A}{\underset{-}{\otimes}} x^*\|_* \leqslant \|x \otimes^A x^*\|^* \leqslant c_A \|x \overset{A}{\underset{-}{\otimes}} x^*\|_*\,.$$

Proof. Since $\tau(A) \cap \mathbf{D}^m$ is dominating, Theorem 2.1 implies $\|h\|_\infty \leqslant \|\Phi_*^A(h)\|$, thus by [5], Theorem 2.7, Φ_*^A will map homeomorphically $H^\infty(\mathbf{D}^m)$ onto $\mathscr{A}_A$. The rest of the proof is obvious.

§ 3. HYPERINVARIANT SUBSPACES

Throughout this section $A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m$ and $S \in \mathscr{L}(\mathscr{X})$ will denote a polynomially bounded m-tuple, resp. an operator which is not a scalar multiple of the identity and $\sup_{n \geqslant 1} \|S^n\| < \infty$. A proper *hyperinvariant subspace* of S will be a closed linear manifold, different from $\{0\}$ and $\mathscr{X}$, invariant for any operator commuting with S. We shall say that S has the *single-valued extension property* if the equation

$$(\lambda - S) f(\lambda) = 0$$

has the only analytic $\mathscr{X}$-valued solution $f = 0$ (see [12], § 3, [10], Chapter I, Definition 1.1). If S has the single-valued extension property then for any $x \in \mathscr{X}$ there exists a unique analytic function x_S valued in $\mathscr{X}$, with a maximal domain $\rho_S(x) \supset \rho(S)$ such that

$$(\lambda - S)\, x_S(\lambda) = x, \quad \lambda \in \rho_S(x).$$

The open set $\rho_S(x)$ is the resolvent set of x with respect to S and $\sigma_S(x) = \mathbf{C} \setminus \rho_S(x)$ is the spectrum of x with respect to S; $\sigma_S(x)$ is a compact subset of $\sigma(S)$, void if and only if $x = 0$ ([10], Chapter I, Proposition 1.2). For any closed subset $\sigma \subset \sigma(S)$ put

$$\mathscr{X}_S(\sigma) = \{x \in \mathscr{X} : \sigma_S(x) \subset \sigma\}\,.$$

Then $\mathscr{X}_S(\sigma)$ is a linear manifold invariant for any operator which commutes with S ([10], Chapter I, Proposition 1.2). It is obvious that if S has not the single-valued extension property, then the point spectrum of S is nonvoid and consequently S has a proper hyperinvariant subspace. If the point spectrum of S is void then S has the single valued extension property.

For any $0 \leqslant \theta < 2\pi$, $0 < \varepsilon < \pi$ put

$$\sigma(\theta, \varepsilon) = \{e^{i\varphi} : \varphi \in [\theta - \varepsilon, \theta + \varepsilon)\}.$$

3.1. LEMMA. *If S is power bounded, $\sigma_p(S^*) = \emptyset$ and $\{S^n\}_{n=1}^{\infty}$ does not tend strongly to 0, then the set*

$$\{\theta \in [0, 2\pi) : \mathscr{X}^*_{S^*}(\sigma(\theta, \varepsilon)) \neq \{0\}, (\forall)\, 0 < \varepsilon < \pi\}$$

is infinite.

Proof. Let $x \in \mathscr{X}$ be such that $\overline{\lim_{n\to\infty}} \|S^n x\| > 0$ and $x_n^* \in \mathscr{X}^*$, $\|x_n^*\| = 1$, $\langle S^n x, x_n^* \rangle = \|S^n x\|$. Since the unit ball in $\mathscr{X}^*$ is weak* compact we may suppose that we have

$$w^*\text{-}\lim_{n\to\infty} S^{*k_n} x^*_{k_n} = x^*.$$

The assumption $\sigma_p(S^*) = \emptyset$ implies the existence of the limit

$$w^*\text{-}\lim_{n\to\infty} S^{*k_n - r} x^*_{k_n} = y^*_r, \quad r = 0,1,\ldots.$$

Since we have $\|y_r^*\| \leqslant \sup_{n\geqslant 1} \|S^n\|$ we can define the analytic function $x^*{}_*(\cdot) : \{\lambda : |\lambda| \neq 1\} \to \mathscr{X}^*$ by the equation

$$x^*_{S^*}(\lambda) = \begin{cases} (\lambda - S^*)^{-1} x^*, & |\lambda| > 1 \\ -\sum_{r=0}^{\infty} \lambda^r y^*_{r+1}, & |\lambda| < 1. \end{cases}$$

Using the relation $S^* y^*_{r+1} = y^*_r$ we derive easily

$$(\lambda - S^*)\, x^*_{S^*}(\lambda) = x^*, \ \|x^*_{S^*}(\lambda)\| \leqslant (\sup_{n\geqslant 1} \|S^n\|)\, |1 - |\lambda||^{-1},$$

consequently $\sigma_{S^*}(x^*) \subset \{\lambda : |\lambda| = 1\}$. For any $\theta \in [0, 2\pi)$, such that $e^{i\theta} \in \sigma_{S^*}(x^*)$ and any $0 \leqslant \varepsilon < \pi$, $0 < \eta \leqslant 1$ put

$$\delta(\theta, \varepsilon, \eta) = \{t e^{i\varphi} : 1 - \eta \leqslant t \leqslant 1 + \eta, \ \varphi \in [\theta - \varepsilon, \ \theta + \varepsilon]\}$$

$$\Gamma(\theta, \varepsilon, \eta) = \partial\delta(\theta, \varepsilon, \eta)$$

and define the analytic function

$$f_{\theta,\varepsilon}(\lambda) = (\lambda - e^{i(\theta-\varepsilon)})^2\, (\lambda - e^{i(\theta+\varepsilon)})^2, \quad \lambda \in \mathbf{C}.$$

Because obviously $f_{\theta,\varepsilon} x_{S^*}(\cdot)$ is continuous on $\Gamma(\theta, \varepsilon, \eta)$ we may consider the integral

$$x^*_{\theta,\varepsilon} = \frac{1}{2\pi i} \int_{\Gamma(\theta,\varepsilon,\eta)} f_{\theta,\varepsilon}(\lambda)\, x^*_{S^*}(\lambda)\, d\lambda,$$

which is independent on η. If we put

$$S^*(\theta, \varepsilon) = (S^* - e^{i(\theta-\varepsilon)})^2 (S^* - e^{i(\theta+\varepsilon)})^2,$$

$$y^*_{\theta, \varepsilon, \eta}(\lambda) = \frac{1}{2\pi i}\int_{\Gamma(\theta,\varepsilon,\eta)} \frac{f_{\theta,\varepsilon}(\zeta)x^*_{S^*}(\zeta)\,d\zeta}{\lambda-\zeta}, \quad \lambda \notin \Gamma(\theta, \varepsilon, \eta)$$

$$z^*_{\theta, \varepsilon, \eta}(\lambda) = \frac{1}{2\pi i}\int_{|\zeta|=1+\eta} \frac{f_{\theta,\varepsilon}(\zeta)x^*_{S^*}(\zeta)\,d\zeta}{\lambda-\zeta} - y^*_{\theta, \varepsilon, \eta}(\lambda), \quad \lambda \in \operatorname{int}\delta(\theta, \varepsilon, \eta)$$

then $y^*_{\theta, \varepsilon, \eta}(\cdot)$, $z^*_{\theta, \varepsilon, \eta}(\cdot)$ are analytic functions and

$$(\lambda - S^*)\, y^*_{\theta, \varepsilon, \eta}(\lambda) = x^*_{\theta, \varepsilon}, \quad \lambda \notin \delta(\theta, \varepsilon, \eta),$$

$$(\lambda - S^*)\, z^*_{\theta, \varepsilon, \eta}(\lambda) = S^*(\theta, \varepsilon)\, x^* - x^*_{\theta, \varepsilon}, \quad \lambda \in \operatorname{int}\delta(\theta, \varepsilon, \eta).$$

Hence we derive

$$\sigma_{S^*}(x^*_{\theta, \varepsilon}) \subset \sigma(\theta, \varepsilon),$$

$$\sigma_{S^*}(S^*(\theta, \varepsilon)\, x^* - x^*_{\theta, \varepsilon}) \subset \{e^{i\varphi} : \varphi \notin (\theta - \varepsilon, \ \theta + \varepsilon)\}.$$

But we also have $e^{i\theta} \in \sigma_{S^*}(S^*(\theta, \varepsilon)\, x^*)$. Indeed if $e^{i\theta} \in \rho_{S^*}(S^*(\theta, \varepsilon)\, x^*)$, then the function $S^*(\theta, \varepsilon)\, x^*_{S^*}(\cdot)$ has an analytic extension in a domain containing $e^{i\theta}$, while $x^*_{S^*}(\cdot)$ has not such an extension. We can find a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \rho_{S^*}(x^*)$ such that $\lim_{n\to\infty} \lambda_n = e^{i\theta}$, $\|x^*_{S^*}(\lambda_n)\| \geqslant n$ (if $x^*_{S^*}(\cdot)$ is bounded near $e^{i\theta}$, then using the assumption $\sigma_p(S^*) = \emptyset$ and the weak* compactness of the unit ball of $\mathscr{X}^*$, we can extend $x^*_{S^*}(\cdot)$ to a weak* continuous function in a neighbourhood of $e^{i\theta}$ and such an extension is necessarily analytic contradicting $e^{i\theta} \in \sigma_{S^*}(x^*)$). Thus we have

$$\lim_{n\to\infty} (\lambda_n - S^*)\, x^*_{S^*}(\lambda_n)\, \|x^*_{S^*}(\lambda_n)\|^{-1} = x^* \lim_{n\to\infty} \|x^*_{S^*}(\lambda_n)\|^{-1} = 0,$$

while

$$\lim_{n\to\infty} \|S^*(\theta, \varepsilon)\, x^*_{S^*}(\lambda_n)\|\, \|x^*_{S^*}(\lambda_n)\|^{-1} = \lim_{n\to\infty} |f(\lambda_n)| = |f(e^{i\theta}))| > 0,$$

$$\lim_{n\to\infty} \|S^*(\theta, \varepsilon)\, x^*_{S^*}(\lambda_n)\|\, \|x^*_{S^*}(\lambda_n)\|^{-1} = \lim_{n\to\infty} \|(S^*(\theta, \varepsilon)\, x^*)_{S^*}(\lambda_n)\|\, \|x^*_{S^*}(\lambda_n)\|^{-1} = 0$$

and the relation $e^{i\theta} \in \sigma_{S^*}(S^*(\theta, \varepsilon)\, x^*)$ follows. Now the inclusions

$$\sigma(S^*(\theta, \varepsilon)\, x^* - x^*_{\theta, \varepsilon}) \subset \{e^{i\varphi} : \varphi \notin (\theta - \varepsilon, \ \theta + \varepsilon)\}$$

$$\sigma_{S^*}(S^*(\theta, \varepsilon)\, x^*) \subset \sigma_{S^*}(x^*_{\theta, \varepsilon}) \cup \sigma_{S^*}(S^*(\theta, \varepsilon)\, x^* - x^*_{\theta, \varepsilon})$$

imply $x_{\theta,\varepsilon} \neq 0$, $\mathscr{X}^*_{S^*}(\sigma(\theta, \varepsilon)) \neq \{0\}$. To conclude the proof it suffices to show that $e^{i\theta}$ is not an isolated point in $\sigma_{S^*}(x^*)$. But if $e^{i\theta}$ is isolated in $\sigma_{S^*}(x^*)$ then $x^*_{\theta,\varepsilon}$ does not depend on ε if ε is enough small, thus

$$0 \neq x^*_\theta = \lim_{\varepsilon\to 0} x^*_{\theta,\varepsilon},$$

$$\lim_{\varepsilon\to 0} \|x^*_{\theta,\varepsilon}\| \leqslant \frac{1}{2\pi} \lim_{\varepsilon\to 0} \int_{\Gamma(\theta,\varepsilon,\varepsilon)} |f_{\theta,\varepsilon}(\lambda)| \cdot \|x^*_{S^*}(\lambda)\| \, |d\lambda| \leqslant$$

$$\leqslant \frac{1}{2\pi} \sup_{n\geqslant 1} \|S^n\| \lim_{\varepsilon\to 0} \int_{\Gamma(\theta,\varepsilon,\varepsilon)} |f_{\theta,\varepsilon}(x)| \, |1 - |\lambda||^{-1} |d\lambda| = 0$$

and this contradiction shows that $e^{i\theta}$ has to be an accumulation point in $\sigma_{S^*}(x^*)$.

3.2. THEOREM. *If S is power bounded and neither $\{S^n\}_{n=1}^\infty$ nor $\{S^{*n}\}_{n=1}^\infty$ tend strongly to 0 then S^* has a proper hyperinvariant subspace.*

Proof. If $\sigma_p(S^*) \neq \emptyset$ then $\ker(S^* - \mu)$, $\mu \in \sigma_p(S^*)$ is a proper hyperinvariant subspace of S^*. If $\sigma_p(S^{**}) \neq \emptyset$ then $\overline{(S^* - \mu)\mathscr{X}^*}$ is a proper hyperinvariant subspace of S^*. If $\sigma_p(S^*) = \sigma_p(S^{**}) = \emptyset$, applying Lemma 3.1 we can find two compact disjoint sets σ, δ such that

$$\mathscr{X}^*_{S^*}(\sigma) \neq \{0\}, \qquad \mathscr{X}^{**}_{S^{**}}(\delta) \neq \{0\}.$$

For any $x^* \in \mathscr{X}^*_{S^*}(\sigma)$, $x^{**} \in \mathscr{X}^{**}_{S^{**}}(\delta)$ the function

$$f(\lambda) = \begin{cases} \langle x^*_{S^*}(\lambda), x^{**}\rangle, & \lambda \notin \sigma, \\ \langle x^*, x^{**}_{S^{**}}(\lambda)\rangle, & \lambda \in \sigma, \end{cases}$$

is analytic in $\mathbf{C}$ because

$$\langle x^*_{S^*}(\lambda), x^{**}\rangle = \langle x^*, x^{**}_{S^{**}}(\lambda)\rangle = \langle R(\lambda; S^*)\, x^*, x^{**}\rangle, \ |\lambda| > 1.$$

But we have $\lim_{\lambda\to\infty} f(\lambda) = 0$, thus by Liouville's Theorem we derive $f(\lambda) \equiv 0$, $\langle x^*, x^{**}\rangle = \lim_{\lambda\to\infty} \lambda f(\lambda) = 0$. This implies that $\overline{\mathscr{X}^*_{S^*}(\sigma)}$ is a proper hyperinvariant subspace of S^*.

3.3. REMARK. In case $\mathscr{X}$ is reflexive, Theorem 3.2 becomes a consequence of [10], Chapter V, Theorem 1.9.

3.4. PROPOSITION. *Let A be polynomially bounded. If $\tau(A) \cap \mathbf{D}^m$ is dominating and the commutant of $\{A_k^*\}_{k=1}^m$ has no proper invariant subspace then $A \in C_{0\vee 0}$, consequently Φ_*^A is a homeomorphism.*

Proof. Since A_k^* has no proper hyperinvariant subspace, applying Theorem 3.2 we infer $A \in C_{0\vee 0}$. But the inclusion $C_{0\vee 0} \subset C^0$ and Theorem 2.3 imply that Φ_*^A is a homeomorphism.

3.5. THEOREM. *Let A be polynomially bounded. If $(\tau_l(A)\setminus\tau_{le}(A)) \cup (\tau_r(A)\setminus\tau_{re}(A) \neq \emptyset$ then the commutant of $\{A_k\}_{k=1}^m$ has a proper invariant subspace.*

Proof. If $\omega \in \tau_l(A)\setminus\tau_{le}(A)$ we can find two subspaces $\mathscr{X}', \mathscr{X}''$ of $\mathscr{X}$ such that $\inf\{|||(A-\omega)x||| : x \in \mathscr{X}'', \|x\| = 1\} > 0$ and

$$\dim \mathscr{X}' = \operatorname{codim} \mathscr{X}'' < \infty,\ \mathscr{X}' \cap \mathscr{X}'' = \{0\},\ \mathscr{X}' + \mathscr{X}'' = \mathscr{X}.$$

Let $\{x_k\}_{k=1}^\infty \subset \mathscr{X}$ be such that $\|x_k\| = 1$, $\lim_{k\to\infty} |||(A-\omega)x_k||| = 0$. If $x_k = x_k' + x_k''$, $x_k' \in \mathscr{X}'$, $x_k'' \in \mathscr{X}''$ we may suppose that $\{x_k'\}_{k=1}^\infty$ converges strongly to x', consequently $\{x_k''\}_{k=1}^\infty$ converges strongly to x'' (via $(A-\omega)|\mathscr{X}''$ is bounded from below with respect to $|||\cdot|||$). Since we have $|||(A-\omega)(x'+x'')||| = 0$ we deduce that $\bigcap_{k=1}^m \ker(A_k - \omega_k)$ is a proper subspace invariant for the commutant of $\{A_k\}_{k=1}^m$. If $\omega \in \tau_r(A)\setminus\tau_{re}(A)$ we apply the above reasoning to A^* to show that $\overline{\sum_{k=1}^m ((A_k - \omega_k)\mathscr{X})}$ is a proper subspace invariant for the commutant of $\{A_k\}_{k=1}^m$.

§4. INVARIANT SUBSPACES FOR POLYNOMIALLY BOUNDED m-TUPLES

As before $A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m$ will denote a polynomially bounded m-tuple. For any $\sigma \subset \mathscr{X}$, $\delta \subset \mathscr{X}^*$ we shall put

$$\sigma \textcircled{A} \delta = \{x \underset{-}{\overset{A}{\otimes}} x^* : x \in \sigma,\ x^* \in \delta\}.$$

The closed ball in $\mathscr{X}$ with center in x and radius $b \geqslant 0$ will be denoted by $B(x, b, \mathscr{X})$. Consider the following possible properties of A:

(α) *There exist $b_A \geqslant 1$, $0 \leqslant r_A < 1$ such that for any $\varphi \in \mathscr{X} \underset{-}{\overset{A}{\otimes}} \mathscr{X}^*$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$ we have*

$$\operatorname{dist}(\varphi, B(x, b_A b^{1/2}, \mathscr{X}) \textcircled{A} B(x^*, b_A b^{1/2}, \mathscr{X}^*)) \leqslant r_A b,$$

where $b = \|\varphi - x \underset{-}{\overset{A}{\otimes}} x^*\|_*$.

(β) $\mathscr{X} \underset{-}{\overset{A}{\otimes}} \mathscr{X}^* = \{x \underset{-}{\overset{A}{\otimes}} x^* : x \in \mathscr{X}, x^* \in \mathscr{X}^*\}$,

(γ) *$A \in C^0$, $\tau(A) \cap \mathbf{D}^m$ is dominating and there exists $b_A \geqslant 1$ such that for any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\{\omega^{(k)}\}_{k=1}^n \subset \tau_{le}(A) \cap \mathbf{D}^m$, $\{c_k\}_{k=1}^n \subset \mathbf{C}$ we have*

$$\operatorname{dist}(\varphi, B(x, b_A b^{1/2}, \mathscr{X}) \textcircled{A} B(x^*, b_A b^{1/2}, \mathscr{X}^*)) = 0,$$

where $\varphi = x \underset{-}{\overset{A}{\otimes}} x^* + \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}}, \quad b = \sum_{k=1}^{n} |c_k|.$

(δ) $A \in C^0$, $\tau(A) \cap \mathbf{D}^m$ *is dominating and there exists* $b_A \geqslant 1$ *such that for any* $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\{\omega^{(k)}\}_{k=1}^{n} \subset \tau_{re}(A) \cap \mathbf{D}^m$, $\{c_k\}_{k=1}^{n} \subset \mathbf{C}$ *we have the same relation as in* (γ).

REMARK. By Theorem 2.3 if A has property (γ) or (σ) then Φ^A_* maps homeomorphically $H^\infty(\mathbf{D}^m)$ onto $\mathscr{A}_A$ and $\mathscr{E}^A_\omega$ is well defined.

4.1. THEOREM. *If* $A \in C^0$, $\tau(A) \cap \mathbf{D}^m$ *is dominating and* A *has property* (β) *then* A *has a proper invariant subspace.*

Proof. For any $\omega \in \mathbf{D}^m$, the functional $\mathscr{E}^A_\omega$, defined by Theorem 2.3, is of the form

$$\mathscr{E}^A_\omega(\Phi^A_*(h)) = \langle \Phi^A_*(h)\, x_\omega, x^*_\omega \rangle, \; h \in H^\infty(\mathbf{D}^m),$$

consequently

$$\langle (A_1 - \omega_1)\, \Phi^A_*(h)\, x_\omega,\; x^*_\omega \rangle = 0.$$

This implies that either $\ker\,(A_1 - \omega_1)$ or $\overline{\mathscr{A}_A(A_1 - \omega_1)x_\omega}$ is a proper invariant subspace of $\mathscr{A}_A$.

4.2. PROPOSITION. *The following implications hold true:*

(1) (α) $\Rightarrow$ (β)
(2) *if* $\tau_{le}(A) \cap \mathbf{D}^m$ *is dominating then* (γ) $\Rightarrow$ (α)
(3) *if* $\tau_{re}(A) \cap \mathbf{D}^m$ *is dominating then* (δ) $\Rightarrow$ (α)
(4) *if* $\tau_l(A) = \tau_{le}(A)$, $\tau_r(A) = \tau_{re}(A)$ *then* ((γ) + (δ)) $\Rightarrow$ (α).

Proof. (1) Let $\varphi \in \mathscr{X} \underset{-}{\overset{A}{\otimes}} \mathscr{X}^*$ be given. We can find by induction two sequences $\{x_n\}_{n=0}^{\infty} \subset \mathscr{X}$, $\{x_n^*\}_{n=0}^{\infty} \subset \mathscr{X}^*$ such that

$$x_0 = 0, \; x_0^* = 0, \; \|\varphi - x_n \underset{-}{\overset{A}{\otimes}} x_n^*\|_* \leqslant r^n \|\varphi\|_* \qquad 0 < r < 1,$$

$$\|x_{n+1} - x_n\| \leqslant b_A r^{n/2}\, \|\varphi\|_*^{1/2}, \; \|x_{n+1}^* - x_n^*\| \leqslant b_A r^{n/2} \|\varphi\|_*^{1/2}.$$

If we put $x = \lim_{n\to\infty} x_n$, $x^* = \lim_{n\to\infty} x_n^*$ we have $\varphi = x \underset{-}{\overset{A}{\otimes}} x^*$.

(2) Let $\varphi \in \mathscr{X} \underset{-}{\overset{A}{\otimes}} \mathscr{X}^*$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\varepsilon > 0$, be given and put $b = \|\varphi - x \underset{-}{\overset{A}{\otimes}} x^*\|$. Because $\tau_{le}(A) \cap \mathbf{D}^m$ is dominating and we have

$$\|(\varphi - x \underset{-}{\overset{A}{\otimes}} x^*) \circ \Phi^A_*\|_* \leqslant c_A b,$$

applying [5], Proposition 2.8 we can find $\{c_k\}_{k=1}^n \subset \mathbf{C}$, $\{\omega^{(k)}\}_{k=1}^n \subset \tau_{le}(A) \cap \mathbf{D}^m$, such that

$$\sum_{k=1}^n |c_k| \leqslant c_A b, \left\| (\varphi - x \overset{A}{\underset{-}{\otimes}} x^*) \circ \Phi_*^A - \sum_{k=1}^n c_k \mathscr{E}_{\omega^{(k)}}^A \right\|_* < \varepsilon.$$

Hence we derive

$$\left\| \varphi - x \overset{A}{\underset{-}{\otimes}} x^* - \sum_{k=1}^n c_k \mathscr{E}_{\omega^{(k)}}^A \right\|_* < \varepsilon \, \|(\Phi_*^A)^{-1}\|.$$

Because we assume that A has the property (γ) we can find $y \in B(x, b_A c_A^{1/2} b^{1/2}, \mathscr{X})$, $y^* \in B(x^*, b_A c_A^{1/2} b^{1/2}, \mathscr{X}^*)$ such that

$$\left\| x \overset{A}{\underset{-}{\otimes}} x^* + \sum_{k=1}^n c_k \mathscr{E}_{\omega^{(k)}}^A - y \overset{A}{\underset{-}{\otimes}} y^* \right\|_* < \varepsilon;$$

consequently we have

$$\operatorname{dist}(\varphi, B(x, b_A c_A^{1/2} b^{1/2}, \mathscr{X}) \; Ⓐ \; B(x^*, b_A c_A^{1/2} b^{1/2}, \mathscr{X}^*)) \leqslant$$

$$\leqslant \|\varphi - y \overset{A}{\underset{-}{\otimes}} y^*\|_* < \varepsilon(1 + \|(\Phi_*^A)^{-1}\|).$$

Choose $\varepsilon = b[2(1 + \|(\Phi_*^A)^{-1}\|)]^{-1}$, $r = 2^{-1}$.

(3) We repeat the proof of (2), with $\{\omega^{(k)}\}_{k=1}^n \subset \tau_{re}(A) \cap \mathbf{D}^m$.

(4) Let $\varphi \in \mathscr{X} \underset{-}{\otimes} \mathscr{X}^*$, $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\varepsilon > 0$ be given and put $b = \|\varphi - x \overset{A}{\otimes} x^*\|_*$. As in (2) we can find $\{c_k\}_{k=1}^n \subset \mathbf{C}$, $\{\omega^{(k)}\}_{k=1}^n \subset \tau(A) \cap \mathbf{D}^m$ such that

$$\sum_{k=1}^n |c_k| \leqslant c_A b, \left\| \varphi - x \overset{A}{\underset{-}{\otimes}} x^* - \sum_{k=1}^n c_k \mathscr{E}_{\omega^{(k)}}^A \right\|_* < \varepsilon.$$

If $\{\omega^{(k)}\}_{k \leqslant n_1} \subset \tau_{le}(A)$, $\{\omega^{(k)}\}_{k > n_1} \subset \tau_{re}(A)$ because $\tau(A) = \tau_{le}(A) \cup \tau_{re}(A)$ by assumption, applying (γ) and (δ) we can find first $y \in B(x, b'_A c_A^{1/2} b^{1/2}, \mathscr{X})$, $y^* \in B(x^*, b'_A c_A^{1/2} b^{1/2}, \mathscr{X}^*)$, such that

$$\left\| x \overset{A}{\underset{-}{\otimes}} x^* + \sum_{k \leqslant n_1} c_k \mathscr{E}_{\omega^{(k)}}^A - y \overset{A}{\underset{-}{\otimes}} y^* \right\|_* < \varepsilon$$

and then $z \in B(y, b''_A c_A^{1/2} b^{1/2}, \mathscr{X})$, $z^* \in B(y^*, b''_A c^{1/2} b^{1/2}, \mathscr{X}^*)$ such that

$$\left\| y \overset{A}{\underset{-}{\otimes}} y^* + \sum_{k > n_1} c_k \mathscr{E}_{\omega^{(k)}}^A - z \overset{A}{\underset{-}{\otimes}} z^* \right\|_* < \varepsilon,$$

where b'_A is provided by (γ) and b''_A is provided by (δ). If we put $b_A = c_A^{1/2}(b'_A + b''_A)$ we have

$$\operatorname{dist}(\varphi, B(x, b_A b^{1/2}, \mathscr{X}) \; \textcircled{A} \; B(x^*, b_A b^{1/2}, \mathscr{X}^*)) \leqslant$$

$$\leqslant \|\varphi - \underline{z} \overset{A}{\otimes} \underline{z}^*\|_* < \varepsilon + \left\| \underline{x} \overset{A}{\otimes} \underline{x}^* + \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega(k)} - \underline{z} \overset{A}{\otimes} \underline{z}^* \right\|_* \leqslant$$

$$\leqslant \varepsilon + \left\| \underline{x} \overset{A}{\otimes} \underline{x}^* + \sum_{k \leqslant n_1} c_k \mathscr{E}^A_{\omega(k)} - \underline{y} \overset{A}{\otimes} \underline{y}^* \right\|_* + \left\| \underline{y} \overset{A}{\otimes} \underline{y}^* + \sum_{k > n_1} c_k \mathscr{E}^A_{\omega(k)} - z \overset{A}{\otimes} z^* \right\|_* < 3\varepsilon.$$

Choose $\varepsilon = b/6$, $r_A = 2^{-1}$.

4.3. LEMMA. *Let $\mathscr{Y}$ be an n-dimensional subspace of $\mathscr{X}$ with a basis $\{e_k\}_{k=1}^n$ of unit vectors such that*

$$\left\| \sum_{k=1}^{n} e^{i\theta_k} \lambda_k e_k \right\| \leqslant a \left\| \sum_{k=1}^{n} \lambda_k e_k \right\|$$

for some fixed positive constant $a \geqslant 1$ and any $\lambda_k \in \mathbf{C}$, $\theta_k \in [0, 2\pi)$. Let $\{c_k\}_{k=1}^n \subset \mathbf{C}$ be such that $\sum_{k=1}^{n} |c_k| = 1$. Then there exist $\{\mu_k\}_{k=1}^n \subset \mathbf{C}$ and $y^* \in Y^*$ such that

$$\left\| \sum_{k=1}^{n} \mu_k e_k \right\| \leqslant 1, \; \|y^*\| \leqslant a, \; \mu_k y^*(e_k) = c_k.$$

Proof. Assume first that $a = 1$ and the norm in $\mathscr{Y}$ is strictly convex and smooth (see [11], Ch. VII, § 2). If we put

$$\sigma = \left\{ \sum_{k=1}^{n} a_k e_k : a_k \geqslant 0, \left\| \sum_{k=1}^{n} a_k e_k \right\| = 1 \right\}$$

then we know that for any $x \in \sigma$ there exists a unique $y_x^* \in \mathscr{Y}^*$ such that $\|y_x^*\| = y_x^*(x) = 1$ (via the smoothness of the norm in $\mathscr{Y}$). Because the norm in $\mathscr{Y}$ is invariant under rotations on the axis determined by the basis $\{e_k\}_{k=1}^n$, we derive easily $y_x^*(e_k) \geqslant 0$, $x \in \sigma$, $1 \leqslant k \leqslant n$. Let $e_k^* \in \mathscr{Y}^*$ be such that

$$\|e_k^*\| = e_k^*(e_k) = 1, \; e_k^*(e_j) = 0, \; k \neq j.$$

Then we have

$$\left\| \sum_{k=1}^{n} e^{i\theta_k} \lambda_k e_k^* \right\| = \left\| \sum_{k=1}^{n} \lambda_k e_k^* \right\|$$

for any $\lambda_k \in \mathbf{C}$, $\theta_k \in [0, 2\pi)$ and the functions

$$\lambda_j \to \left\| \sum_{k=1}^{n} \lambda_k e_k \right\|, \; \lambda_j \to \left\| \sum_{k=1}^{n} \lambda_k e_k^* \right\|$$

increase with $|\lambda_j|$. This, together with the strict convexity of the norm in both $\mathscr{Y}$ and $\mathscr{Y}^*$, imply the following equivalence:

$$y_x^*(e_k) = 0 \Leftrightarrow e_k^*(x) = 0.$$

Suppose our lemma holds true if the dimension of the subspace does not exceed $n-1$ ($\geqslant 1$). For any $t \in [0, 1]$ put

$$e_k(t) = \begin{cases} e_k & , \ k < n-1, \\ te_{n-1} + f(t)\, e_n, & k = n-1, \end{cases}$$

where $f(t) \in [0, 1]$, $\|e_{n-1}(t)\| = 1$. The function f is continuous and $f(0) = 1, f(1) = 0$. Using the above assumption on the dimension, the Hahn-Banach Theorem and the invariance under rotations on axis we can find $x(t) \in \sigma$, $x(t) = \sum_{k=1}^{n-1} a_k(t)\, e_k(t)$, $y_t^* \in \mathscr{Y}^*$ ($y_t^* = y_{x(t)}^*$) such that

$$\|x(t)\| = \|y_t^*\| = 1,\ a_k(t)\, y_t^*(e_k(t)) = |c_k|,\ k < n-1,$$

$$a_{n-1}(t)\, y_t^*(e_{n-1}(t)) = |c_{n-1}| + |c_n|.$$

The vectors $x(t)$ and y_t^* are uniquely determined because if $x'(t)$, $y_t'^*$ is another solution of the previous equations and $x'(t) = \sum_{k=1}^{n-1} a_k'(t) e_k(t)$ we have

$$y_t'^*(x(t)) + y_t^*(x'(t)) = \sum_{k=1}^{n-1} [a_k(t)\, y_t'^*(e_k(t)) + a_k'(t)\, y_t^*(e_k(t))] \leqslant 2.$$

But since $|c_k| = a_k(t)\, y_t^*(e_k(t)) = a_k'(t)\, y_t'^*(e_k(t))$, if we put

$$r_k(t) = a_k'(t)/a_k(t) = y_t^*(e_k(t))/y_t'^*(e_k(t)) \quad \text{if } a_k(t) \neq 0$$

we derive

$$y_t'^*(x(t)) + y_t^*(x'(t)) = \sum_{a_k(t) \neq 0} (r_k(t) + 1/r_k(t))\, a_k(t)\, y_t^*(e_k(t)) \leqslant 2$$

which is possible only if $r_k(t) = 1$, $x(t) = x'(t)$, $y_t^* = y_t'^*$. This implies the continuity of the functions

$$t \to a_k(t), \quad t \to y_t^*$$

and in particular the function

$$t \to y_t^*(a_{n-1}(t)\, te_{n-1})$$

will be continuous. Since $y_0^*(0) = 0$, $y_1^*(a_{n-1}(1)\, e_{n-1}) = a_{n-1}(1)\, y_1^*(e_{n-1}(1)) = |c_{n-1}| + |c_n|$ we can find $s \in [0, 1]$ such that $y_s^*(a_{n-1}(s)\, se_{n-1}) = |c_{n-1}|$. If $c_k = e^{i\theta_k}|c_k|$ and we put

$$\mu_k = e^{i\theta_k} a_k(s),\ k < n - 1,\ \mu_{n-1} = e^{i\theta_{n-1}} a_{n-1}(s)\, s,$$
$$\mu_n = e^{i\theta_n} a_{n-1}(s) f(s),\ y^* = y_s^*$$

we have a solution for the conditions in our lemma. The proof will be completed by induction if we observe that in case $\dim \mathscr{Y} = 1$, the lemma is trivial.

Further if $a = 1$ and $\|\cdot\|$ is not strictly convex and smooth then for any $0 < \varepsilon < 1$ let $\|\cdot\|'_\varepsilon$ denote the norm in $\mathscr{Y}$ determined by duality with the norm in $\mathscr{Y}^*$ determined by the equation

$$\left\| \sum_{k=1}^{n} \lambda_k e_k^* \right\|'_\varepsilon = (1 - \varepsilon) \left\| \sum_{k=1}^{n} \lambda_k e_k^* \right\| + \varepsilon \left(\sum_{k=1}^{n} |\lambda_k|^2 \right)^{1/2}$$

and define the norm $\|\cdot\|_\varepsilon$ in $\mathscr{Y}$ by the equation

$$\left\| \sum_{k=1}^{n} \lambda_k e_k \right\|_\varepsilon = (1 - \varepsilon) \left\| \sum_{k=1}^{n} \lambda_k e_k \right\|'_\varepsilon + \varepsilon \left(\sum_{k=1}^{n} |\lambda_k|^2 \right)^{1/2}.$$

The norm $\|\cdot\|'_\varepsilon$ in $\mathscr{Y}$ is smooth, because $\|\cdot\|'_\varepsilon$ in $\mathscr{Y}^*$ is strictly convex and the norm $\|\cdot\|_\varepsilon$ in $\mathscr{Y}$ is both smooth and strictly convex as well as invariant under rotations on axis. Thus we can find a solution $x_\varepsilon, y_\varepsilon^*$. Finally using the compactness of the closed unit ball in finite dimensional spaces and the relations $\lim_{\varepsilon \to 0} \|z\|_\varepsilon = \|z\|$, $\lim_{\varepsilon \to 0} \|z^*\|_\varepsilon = \|z^*\|$, uniformly with respect to z and z^* in compact sets, we can find a solution $\{\mu_k\}_{k=1}^n$, y^* for the conditions in our lemma as a limit of ε-solutions.

Now assume $a > 1$. If we introduce the norm

$$\left\| \sum_{k=1}^{n} \lambda_k e_k \right\|_0 = \sup \left\{ \left\| \sum_{k=1}^{n} e^{i\theta_k} \lambda_k e_k \right\| : \theta_k \in [0, 2\pi) \right\}$$

and $x = \sum_{k=1}^{n} \mu_k e_k$, $y^* \in \mathscr{Y}^*$, $\|x\|_0 \leqslant 1$, $\|y^*\|_0 \leqslant 1$ is a solution, we have $\|x\| \leqslant 1$, $\|y^*\| \leqslant a$, thus the assumption $a = 1$ is not a restriction.

Now recall that a *scalar operator* (in the sense of Dunford, [12], § 3) acting in $\mathscr{X}$ is an integral of the form $\int \lambda E(d\lambda)$, where E denotes a strongly countably additive spectral measure with compact support, with domain the Boolean algebra all Borel subsets in **C** and value projections in $\mathscr{X}$. For any continuous complex function f defined on $\sigma\left(\int \lambda\, E(d\lambda)\right)$ we have

$$\left\| \int f(\lambda)\, E(d\lambda) \right\| \leqslant v(E) \sup |f(\lambda)|,$$

where $v(E)$ is a positive constant depending on E.

4.4. THEOREM. *Suppose there exist $K \in \mathscr{L}(\mathscr{X})$, $K' \in \mathscr{L}(\mathscr{X}^*)$, compact injective scalar operators. If $A \in C_{0\cdot} \cap C_{\cdot 0}$ and $\tau(A) \cap \mathbf{D}^m$ is dominating then A has a proper invariant subspace.*

Proof. Because we have in particular $A \in C^0$, applying Theorem 3.5, Theorem 4.1 and Proposition 4.2 we shall show that A has properties (γ) and (δ). Let $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\{c_k\}_{k=1}^n \subset \mathbf{C}$, $\{\omega^{(k)}\}_{k=1}^n \subset \tau_{1e}(A) \cap \mathbf{D}^m$, $\varepsilon > 0$, be given and let E denote the spectral measure of K. Because K is compact and injective, $E(\sigma)$ is a finite rank projection whenever σ is a finite set and $\|E(\sigma)x - x\|$ can be made arbitrarily small if σ is suitably chosen (see [12], § 3). By Corollary 1.9 we also know that

$$\{\Phi_*^A(h)\,x : h \in H^\infty(\mathbf{D}^m),\ \|h\|_\infty < 1\}$$

is a compact set in the strong topology of $\mathscr{X}$.

We can choose finite disjoint subsets $\{\sigma_k\}_{k=1}^n$ of $\sigma(K)$ and pick e_k such that e_k $(\ker x^*) \cap E(\sigma_k)\mathscr{X}$, $\|e_k\| = 1$, $1 \leqslant k \leqslant n$ and

$$|||(A - \omega^{(k)})\,e_k||| \leqslant \varepsilon,\quad \left\| E\left(\bigcup_{k=1}^n \sigma_k\right) \Phi_*^A(h)\,x \right\| \leqslant \varepsilon \|h\|_\infty.$$

Since we have

$$\left\| \sum_{k=1}^n \mathrm{e}^{\mathrm{i}\theta_k} \lambda_k e_k \right\| \leqslant v(E) \left\| \sum_{k=1}^n \lambda_k e_k \right\|$$

applying Lemma 4.3 and Hahn-Banach Theorem, we can find $\{\mu_k\}_{k=1}^n \subset \subset \mathbf{C}, z^* \in E\left(\bigcup_{k=1}^n \sigma_k\right)^* \mathscr{X}^*$ such that

$$\left\| \sum_{k=1}^n \mu_k e_k \right\| \leqslant 1,\ \|z^*\| \leqslant v(E)^2,\ \mu_k z^*(e_k) = c_k \left(\sum_{j=1}^n |c_j| \right)^{-1}$$

$$|||(A - \omega^{(k)})\,e_k||| \leqslant \varepsilon,\quad \left\| E\left(\bigcup_{k=1}^n \sigma_k\right) \Phi_*^A(h)\,x \right\| \leqslant \varepsilon \|h\|_\infty\ .$$

If we put

$$w = v(E) \left(\sum_{k=1}^n |c_k| \right)^{1/2} \sum_{k=1}^n \mu_k e_k,\ w^* = v(E)^{-1} \left(\sum_{k=1}^n |c_k| \right)^{1/2} z^*,$$

$$b_A = v(E),\ u = x + w,\ u^* = x^* + w^*$$

we have $u \in B(x, b_A \left(\sum_{k=1}^n |c_k| \right)^{1/2}, X)$, $u^* \in B\left(x^*,\ b_A \left(\sum_{k=1}^n |c_k| \right)^{1/2},\ X^*\right)$.

Since

$$\mu_k z^*(e_k) = c_k(\sum |c_k|)^{-1} \text{ and } \|\mu_k\| \leqslant v(E),$$

we have

$$\left\| \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - w \underset{-}{\overset{A}{\otimes}} w^* \right\|_* \leqslant \sup_{\|h\|_\infty \leqslant 1} \left| \left(\sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - w \underset{-}{\overset{A}{\otimes}} w^* \right) (\Phi^A_*(h)) \right| =$$

$$= \left(\sum_{k=1}^{n} |c_k| \right) \sup_{\|h\|_\infty \leqslant 1} \left| \sum_{k=1}^{n} \mu_k \langle (h(\omega^{(k)}) - \Phi^A_*(h))\, e_k, z^* \rangle \right| \leqslant$$

$$\leqslant v(E)^2 \left(\sum_{k=1}^{n} |c_k| \right) \sup_{\|h\|_\infty \leqslant 1} \left(\sum_{k=1}^{n} |\mu_k| \, \|(h(\omega^{(k)}) - \Phi^A_*(h))\, e_k\| \right) \leqslant$$

$$\leqslant v(E)^3 \left(\sum_{k=1}^{n} |c_k| \right) \left(\sum_{k=1}^{n} \sup_{\|h\|_\infty \leqslant 1} \|(h(\omega^{(k)}) - \Phi^A(h))\, e_k\| \right).$$

Since $z^* \in E\left(\sum_{k=1}^{n} \sigma_k \right)^* x^*$, we have

$$\|x \underset{-}{\overset{A}{\otimes}} w^*\|_* \leqslant \sup_{\|h\|_\infty \leqslant 1} |\langle \Phi^A_*(h)\, x, w^* \rangle| =$$

$$= v(E)^{-1} \left(\sum_{k=1}^{n} |c_k| \right)^{1/2} \sup_{\|h\|_\infty \leqslant 1} \left| \left\langle E\left(\bigcup_{k=1}^{n} \sigma_k \right) \Phi^A_*(h)\, x, z^* \right\rangle \right| \leqslant$$

$$\leqslant \varepsilon v(E) \left(\sum_{k=1}^{n} |c_k| \right)^{1/2}.$$

Finally since $\langle e_k, x^* \rangle = 0$, we have

$$\|w \underset{-}{\overset{A}{\otimes}} x^*\|_* \leqslant v(E) \left(\sum_{k=1}^{n} |c_k| \right)^{1/2} \sup_{\|h\|_\infty \leqslant 1} \left| \sum_{k=1}^{n} \mu_k \langle \Phi^A_*(h) e_k, x^* \rangle \right| \leqslant$$

$$\leqslant v(E)^2 \left(\sum_{k=1}^{n} |c_k| \right)^{1/2} \left(\sum_{k=1}^{n} \sup_{\|h\|_\infty \leqslant 1} |\langle \Phi^A_*(h)\, e_k, x^* \rangle| \right) =$$

$$= v(E)^2 \left(\sum_{k=1}^{n} |c_k| \right)^{1/2} \left(\sum_{k=1}^{n} \sup_{\|h\|_\infty \leqslant 1} |\langle (\Phi^A_*(h) - h(\omega^{(k)}))\, e_k, x^* \rangle| \right) \leqslant$$

$$\leqslant v(E)^2 \left(\sum_{k=1}^{n} |c_k| \right)^{1/2} \|x^*\| \left(\sum_{k=1}^{n} \sup_{\|h\|_\infty \leqslant 1} \|(\Phi^A_*(h) - h(\omega^{(k)}))\, e_k \right).$$

Now using the relations

$$\left\| x \underset{-}{\overset{A}{\otimes}} x^* + \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - u \underset{-}{\overset{A}{\otimes}} u^* \right\|_* =$$

$$= \left\| \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - w \underset{-}{\overset{A}{\otimes}} w^* - x \underset{-}{\overset{A}{\otimes}} w^* - w \underset{-}{\overset{A}{\otimes}} x^* \right\|_* \leqslant$$

$$\leqslant \left\| \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - w \underset{-}{\overset{A}{\otimes}} w^* \right\|_* + \| x \underset{-}{\overset{A}{\otimes}} w^* \|_* + \| w \underset{-}{\overset{A}{\otimes}} x^* \|_*$$

and Theorem 2.1 we deduce that $\left\| x \underset{-}{\overset{A}{\otimes}} x^* + \sum_{k=1}^{n} c_k \mathscr{E}^A_{\omega^{(k)}} - u \underset{-}{\overset{A}{\otimes}} u^* \right\|_*$ can be made arbitrarily small, thus A has the property (γ).

The first part of the proof implies that A^* has property (γ), too, thus if $\{\omega^{(k)}\}_{k=1}^n \subset \tau_{re}(A)$, we can find

$$u^{**} \in B\left(x,\, b_A \left(\sum_{k=1}^{n} |c_k| \right)^{1/2},\, \mathscr{X}^{**}\right), \quad u^* \in B\left(x^*,\, b_A \left(\sum_{k=1}^{n} |c_k| \right)^{1/2},\, \mathscr{X}^*\right)$$

such that

$$\sup_{\|h\|_\infty \leqslant 1} \left| \langle \Phi^A_*(h)\, x, x^* \rangle + \sum_{k=1}^{n} c_k h(\omega^{(k)}) - u^{**}(\Phi^A_*(h)^* u^*) \right| \leqslant \varepsilon.$$

Because

$$\{\Phi^A_*(h)^* u^* : h \in H^\infty(\mathbf{D}^m),\ \|h\|_\infty \leqslant 1\}$$

is a compact set in the strong topology of $\mathscr{X}^*$ and the unit ball of $\mathscr{X}$ is weak* dense in the unit ball of $\mathscr{X}^{**}$, we may suppose $u^{**} \in \mathscr{X}$. This shows that A has the property (δ).

4.5. Corollary. *Suppose $\mathscr{X}$ is reflexive and there exists $K \in \mathscr{L}(\mathscr{X})$ a compact injective scalar operator. If $A \in C_{0\cdot} \cap C_{\cdot 0}$ and $\tau(A) \cap \mathbf{D}^m$ is dominating then A has a proper invariant subspace.*

Proof. By [12], Theorem 18, (iv), $K^* \in \mathscr{L}(\mathscr{X}^*)$ is a compact scalar operator. It is obvious that K has dense range, thus K^* is injective and we can apply Theorem 4.4.

4.6. Corollary. *Suppose $\mathscr{X}$ has an unconditional basis and either $\mathscr{X}$ is reflexive or $\mathscr{X}^*$ has an unconditional basis. If $A \in C_{0\cdot} \cap C_{\cdot 0}$ and $\tau(A) \cap \mathbf{D}^m$ is dominating then A has a proper invariant subspace.*

Proof. Let $\{x_n\}$ be an unconditional basis in $\mathscr{X}$ (see [26], Definition 14.1, p. 396). Put $\sigma = \{0\} \cup \{1/n\}_{n \geqslant 1}$. Using [26], Theorem 17.1 we can define a spectral measure E on the Borel subsets of σ as follows: for any $\delta \subset \sigma$ and $x \in \mathscr{X}$ we put

$$E(\delta)\, x = \sum_{k^{-1} \in \delta} \lambda_k x_k, \text{ if } x = \sum_{k \geqslant 1} \lambda_k x_k.$$

It is easy to check that $K = \int \lambda E(\mathrm{d}\lambda) \in \mathscr{L}(\mathscr{X})$ is a compact injective scalar operator thus either by Corollary 4.5 or by Theorem 4.4 we deduce that A has a proper invariant subspace.

4.7. THEOREM. *Suppose there exists $K \in \mathscr{L}(\mathscr{X})$ a compact injective scalar operator (or suppose, in particular, that $\mathscr{X}$ has an unconditional basis). If $A \in C_{0\cdot}$ and $\tau_1(A) \cap \mathbf{D}^m$ is dominating then A has a proper invariant subspace.*

Proof. Because $A \in C_{0\cdot}$, the proof of Theorem 4.4 shows that A has the property (γ) and by Theorem 3.5, Theorem 4.1 and Proposition 4.2, A has a proper invariant subspace.

4.8. THEOREM. *Suppose there exist $K' \in \mathscr{L}(\mathscr{X}^*)$ a compact injective scalar operator (or suppose, in particular, that $\mathscr{X}^*$ has an unconditional basis). If $A \in C_{\cdot 0}$ and $\tau_r(A) \cap \mathbf{D}^m$ is dominating then A has a proper invariant subspace.*

Proof. Because $A \in C_{\cdot 0}$ we derive as in the proof of Theorem 4.7 that A has the property (δ) and that A has a proper invariant subspace.

Recall now that $\mathscr{X}$ is called *uniformly convex* if whenever $\{x_n\}_{n=1}^{\infty} \subset \mathscr{X}$, $\{y_n\}_{n=1}^{\infty} \subset \mathscr{X}$ are sequences such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1/2\,(\lim_{n\to\infty} \|x_n + y_n\|) = 1$$

we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

A uniformly convex space is reflexive (see [11], pag. 188—189).

Consider the following possible properties of $A \in C^0$, if Φ^A_* is a homeomorphism:

(γ') For any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\varepsilon > 0$, $b > 0$, there exists $0 \leqslant r < 1$ such that

$$\operatorname{dist}(\varphi, B(x,\ b^{1/2},\ \mathscr{X}) \textcircled{A} B(x^*,\ b^{1/2},\ \mathscr{X}^*)) < \varepsilon$$

whenever $\varphi = x \underset{-}{\overset{A}{\otimes}} x^* + \sum_{l=1}^{n} c_l \mathscr{E}^A_{\omega^{(l)}}$, $\sum_{l=1}^{n} |c_l| = b$, $\{c_l\}_{l=1}^{n} \subset \mathbf{C}$, $\{\omega^{(l)}\}_{l=1}^{n} \subset \tau_{le}(A) \cap \mathbf{D}^m$,

$\min_{1 \leqslant k \leqslant m} |\omega_k^{(l)}| \geqslant r, \quad 1 \leqslant l \leqslant n.$

(δ') For any $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $\varepsilon > 0$, $b > 0$ there exists $0 \leqslant r < 1$ such that

$$\operatorname{dist}\,(\varphi, B(x, b^{1/2}, \mathscr{X}) \textcircled{A} B(x^*, b^{1/2}, \mathscr{X}^*)) < \varepsilon$$

whenever $\varphi = x \overset{A}{\otimes} x^* + \sum_{l=1}^{n} c_l \mathscr{E}^A_{\omega^{(l)}}$, $\sum_{l=1}^{n} |c_l| = b$, $\{c_l\}_{l=1}^n \subset \mathbf{C}$, $\{\omega^{(l)}\}_{l=1}^n \subset \tau_{re}(A) \cap \mathbf{D}^m$,

$\min_{1 \leqslant k \leqslant m} |\omega_k^{(l)}| \geqslant r, \quad 1 \leqslant l \leqslant n.$

4.9. PROPOSITION. *Suppose $A \in C^0$ and Φ^A_* is a homeomorphism. Then the following implications hold:*

(i) *if $\tau_{le}(A) \cap \mathbf{D}^m$ is dominating, (γ') $\Rightarrow$ (α)*
(ii) *if $\tau_{re}(A) \cap \mathbf{D}^m$ is dominating, (δ') $\Rightarrow$ (α)*
(iii) *if $\tau_e(A) \cap \mathbf{D}^m$ is dominating, ((γ') + (δ')) $\Rightarrow$ (α).*

Proof. The condition $\min_{1 \leqslant k \leqslant m} |\omega_k^{(l)}| \geqslant r$, $1 \leqslant l \leqslant n$, makes no obstruction if we repeat the proof of Proposition 4.2, (2), (3), (4).

4.10. LEMMA. *Suppose there exists a compact injective scalar operator $K \in \mathscr{L}(\mathscr{X})$ with the spectral measure E such that $v(E) = 1$ (i.e. $\| \int f(\lambda) E(\mathrm{d}\lambda) \| = \sup_{\lambda \in \sigma(K)} |f(\lambda)|$, for any continuous complex function f defined on $\sigma(K)$) and $\mathscr{X}^*$ is uniformly convex. If $A \in C^0$, $A_1 \in C_{0 \cdot}$, $\|A_1\| = 1$ and Φ^A_* is a homeomorphism then A has the property (γ').*

Proof. We imitate that part of the proof of Theorem 4.4 in which it is shown that A has the property (γ). Using the same notation, everything goes smooth except the fact that $\|x \overset{A}{\otimes} w^*\|_*$ can be made arbitrarily small, because the argument involving compactness is not available. We shall show that $\| x \overset{A}{\otimes} w^* \|_*$ can be made small if $0 \leqslant r < 1$ is properly chosen and $|\omega_1^{(l)}| \geqslant r$, $1 \leqslant l \leqslant n$. Since $w^* = b^{1/2} z^*$ because of $v(E) = 1$ and z^* is produced by Lemma 4.3 we have

$$\|z^*\| = 1 = z^* \left(\sum_{l=1}^{n} |\mu_l| e_l \right)$$

(see the proof of Lemma 4.3). Let $\eta > 0$ be given and put

$$\mathscr{U} = \{S \in \mathscr{L}(\mathscr{X}) \colon A_1 S = S A_1,\ \|S\| \leqslant 1\},$$

$$\mathscr{U}_\eta = \{S \in \mathscr{U} \colon \|Sx\| \leqslant \eta\},$$

$$\mathscr{U}'_\eta = \{S \in \mathscr{U} \colon \|Sx\| > \eta\}.$$

Since $\|x \overset{A}{\otimes} w^*\|_* \leqslant \left(\sum_{k=1}^{n} |c_k|\right)^{1/2} \sup_{\|h\|_\infty \leqslant 1} |\langle \Phi_*^A(h)x, x^*\rangle|$ and each $\Phi_*^A(h)$ commutes with A_1 it suffices to show that we have

$$|z^*(Sx)| \leqslant \eta, \ S \in \mathscr{U},$$

if $|\omega_1^{(l)}| \geqslant r$ and r is enough close to 1. Since the above relation holds true if $S \in \mathscr{U}_\eta$, pick $S \in \mathscr{U}'_S$. If we put

$$\mathrm{d}_S = \inf \left\{ \left\| \lambda\, Sx - \sum_{l=1}^{n} |\mu_l| e_l \right\| : \lambda \in \mathbf{C} \right\}$$

and if we choose $\lambda_S \in \mathbf{C}$ such that

$$\mathrm{d}_S = \left\| \lambda_S Sx - \sum_{l=1}^{n} |\mu_l| e_l \right\|$$

we have $|\lambda_S| \leqslant 2\eta^{-1}$ and since $\|A_1\| = 1$ and $\|S\| \leqslant 1$ for any natural number k

$$\mathrm{d}_S \geqslant \left\| A_1^k \left(\lambda_S Sx - \sum_{l=1}^{n} |\mu_l|\, e_l \right) \right\| \geqslant$$

$$\geqslant \left\| \sum_{l=1}^{n} |\mu_l| (\omega_1^{(l)})^k e_l \right\| - 2\eta^{-1} \|A_1^k x\| - \left\| \sum_{l=1}^{n} |\mu_l| (A_1^k - (\omega_1^{(l)})^k) e_l \right\|.$$

Now letting k to be enough large and then letting r to be close to 1 and choosing properly $\{e_l\}_{l=1}^n$, it becomes obvious that we may suppose that $(1 - \mathrm{d}_S)$ is arbitrarily small, uniformly with respect to S. By Hahn-Banach Theorem we can find $z_S^* \in \mathscr{X}^*$ such that

$$\|z_S^*\| \leqslant \mathrm{d}_S^{-1}, \ z_S^*(Sx) = 0, \ z_S^* \left(\sum_{l=1}^{n} |\mu_l|\, e_l \right) = 1.$$

Finally the assumption on the uniform convexity of $\mathscr{X}^*$ implies that $\|z^* - z_S^*\|$ is small if $|1 - \mathrm{d}_S|$ is small and this concludes the proof.

4.11. THEOREM. *Let A be a polynomially bounded m-tuple. Suppose there exists a compact injective scalar operator $K \in \mathscr{L}(\mathscr{X})$ with the spectral measure E such that $v(E) = 1$. Then A has a proper invariant subspace in case at least one of the following conditions is fulfilled:*

(i) *$\mathscr{X}$ and $\mathscr{X}^*$ are uniformly convex, $\tau(A) \cap \mathbf{D}^m$ is dominating and there exist $1 \leqslant k \leqslant m$, $1 \leqslant j \leqslant m$ such that $A_k \in C_{0\cdot}$, $A_j \in C_{\cdot 0}$, $\|A_k\| = \|A_j\| = 1$,*

(ii) *$\mathscr{X}^*$ is uniformly convex, $\tau_1(A) \cap \mathbf{D}^m$ is dominating and there exists $1 \leqslant k \leqslant m$ such that $A_k \in C_{0\cdot}$, $\|A_k\| = 1$.*

(iii) $\mathscr{X}$ *is uniformly convex*, $\tau_r(A) \cap \mathbf{D}^m$ *is dominating and there exists* $1 \leqslant j \leqslant m$ *such that* $A_j \in C_{\cdot 0}$, $\|A_j\| = 1$.

Proof. By Proposition 3.4 and Theorem 4.5 we may assume $A \in C^0$, Φ_*^A is a homeomorphism and $\tau_1(A) = \tau_{1e}(A)$, $\tau_r(A) = \tau_{re}(A)$. Using Lemma 4.10 we infer that A has the property (γ') in cases (i), (ii) and the property (δ') in cases (i), (iii). To conclude the proof we apply Proposition 4.9,Theorem 4.1 and Proposition 4.2.

4.12. THEOREM. *Suppose* $\mathscr{X}$ *is a Hilbert space and* T, $S \in \mathscr{L}(\mathscr{X})$ *are commuting contractions. Put* $B = (T, S)$. *Then* B *has a proper invariant subspace in case at least one of the following conditions is fulfilled:*

(i) $\tau(B) \cap \mathbf{D}^2$ *is dominating and* $T \in C_{0\cdot}$, $S \in C_{\cdot 0}$,

(ii) $\tau(B) \cap \mathbf{D}^2$ *is dominating and* $T \in C_{0\cdot} \cap C_{\cdot 0}$,

(iii) $\tau_1(B) \cap \mathbf{D}^2$ *is dominating and* $T \in C_{0\cdot}$,

(iv) $\tau_r(B) \cap \mathbf{D}^2$ *is dominating and* $T \in C_{\cdot 0}$.

Proof. By the theorem of Ando [2] (see also [28], Ch. I. § 6) B follows to be polynomially bounded, thus we can apply Theorem 4.11.

4.13. THEOREM. *Suppose there exists* $K \in \mathscr{L}(\mathscr{X})$ *a compact injective scalar operator (in particular suppose that* $\mathscr{X}$ *has an unconditional basis) and* $\mathscr{X}$ *is reflexive Then any polynomially bounded operator* $T \in \mathscr{L}(\mathscr{X})$ *such that* $\sigma(T) \cap \mathbf{D}$ *is dominating has a proper invariant subspace.*

Proof. It is easy to see that if T has no proper hyperinvariant subspace then we have

$$\sigma(T) = \tau_{1e}(T) = \tau_{re}(T)$$

and by Proposition 3.4 we may assume $T \in C_{0\cdot}$. Now we can apply Theorem 4.7.

§ 5. INVARIANT SUBSPACES AND QUASISCALAR OPERATORS

Let $\mathscr{Y}$ be a complex Banach space including $\mathscr{X}$ as a subspace and let

$$A = (A_1, \ldots, A_m) \in \mathscr{L}(\mathscr{X})^m, \quad B = (B_1, \ldots, B_m) \in \mathscr{L}(\mathscr{Y})^m$$

be given. We shall call B an *extension* of A if

$$B_k\mathscr{X} \subset \mathscr{X}, \; B_k|\mathscr{X} = A_k, \; 1 \leqslant k \leqslant m.$$

If $\mathscr{M}$ is a compact Hausdorff space we denote by $\mathscr{C}(\mathscr{M})$ the supnorm algebra of all continuous complex functions defined on $\mathscr{M}$. A *spectral distribution*

$$\mathscr{V} : \mathscr{C}(\mathscr{M}) \to \mathscr{L}(\mathscr{Y})$$

will be a linear multiplicative continuous map such that $\mathscr{V}(1) = I$(= the identity operator). If B is a commutative m-tuple, then the carrier space $\mathscr{M}_B$ of the Banach algebra generated by $\{B_k\}_{k=1}^m$ and I, can be canonically identified with a compact subset of $\mathbf{C}^m$ (see [22], Ch. III, § 1). We shall call B a *quasiscalar* m-tuple if B is commutative and there exists a spectral distribution $\mathscr{V}_B : \mathscr{C}(\mathscr{M}_B) \to \mathscr{L}(\mathscr{Y})$ such that

$$\mathscr{V}_B(p) = p(B_1, \ldots, B_m)$$

for any polynomial p. It is obvious that a quasiscalar operator is a generalized scalar operator of order 0, in the sense of [10], Ch. IV, § 1. If $\mathscr{Y}$ is reflexive then any quasiscalar operator is a scalar operator (see [12], Theorem 18).

Let $S \in \mathscr{L}(\mathscr{Y})$ be given. A *spectral maximal space* of S will be an invariant subspace $\mathscr{Z}$ of S, such that $\mathscr{Z}_0 \subset \mathscr{Z}$ whenever $\mathscr{Z}_0$ is an invariant subspace of S and $\sigma(S|\mathscr{Z}_0) \subset \sigma(S|\mathscr{Z})$ (see [10], Ch. I, § 3). We shall call S a *decomposable* operator if for any open covering $\{G_k\}_{k=1}^n$ of $\sigma(S)$ there exists a system $\{\mathscr{Z}_k\}_{k=1}^n$ of spectral maximal spaces of S such that (see [10], Ch. II, Definition 1.1)

$$\mathscr{Y} = \sum_{k=1}^{n} \mathscr{Z}_k, \ \sigma(S|\mathscr{Z}_k) \subset G_k, \quad 1 \leqslant k \leqslant n.$$

If S is decomposable then S has the single-valued extension property and for any closed set σ,

$$\mathscr{Y}_S(\sigma) = \{y \in \mathscr{Y} : \ \sigma_S(y) \subset \sigma\}$$

is a spectral maximal space of S such that $\sigma(S|\mathscr{Y}_S(\sigma)) \subset \sigma$ ([10], Ch. II). If S is a quasiscalar operator then by [10], Ch. III, Theorem 1.19, S follows to be decomposable. An operator $T \in \mathscr{L}(\mathscr{X})$ will be called *subdecomposable*, resp. *subquasiscalar*, resp. *subscalar* if it has a decomposable, resp. quasiscalar, resp. scalar extension $S \in \mathscr{L}(\mathscr{Y})$. Analogously, we call A a subquasiscalar m-tuple if it has a quasiscalar extension $B \in \mathscr{L}(\mathscr{Y})^m$.

5.1. PROPOSITION. *Let $T \in \mathscr{L}(\mathscr{X})$ be such that T is power bounded and T^* is subdecomposable. Then either T^* has a proper hyperinvariant subspace or $T \in C_{0\cdot}$ or T is a scalar multiple of the identity.*

Proof. Assume $T \notin C_{0\cdot}$, $\sigma_p(T^*) = \emptyset$. Applying Lemma 3.1 we can find two compact sets $\sigma, \delta \subset \sigma(T^*)$ such that

$$\sigma \cap \delta = \emptyset, \ \mathscr{X}^*_{T^*}(\sigma) \neq \{0\}, \ \mathscr{X}^*_{T^*}(\delta) \neq \{0\}.$$

If $S \in \mathscr{L}(\mathscr{Y})$ is a decomposable extension of T^*, we have

$$\overline{\mathscr{X}^*_{T^*}(\sigma)} \cap \mathscr{X}^*_{T^*}(\delta) \subset \mathscr{Y}_S(\sigma) \cap \mathscr{Y}_S(\delta) = \{0\},$$

thus $\overline{\mathscr{X}^*_{T^*}(\sigma)}$ is a proper hyperinvariant subspace of T^*.

If $\sigma_p(T^*)\neq\emptyset$ then $\ker(T^*-\lambda)$, $\lambda\in\sigma_p(T^*)$ is either a proper hyperinvariant subspace of T^* or $T=\lambda I$.

5.2. COROLLARY. *Let $T\in\mathscr{L}(\mathscr{X})$ be power-bounded and subdecomposable. If $\mathscr{X}$ is reflexive then either T has a proper hyperinvariant subspace or $T\in C_{\cdot 0}$ or T is a scalar multiple of the identity.*

Proof. Since $T^{**}=T$ we apply Proposition 5.1.

5.3. THEOREM. *Suppose A is subquasiscalar, $\sigma(A_k)\subset\mathbf{D}$, $1\leqslant k\leqslant m$ and $\tau_1(A)\cap\mathbf{D}^m$ is dominating. Then A^* has a proper invariant subspace.*

Proof. Let $B=(B_1,\ldots,B_m)\in\mathscr{L}(\mathscr{Y})^m$ be a quasiscalar extension of A such that $\mathscr{Y}$ is generated by the set

$$\{\mathscr{V}_B(f)x_0 : x_0\in\mathscr{X},\ f\in\mathscr{C}(\mathscr{M}_B)\}.$$

Proceeding as in [10], Ch. III, § 1, we derive easily $\mathscr{M}_B\subset\overline{\mathbf{D}^m}$ and this implies that A is polynomially bounded. By Proposition 3.4 we may assume $A\in C^0$ thus applying Theorem 3.5, Theorem 4.1 and Proposition 4.2 we shall show that A has the property (γ). Let $x\in\mathscr{X}$, $x^*\in\mathscr{X}^*$, $\{c_l\}_{l=1}^n\subset\mathbf{C}$, $\{\omega^{(l)}\}_{l=1}^n\subset\tau_{l_c}(A)\cap\mathbf{D}^m$, be given. Choose $0<\eta<1$ such that $\max\limits_{1\leqslant k\leqslant m}|\omega_k^{(j)}-\omega_k^{(l)}|>2\eta$, $j\neq l$. For any $\varepsilon>0$ there are $e_l\in\ker x^*$, $1\leqslant l\leqslant n$, such that

$$\|e_l\|=1,\ \ |||(A-\omega^{(l)})e_l|||<\varepsilon.$$

Take $f_\eta\in C(\mathbf{C})$ such that

$$0\leqslant f_\eta\leqslant 1,\ f_\eta(\lambda)=1,\ |\lambda|\leqslant\eta,\ f_\eta(\lambda)=0,\ |\lambda|\geqslant 2\eta,$$

and define $f_\eta^{(l)}, g_\eta^{(l)}\in\mathscr{C}(\mathscr{M}_B)$, $1\leqslant l\leqslant\eta$ by

$$f_\eta^{(l)}(\omega)=f_\eta(\omega_1^{(l)}-\omega_1)\ldots f_\eta(\omega_m^{(l)}-\omega_m),\quad \omega\in\mathscr{M}_B,$$

$$g_\eta^{(l)}(\omega)=(\omega_1^{(l)}-\omega_1)^{-1},\ |\omega_1^{(l)}-\omega_1|>\eta/2,\quad \omega\in\mathscr{M}_B.$$

Since $1-f_\eta^{(1)}(\omega)=(1-f_\eta^{(1)}(\omega))\cdot g_\eta^{(1)}(\omega)(\omega_1-\omega)$, we have

$$\|\mathscr{V}_B(1-f_\eta^{(l)})e_1\|=\|\mathscr{V}_B((1-f_\eta^{(l)})\,g_\eta^{(l)})(\omega_1^{(l)}-A_1)e_l\|\leqslant\frac{2\varepsilon}{\eta}\|\mathscr{V}_B\|.$$

Let $e_l'=\mathscr{V}_B(f_\eta^{(l)})e_l$. Since $f_\eta^{(j)}\cdot f_\eta^{(l)}=0$, $j\neq l$ and $\mathscr{V}_B$ is multiplicative it is easy to see that

$$\left\|\sum_{l=1}^n e^{i\theta_l}\lambda_l e_l'\right\|\leqslant\|\mathscr{V}_B\|\left\|\sum_{l=1}^n\lambda_l e_l'\right\|,$$

thus applying Lemma 4.3 we can find $\{\mu_l\}_{l=1}^n\subset\mathbf{C}$, $y^*\in\mathscr{V}^*$ such that

$$\left\|\sum_{l=1}^n\mu_l e_l'\right\|\leqslant 1,\ \|y^*\|\leqslant\|\mathscr{V}_B\|,\quad \mu_l y^*(e_l')=c_l\left(\sum_{k=1}^n|c_k|\right)^{-1}.$$

Let z_0^* be the restriction of y^* to $\mathscr{X}$ and define $z^*\in\mathscr{X}^*$ by the equation

$$z^*(x_0)=\left\langle\mathscr{V}_B\left(\sum_{l=1}^n f^{(l)}\right)x_0,\ y^*\right\rangle,\quad x_0\in\mathscr{X}.$$

We can choose ε (depending on η and n) such that $\sum_{l=1}^{n} |\mu_l| \, \|e_l' - e_l\|$, $\sum_{l=1}^{n} \left| \mu_l y^*(e_l) - c_l \left(\sum_{k=1}^{n} |c_k| \right)^{-1} \right|$ and $\left\| v_B \left(1 - \sum_{l=1}^{n} f_{\eta}^{(l)} \right) \left(\sum_{k=1}^{n} \mu_k c_k \right) \right\|$ are arbitrary small.

If we put

$$w = \|\mathscr{V}_B\| \left(\sum_{l=1}^{n} c_l \right)^{1/2} \sum_{l=1}^{n} \mu_l e_l, \quad w^* = \|\mathscr{V}_B\|^{-1} \left(\sum_{l=1}^{n} |c_l| \right)^{1/2} z^*,$$

$$b_A = 2\,\|\mathscr{V}_B\|, \quad u = x + w, \quad u^* = x^* + w^*$$

we have

$$u \in B\left(x,\, b_A \left(\sum_{l=1}^{n} c_l \right)^{1/2},\, \mathscr{X}\right), \quad u^* \in B\left(x^*, b_A \left(\sum_{l=1}^{n} |c_l| \right)^{1/2},\, \mathscr{X}^*\right)$$

$$\left\| x \underset{-}{\overset{A}{\otimes}} x^* + \sum_{l=1}^{n} c_l \mathscr{E}^A_{\omega^{(l)}} - u \underset{-}{\overset{A}{\otimes}} u^* \right\|_* \leqslant$$

$$\leqslant \left\| \sum_{l=1}^{n} c_l \mathscr{E}^A_{\omega^{(l)}} - w \underset{-}{\overset{A}{\otimes}} w^* \right\|_* + \|x \underset{-}{\overset{A}{\otimes}} w^*\|_* + \|w \underset{-}{\overset{A}{\otimes}} x^*\|_* .$$

But using the relation

$$\left\| \left(\sum_{l=1}^{n} \mu_l e_l \right) \underset{-}{\overset{A}{\otimes}} (z^* - z_0^*) \right\|_* \leqslant c_A \left\| \mathscr{V}_B \left(1 - \sum_{l=1}^{n} f_{\eta}^{(l)} \right) \sum_{l=1}^{n} \mu_l e_l \right\| \, \|y^*\|$$

we may suppose that $\left\| \left(\sum_{l=1}^{n} \mu_l e_l \right) \underset{-}{\overset{A}{\otimes}} (z^* - z_0^*) \right\|_*$ is arbitrarily small and following the proof of Theorem 4.4 we can make both $\left\| \sum_{l=1}^{n} c_l \mathscr{E}^A_{\omega^{(l)}} - w \underset{-}{\overset{A}{\otimes}} w^* \right\|_*$ and $\|w \underset{-}{\overset{A}{\otimes}} x^*\|_*$ arbitrarily small. To conclude the proof we shall prove that $\|x \underset{-}{\overset{A}{\otimes}} w^*\|_*$ becomes small if η is small and $\sigma_p(A_1^*) = \emptyset$. Indeed if we suppose the contrary, we can find $1 \leqslant l \leqslant n$, $\{h_r\}_{r=1}^{\infty} \subset H^{\infty}(\mathbf{D}^m)$, $\|h_r\| \leqslant 1$, $\{\eta_r\}_{r=1}^{\infty}$ tending to 0 such that

$$\lim_{r \to \infty} \langle \mathscr{V}_B(f_{\eta_r}^{(l)}) \Phi^A(h_r) x_0, y^* \rangle = y_0^*(x_0), \quad y_0^* \in \mathscr{X}^*, \; y_0^* \neq 0.$$

Since we have

$$|((A_1 - \omega_1^{(l)})^* y_0^*)(x_0)| = \lim_{r \to \infty} |\langle \mathscr{V}_B(f_{\eta_r}^{(l)}) \Phi^A(h_r)(B_1 - \omega_1^{(l)}) x_0, y^* \rangle| \leqslant$$

$$\leqslant \lim_{r \to \infty} 2\, c_A \|\mathscr{V}_B\| \, \eta_r \|h_r\|_{\infty} \|x_0\| \, \|z_0^*\| = 0$$

we derive $y_0^* \in \ker (A_1 - \omega_1^{(l)})^*$, whence it follows $y_0^* = 0$.

5.4. THEOREM. *Suppose A is subquasiscalar, $\sigma(A_k) \subset \bar{\mathbf{D}}$, $1 \leqslant k \leqslant m$ and $\tau(A) \cap \mathbf{D}^m$ is dominating. If there exists a compact injective scalar operator $K \in \mathscr{L}(\mathscr{X})$ (in particular if $\mathscr{X}$ has an unconditional basis) and $\mathscr{X}$ is reflexive then A has a proper invariant subspace.*

Proof. By the proof of Theorem 5.3 we know that A is polynomially bounded. Using Corollary 5.2 we may suppose $A \in C_{\cdot 0}$. If we use the notation of Theorem 5.3 we observe that in case $|||(A-\omega^{(l)})e_l|||$ is small and

$$e_l^* \in \mathscr{X}^*, \ \|e_l^*\| = \|e_l\| = e_l^*(e_l) = 1$$

then $e_l^*(\mathscr{V}_B(f_\eta^{(l)})e_l)$ is close to 1. Defining $x_l^* \in \mathscr{X}^*$ by the equation

$$x_l^*(x) = \langle \mathscr{V}_B(f_\eta^{(l)})x, \ y_l^* \rangle, \ x \in \mathscr{X},$$

where $y_l^* \in \mathscr{Y}^*$ is an extension of e_l^* we may suppose $\|x_l^*\| \geqslant 1/2$. But referring again to the last lines in the proof of Theorem 5.3 we can make $|||(A^*-\omega^{(l)})x_l^*|||$ arbitrarily small and hence we derive the inclusion $\tau_l(A) \subset \tau_r(A)$. To conclude the proof we apply Theorem 4.8.

5.5. THEOREM. *Suppose there exist $K \in \mathscr{L}(\mathscr{X})$ a compact injective scalar operator (or suppose, in particular, that $\mathscr{X}$ has an unconditional basis) and $\mathscr{X}^*$ is uniformly convex. If A is polynomially bounded, $\tau(A) \cap \mathbf{D}^m$ is dominating and each A_k is subscalar with a scalar extension $B_k \in \mathscr{L}(\mathscr{Y}_k)$ such that $\|\mathscr{V}_{B_k}\| = 1$ then A has a proper invariant subspace.*

Proof. As in the proof of Theorem 5.4 we may suppose $A \in C_{\cdot 0}$ and then show the inclusion $\tau_l(A) \subset \tau_r(A)$. For simplicity we shall prove the implication

$$0 \in \tau_l(A) \Rightarrow 0 \in \tau_r(A).$$

Thus if $|||Ae|||$ is small and

$$e^* \in \mathscr{X}^*, \ \|e^*\| = \|e\| = e^*(e) = 1$$

then $\|e - \mathscr{V}_{B_k}(f_\eta)e\|$, $1 \leqslant k \leqslant m$ is small (see the proof and the notation of Theorem 5.3). If $y_k^* \in \mathscr{Y}_k^*$ is an extension of e^* and we define $x_k^* \in \mathscr{X}^*$ by the equation

$$x_k^*(x) = \langle \mathscr{V}_{B_k}(f_\eta)x, \ y_k^* \rangle, \ x \in \mathscr{X}$$

we have

$$\|x_k^*\| \leqslant 1, \ |x_k^*(e) - 1| \leqslant \|\mathscr{V}_{B_k}(f_\eta)e - e\|$$

consequently, in view of the uniform convexity of $\mathscr{X}^*$, we may suppose that $\|e^* - x_k^*\|$ is small. Since we have

$$\|A_k^*e^*\| \leqslant \|A_k^*x_k^*\| + \|A_k^*\| \, \|e^* - x_k^*\|,$$
$$|A_k^*x_k^*(x)| = |\langle \mathscr{V}_{B_k}(f_\eta) B_k x, \ y_k^* \rangle| \leqslant 2\eta \, \|x\|, \ x \in \mathscr{X},$$

the proof is concluded.

5.6. COROLLARY. *Suppose $\mathscr{X}$ is a Hilbert space, A is polynomially bounded $\tau(A) \cap \mathbf{D}^m$ is dominating and A_k is subnormal, $1 \leqslant k \leqslant m$. Then A has a proper invariant subspace.*

Proof. Since A_k is subnormal it has a normal extension $B_k \in \mathscr{L}(\mathscr{Y}_k)$, where $\mathscr{Y}_k$ is a Hilbert space. Now we apply Theorem 5.5.

5.7. COROLLARY. *Suppose A has a normal extension $B \in \mathscr{L}(\mathscr{Y})^m$,* (i.e. *$\mathscr{Y}$ is a Hilbert space and $\{B_k\}_{k=1}^m$ are commuting normal operators) $\sigma(A_k) \subset \overline{\mathbf{D}}$, $1 \leqslant k \leqslant m$ and $\tau(A) \cap \mathbf{D}^m$ is dominating. Then A has a proper invariant subspace.*

Proof. Since A is polynomially bounded (see the proof of Theorem 5.3) we apply Corollary 5.6.

5.8. THEOREM. *If $T \in \mathscr{L}(\mathscr{X})$ is subquasiscalar and $\partial \mathbf{D} \subset \sigma(T) \subset \bar{\mathbf{D}}$ then T^* (and in particular T if T is subscalar and $\mathscr{X}$ is reflexive) has a proper invariant subspace.*

Proof. If T has no proper invariant subspace then we derive easily $\sigma_l(T) = \tau_l(T) = \sigma(T)$. Now if $\sigma(T) \cap \mathbf{D}$ is not dominating in $\mathbf{D}$ then T has a hyperinvariant subspace by [3], Lemma 2.1 (though the lemma is stated for operators in Hilbert spaces its proof is valid in Banach spaces). We conclude the proof applying Theorem 5.3.

5.9. THEOREM. *Let m be a finite, positive, Borel measure in $\mathbf{C}$ with compact support and let T denote the restriction to an invariant subspace of the multiplication by the argument in $L^p(m)$, $1 < p < \infty$. Then T has a proper invariant subspace.*

Proof. Proceeding exactly as in [4] we reduce to the case $\partial \mathbf{D} \subset \sigma(T) \subset \bar{\mathbf{D}}$ and then we apply Theorem 5.8.

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