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HOMOGENEOUS C^*-EXTENSIONS OF $C(X) \otimes K(H)$. PART II

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In the first part of this paper (*J. Operator Theory*, 1 (1979), 55–108) we began studying a generalization of the Brown-Douglas-Fillmore theory of extension, in which the ideal $K(H)$ of compact operators is replaced by $C(X)\otimes K(H)$.

For X a finite-dimensional compact metrizable space and A a separable nuclear C^*-algebra with unit, the equivalence classes of certain extensions which we called homogeneous extensions, of $C(X)\otimes K(H)$ by A, gave rise to a group $\mathrm{Ext}(X, A)$.

Based on the results about $\mathrm{Ext}(X, A)$ obtained in the first part of this paper, we shall develop here further topological properties of $\mathrm{Ext}(X, A)$. This includes the study of a certain $\mathrm{K}(X)$-module structure on $\mathrm{Ext}(X, A)$, the long exact sequences for each of the two variables, the periodicity theorem and a result showing that taking suspensions in one of the variables has the same effect on $\mathrm{Ext}(X, A)$ as taking suspensions in the other variable.

Much of the material in this paper is derived from standard techniques in algebraic topology and from the adaptions of these techniques due to L. G. Brown [9] for extending the Brown-Douglas-Fillmore theory from commutative to non commutative C^*-algebras.

Since we have chosen to make this paper rather selfcontained, some of it is almost expository.

Before passing to a more detailed description of the content of the present paper let us briefly recall how far we had come in studying the properties of $\mathrm{Ext}(X, A)$ in Part I.

After dealing with the $\mathrm{Ext}(X, A)$ is a group question, we obtained results on homotopy-invariance and short exact sequences. Thus we proved that for a nuclear C^*-algebra A and a two-sided closed ideal J of A there is an exact sequence

$$\mathrm{Ext}(X, A/J) \to \mathrm{Ext}(X, A) \to \mathrm{Ext}(X, \tilde{J}).$$

Similarly, for Y a closed subset of X we obtained an exact sequence

$$\mathrm{Ext}(X, Y; A) \to \mathrm{Ext}(X, A) \to \mathrm{Ext}(Y, A).$$

For the homotopy-invariance properties besides the usual assumptions on the C^*-algebras and compacta we had to assume generalized quasidiagonality (abbreviated g.q.d.) of the C^*-algebras.

If A is g.q.d. and $\rho_k \colon A \to B$ $(k = 1,2)$ are homotopic unital $*$-homomorphisms we proved that the corresponding group-homomorphisms:

$$\rho_{k*} \colon \mathrm{Ext}(X, B) \to \mathrm{Ext}(X, A) \quad (k = 1, 2)$$

coincide.

Similarly, also for A g.q.d., we proved that if $f_k \colon X \to Y$ $(k = 1, 2)$ are homotopic continuous maps, then the group-homomorphisms

$$f_k^* \colon \mathrm{Ext}(Y, A) \to \mathrm{Ext}(X, A) \quad (k = 1, 2)$$

coincide.

Part II has three sections § 7-§ 9, with numbers continuing those of the first part. The list of references, for the readers convenience, is a concatenation of the list of references of Part I and of an additional list of references.

In more detail the content of the three sections of the present paper is as follows.

In § 7 it is shown that there is a natural isomorphism $\mathrm{Ext}(X, C(S^1)) \simeq \mathrm{K}(X)$ and there is a homomorphism $\mathrm{K}(X) \to \mathrm{Ext}(X, A)$ related to weak equivalence.

We also exibit a natural $\mathrm{K}(X)$-module structure on $\mathrm{Ext}(X, A)$ and we show that the action of fiber-preserving automorphisms of $C_n(X, K(H))$ on $\mathrm{Ext}(X, A)$ corresponds to multiplication by line-bundles.

In § 8 using the short exact sequences and the homotopy-invariance established in Part I, long exact sequences for $\mathrm{Ext}(X, A)$ are derived. In the X-variable this is absolutely standard and the proofs are omitted. For the A-variable the proofs are given, but this is only a more detailed exposition of L. G. Brown's adaptation for the non-commutative case of the usual proofs.

The reader who wants some intuitive background should read the derivation of the long exact sequence given here in parallel with the derivation of the long exact sequence for the usual Ext in the commutative case in [12] and think of how the constructions at the level of spaces translate into constructions at the level of the corresponding C^*-algebras of continuous functions on those spaces.

In § 9 the periodicity theorems in the X-variable and in the A-variable are obtained. We show that there is an interchange isomorphism which expresses the fact that taking suspensions in the X-variable or in the A-variable has the same effect on $\mathrm{Ext}(X, x_0, A)$.

This makes the periodicity theorems in the two variables equivalent and our proof will be half in the X-variable and half in the A-variable.

The first half is an adaptation of a half of the proof for K-theory [5] and the second half follows closely a half of the proof of the periodicity theorem for Ext given by Brown-Douglas-Fillmore [12].

We should mention that it had been noted by L.G. Brown in [9] that the proof of the periodicity theorem in [12] could be adapted for the non-commutative case.

The assumptions under which we give the periodicity theorem are that X be a pointed finite-dimensional compact metrizable space and A be a nuclear g.q.d. C^*-algebra, having a rank-one homomorphism.

A curious corollary of the periodicity theorem is given in 9.12.

We would like to mention that further topological properties of $\mathrm{Ext}(X, A)$ have been obtained by C. Schochet [59].

On the other hand, G. G. Kasperov has announced in the short note [50] results for a related much more general two-variables Ext-functor, both variables of which are non-commutative C^*-algebras. His results are obtained, in part, by connecting the Ext-functor to a generalization of his previous work on K-homology [51].

§ 7.

In this section we discuss certain relations between $\mathrm{Ext}(X, A)$ and $\mathrm{K}(X)$.

First, we consider the $\mathrm{K}(X)$-valued index for unitary elements of $C_{*s}(X, L(H))/C_n(X, K(H))$, an adaption of Atiyah's results in the Appendix of [5]. This index gives a natural isomorphism $\mathrm{Ext}(X, C(S^1)) \simeq \mathrm{K}(X)$ and also a natural homomorphism $\mathrm{K}(X) \to \mathrm{Ext}(X, A)$ defining the weak equivalence relation on $\mathrm{Ext}(X, A)$. Second, a natural $\mathrm{K}(X)$-module structure on $\mathrm{Ext}(X, A)$ is defined. It is shown that the action of fiber-preserving automorphisms of $C_n(X, K(H))$ on $\mathrm{Ext}(X, A)$ can be expressed by means of the action of the multiplicative group of classes of line-bundles in $\mathrm{K}(X)$.

The material in this section consists, to a large extent, of adaptations of knonw facts, included for the sake of some completeness.

By $P_0(X, H), P(X, H)$ we shall denote the orthogonal projections in $C_n(X, K(H))$ and respectively in $C_{*s}(X, L(H))$.

7.1. LEMMA. *Let $P_i \in P(X, H)$, $(i = 1, 2)$, be such that $\dim P_i(x)H = \infty$ for all $x \in X$, $i = 1, 2$. Then there is $V \in C_{*s}(X, L(H))$ such that $V^*V = P_1$, $VV^* = P_2$.*

Proof. The projections P_i determine continuous fields of Hilbert spaces $((P_i(x)H)_{x\in X}, \Gamma_i)$ where $\Gamma_i \subset \prod_{x\in X} (P_i(x)H)$ is the set of $(P_i(x)f(x))_{x\in X}$ where f runs over $C(X, H)$. Then by ([21], 10.8.7.) these two continuous fields of Hilbert spaces are isomorphic. So there are unitary operators W_x from $P_1(x)H$ to $P_2(x)H$ such that

$$\{(W_x h_x)_{x\in X} | (h_x)_{x\in X} \in \Gamma_1\} = \Gamma_2.$$

Define $V(x) \in L(H)$ by $V(x)\, h = W_x P_1(x)\, h$. Then it follows easily that $V = (X \ni x \to \to V(x) \in L(H)) \in C_{*s}(X, L(H))$ and $V^*V = P_1$, $VV^* = P_2$. Q.E.D.

For $P \in P_0(X, H)$, the subset $\bigcup_{x \in X} (\{x\} \times P(x)H)$ of $X \times H$ together with the natural projection onto X defines a locally trivial vector bundle over X.

Let Vect(X) be the semigroup of isomorphism classes of locally trivial vector bundles over X endowed with the direct sum operation. For $P \in P_0(X, H)$ the equivalence class of the corresponding vector bundle will be denoted by $[P] \in \mathrm{Vect}(X)$ and the stable equivalence class by $[P]_{\mathrm{K}} \in \mathrm{K}(X)$.

The next two lemmas are quite standard; their proofs will be omitted.

7.2. LEMMA. *The map* $P_0(X, H) \ni P \to [P] \in \mathrm{Vect}(X)$ *is onto. Moreover for* $P_1, P_2 \in P_0(X, H)$ *the following conditions are equivalent:*

(i) $[P_1] = [P_2]$

(ii) *there is a unitary* $U \in I + C_n(X, K(H))$ *such that* $UP_1U^* = P_2$.

(iii) *there is* $V \in C_n(X, K(H))$ *such that* $V^*V = P_1$ *and* $VV^* = P_2$.

(iv) *there is* $W \in C_{*s}(X, L(H))$ *such that* $W(x)\, P_1(x)\, H = P_2(x)\, H$ *and* $\mathrm{Ker}\, W(x) \cap P_1(x)\, H = 0$ *for all* $x \in X$.

7.3. LEMMA. *Let* $P_1, P_2 \in P_0(X, H)$ *be such that* $\|P_1 - P_2\| < 1$. *Then we have* $[P_1] = [P_2]$.

The next lemma enables us to define the $\mathrm{K}(X)$-valued index.

7.4. LEMMA. *Let* $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ *be unitary. Then there is a partial isometry* $W \in C_{*s}(X, L(H))$ *such that* $p(W) = U$. *Moreover*

$$[I - W^*W]_{\mathrm{K}} - [I - WW^*]_{\mathrm{K}} \in \mathrm{K}(X)$$

is independent of the particular choice of W *(i.e. depends only on* U*).*

Proof. Consider $P_j \in P_0(X, H)$, $P_1 \leqslant P_2 \leqslant \ldots$, an approximate unit of $C_n(X, K(H))$ and let $V \in C_{*s}(X, L(H))$ be such that $p(V) = U$. Then

$$(I - P_j)\, V^*V(I - P_j) = (I - P_j) + (I - P_j)\,(V^*V - I)\,(I - P_j).$$

Since $V^*V - I \in C_n(X, K(H))$ there is some $j \in \mathbf{N}$ such that

$$\|(I - P_j)\,(V^*V - I)\,(I - P_j)\| < 1.$$

Set

$$W = V(I - P_j)\,(P_j + (I - P_j)\, V^*V(I - P_j))^{-1/2}$$

which is a partial isometry with $p(W) = U$.

For the second assertion, let W_1, W_2 be partial isometries such that $p(W_1) = p(W_2) = U$. Let $j_0 \in \mathbf{N}$ be such that for $j \geqslant j_0$ we have

$$\|(I - W_i^*W_i)\,(I - P_j)\| < 1 \quad (i = 1, 2).$$

Then for $j \geqslant j_0$, $P_j + (I - P_j)\, W_i^*W_i(I - P_j)$ will be invertible and we may define partial isometries $L_{ij} = W_i^*\, W_i(I - P_j)\,(P_j + (I - P_j)\, W_i^*W_i(I - P_j))^{-1/2}$ and projections $E_{ij} = L_{ij}L_{ij}^*$.

Then $W_{ij} = W_i E_{ij}$ will be partial isometries with the following properties:

$$p(W_{ij}) = U$$

$$\lim_{j\to\infty} \|E_{ij} - (I - P_j)\| = 0$$

$$\lim_{j\to\infty} \| W_{1j} - W_{2j}\| = 0.$$

Then for $j \geqslant j_0$ great enough, we have

$$\|W_{1j}^* W_{1j} - W_{2j}^* W_{2j}\| < 1 \text{ and } \| W_{1j} W_{1j}^* - W_{2j} W_{2j}^*\| < 1$$

so that Lemma 7.3. gives

$$[I - W_{1j}^* W_{1j}]_{\mathrm{K}} - [I - W_{1j} W_{1j}^*]_{\mathrm{K}} = [I - W_{2j}^* W_{2j}]_{\mathrm{K}} - [I - W_{2j} W_{2j}^*]_{\mathrm{K}}.$$

Thus it will be sufficient to prove that:

$$(*) \quad \begin{aligned} &[I - W_{ij}^* W_{ij}]_{\mathrm{K}} - [I - W_i^* W_i]_{\mathrm{K}} = \\ &= [I - W_{ij} W_{ij}^*]_{\mathrm{K}} - [I - W_i W_i^*]_{\mathrm{K}}. \end{aligned}$$

Definind $R_{ij} = W_i^* W_i - W_{ij}^* W_{ij}$, $S_{ij} = W_i W_i^* - W_{ij} W_{ij}^*$, we have R_{ij} $S_{ij} \in P_0(X, H)$ since $E_{ij} \leqslant W_i^* W_i$. Then $W_i R_{ij}$ is a partial isometry with $(W_i R_{ij})^*(W_i R_{ij}) = R_{ij}$ and $(W_i R_{ij})(W_i R_{ij})^* = S_{ij}$ so that by Lemma 7.2 we have $[R_{ij}] = [S_{ij}]$. Now $(*)$ follows from:

$$[I - W_{ij}^* W_{ij}] = [I - W_i^* W_i] + [R_{ij}]$$

$$[I - W_{ij} W_{ij}^*] = [I - W_i W_i^*] + [S_{ij}]. \qquad \text{Q.E.D.}$$

Now we can define *the index of a unitary element* $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ by

$$\text{index } U = [I - W^* W]_{\mathrm{K}} - [I - WW^*]_{\mathrm{K}}$$

where W is any partial isometry with $p(W) = U$.

The next lemma gives the main properties of the index.

7.5. LEMMA. *The index-map from the unitary group of $C_{*s}(X, L(H))/C_n(X, K(H))$ to* $\mathrm{K}(X)$ *is onto. Also,* index $U = 0$ *if and only if there is a unitary $V \in C_{*s}(X, L(H))$ such that $p(V) = U$. For U_1, U_2 unitaries in $C_{*s}(X, L(H))/C_n(X, K(H))$ we have:*

(i) $\text{index}(U_1 \oplus U_2) = \text{index } U_1 + \text{index } U_2$

(ii) $\text{index } U_1 U_2 = \text{index } U_1 + \text{index } U_2$

(iii) $\|U_1 - U_2\| < 1$ *implies that* $\text{index } U_1 = \text{index } U_2$.

Proof. Given $\alpha \in \mathrm{K}(X)$, byLemma 7.2 there are $P_1, P_2 \in P_0(X, H)$ such that $\alpha = [P_1]_{\mathrm{K}} - [P_2]_{\mathrm{K}}$. Then because of Lemma 7.1 there is $V \in C_{*s}(X, L(H))$ such that $V^*V = I - P_1$, $VV^* = I - P_2$. Clearly $p(V)$ is unitary and index $p(V) = \alpha$.

If V is a unitary of $C_{*s}(X, L(H))$ then index $p(V) = 0$. Conversely let $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ be unitary with index$U = 0$ and let $W \in C_{*s}(X, L(H))$ be a partial isometry with $p(W) = U$. In view of Lemma 7.2 there is $Q \in P_0(X, H)$ such that $[I - W^*W] + [Q] = [I - WW^*] + [Q]$. Using Lemma 7.1 there is $S \in C_{*s}(X, L(H))$ such that $S^*S = I$ and $SS^* = W^*W$. Then $V_1 = W(\mathrm{I} - SQS^*)$ is a partial isometry, $p(V_1) = U$ and $[I - V_1^*V_1] = [I - W^*W] + [Q] = [I - WW^*] + [Q] = [I - V_1V_1^*]$.

Hence using Lemma 7.2 there is $L \in C_n(X, K(H))$ such that $I - V_1^*V_1 = L^*L$, $I - V_1V_1^* = LL^*$. Defining $V = V_1 + L$ we have $p(V) = U$ and V is unitary.

Concerning assertions (i)-(iii) we remark that (i) is quite trivial and we shall first prove (iii) and then (ii).

To prove (iii) we shall first prove that $\|U_1 - U_2\| < 1/2$ implies that index$U_1 = $ indexU_2. Indeed by the proof of the first part of Lemma 7.4 there are partial isometries $W_i (i = 1, 2)$ with $p(W_i) = U_i$ and $\|W_1 - W_2\| < 1/2$. Then $\|(I - W_1^*W_1) - (I - W_2^*W_2))\| < 1$ and $\|(I - W_1W_1^*) - (I - W_2W_2^*)\| < 1$ so that index $U_1 = $ indexU_2 follows from Lemma 7.3.

Now if $\|U_1 - U_2\| < 1$ we have $\|U_1U_2^* - I\| < 1$ and hence there is a hermitian element $A \in C_{*s}(X, L(H))/C_n(X, K(H))$ such that $\exp(\mathrm{i}A) = U_1U_2^*$ so that U_1 and U_2 may be joined by the continuous curve $\exp(\mathrm{i}tA)U_2$ $(t \in [0, 1])$. But in view of the previously proved fact the index is locally constant and hence index $U_1 = $ index U_2.

To prove that index$U_1U_2 = $ index$U_1 +$ indexU_2 in view of (i) and (iii) it will be sufficient to prove that $U_1U_2 \oplus I$ and $U_1 \oplus U_2$ can be joined by a norm-continuous curve of unitaries. This is done with the usual trick:

$$U(t) = \begin{pmatrix} U_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} U_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Q.E.D.

$t \in [0, \pi/2]$.

Identify S^1 with the unit-circle $\{z \in \mathbf{C} \mid |z| = 1\}$; the function $\chi \in C(S^1)$, given by $\chi(z) = z$, is unitary and generates $C(S^1)$.

The following proposition is an immediate consequence of Lemma 7.5.

7.6. PROPOSITION. *The map* $\mathrm{Ext}(X, C(S^1)) \ni [\tau] \to$ index $\tau(\chi) \in \mathrm{K}(X)$ *is well defined and is an isomorphism of* Ext $(X, C(S^1))$ *onto* $\mathrm{K}(X)$.

We pass now to the discussion of the weak equivalence of homogeneous X-extensions.

Two homogeneous X-extensions by A, defined by unital $*$-monomorphisms $\tau_i \colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$ $(i = 1, 2)$ are said to be *weakly equivalent* if there is a unitary $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ such that $U\tau_1(a) = \tau_2(a)\, U$ for all $a \in A$. To emphasize the distinction between weak equivalence and equivalence the latter will be also called strong equivalence. The semigroup of weak equivalence classes of homogeneous X-extensions by A will be denoted by $\mathrm{Ext}_w(X, A)$ and there is a natural homomorphism $\mathrm{Ext}(X, A) \to \mathrm{Ext}_w(X, A)$. We shall write $[\tau]_w$ for the weak equivalence class of τ.

Assume $[\tau_i] \in \mathrm{Ext}(X, A)$ $(i = 1, 2)$ are weakly equivalent and let $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ be a unitary implementing the weak equivalence. Using Lemma 7.5 it is easily seen that the strong equivalence class $[\tau_2]$ depends only on $[\tau_1]$ and index U. Since the class of trivial X-extensions by A is a natural element in $\mathrm{Ext}(X,A)$ it follows that $[\tau_1]$ and $[\tau_2]$ are weakly equivalent if and only if there is $[\sigma]$ weakly equivalent to the trivial extensions such that $[\tau_1] + [\sigma] = [\tau_2]$. Assume now $[\sigma] \in \mathrm{Ext}(X, A)$ is trivial and let $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ be unitary and define $\sigma_1(a) = U\sigma(a)\, U^*$ $(a \in A)$; then ,since $[\sigma_1]$ depends only on index$U$, there is a map ε: $\mathrm{K}(X) \to \mathrm{Ext}(X, A)$ such that $\varepsilon\,(\mathrm{index}\, U) = [\sigma_1]$. In view of the properties of the index, ε is a homomorphism and the diagram

$$\mathrm{K}(X) \xrightarrow{\varepsilon} \mathrm{Ext}(X, A) \to \mathrm{Ext}_w(X, A) \to 0$$

is an exact sequence, in the sense that $[\tau_1]_w = [\tau_2]_w$ if and only if $[\tau_1] = [\tau_2] + \varepsilon(\alpha)$ for some $\alpha \in \mathrm{K}(X)$. Of course if $\mathrm{Ext}(X, A)$ is a group then $\mathrm{Ext}_w(X, A)$ is also a group and exactness of the above sequence has the usual meaning.

If A, B are C^*-algebras with unit and $f\colon A \to B$ is a unit-preserving $*$-homomorphism then it is easily seen that the diagram

$$\begin{array}{ccc} & \overset{\varepsilon}{\nearrow} & \mathrm{Ext}(X, A) \\ \mathrm{K}(X) & & \big\uparrow f_* \\ & \underset{\varepsilon}{\searrow} & \mathrm{Ext}(X, B) \end{array}$$

is commutative.

In particular if A has a one-dimensional representation then $\varepsilon = 0$ and hence $\mathrm{Ext}(X, A) = \mathrm{Ext}_w(X, A)$.

The next question we shall discuss is *a natural* $\mathrm{K}(X)$*-module structure on* $\mathrm{Ext}(X, A)$. Of course *we shall assume A is nuclear,* so that $\mathrm{Ext}(X, A)$ is a group.

In the remaining part of this section, for $T_i \in C_{*s}(X, L(H))$ $(i = 1, 2)$ we shall use the notation

$$T_1 \otimes T_2 = (X \ni x \to T_1(x) \otimes T_2(x) \in L(H \otimes H)) \in C_{*s}(X, L(H \otimes H)).$$

Consider $P \in P_0(X, H)$ such that $P(x) \neq 0$, $(\forall)\, x \in X$. In view of Lemma 7.1 there is some $V \in C_{*s}(X, L(H \otimes H, H))$ such that $V^*V = I_H \otimes P$ and $VV^* = I_H$.

For $T \in C_{*s}(X, L(H))$ define

$$\mu(T) = V(T \otimes P) V^* \in C_{*s}(X, L(H)).$$

Then μ is a unital $*$-monomorphism of $C_{*s}(X, L(H))$ into itself and

$$\mu(T) \in C_n(X, K(H)) \Leftrightarrow T \in C_n(X, K(H)).$$

It is easily seen that for some other $P' \in P_0(X, H)$, with $[P'] = [P]$ and some other V' corresponding to P', the corresponding homomorphism μ' differs from μ by an inner automorphism of $C_{*s}(X, L(H))$, i.e. there is a unitary $U \in C_{*s}(X, L(H))$ such that $\mu'(T) = U\mu(T) U^*$ for all $T \in C_{*s}(X, L(H))$. Note also that for $Q \in P_0(X, H)$ we have

$$[\mu(Q)] = [P \otimes Q].$$

Let $\tilde{\mu}$ be the unital $*$-monomorphism of $C_{*s}(X, L(H))/C_n(X, L(H))$ induced by μ. Then for $\tau: A \to C_{*s}(X, L(H))/C_n(X, K(H))$ defining a homogeneous X-extension by A it is easily seen that $\tilde{\mu} \circ \tau$ also defines a homogeneous X-extension by A and $[\tilde{\mu} \circ \tau]$ depends only on $[\tau]$ and $[P]$. Also for $U \in C_{*s}(X, L(H))/C_n(X, K(H))$ a unitary we have

$$\operatorname{index} \tilde{\mu}(U) = [P]_{\mathrm{K}} \operatorname{index} U.$$

Thus for $P \in P_0(X, H)$ with $P(x) \neq 0$, $(\forall)\, x \in X$ we may define

$$[P] \cdot [\tau] = [\tilde{\mu} \circ \tau].$$

It is quite standard to verify that for $P_i \in P_0(X, H)$, $P_i(x) \neq 0$ $(\forall)\, x \in X$ $(i = 1, 2)$ and $[\tau_i] \in \operatorname{Ext}(X, A)$ $(i = 1, 2)$ we have

$$[P_1] \cdot [\tau_1] + [P_2] \cdot [\tau_1] = ([P_1] + [P_2]) \cdot [\tau_1]$$

$$[P_1] \cdot ([\tau_1] + [\tau_2]) = [P_1] \cdot [\tau_1] + [P_1] \cdot [\tau_2]$$

$$[P_1] \cdot ([P_2] \cdot [\tau_2]) = [P_1 \otimes P_2] \cdot [\tau_2]$$

$$[P_1] \cdot \varepsilon(\alpha) = \varepsilon([P_1]_{\mathrm{K}} \cdot \alpha).$$

From these properties we immediately infer that $([P], [\tau]) \to [P] \cdot [\tau]$ can be uniquely extended to a bilinear map $\mathrm{K}(X) \times \operatorname{Ext}(X, A) \to \operatorname{Ext}(X, A)$ which defines a $\mathrm{K}(X)$-module structure on $\operatorname{Ext}(X, A)$. Moreover $\varepsilon: \mathrm{K}(X) \to \operatorname{Ext}(X, A)$ is a homomorphism of $\mathrm{K}(X)$-modules.

The isomorphism problem for the algebras arising from homogeneous extensions of $C_n(X, K(H))$ naturally leads to the study on the action of the automorphism group of $C_n(X, K(H))$ on $\operatorname{Ext}(X, A)$. All the facts concerning this group, which we shall use, are well known and can be found for instance in Section 2 of [53]. Thus, every automorphism of $C_n(X, K(H))$ is the composition of an automorphism induced by an automorphism of the base space X and a fiber-preserving automorphism; i.e. an automorphism acting trivially on X (the spectrum of $C_n(X, K(H))$. The group of fiber-preserving automorphism will be denoted by $\mathrm{Aut}_{C(X)}(C_n(X, K(H)))$ since is is easily seen to consist of those automorphism which preserve the $C(X)$-module structure of $C_n(X, K(H))$.

We will be interested only in the action of $\mathrm{Aut}_{C(X)}(C_n(X, K(H))$ on $\mathrm{Ext}(X, A)$ and we shall point out below that this action can be expressed in terms of the $\mathrm{K}(X)$-module structure of $\mathrm{Ext}(X, A)$.

Since the inner automorphisms, i.e. the automorphisms $\mathrm{Inn}(C_n(X, K(H)))$ induced by unitaries of $C_{*s}(X, L(H))$, act trivially on $\mathrm{Ext}(X, A)$ it will be actually the factor group

$$\mathrm{Out}(X) = \mathrm{Aut}_{C(X)}(C_n(X, K(H)))/\mathrm{Inn}(C_n(X, K(H)))$$

which will act on $\mathrm{Ext}(X, A)$.

Now locally, every fiber-preserving automorphism α is given by a unitary. That is, there is an open cover $\{\omega_i\}_{i\in I}$ of X and there are unitaries $U_i \in C_{*s}(X, L(H))$ such that $\alpha(T) = U_i T U_i^*$ for $T \in C_n(X, K(H))$ with $\mathrm{supp}\, T \subset \omega_i$. Moreover for $x \in \omega_i \cap \omega_j$, $U_i^*(x)\, U_j(x) = \lambda_{ij}(x) I$ where $\lambda_{ij}(x) \in \mathbf{C}$, $|\lambda_{ij}(x)| = 1$. We get thus a 1-cocycle $(\lambda_{ij})_{\omega_i \cap \omega_j} \neq \emptyset$. The automorphism α is inner if and only if the cohomology class of this cocycle in $\mathrm{H}^1(X, \mathbf{T})$ is zero. The product $\alpha \circ \beta$ has as cocycle the product of the corresponding cocycles. Thus we have an injective homomorphism of $\mathrm{Out}(X) \to \mathrm{H}^1(X, \mathbf{T})$.

This is also surjective because of the contractibility of the unitary group $U(H)$ endowed with the $*$-strong topology.

Now there is also a bijection from $\mathrm{H}^1(X, \mathbf{T})$ to equivalence classes of line bundles over X. In fact, if (λ_{ij}) is the cocycle obtained from a fiber-preserving automorphism α, then a corresponding line-bundle is the line-bundle given by $\alpha(P_0)$ where P_0 is some constant rank-one projection.

Since there is a cocycle corresponding to both α and to the automorphism constructed from $\alpha(P_0)$, we infer that these automorphisms differ only by an inner automorphism. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(X) & \xrightarrow{\ \sim\ } & \mathrm{H}^1(X, \pi) \to \mathrm{K}(X) \\ & \searrow \quad \swarrow & \\ & \mathrm{End}(\mathrm{Ext}(X, A)) & \end{array}$$

In summary, the elements $[\tau_1], [\tau_2] \in \mathrm{Ext}(X, A)$ are conjugated by some fiber-preserving automorphism if and only if $[\tau_1] = \beta[\tau_2]$ where $\beta \in \mathrm{K}(X)$ is the class of some line-bundle.

§ 8.

Using the short exact sequences and homotopy-invariance results of sections 4, 5, 6 we shall obtain in this section one-sided long exact sequences for $\mathrm{Ext}(X, x_0; A)$. For the X-variable this is standard algebraic topology and the corresponding result will be mentioned without proof at the end of this section. The same techniques were used for commutative A by Brown-Douglas-Fillmore, and L. G. Brown [9] has supplied the necessary definitions for suspensions, mapping cylinders etc., to make the same machinery work also in the non-commutative case. Since the presentation in [9] is somewhat sketchy, we give below for the reader's convenience a more detailed presentation of the proof of the long exact sequence in the A-variable.

Let A, B be two unital C^*-algebras and $\rho: A \to B$ a unital $*$-homomorphism. Then we shall consider the unital C^*-algebras:

$$\mathrm{Z}(\rho) = \{\xi \oplus x \in C([0, 1]), B) \oplus A \mid \xi(1) = \rho(x)\}$$

and

$$\mathrm{C}(\rho) = \{\xi \oplus x \in C([0, 1], B) \oplus A \mid \xi(1) = \rho(x), \xi(0) \in \mathbf{C} \cdot 1_B\}.$$

These C^*-algebras are the analogous of the mapping cylinder and of the mapping cone from algebraic topology.

Consider further

$$\mathrm{C}A = \{\xi \in C([0, 1], A) \mid \xi(0) \in \mathbf{C}1_A\}$$

and

$$\mathrm{S}A = \{\xi \in C([0, 1], A) \mid \xi(0) \in \mathbf{C}1_A,\ \xi(1) \in \mathbf{C}1_A\}$$

which correspond to the cone and suspension.

Since we shall need the short exact sequence in Theorem 4.1 *all C^*-algebras in this section will be assumed nuclear.* Clearly $\mathrm{Z}(\rho)$, $\mathrm{C}(\rho)$, $\mathrm{S}A$, $\mathrm{C}A$ will also be nuclear.

Further, in order to use the homotopy-invariance results in §5 we shall consider C^*-algebras A having composition-series $(J_\rho)_{0 \leqslant \rho \leqslant \alpha}$ with quasi-diagonal quotients $\widetilde{J}_{\rho+1}/J_\rho$, a property we shall call *generalized quasidiagonality* (abbreviated g.q.d.). Also, *throughout this section all C^*-algebras will be assumed to be g.q.d.* .

It is easy to see that direct sums and subalgebras of g.q.d. C^*-algebras are still g.q.d.. Also for A g.q.d. we have that $C([0,1], A)$ is g.q.d. (consider the composition series $(C([0,1], I_\rho))_{0 \leqslant \rho \leqslant \alpha}$). As a consequence we infer that $\mathrm{Z}(\rho)$, $\mathrm{C}(\rho)$, $\mathrm{S}A$, $\mathrm{C}A$ will also be g.q.d. .

One caution is necessary: when A is g.q.d., it does not follow that a quotient A/J is also g.q.d. (in fact every separable C^*-algebra is a quotient of a quasidiagonal C^*-algebra), so A/J will be assumed in what follows to be g.q.d. . It is also easy to see that if A/J and $\widetilde{J}$ are g.q.d. then A is also g.q.d. .

Under the above assumptions we begin the proof of the long exact sequence which is based on several lemmas.

8.1. LEMMA. *Let (X, x_0) be a pointed, finite-dimensional compact metrizable space $\rho: A \to B$ a unital $*$-homomorphism. Then*

$$\mathrm{Ext}(X, x_0; B) \xrightarrow{\rho_*} \mathrm{Ext}(X, x_0; A) \xrightarrow{q_*} \mathrm{Ext}\,(X, x_0; \mathrm{C}(\rho))$$

is an exact sequence, where q: $\mathrm{C}(\rho) \to A$ is the natural projection.

Proof. We shall consider the C^*-algebra

$$\mathrm{C}_0(\rho) = \{\xi \oplus x \in \mathrm{C}(\rho) \mid \xi(0) = 0\}$$

and the following $*$-homomorphisms:

i: $\mathrm{C}_0(\rho) \to \mathrm{Z}(\rho)$ the inclusion,

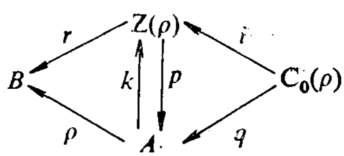
p: $\mathrm{Z}(\rho) \to A$ the natural projection,

r: $\mathrm{Z}(\rho) \to B$ defined by

$$r(\xi \oplus x) = \xi(0) \quad \text{for} \quad \xi \oplus x \in \mathrm{Z}(\rho),$$

k: $A \to \mathrm{Z}(\rho)$ given by $k(x) = f_x \oplus x$, where $f_x \in C\,([0,1], B)$ is the constant function equal $\rho(x)$.

Then the diagram



is commutative (i.e. $r \circ k = \rho$, $p \circ i = q$).

Applying Theorem 4.1 to the exact sequence

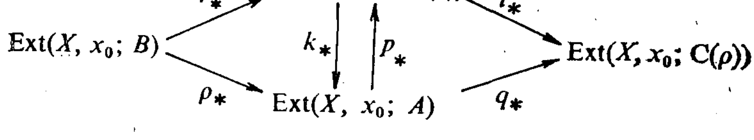
$$0 \to \mathrm{C}_0(\rho) \xrightarrow{i} \mathrm{Z}(\rho) \xrightarrow{r} B \to 0$$

we obtain an exact sequence

$$\mathrm{Ext}(X, x_0; B) \xrightarrow{r_*} \mathrm{Ext}(X, x_0;\ \mathrm{Z}(\rho)) \xrightarrow{i_*} \mathrm{Ext}(X, x_0;\ \mathrm{C}(\rho)).$$

(Note that $\widetilde{\mathrm{C}_0(\rho)}$ is $\mathrm{C}(\rho)$.)

Now $p \circ k = \mathrm{id}_A$ so that $k_* \circ p_* = \mathrm{id}$. If we can show that $k \circ p$ is homotopic to $\mathrm{id}_{Z(\rho)}$ then by the homotopy-invariance the maps k_* and p_* are isomorphisms. We infer that the exactness of the top row in the diagram



will imply the exactness of the bottom row, which is the desired result.

Thus, consider

$$G_s\colon Z(\rho) \to Z(\rho), \quad s \in [0,1]$$

the $*$-homomorphisms defined by

$$G_s(\xi \oplus x) = \xi_s \oplus x$$

where $\xi_s(t) = \xi(1-(1-s)(1-t))$.

Then $G_0 = \mathrm{id}_{Z(\rho)}$, $G_1 = k \circ p$ and obviously G_s depends continuously on s in the point-norm topology. Q.E.D.

8.2. LEMMA. *Let (X, x_0) be a pointed finite-dimensional compact metrizable space, assume the C^*-algebra A is unital and $J \subset A$ is a closed two-sided ideal such that A/J is contractible (of course A, $\widetilde{J}$, A/J are nuclear, g.q.d.). Then the inclusion $i\colon \widetilde{J} \to A$ induces an isomorphism*

$$i_*\colon \mathrm{Ext}(X, x_0; A) \to \mathrm{Ext}(X, x_0; \widetilde{J}).$$

Proof. Let $p\colon A \to A/J$ be the canonical surjection. By homotopy-invariance we have

$$\mathrm{Ext}(X, x_0; A/J) = \{0\}$$

and hence by Theorem 4.1 we infer that i_* is injective.

By Lemma 8.1 there is an exact sequence

$$\mathrm{Ext}(X, x_0; A) \xrightarrow{i_*} \mathrm{Ext}(X, x_0; \widetilde{J}) \to \mathrm{Ext}(X, x_0; \mathrm{C}(i)).$$

So to prove that i_* is surjective it will be sufficient to prove that

$$\mathrm{Ext}(X, x_0; \mathrm{C}(i)) = \{0\}.$$

Consider $\varphi\colon \mathrm{C}(i) \to \mathrm{S}(A/J)$ defined by $\varphi(\xi \oplus x) = \zeta$, where $\zeta(t) = p(\xi(t))$ for $t \in [0,1]$. It is easily seen that φ is a surjection. Thus, there is an exact sequence

$$\mathrm{Ext}(X, x_0; S(A/J)) \to \mathrm{Ext}(X, x_0; \mathrm{C}(i)) \to \mathrm{Ext}(X, x_0; \widetilde{\mathrm{Ker}\,\varphi}).$$

It follows that it will be sufficient to prove that $\mathrm{S}(A/J)$ and $\widetilde{\mathrm{Ker}\,\varphi}$ are contractible C^*-algebras.

It is easily seen that $\mathrm{S}(A/J)$ is contractible (suspensions of contractible C^*-algebras are contractible). Indeed, the contractibility of A/J means that there exists a continuous family $(\Phi_s)_{s\in[0,1]}$ of $*$-homomorphisms of A/J into A/J such that $\Phi_0 = \mathrm{id}_{A/J}$ and Φ_1 is one-dimensional. Then defining $\Psi_s\colon \mathrm{S}(A/J) \to \mathrm{S}(A/J)$ by $(\Psi_s\xi)(t) = \Phi_{2s}(\xi(t))$ for $0 \leqslant s \leqslant 1/2$ and $(\Psi_s\xi)(t) = \Phi_1(\xi(2(1-s)t))$ for $1/2 \leqslant s \leqslant 1$ we see that $(\Psi_s)_{s\in[0,1]}$ implements the contractibility of $\mathrm{S}(A/J)$.

To show that $\widetilde{\mathrm{Ker}\,\varphi}$ is contractible remark first that $\mathrm{Ker}\varphi$ consists of all elements of the form $\xi \oplus x$, where $\xi \in C([0,1], A)$, $x \in \widetilde{J}$, $x = \xi(1)$, $\xi(t) \in J$ for all $t \in [0,1]$ and $\xi(0) = 0$. Thus $\mathrm{Ker}\varphi$ is isomorphic to

$$B = \{\xi \in C([0,1], J) \mid \xi(0) = 0\}.$$

Defining $G_s\colon \widetilde{B} \to \widetilde{B}$ by $G_s(\lambda e + \xi) = \lambda e + \xi'$ where $\xi'(t) = \xi(st)$ for $t \in [0, 1]$ it is easily seen that $(G_s)_{s\in[0,1]}$ implements the contractibility of $\widetilde{B}$.

Q.E.D.

Consider now A, J and $q\colon A \to A/J$ as in the preceding lemma and let us define $c_A\colon CA \to A$ by $c_A(\xi) = \xi(1)$ and

$$CA \bigcup CA/J = \{\xi \oplus \zeta \in CA \oplus CA/J \mid qc_A(\xi) = c_{A/J}(\zeta)\}.$$

Consider further the diagram

$$(*) \qquad \begin{array}{ccc} CA \bigcup CA/J & \xleftarrow{\ r\ } & SA/J \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle Sq} \\ SA & \xleftarrow{\ f\ } & SA \end{array}$$

where

$$s(\xi) = \xi \oplus q(\xi(1)) \qquad \text{for } \xi \in SA$$

$$(f(\xi))(t) = \xi(1-t) \qquad \text{for } \xi \in SA,\ t \in [0,1],$$

and

$$r(\xi) = \lambda 1_A \oplus \xi \quad \text{where } \xi \in SA/J$$

and

$$\xi(1) = \lambda 1_{A/J}.$$

With these preparations we can now state the next lemma.

8.3. LEMMA. *The diagram* $(*)$ *is commutative up to homotopy* (i.e. $s \circ f$ *is homotopic to* $r \circ Sq$).

Proof. Let $H_s\colon SA \to CA \bigcup CA/J$, $s \in [0,1]$, be defined by $H_s(\xi) = \xi_1 \oplus \xi_2$ where $\xi_1(t) = \xi(1 - st)$, $\xi_2(t) = q(\xi((1-s)t))$, $t \in [0, 1]$. Then $H_0 = r \circ Sq$, $H_1 = s \circ f$.

8.4. THEOREM. *Let* (X, x_0) *be a pointed finite-dimensional compact metrizable space and let* J *be a closed two-sided ideal of a unital* C^**-algebra* A $(\widetilde{J}, A, A/J$ *are nuclear, g.q.d.). Then there is a natural exact sequence*

$$\mathrm{Ext}(X, x_0;\ A/J) \to \mathrm{Ext}(X, x_0;\ A) \to \mathrm{Ext}(X, x_0;\ \widetilde{J}) \to$$

$$\to \mathrm{Ext}(X, x_0; \mathrm{S}A/J) \to \mathrm{Ext}(X, x_0; \mathrm{S}A) \to \mathrm{Ext}(X, x_0; \mathrm{S}\widetilde{J}) \to$$

$$\to \mathrm{Ext}(X, x_0; \mathrm{S}^2 A/J) \to \ldots .$$

Proof. Consider

$$A \bigcup CA/J = \{x \oplus \xi \in A \oplus C([0, 1], A/J) \mid \xi(0) \in \mathbf{C}1_{A/J},\ \xi(1) = q(x)\}$$

and consider

$$\gamma\colon A \bigcup CA/J \to A$$

$$\delta\colon A \bigcup CA/J \to CA/J$$

the $*$-homomorphisms given by the projections. We have:

$$\operatorname{Ker}\gamma = \{0 \oplus \xi \in A \oplus C([0,1],\ A/J) \mid \xi(0) \in \mathbf{C}1_{A/J},\ \xi(1) = 0\}$$

$$\operatorname{Ker}\delta = \{x \oplus 0 \in A \oplus C([0,1],\ A/J) \mid x \in J\}.$$

Hence defining $\mathrm{S}_0 B = \{\xi \in C([0,1],\ B) \mid \xi(0) \in \mathbf{C}\cdot 1_B,\ \xi(1) = 0\}$ we have obvious isomorphisms $\operatorname{Ker}\delta \simeq J$, $\operatorname{Ker}\gamma \simeq \mathrm{S}_0 A/J$. Thus there are exact sequences

$$0 \to \mathrm{S}_0 A/J \xrightarrow{h} A \bigcup \mathrm{C}A/J \xrightarrow{\gamma} A \to 0$$

$$0 \to J \xrightarrow{k} A \bigcup \mathrm{C}\,A/J \xrightarrow{\delta} \mathrm{C}A/J \to 0.$$

Moreover $\gamma \circ k$ is just the inclusion i of J into A. Also, since $\mathrm{C}A/J$ is contractible, Lemma 8.2 shows that k_* is an isomorphism between $\operatorname{Ext}(X, x_0;\ A\bigcup \mathrm{C}A/J)$ and $\operatorname{Ext}(X, x_0; \widetilde{J})$. With these preparations we can now define the connecting homomorphism by $\partial = h_* \circ k_*^{-1}$. This gives a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ext}(X, x_0;\ A) & \xrightarrow{\gamma_*} & \operatorname{Ext}(X, x_0;\ A \bigcup \mathrm{C}A/J) & \xrightarrow{h_*} & \operatorname{Ext}(X, x_0;\ \widetilde{\mathrm{S}_0\, A/J}) \\
& \searrow^{i_*} & \downarrow k_* & \nearrow_{\partial} & \\
& & \operatorname{Ext}(X,\ x_0; \widetilde{J}) & &
\end{array}$$

Since k_* is an isomorphism, remarking that $\mathrm{S}_0\widetilde{A}/J$ is $\mathrm{S}A/J$ and that exactness of the top row implies exactness of the bottom row, we have thus proved exactness of the long sequence at $\operatorname{Ext}(X, x_0; \widetilde{J})$.

We pass now to the exactness at $\operatorname{Ext}(X, x_0;\ \mathrm{S}A/J)$. We shall use here the notations of Lemma 8.3. Consider also the $*$-homomorphism

$$l\colon\ \mathrm{C}A \bigcup \mathrm{C}A/J \to A \bigcup \mathrm{C}A/J,\ l(\xi \oplus \zeta) = c_A(\xi) \oplus \zeta\,.$$

Then $l \circ r = h$. Using Lemma 8.3 we have that the triangles in the diagram

$$(**)\qquad \begin{array}{ccccc}
& & \operatorname{Ext}(X,\ x_0;\ \mathrm{C}A\bigcup \mathrm{C}A/J) & & \\
& \nearrow^{l_*} & & \searrow^{(s\circ f)_*} & \\
\operatorname{Ext}(X,\ x_0;\ A \bigcup \mathrm{C}\,A/J) & & \downarrow r_* & & \operatorname{Ext}(X,\ x_0;\ \mathrm{S}A) \\
& \searrow_{h_*} & & \nearrow_{(\mathrm{S}q)_*} & \\
& & \operatorname{Ext}(X,\ x_0;\ \mathrm{S}A/J) & &
\end{array}$$

are commutative.

Now r_* is an isomorphism because there is an exact sequence

$$0 \to S_0A/J \xrightarrow{r} CA \bigcup CA/J \to CA \to 0$$

where CA is contractible.

Moreover denoting $S_0'A = f(S_0A)$ we have the exact sequence

$$0 \to S_0'A \xrightarrow{s \circ f} CA \bigcup CA/J \xrightarrow{l} A \bigcup CA/J \to 0.$$

Thus the top row in diagram (**) is exact and r_* being an isomorphism the bottom row will also be exact. Since $\partial = h_* \circ k_*^{-1}$ where k_* is an isomorphism it follows that the exactness of the bottom row in (**) is in fact equivalent to the exactness at $\mathrm{Ext}(X, x_0;\, SA/J)$ of the long exact sequence.

We haven't mentioned until now exactness at $\mathrm{Ext}(X, x_0;\, A)$, which is the content of Theorem 4.1.

In order to obtain exactness also for the rest of the sequence from what has been already proved, there is still a point to be established, namely that the inclusion $\mathrm{Ker} S^n q \subset S^n\widetilde{J}$ induces an isomorphism between $\mathrm{Ext}(X, x_0;\, S^n\widetilde{J})$ and $\mathrm{Ext}(X, x_0;\, \mathrm{Ker}\, S^n q)$. But this follows from Lemma 8.2 applied to the exact sequence $0 \to \mathrm{Ker}\, S^n q \to S^n\widetilde{J} \to S^n\mathbf{C} \to 0$.

This ends the proof of the exactness of the long sequence.

The naturality of the long exact sequence refers to $*$-homomorphisms $\rho: A \to A'$ and ideals $J \subset A$, $J' \subset A'$ such that $\rho(J) \subset J'$. Then ρ induces also a $*$-homomorphism $A/J \to A'/J'$ and the naturality property is the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}(X, x_0;\, A'/J') & \to & \mathrm{Ext}(X, x_0;\, A') & \to & \mathrm{Ext}(X, x_0;\, \widetilde{J'}) & \to & \mathrm{Ext}(X, x_0;\, SA'/J') \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}(X, x_0;\, A/J) & \to & \mathrm{Ext}(X, X_0;\, A) & \to & \mathrm{Ext}(X, x_0;\, \widetilde{J}) & \to & \mathrm{Ext}(X, x_0;\, SA/J) \to
\end{array}$$

The easy verification is left to the reader. Q.E.D.

A consequence of Theorem 8.4. which we shall use is given in the next lemma.

8.5. LEMMA. *Let (X, x_0) be a pointed finite-dimensional compact metrizable space and let $h: A \to B$ be a surjective $*$-homomorphism (A, B nuclear, g.q.d.). Assume moreover there is a $*$-homomorphism $j: B \to A$ such that $h \circ j = \mathrm{id}_B$. Then we have the split exact sequence*:

$$0 \to \mathrm{Ext}(X, x_0;\, B) \underset{j_*}{\overset{h_*}{\rightleftarrows}} \mathrm{Ext}(X, x_0;\, A) \to \mathrm{Ext}(X, x_0;\, \widetilde{\mathrm{Ker}\, h}) \to 0$$

so that $\mathrm{Ker}\, j_*$ *is naturally isomorphic to* $\mathrm{Ext}(X, x_0;\, \widetilde{\mathrm{Ker}\, h})$.

Proof. Since $h \circ j = \mathrm{id}_B$ it follows that $j_* \circ h_* = \mathrm{id}_{\mathrm{Ext}(X, x_0; B)}$ and $(Sj_*) \circ (Sh)_* = \mathrm{id}_{\mathrm{Ext}(X, x_0; SB)}$. Thus h_* and $(Sh)_*$ are injective and the Lemma follows from Theorem 8.4. Q.E.D.

Since we want to discuss reduced suspensions, we shall consider "pointed" C^*-algebras and their "smash-product".

But first we need some remarks concerning the fact that the tensor product of nuclear g.q.d. C^*-algebras is still nuclear g.q.d.

It is known that the tensor product of two nuclear C^*-algebras is still a nuclear C^*-algebra. Moreover if an exact sequence

$$0 \to J \to A \to A/J \to 0$$

is tensored by a nuclear C^*-algebra, the new sequence will again be exact (use [58]).

Now, the spatial tensor product of two quasidiagonal C^*-algebras is easily seen to be again a quasidiagonal C^*-algebra. If the quasidiagonal C^*-algebras are also nuclear, then there is a unique tensor product, which must then be quasidiagonal.

Consider A, B nuclear g.q.d. C^*-algebras with composition series $(I_\alpha)_{\alpha \in \mathscr{I}}$, $(J_\beta)_{\beta \in \mathscr{J}}$ where $\mathscr{I}$, $\mathscr{J}$ are well-ordered sets. Then $A \otimes B$ is nuclear and is also g.q.d. as can be seen using the composition series $(I_\alpha \otimes B + I_{\alpha+1} \otimes J_\beta)_{(\alpha,\beta) \in \mathscr{I} \times \mathscr{J}}$ where $\mathscr{I} \times \mathscr{J}$ has been given the lexicographical order.

The "pointed" C^*-algebras we shall consider will be unital C^*-algebras A together with a specified one-dimensional unital $*$-homomorphism $\chi : A \to \mathbf{C}$. Then it is natural to define the "smash-product" of (A_1, χ_1) and (A_2, χ_2) as $(\widetilde{\mathrm{Ker}\, \chi_1 \otimes \mathrm{Ker}\, \chi_2})$ together with the one-dimensional $*$-homomorphism χ such that $\mathrm{Ker}\, \chi = \mathrm{Ker}\, \chi_1 \otimes \otimes \mathrm{Ker}\, \chi_2$.

Consider now unital nuclear g.q.d. C^*-algebras A, A_1, A_2 and unital $*$-homomorphisms $\chi_k \colon A_k \to \mathbf{C}$ $(k = 1,2)$. Consider further the following four $*$-homomorphisms corresponding to natural inclusions

$$j_1 \colon A \otimes A_2 \to A \otimes A_1 \otimes A_2$$

$$j_2 \colon A \otimes A_1 \to A \otimes A_1 \otimes A_2$$

$$i \colon A \otimes (\widetilde{\mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\chi_2}) \to A \otimes A_1 \otimes A_2$$

$$j \colon A \to A \otimes (\widetilde{\mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\chi_2})$$

and consider the left inverses of j_1, j_2, j obtained by tensoring the distinguished one-dimensional representations of A_1, A_2 and $\widetilde{\mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\chi_2}$:

$$h_1 \colon A \otimes A_1 \otimes A_2 \to A \otimes A_2$$

$$h_2 \colon A \otimes A_1 \otimes A_2 \to A \otimes A_1$$

$$h \colon A \otimes (\widetilde{\mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\chi_2}) \to A.$$

With this notation we have the following lemma.

8.6. LEMMA. *The map i_* gives a natural isomorphism of* $\mathrm{Ker} j_{1*} \cap \mathrm{Ker} j_{2*}$ *onto* $\mathrm{Ker}\ j_*$.

Proof. Applying Lemma 8.5 to $h_2 \circ j_2 = \mathrm{id}_{A\otimes A_1}$ we see that, denoting by k_1 the inclusion of $\widetilde{A\otimes A_1 \otimes \mathrm{Ker}\,\chi_2}$ into $A\otimes A_1 \otimes A_2$, the map k_{1*} gives an isomorphism of $\mathrm{Ker} j_{2*}$ onto $\mathrm{Ext}\,(X, x_0;\ \mathrm{Ker}\,\tilde{h}_2) = \mathrm{Ext}\,(X, x_0;\ \widetilde{A\otimes A_1 \otimes \mathrm{Ker}\,\chi_2})$.

Denote by j_1' the inclusion of $\widetilde{A\otimes \mathrm{Ker}\,\chi_2}$ into $\widetilde{A\otimes A_1 \otimes \mathrm{Ker}\,\chi_2}$ and by h_1': $\widetilde{A\otimes A_1\otimes \mathrm{Ker}\,\chi_2} \to \widetilde{A\otimes \mathrm{Ker}\,\chi_2}$ the left inverse of j_1' which is the restriction of h_1. Let also k_2 be the inclusion of $\widetilde{A\otimes \mathrm{Ker}\,\chi_1 \otimes \mathrm{Ker}\,\chi_2}$ into $\widetilde{A\otimes A_1 \otimes \mathrm{Ker}\,\chi_2}$. Applying Lemma 8.5 to (h_1', j_1') and remarking that $\mathrm{Ker} h_1' = A\otimes \mathrm{Ker}\,\chi_1 \otimes \mathrm{Ker}\,\chi_2$ it follows that k_{2*} gives an isomorphism of $\mathrm{Ker} j_{1*}'$ onto $\mathrm{Ext}(X, x_0;\ \widetilde{A\otimes \mathrm{Ker}\,\chi_1 \otimes \mathrm{Ker}\,\chi_2})$. Thus we infer that $(k_1 \circ k_2)_*$ gives an isomorphism of $\mathrm{Ker}(k_1 \circ j_1')_* \cap \mathrm{Ker}\,j_{2*}$ onto $\mathrm{Ext}\ (X,x_0;\ \widetilde{A\otimes \mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\chi_2})$.

On the other hand, applying Lemma 8.5 to (h, j) and denoting by k the inclusion of $\mathrm{Ker}\,\tilde{h} = \widetilde{A\otimes \mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\,\chi_2}$ into $A\otimes(\widetilde{\mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\,\chi_2})$ we have that k_* gives an isomorphism of $\mathrm{Ker} j_*$ onto $\mathrm{Ext}(X,\ x_0;\ \widetilde{A\otimes \mathrm{Ker}\chi_1 \otimes \mathrm{Ker}\,\chi_2})$.

Since $i \circ k = k_1 \circ k_2$ and $k_* \,|\mathrm{Ker}\,j_*$ is an isomorphism, it follows that in order to conclude the proof it will be sufficient to show that

$$\mathrm{Ker}(k_1 \circ j_1')_* \cap \mathrm{Ker}\,j_{2*} = \mathrm{Ker}\,j_{1*} \cap \mathrm{Ker}\,j_{2*}.$$

Consider the split exact sequence

$$0 \to A\otimes \mathrm{Ker}\,\chi_2 \xrightarrow{t} A\otimes A_2 \underset{r}{\overset{s}{\rightleftarrows}} A \to 0$$

where t, r are the canonical inclusions and $s = 1_A \otimes \chi_2$. Then $(k \circ j_1')_* = (j_1 \circ t)_*$. Thus, if $\alpha \in \mathrm{Ker}(k_1 \circ j_1')_*$ we have $j_{1*}\alpha = s_*\beta$ where $\beta = (j_1 \circ r)_*\alpha$. Now, $j_1 \circ r$ is the natural inclusion of A into $A\otimes A_1 \otimes A_2$. Hence, if α is also in $\mathrm{Ker}\,j_{2*}$ then $(j_1 \circ r)_*\alpha = 0$, so that we get the desired conclusion. Q.E.D.

Let us also indicate a generalization of Lemma 8.6, the proof of which can be based on using Lemma 8.6 and Lemma 8.5 several times and which will be omitted.

8.7. LEMMA. *Let A be a g.q.d. nuclear C^*-algebra and let $(A_k,\ \chi_k)$ $(k = 1, \ldots, n)$ be "pointed" g.q.d. nuclear C^*-algebras. Consider j_k the inclusion of $A\otimes A_1 \otimes \ldots \otimes A_{k-1} \otimes A_{k+1} \otimes \ldots \otimes A_n$ into $A\otimes A_1 \otimes \ldots \otimes A_n$, j the inclusion of A into $A\otimes(\widetilde{\mathrm{Ker}\,\chi_1 \otimes \ldots \otimes \mathrm{Ker}\,\chi_n})$ and i the inclusion of $A\otimes(\widetilde{\mathrm{Ker}\chi_1 \otimes \ldots \otimes \mathrm{Ker}\,\chi_n})$ into $A\otimes A_1 \otimes \ldots \otimes A_n$. Then i_* gives a natural isomorphism of* $\mathrm{Ker}\,j_{1*} \cap \ldots \cap \mathrm{Ker}\,j_{n*}$ *onto* $\mathrm{Ker}\ j_*$.

Remark that in case $A = \mathbf{C}$ the preceding lemma gives a description of $\mathrm{Ext}(X,\ x_0;\ \mathrm{Ker}\,\widetilde{\chi_1 \otimes \ldots \otimes \mathrm{Ker}}\,\chi_n)$ as a subgroup of $\mathrm{Ext}\ (X,\ x_0;\ A_1 \otimes \ldots \otimes A_n)$.

For (A, χ) a "pointed" nuclear, g.q.d. C^*-algebra we shall consider the "reduced suspension" $\mathrm{S}(A, \chi)$ of (A, χ) which is:

$$\tilde{\mathrm{S}}(A, \chi) = \{f \in \mathrm{S}A \mid \chi f(t) \text{ is constant}\}$$

or equivalently

$$\bar{\mathrm{S}}(A, \chi) = \{f \in C([0,1], A) \mid \chi f(t) = 0,\ \forall\, t \in [0,1], f(0) = f(1) = 0\}^{\sim}$$

which makes $\bar{\mathrm{S}}(A, \chi)$ a "pointed" C^*-algebra. The "reduced suspension" of (A, χ) defined in this way coincides with L. G. Brown's suspension [9] of $\ker \chi$ with a unit adjoined. Denoting by J the ideal

$$\{f \in C([0,1], A) \mid \chi f(t) = 0,\ \forall\, t \in [0,1], f(0) = f(1) = 0\},$$

we have an exact sequence

$$0 \to J \to \mathrm{S}A \to C([0,1]) \to 0$$

and since $C([0,1])$ is contractible and $\tilde{J} = \bar{\mathrm{S}}(A, \chi)$ it follows that the inclusion $\bar{\mathrm{S}}(A,\chi) \subset \subset \mathrm{S}A$ gives a natural isomorphism of $\mathrm{Ext}(X, x_0; \mathrm{S}A)$ and $\mathrm{Ext}(X, x_0; \bar{\mathrm{S}}(A, \chi))$.

More generally it is easily seen that this holds also for iterated suspensions, i.e. there is a natural isomorphism between $\mathrm{Ext}(X, x_0; \mathrm{S}^n A)$ and $\mathrm{Ext}(X, x_0; \bar{\mathrm{S}}^n(A, \chi))$. Thus, for "pointed" C^*-algebras, as far as only Ext is involved, we can always replace the usual suspensions by reduced suspensions.

Now, the "reduced suspension" can be viewed as a "smash product". Indeed, consider $(C(S^1), \varepsilon)$ where $\varepsilon\colon C(S^1) \to \mathbf{C}$ is any character of $C(S^1)$. Then there is a natural isomorphism between $\bar{\mathrm{S}}^n A$ and $\widetilde{\mathrm{Ker}\chi \otimes \underbrace{\mathrm{Ker}\varepsilon \otimes \ldots \otimes \mathrm{Ker}\varepsilon}_{n\text{-times}}}$. Consider

$$h_0\colon \underbrace{C(S^1) \otimes \ldots C(S^1)}_{n\text{-times}} \to A \otimes \underbrace{C(S^1) \otimes \ldots \otimes C(S^1)}_{n\text{-times}},$$

the natural inclusion and consider also

$$h_j\colon A \otimes \underbrace{C(S_1) \otimes \ldots \otimes C(S^1)}_{(n-1)\text{-times}} \to A \otimes \underbrace{C(S^1) \otimes \ldots \otimes C(S^1)}_{n\text{-times}} \quad (1 \leqslant j \leqslant n)$$

the injection obtained by omitting the j-th $C(S^1)$-factor. Then we have the following consequence of Lemma 8.7 (see also the remarks after this lemma).

8.8. COROLLARY. *There is a natural isomorphism of* $\mathrm{Ext}(X, x_0; \bar{\mathrm{S}}^n(A, \chi))$ *onto* $\mathrm{Ker} h_{0*} \cap \mathrm{Ker} h_{1*} \cap \ldots \cap \mathrm{Ker} h_{n*}$.

This concludes our discussion in this section concerning the A-"variable" and we shall now briefly summarize the coresponding facts for the X-"variable" (without proofs).

Thus, let X be a finite-dimensional compact metrizable space, $Y \subset X$ a closed subset and $x_0 \in Y$ a common basepoint for X and Y. We denote by $\mathrm{S}X$ the suspension of X, by $\bar{\mathrm{S}}X$ the reduced suspension, and by $q\colon \mathrm{S}X \to \bar{\mathrm{S}}X$ the canonical surjective map.

Then for a nuclear, g.q.d. C^*-algebra A we have:

1°. $$q^*\colon \mathrm{Ext}\,(\bar{S}X, \sigma; A) \to \mathrm{Ext}\,(SX, \sigma_0; A)$$
is an isomorphism (σ, σ_0 are basepoints).

2°. *There is a natural exact sequence*

$$\mathrm{Ext}\,(Y, x_0; A) \leftarrow \mathrm{Ext}\,(X, x_0; A) \leftarrow \mathrm{Ext}\,(X, Y; A) \leftarrow$$
$$\leftarrow \mathrm{Ext}\,(SY, x_0; A) \leftarrow \mathrm{Ext}\,(SX, x_0; A) \leftarrow \mathrm{Ext}\,(SX, SY; A) \leftarrow \dots$$
$$\leftarrow \dots$$

3°. *The group* $\mathrm{Ext}(S^nX, x_0; A)$ *is naturally isomorphic to the subgroup of* $\mathrm{Ext}(S^n\times X, (\sigma, x_0); A)$ *(where $\sigma \in S^n$, $x_0 \in X$ are basepoints) which have trivial restrictions to both $S^n \times \{x_0\}$ and $\{\sigma\}\times X$.*

§ 9.

This section deals with periodicity for Ext (X, A). First we establish some properties of a certain clutching construction. Besides its use in the proof of the periodicity theorem, this yields the fact that, roughly speaking, taking suspensions in the X-variable or in the A-variable has the same effect on Ext $(X, x_0; A)$. This makes the periodicity theorems in the X-variable and in the A-variable equivalent.

The proof of periodicity that we give has two parts. Half is an adaption of a half of the proof for K-theory in [5]; half is almost a repetition of half of the proof for the usual Ext given in [12].

In addition to the usual assumptions: X finite-dimensional, A nuclear, g.q.d. we will obtain our results under the additional assumption that A has a one-dimensional representation.

For the clutching construction, consider X a finite-dimensional compact metrizable space, S^1 the one-dimensional sphere identified with $\{z \in \mathbf{C} \mid |z| = 1\}$ and let

$$r\colon X\times[0,1] \to X, \qquad r(x, h) = x$$
$$s\colon X\times[0,1] \to X\times S^1, \qquad s(x, h) = (x, \exp(2\pi i h))$$
$$t\colon X \to X\times S^1, \qquad t(x) = (x, 1)$$
$$i_h\colon X \to X\times[0,1], \qquad i_h(x) = (x, h),\ h \in [0,1].$$

Let further $\rho\colon A \to C_{*s}(X, L(H))$ be a unital $*$-homomorphism such that $p\circ\rho$ defines a trivial X-extension by A and let $U \in C_{*s}(X, L(H))$ be a unitary such that $[U, \rho(A)] \subset C_n(X, K(H))$. We shall use ρ and U to construct an $X\times S^1$-extension by A and moreover we shall prove that the corresponding element of Ext $(X\times S^1, A)$ depends only on the class in $\mathrm{Ext}(X, A\otimes C(S^1))$ naturally defined by ρ and U.

Consider $\widetilde{U} \in C_{*s}(X\times[0,1], L(H))$ any unitary such that $\widetilde{U}(x, 0) = U(x)$ and $\widetilde{U}(x, 1) = I$ (such unitaries exists since the unitary group of $L(H)$ is contractible in the $*$-strong topology). Since $U(x)\,\rho(a)U^{-1}(x) - \rho(a) \in C_n(X, K(H))$ we can find $R(a) \in C_{*s}(X\times S^1, L(H))$ such that

$$R(a)\circ s - \widetilde{U}(\rho(a)\circ r)\,\widetilde{U}^{-1} \in C_n(X\times[0,1], K(H)).$$

Clearly $p(R(a))$ depends only on ρ and $\widetilde{U}$ and the map $a \to p(R(a))$ is a $*$-homomorphism which defines a $X\times S^1$-extension by A.

Let us now make some remarks concerning the preceding construction:

1) Let $\widetilde{U}' \in C_{*s}(X\times[0,1], L(H))$ be another unitary such that $\widetilde{U}'(x, 0) = U(x)$ and $\widetilde{U}'(x, 1) = I$ and let $R'(a)$ be defined in the same way as $R(a)$ using $\widetilde{U}'$ instead of $\widetilde{U}$. Then there is a unitary $V \in C_{*s}(x\otimes S^1, L(H))$ such that $V\circ s = \widetilde{U}'\,\widetilde{U}^{-1}$. We have $VR(a) - R'(a)V \in C_n(X\times S^1, K(H))$, so that the $X\times S^1$ -extensions by A defined by $a \to p(R'(a))$ and $a \to p(R(a))$ are equivalent. Thus the class of the $X\times S^1$-extension by A defined by $a \to p(R(a))$ depends only on ρ and U. Denote it by $\alpha(\rho, U)$.

2) Assuming that $\rho'\colon A \to C_{*s}(X, L(H))$ is another unital $*$-homomorphism such that $p\circ\rho = p\circ\rho'$, it is straight-forward to check that $\alpha(\rho, U) = \alpha(\rho', U)$.

3) Consider ρ and U as above and let $U' \in C_{*s}(X, L(H))$ be a unitary such that $U' - U \in C_n(X, K(H))$. To prove that $\alpha(\rho, U') = \alpha(\rho, U)$ we shall need the fact that ρ *has a one-dimensional representation*. Consider $W = U^{-1}U' \in I + C_n(X, K(H))$.

We shall construct a unitary $\widetilde{W} \in C_{*s}(X\times[0,1], L(H))$ such that $\widetilde{W}(x, 0) = W(x)$, $\widetilde{W}(x, 1) = I$ and $[\widetilde{W}, \rho(a)\circ r] \in C_n(X\times[0,1], K(H))$ for all $a \in A$. This will then easily give the desired conclusion by taking $\widetilde{U}' = \widetilde{U}\,\widetilde{W}$, so all we have to do is to construct $\widetilde{W}$. Since $W \in I + C_n(X, K(H))$ there is a projection $P \in C_{*s}(X, L(H))$ with $\dim P(x) = \infty$ and $\dim(I - P)(x) = \infty$ for all $x \in X$, such that $\|W - (PWP + (I - P))\| < 1/2$. Define $\widetilde{W}(x, h)$ for $h \in [0,1/3]$ the unitary obtained from the polar decomposition of $(1 - 3h)W(x) + 3h(P(x)W(x)P(x) + (I - P(x)))$.

Consider now $\chi\colon A \to \mathbf{C}$ a one-dimensional unital $*$-homomorphism. It follows from Theorem 2.10 that there is a projection $Q \in C_{*s}(X, L(H))$ with $\dim Q(x) = \dim(I - Q)(x) = \infty$ for all $x \in X$, such that $\rho(a)Q - \chi(a)Q \in C_n(X, K(H))$ for all $a \in A$. Using Lemma 7.1 we easily construct a unitary $V \in C_{*s}(X, L(H))$ such that $VPV^* = Q$ and because of the contractibility of the unitary group of $L(H)$ with respect to the $*$-strong topology there is a unitary $\widetilde{V} \in C_{*s}(X\times[1/3,2/3], L(H))$ such that $\widetilde{V}(x, 1/3) = I$, $\widetilde{V}(x, 2/3) = V(x)$. Define $\widetilde{W}$ on $X\times[1/3, 2/3]$ by $\widetilde{W}(x, h) = \widetilde{V}(x, h)\widetilde{W}(x, 1/3)\widetilde{V}^*(x, h)$. Remark that $\widetilde{W}(x, 2/3)(I - Q(x)) = I - Q(x)$. By Lemma 7.1 there is some $S \in C_{*s}(x, L(H))$ such that $SS^* = Q$, $S^*S = I$. Then $S^*(x)\,W(x, 2/3)\,S(x)$ is unitary. Let $\widetilde{M} \in C_{*s}(X\times[2/3,1], L(H))$ be any unitary such that $\widetilde{M}(x, 2/3) = S^*(x)W(x, 2/3)\,S(x)$ and $\widetilde{M}(x, 1) = I$. Define $\widetilde{W}$ on $X\times[2/3,1]$

by $\widetilde{W}(x, h) = S(x)\widetilde{M}(x, h)\, S^*(x) + (I - Q(x))$. It is now easy to check that $\widetilde{W}$ has the desired properties. Thus we have proved that

$$\alpha(\rho, U) = \alpha(\rho, U').$$

4) For (ρ', U') another pair with the same properties as (ρ, U) it is immediate that $\alpha(\rho\oplus\rho', U\oplus U') = \alpha(\rho, U) + \alpha(\rho', U')$. Also if $V \in C_{*s}(X, L(H))$ is a unitary such that $V\rho(a) = \rho'(a)V$, $(\forall)\, a \in A$ and $VU = U'V$ it is quite straightforward that $\alpha(\rho, U) = \alpha(\rho', U')$. This last remark together with 2) and 3) shows that more generally if there is a unitary $V \in C_{*s}(X, L(H))$ such that $V\rho\,(a) - \rho'(a)V \in C_n(X, K(H))$ then $\alpha(\rho, U) = \alpha(\rho', U')$ (of course we assume that A has a one-dimensional representation).

The pair (ρ, U) defines a $*$-homomorphism of $A\otimes C(S^1)$ into $C_{*s}(X, L(H))/$ $/C_n(X, K(H))$. If this $*$-homomorphism defines a trivial X-extension by $A\otimes C(S^1)$ it is immediate (in view of the additivity of α and of the equivalence of trivial extensions) that $\alpha(\rho, U) = 0$.

5) Consider $\pi_1\colon A \to A\otimes C(S^1)$, $\pi_2\colon C(S^1) \to A\otimes C(S^1)$ the natural homomorphisms and let $[\sigma] \in \mathrm{Ext}(X, A\otimes C(S^1))$ be such that $\pi_{1*}[\sigma] = 0$, $\pi_{2*}[\sigma] = 0$. Let (ρ, U) be a pair as above defining σ. Then 4) implies that $\alpha(\rho, U)$ depends only on $[\sigma]$. Thus we may write $\alpha([\sigma])$ for such $\alpha(\rho, U)$ and by 4) it follows that α is a homomorphism from

$$\{[\sigma] \in \mathrm{Ext}(X, A\otimes C(S^1))\ |\pi_{1*}[\sigma] = 0,\ \pi_{2*}[\sigma] = 0\}$$

into $\mathrm{Ext}(X\otimes S^1, A)$. Also clearly $t^*\alpha([\sigma]) = 0$.

Moreover if (ρ, U) is any pair for which we did define $\alpha(\rho, U)$, then (ρ, U) gives rise to a unital $*$-homomorphism of $A\otimes C(S^1)$ into $C_{*s}(X, L(H))/C_n(X, K(H))$. For (ρ', U') a trivial X-extension by $A\otimes C(S^1)$ we may consider $\alpha(\rho\oplus\rho', U\oplus U')$. By 4) it follows that $\alpha(\rho, U) = \alpha(\rho\oplus\rho', U\oplus U')$. Thus we may add to (ρ, U) some trivial X-extension by $A\otimes C(S^1)$ and this leaves $\alpha\,(\rho, U)$ unchanged.

6) We shall prove that

$$\alpha\colon\{[\sigma] \in \mathrm{Ext}(X, A\otimes C(S^1)) \mid \pi_{1*}[\sigma] = 0,\ \pi_{2*}[\sigma] = 0\} \to$$
$$\to\{[\tau] \in \mathrm{Ext}(X\times S^1, A)\ |t^*[\tau] = 0\}$$

is injective.

Proof. Assume $\alpha([\sigma]) = 0$ and let (ρ, U) be a pair defining $[\sigma]$. In view of the construction of $\alpha(\rho, U)$, this means that there is a unitary $\widetilde{V} \in C_{*s}\,(X\times[0,1], L(H))$ such that $\widetilde{V}(x, 0) = \widetilde{V}(x, 1)$ for all $x \in X$ and

$$\widetilde{V}\,\widetilde{U}(\rho(a)\circ r)\,\widetilde{U}^{-1}\widetilde{V}^{-1} - \rho(a)\circ r \in C_n(X\times[0,1],\ K(H))$$

for all $a \in A$. Since $\widetilde{U}(x, 1) = I$, denoting $V(x) = \widetilde{V}(x, 0) = \widetilde{V}(x, 1)$ we have

$$[V, \rho(A)] \in C_n(X, K(H)).$$

Hence defining

$$\tilde{U}_1 = (V \circ r)^{-1} \tilde{V}\tilde{U}$$

we have

$$\tilde{U}_1(\rho(a) \circ r)\tilde{U}_1^{-1} - \rho(a) \circ r \in C_n(X \times [0,1], K(H))$$

for $a \in A$ and

$$\tilde{U}_1(X, 0) = U(x), \quad \tilde{U}_1(X, 1) = I.$$

Since $[\tilde{U}_1, \rho(a) \circ r] \in C_n(X \times [0,1], K(H))$, it follows that $\tilde{U}_1$ and the $\rho(a) \circ r$ $(a \in A)$ define a unital $*$-homomorphism of $A \otimes C(S^1)$ into $C_{*s}(X \times [0,1], L(H))/C_n(X \times \times [0,1], K(H))$ and by direct sum with some trivial $X \times [0,1]$-extension by $A \otimes C(S^1)$ we get an element $[\delta] \in \mathrm{Ext}(X \times [0,1], A \otimes C(S^1))$. Since $\tilde{U}_1(x, 0) = U(x)$ and $\tilde{U}_1(x, 1) = I$ it follows that $i_0^*[\delta] = [\sigma]$ and $i_1^*[\delta] = 0$. Thus by the homotopy invariance property we infer $[\sigma] = 0$.

7) We shall prove that

$$\alpha\colon \{[\sigma] \in \mathrm{Ext}(X, A \otimes C(S^1)) \mid \pi_{1*}[\sigma] = 0, \pi_{2*}[\sigma] = 0\} \to$$

$$\to \{[\tau] \in \mathrm{Ext}(X \times S^1, A) \mid t^*[\tau] = 0\}$$

is onto. In view of 6) this will show that α is an isomorphism.

Proof. Assume $[\tau] \in \mathrm{Ext}\,(X \times S^1, A)$ is such that $t^*[\tau] = 0$ and for each $a \in A$ let $R(a) \in C_{*s}(X \times S^1, L(H))$ be such that $\tau(a) = p(R(a))$. Since $t^*[\tau] = 0$ there is a unital $*$-homomorphism $\rho\colon A \to C_{*s}(X, L(H))$ defining a trivial X-extension by A such that $\rho(a) - R(a) \circ t \in C_n(X, K(H))$ for all $a \in A$. Also, from $s \circ i_0 = t$ we infer $i_0^*(s^*[\tau]) = 0$ and hence by homotopy $s^*[\tau] = 0$.

Thus there is a unitary $\tilde{V} \in C_{*s}(X \times [0,1], L(H))$ such that

$$\tilde{V}(\rho(a) \circ r)\tilde{V}^{-1} - (R(a) \circ s) \in C_n(X \times [0,1], K(H)).$$

Because of $(\rho(a) \circ r) \circ i_1 - (R(a) \circ s) \circ i_1 \in C_n(X, K(H))$ this implies $[V, \rho(A)] \subset \subset C_n(X, K(H))$ where $V = \tilde{V} \circ i_1$.

In a similar way we have also $[\tilde{V} \circ i_0, \rho(A)] \subset C_n(X, K(H))$. Hence defining $\tilde{U} = \tilde{V}(V \circ r)^{-1}$ and $U = \tilde{U} \circ i_0$ we have:

$$\tilde{U}(\rho(a) \circ r)\tilde{U}^{-1} - (R(a) \circ s) \in C_n(X \times [0, 1], K(H)),$$

$$[U, \rho(A)] \subset C_n(X, K(H)),$$

$$\tilde{U} \circ i_1 = I.$$

These relations show that $[\tau] = \alpha(\rho, U)$, which proves our assertion.

It is quite straightforward from our construction that α is functional in X and A.

Summing up the preceding discussion, we have proved:

9.1. THEOREM. *The above defined homomorphism*

$$\alpha\colon \{[\sigma]\in \mathrm{Ext}(X,\ A\otimes C(S^1))\mid \pi_{1*}[\sigma]=0,\ \pi_{2*}[\sigma]=0\}\to$$

$$\to \{[\tau]\in \mathrm{Ext}(X\times S^1,\ A)\mid t^*[\tau]=0\}$$

is a natural isomorphism.

In view of Corollary 8.8 there is a natural isomorphism

$$\{[\sigma]\in \mathrm{Ext}\,(X, A\otimes (S^1))\mid \pi_{1*}[\sigma]=0,\ \pi_{2*}[\sigma]=0\} \xrightarrow{\sim} \mathrm{Ext}(X, \bar{\mathrm{S}}(A, X)).$$

Assuming we have a pointed space (X, x_0) and denoting by y_0 the basepoint of $\bar{S}X$, using § 8 and the functoriality of α we easily get:

9.2. COROLLARY. *The isomorphism α induces a natural interchange isomorphism (still denoted by α):*

$$\alpha\colon \mathrm{Ext}\,(X, x_0;\ \bar{\mathrm{S}}(A, X))\to \mathrm{Ext}\,(\bar{S}X, y_0;\ A).$$

For the proof of the periodicity theorem we shall need the following technical result:

9.3. LEMMA. *Let $\eta\in \mathrm{Ext}\,(X\times S^1, A\otimes C(S^1))$ be such that $\pi_{1*}\eta=0$, $\pi_{2*}\eta=0$, $t^*\eta=0$. Then there is a unital $*$-homomorphism $\rho\colon A\to C_{*s}(X\times S^1, L(H))$ defining a trivial $X\times S^1$-extension by A and a unitary $U\in C_{*s}(X\times S^1, L(H))$ with $[U, \rho(A)]\subset \subset C_n\,(X\times S^1, K(H))$ such that the following properties hold:*

(i) *ρ is constant with respect to $X\times S^1$,*

(ii) *$U(x, z)$ with $x\in X$, $z\in S^1$ is norm continuous in z, uniformly with respect to $x\in X$.*

(iii) *ρ and U define a $X\times S^1$-extension by $A\otimes C(S^1)$ of class η.*

Proof. For a tensor product of unital C^*-algebras $A_1\otimes A_2\otimes A_3$, we shall denote by π_i and π_{ij} the natural injections of A_i and $A_i\otimes A_j$ into $A_1\otimes A_2\otimes A_3$. Since $t^*\eta=0$ it follows from Theorem 9.1 that there is some class

$$\theta\in \mathrm{Ext}(X,\ A\otimes C(S^1)\otimes C(S^1))$$

with $\pi_{12*}\theta=0$, $\pi_{3*}\theta=0$ such that $\alpha(\theta)=\eta$. Since $\pi_{1*}\eta=0$, $\pi_{2*}\eta=0$, we infer by the naturality of α that $\pi_{13*}\theta=0$, $\pi_{23*}\theta=0$. Since $\pi_{1*}\theta=0$, $\pi_{2*}\theta=0$, $\pi_{3*}\theta=0$, a X-extension by $A\otimes C(S^1)\otimes C(S^1)$ of class θ can be described by means of a unital $*$-homomorphism $\mu\colon A\to C_{*s}(X, L(H))$ and unitaries U_2, $U_3\in C_{*s}(X, L(H))$. Since $\pi_{13*}\theta=0$ we may suppose μ and U_3 are constant with respect to X and $[\mu(A), U_3]=0$. To construct $\alpha(\theta)$ we must construct $\tilde{U}_3\in C_{*s}(X\times[0,1], L(H))$ with $\tilde{U}_3\circ i_0=U_3$, $\tilde{U}\circ i_1=I$. Since U_3 is constant we can take $\tilde{U}_3\circ i_h=f_h(U_3)$ where $f_h\colon S^1\to S^1$ is given by $f_h\,(\exp\,(2\pi im))=\exp\,(2\pi im(1-h))$ for $m\in[0,1)$. It follows that U_3 is norm-continuous and $[\widetilde{U}_3, \mu(a)\circ r] = 0$ for all $a \in A$. In view of the construction of $\alpha(\theta)$ the norm-continuity of $\widetilde{U}_3$ implies (ii) and the commutation $[\widetilde{U}_3, \mu(a)\circ r] = 0$ together with the fact that μ is constant with respect to x implies (i). Q.E.D.

Consider now: $\rho: A \to C_{*s}(X\times S^1, L(H))$ a unital $*$-homomorphism constant with respect to $X\times S^1$ defining a trivial $X\times S^1$-extension by A and let $U \in C_{*s}(X\times S^1, L(H))$ be a unitary such that $[\rho(A), U] \subset C_n(X\times S^1, K(H))$. Then ρ and U determine a unital $*$-homomorphism of $A \otimes C(S^1)$ into $C_{*s}(X\times S^1, L(H))/$ $/C_n(X\times S^1, K(H))$. By taking the direct sum of this homomorphism with some trivial $X\times S^1$-extension by $A \otimes C(S^1)$ we get an $X\times S^1$-extension by $A \otimes C(S^1)$, the equivalence class of which will be denoted by $T(\rho, U)$. Clearly $\pi_{1*}T(\rho, U) = 0$, $\pi_{2*}T(\rho, U) = 0$. In fact every element $\eta \in \mathrm{Ext}(X\times S^1, A \otimes C(S^1))$ with $\pi_{1*}\eta = 0$, $\pi_{2*}\eta = 0$ is of the form $T(\rho, U)$, where ρ can even be supposed fixed in view of Theorem 2.10.

In case $G \in C_{*s}(X\times S^1, L(H))$ is an invertible element and $[\rho(A), G] \subset$ $\subset C_n(X\times S^1, K(H))$ we may consider $T(\rho, \omega(G))$, where $\omega(G) = G(G^*G)^{-1/2}$ is the unitary part in the polar decomposition of G. Two such invertible elements $G_j \in C_{*s}(X\times S^1, L(H))$ $(j = 1,2)$ with $[\rho(A), G_j] \subset C_n(X\times S^1, K(H))$ will be called homotopic if there is an invertible element $\widetilde{G} \in C_{*s}(X\times S^1\times[0,1], L(H))$ with $\widetilde{G}(x, z, 0) = G_1(x, z)$, $\widetilde{G}(x, z, 1) = G_2(x, z)$ and $[\widetilde{G}, \widetilde{\rho}(A)] \subset C_n(X\times S^1\times[0,1], K(H))$ where $(\widetilde{\rho}(a))(x, z, h) = \rho(a)(x, z)$. It is immediate that if G_1, G_2 are homotopic then $T(\rho, \omega(G_1)) = T(\rho, \omega(G_2))$.

Also, if $[\rho(A), G] = 0$ then $T(\rho, \omega(G)) = 0$.

9.4. Lemma. *We have*

$$T(\rho, \omega(G_1G_2)) = T(\rho, \omega(G_2G_1)) = T(\rho, \omega(G_1)\,\omega(G_2)) =$$

$$= T(\rho, \omega(G_2)\,\omega(G_1)) = T(\rho \oplus \rho, \omega(G_1 \oplus G_2)) =$$

$$= T(\rho, \omega(G_1)) + T(\rho, \omega(G_2)).$$

Proof. It is sufficient to prove that $T(\rho, \omega(G_1G_2)) = T(\rho \oplus \rho, \omega(G_1\oplus G_2))$. We have $T(\rho, \omega(G_1G_2)) = T(\rho \oplus \rho, \omega(G_1G_2 \oplus I))$. Thus it will be sufficient to prove that $G_1\oplus G_2$ and $G_1G_2 \oplus I$ are homotopic. This is established by the usual trick:

$$\widetilde{G}(x, z, h) = \begin{pmatrix} G_1(x, z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos\frac{\pi h}{2} & \sin\frac{\pi h}{2} \\ -\sin\frac{\pi h}{2} & \cos\frac{\pi h}{2} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G_2(x, z) \end{pmatrix} \begin{pmatrix} \cos\frac{\pi h}{2} & -\sin\frac{\pi h}{2} \\ \sin\frac{\pi h}{2} & \cos\frac{\pi h}{2} \end{pmatrix}.$$

Q.E.D.

9.5. LEMMA. *Let* $\eta \in \operatorname{Ext}(X\times S^1, A \otimes C(S^1))$ *be such that* $\pi_{1*}\eta = 0$, $\pi_{2*}\eta = 0$, $t^*\eta = 0$. *Further, let* $\rho: A \to C_{*s}(X\times S^1, L(H))$ *be a unital* $*$*-homomorphism, constant with respect to* $X\times S^1$, *defining a trivial* $X\times S^1$*-extension by* A. *Then there is some integer* n *and* $D_0, D_1, \ldots, D_n \in C_{*s}(X, L(H))$ *satisfying* $[D_j, \rho(a)\circ t] \in C_n(X, K(H))$ *for* $j = 0,1, \ldots, n$ *and for all* $a \in A$, *such that*

$$G(x, z) = \sum_{j=0}^{n} z^j D_j(x)$$

is invertible in $C_{*s}(X\times S^1, L(H))$ *and*

$$T(\rho, \omega(G)) = \eta.$$

Proof. The fact that ρ in the statement of Lemma 9.3 can be given in advance is a consequence of Theorem 2.10. Thus using Lemma 9.3 there is a unitary $U \in C_{*s}(X\times S^1, L(H))$ satisfying the conditions specified in the statement of Lemma 9.3 so that $\eta = T(\rho, U)$. Let $\varphi \geqslant 0$ be a scalar C^∞-function on S^1 with $\int_{S^1} \varphi(\xi)\, d\lambda(\xi) = 1$ ($d\lambda$-Lebesgue measure) and define $G' \in C_{*s}(X\times S^1, L(H))$ be the convolution:

$$G'(x, z) = \int_{S^1} U(x, \xi)\, \varphi(z\xi^{-1})\, d\lambda(\xi).$$

Then, because of property (ii) in the statement of Lemma 9.3, if the support of φ is in some small enough neighborhood of $1 \in S^1$ we shall have $\|U - G'\| \leqslant 1/2$. Also it is easily seen that $[G', \rho(A)] \subset C_n(X\times S^1, K(H))$ and U and G' are homotopic so that $\eta = T(\rho, \omega(G'))$. Since G' is a convolution by a C^∞-function, its Fourier series:

$$G'(x, z) = \sum_{j\in \mathbf{Z}} z^j D'_j(x)$$

with $D'_j \in C_{*s}(X, L(H))$ is uniformly absolutely convergent, i.e.

$$\sum_{j\in \mathbf{Z}} \|D'_j\| < \infty.$$

Moreover it is immediate from the formulae giving the Fourier-coefficients D'_j that

$$[D'_j, \rho(a)\circ t] \in C_n(X, K(H)) \quad \text{for } a \in A.$$

Defining

$$G''(x, z) = \sum_{j=-m}^{m} z^j D'_j(x)$$

for m great enough, in view of the absolute convergence of the Fourier series of G' we shall have $\|G'' - U\| < 1/2$ so that $T(\rho, \omega(G'')) = \eta$. Defining

$$G(x, z) = \sum_{j=0}^{2m} z^j D'_{j-m}(x)$$

we shall prove that

$$T(\rho, \omega(G)) = \eta.$$

Let us consider $Z(x, z) = zI$. Then $[\rho(A), Z^m] = 0$ so that $T(\rho, Z^m) = 0$. Using Lemma 9.4 we have

$$T(\rho, \omega(G)) = T(\rho, \omega(Z^m G'')) = T(\rho, Z^m) + T(\rho, \omega(G'')) =$$

$$= T(\rho, \omega(G'')) = \eta.$$

Q.E.D.

For the next two lemmas $\rho: A \to C_{*s}(X\times S^1, L(H))$ will be a given unital $*$-homomorphism, constant with respect to $X\times S^1$ and defining a trivial $X\times S^1$-extension by A.

9.6. LEMMA. *Let $\eta \in \mathrm{Ext}(X\times S^1, A \otimes C(S^1))$ be such that $\pi_{1*}\eta = 0$, $\pi_{2*}\eta = 0$, $t^*\eta = 0$. Then there are $D_0, D_1 \in C_{*s}(X, L(H))$ satisfying $[D_j, \rho(a)\circ t] \in C_n(X, K(H))$ for $j = 0, 1$ and $a \in A$, so that $G(x, z) = D_0(x) + zD_1(x)$ is invertible in $C_{*s}(X\times S^1, L(H))$ and*

$$T(\rho, \omega(G)) = \eta.$$

Proof. In view of Lemma 9.5 we have $\eta = T(\rho, \omega(G_0))$ where

$$G_0(x, z) = \sum_{j=0}^{n} z^j D_j'(x)$$

and $[D_j', \rho(a)\circ t] \subset C_n(X, K(H))$ for all $a \in A$ and $j = 1, \ldots, n$. Let us consider $Z(x, z) = zI$ and $G_k(x, z) = \sum_{j=0}^{n-k} z^j D_{j+k}'(x)$.

We have the matrix-identity:

$$\begin{pmatrix} D_0' & D_1' & \cdots & D_n' \\ -Z & I & & \\ & -Z & I & \\ & & \ddots & \ddots \\ & & -Z & I \end{pmatrix} =$$

$$= \begin{pmatrix} I & G_1 & G_2 \ldots & G_n \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} \begin{pmatrix} G_0 & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} \begin{pmatrix} I & & & \\ -Z & I & & \\ & \ddots & \ddots & \\ & & -Z & I \end{pmatrix}.$$

Let Γ denote the matrix in the first term of the above identity; we have $\Gamma(x, z) = \Gamma_0(x) + z\Gamma_1(x)$ where $\Gamma_0, \Gamma_1 \in C_{*s}(X, L(H \oplus \ldots \oplus H))$ and $[\Gamma_j, ((\rho \oplus \ldots \ldots \oplus \rho)(a)) \circ t] \subset C_n(X, K(H \oplus \ldots \oplus H))$ for $j = 0, 1$ and $a \in A$. The first and third matrix in the second term of the identity are of the form $I +$ nilpotent, and hence homotopic to I. Thus we have:

$$T(\rho \oplus \ldots \oplus \rho, \omega(\Gamma)) =$$

$$= T(\rho \oplus \ldots \oplus \rho, \omega(G_0 \oplus I \oplus \ldots \oplus I)) = T(\rho, \omega(G_0)) = \eta.$$

By the non-commutative Weyl-von Neumann type theorem there is a constant unitary $V \in C_{*s}(X \times S^1, L(H, H \oplus \ldots \oplus H))$ such that

$$V^{-1}(\rho \oplus \ldots \oplus \rho)(a) V - \rho(a) \in C_n(X \times S^1, K(H))$$

for $a \in A$. Then we may take $G = V^{-1}\Gamma V$, $D_j = V^{-1}\Gamma_j V$, $(j = 0, 1)$. Q.E.D.

9.7. LEMMA. *Let $\eta \in \mathrm{Ext}(X \times S^1, A \otimes C(S^1))$ be such that $\pi_{1*}\eta = 0$, $\pi_{2*}\eta = 0$, $t^*\eta = 0$. Then there are orthogonal projections $P_0, P_1 \in C_{*s}(X, L(H))$ satisfying*

$$[P_j, \rho(a) \circ t] \in C_n(X, K(H)) \quad P_j = P_j^* = P_j^2,\ P_0 + P_1 = I$$

for $j = 0, 1$ and $a \in A$, so that for $G(x, z) = P_0(x) + zP_1(x)$ we have $T(\rho, G) = \eta$.

Proof. In view of Lemma 9.6 we have $\eta = T(\rho, \omega(G'))$ where $G'(x, z) = D_0'(x) + zD_1'(x)$. Defining $\bar{D}_j'(x, z) = D_j'(x)$, we infer from $t^*\eta = 0$ that

$$T(\rho, \omega(\bar{D}_0' + \bar{D}_1')) = 0.$$

Hence using Lemma 9.4 we have

$$T(\rho, \omega((\bar{D}_0' + \bar{D}_1')^{-1} G')) = \eta.$$

Thus taking $D_j = (D_0' + D_1')^{-1}D_j'$ $(j = 0, 1)$ and $G''(x, z) = D_0(x) + zD_1(x)$ we have $T(\rho, \omega(G'')) = \eta$ and $D_0 + D_1 = I$. The invertibility of $D_0 + zD_1$ for all $z \in \mathbf{C}$ with $|z| = 1$ is equivalent to that fact that the spectrum of D_0 does not meet $\{\zeta \in \mathbf{C} |\ \mathrm{Re}\, \zeta = 1/2\}$. Let then Q_0 denote the spectral projection of D_0 for $\{\zeta \in \mathbf{C} |\mathrm{Re}\, \zeta > 1/2\}$ and $Q_1 = I - Q_0$. Since Q_0 is an idempotent we have

$$(Q_0Q_0^*)^2 - Q_0Q_0^* = Q_0(I - Q_0)^*(I - Q_0)\, Q_0^* \geqslant 0$$

so that the spectrum of $Q_0Q_0^*$ is contained in $\{0\} \cup [1, \infty)$. It follows that the spectral projection P_0 of $Q_0Q_0^*$ for $[1, \infty)$ is contained in $C_{*s}(X, L(H))$, $[P_0, \rho(a) \circ t] \in C_n(X, K(H))$ and it is easily seen that $P_0Q_0 = Q_0$ and $Q_0P_0 = P_0$. Thus $S = I -$

$- P_0Q_0(I - P_0)$ is invertible and $SP_0S^{-1} = Q_0$. We define $G'''(x, z) = Q_0(x) + zQ_1(x)$, $P_1 = I - P_0$, $\overline{S}(x, z) = S(x)$ and $G(x, z) = P_0(x) + zP_1(x)$. Taking

$$\Gamma(x, z, h) = (1 - h)\, D_0(x) + hQ_0(x) + z((1 - h)\, D_1(x) + hQ_1(x))$$

it is easily seen that Γ gives a homotopy connecting G'' and G''' so that $\eta = T(\rho, \omega(G'')) = T(\rho, \omega(G'''))$.

Using Lemma 9.4 we have

$$T(\rho, \omega(SG'''\, S^{-1})) = T(\rho, \omega(G'''))$$

so that $\eta = T(\rho, \omega(G))$. Q.E.D.

We define now the maps

$$L_k\colon \mathrm{Ext}(X\times S^1, A) \to \mathrm{Ext}(X\times S^1, A \otimes C(S^1))$$

in the following way: for $[\tau] \in \mathrm{Ext}(X\times S^1, A)$, τ and Z^k determine a homomorphism of $A \otimes C(S^1)$ into $C_{*s}(X\times S^1, L(H))/C_n(X\times S^1, K(H))$ which after adding a trivial $X\times S^1$-extension by $A \otimes C(S^1)$ determines a $X\times S^1$-extension by $A \otimes C(S^1)$. The class of this $X\times S^1$-extension by $A \otimes C(S^1)$ is easily seen to depend only on $[\tau]$ and will be denoted by $L_k[\tau]$. Then L_k is a homomorphism, $\pi_{1*}L_k[\tau] = [\tau]$, $\pi_{2*}L_k[\tau] = 0$ and $t^*L_k[\tau]$ does not depend on k.

Denoting by $u\colon X\times S_1 \to X$, the projection $u(x, z) = x$ for $[\tau] \in \mathrm{Ext}(X, A)$ we define

$$\beta[\tau] = L_1u^*[\tau] - L_0u^*[\tau].$$

It is immediate that

$$\pi_{1*}\beta[\tau] = 0,\ \pi_{2*}\beta[\tau] = 0,\ t^*\beta[\tau] = 0.$$

So we have a homomorphism

$$\beta\colon \mathrm{Ext}(X, A) \to \{\eta \in \mathrm{Ext}(X\times S^1, A \otimes C(S^1))\ |\pi_{1*}\eta = 0,\ \pi_{2*}\eta = 0,\ t^*\eta = 0\}.$$

Now Lemma 9.7 is equivalent to the fact that β is surjective. Indeed, with the notations of Lemma 9.7, taking $P_0' = P_0 \oplus I \oplus 0$, $P_1' = P_1 \oplus 0 \oplus I$, $G'(x, z) = P_0'(x) + z(P_1'(x))$ we have $T(\rho \oplus \rho \oplus \rho, G') = T(\rho, G) + T(\rho, I) + T(\rho, Z) = T(\rho, G) = \eta$ and considering $[\tau_0]$, $[\tau_1]$ the X-extensions by A obtained by "restricting" $t^*(\rho \oplus \rho \oplus \rho)$ to P_0' and P_1' it is immediate that $[\tau_0] + [\tau_1] = 0$ and $\eta = T(\rho \oplus \rho \oplus \rho, G') = \beta[\tau_1]$.

The periodicity theorem is equivalent to the assertion that β is an isomorphism. Since we have already proved that β is surjective we must now prove that it is also injective.

To this end, we shall use a map

$$\gamma\colon \{\eta \in \mathrm{Ext}(X, A \otimes C(S^1) \otimes C(S^1)) \mid \pi_{23*}\eta = 0\} \to \mathrm{Ext}(X, A)$$

related to the periodicity map for Ext, given in [12].

We shall prove that $(\gamma \circ \alpha^{-1} \circ \beta)[\tau] = [\tau]$ for all $[\tau] \in \mathrm{Ext}(X, A)$ which in particular shows that β is injective. In fact after finding a suitable description of $(\alpha^{-1} \circ \beta)[\tau]$ the proof of $(\gamma \circ \alpha^{-1} \circ \beta)[\tau] = [\tau]$ will be an ad literam repetition of the surjectivity of the periodicity map for Ext.

Let $q\colon S^1 \times S^1 \to S^2$ be obtained by collapsing $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ to a point and let P' be a projection in $C(S^1) \otimes C(S^1) \otimes M_2 \simeq C(S^1 \times S^1) \otimes M_2$ corresponding to the pull-back from S^2 to $S^1 \times S^1$ via q of the Hopf line-bundle (an explicit realization of P' will be given later). Let further $P = 1_A \otimes P' \in A \otimes C(S^1) \otimes C(S^1) \otimes M_2$ and let $[\eta] \in \mathrm{Ext}(X, A \otimes C(S^1) \otimes C(S^1))$ be such that $\pi_{23*}[\eta] = 0$. Then $[\eta]$ gives rise to an element $[\eta \otimes \mathrm{id}_{M_2}] \in \mathrm{Ext}(X, A \otimes C(S^1) \otimes C(S^1) \otimes M_2)$ such that $(\eta \otimes \mathrm{id}_{M_2})$ restricted to $1_A \otimes C(S^1) \otimes C(S^1) \otimes M_2$ is trivial. Then $(\eta \otimes \mathrm{id}_{M_2})(P)$ is a projection in C_{*s}/C_n which lifts to a projection in C_{*s}. Since moreover $(\eta \otimes \mathrm{id}_{M_2})(P)$ commutes with $(\eta \otimes \mathrm{id}_{M_2})(\pi_1(A))$ it follows that we can define a homogeneous X-extension by A, by restricted $(\eta \otimes \mathrm{id}_{M_2}) \circ \pi_1$ to $(\eta \otimes \mathrm{id}_{M_2})(P)$.

The class of this X-extension by A in $\mathrm{Ext}(X, A)$ will be denoted by $\gamma([\eta])$.

Of course, since the construction of $\gamma([\eta])$ implies the choice of a lifting of P, $\gamma([\eta])$ will be defined only up to weak-equivalence, but in view our assumption that A has a one-dimensional representation, weak and strong equivalence for homogeneous X-extensions by A coincide. So the map γ is well-defined and is, of course, a homomorphism.

We pass now to the description of $(\alpha^{-1} \circ \beta)[\tau]$.

Thus consider $\tau\colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$ defining an X-extension by A and let further

$$\tau_0\colon A \to C_{*s}(X, L(H_0))/C_n(X, K(H_0))$$

and $B_0 = B_0^* \in C_{*s}(X, L(H))$ be such that τ_0 and $p(B_0)$ define a trivial homogeneous X-extension by $A \otimes C([-1, 1])$. Consider also $R(a), R_0(a)$ such that $p(R(a)) = \tau(a), p(R_0(a)) = \tau_0(a)$ for $a \in A$. Let further $\widetilde{R}(a)$, $\widetilde{B} \in C_{*s}(X \times [0, 1], L(H))$ be defined by $(\widetilde{R}(a))(x, h) = (R(a))(x)$, $\widetilde{B}(x, h) = (2h - 1)I$ and let similarly $\widetilde{R}_0(a)$, $\widetilde{B}_0 \in C_{*s}(X \times [0, 1], L(H_0))$ be defined by $(\widetilde{R}_0(a))(x, h) = (R_0(a))(x)$ and $\widetilde{B}_0(x, h) = B_0(x)$. By $\widetilde{\tau}, \widetilde{\tau}_0$ we shall denote the $X \times [0,1]$ extension by A defined by the $\widetilde{R}(a)$ and $\widetilde{R}_0(a)$. Because of homotopy we infer that the $X \times [0, 1]$ extensions of $A \otimes C([-1,1])$ defined by $(\widetilde{\tau} \oplus \widetilde{\tau}_0, p(\widetilde{B} \oplus \widetilde{B}_0))$ and $(\widetilde{\tau} \oplus \widetilde{\tau}_0, p(I \oplus \widetilde{B}_0))$ are equivalent. Hence there is a unitary $\widetilde{V} \in C_{*s}(X \times [0, 1], L(H \oplus H_0))$ such that

$$\widetilde{V}(\widetilde{R}(a) \oplus \widetilde{R}_0(a)) - (\widetilde{R}(a) \oplus \widetilde{R}_0(a))\widetilde{V} \in C_n(X \times [0, 1], K(H \oplus H_0))$$

and

$$\widetilde{V}(I \oplus \widetilde{B}_0) - (\widetilde{B} \oplus \widetilde{B}_0)\, \widetilde{V} \in C_n(X\times[0, 1], K(H \oplus H_0)).$$

Defining $U = (\widetilde{V}\circ i_0)(\widetilde{V}\circ i_1)^{-1}$ we have:

$$[U, R(a) \oplus R_0(a)] \in C_n(X, K(H \oplus H_0))$$

$$U(I \oplus B_0) - ((-I) \oplus B_0)\, U \in C_n(X, K(H \oplus H_0)).$$

Consider now $W = -\exp(\pi i(I \oplus B_0)) \in C_{*s}(X, L(H \oplus H_0))$. The triple $(\tau \oplus \tau_0, p(W), p(U))$ defines then a unital $*$-homomorphism of $A \otimes C(S^1) \otimes C(S^1)$ into $C_{*s}(X, L(H \oplus H_0))/C_n(X, K(H \oplus H_0))$ which defines (after adding a trivial extension) an element $[\sigma] \in \mathrm{Ext}(X, A \otimes C(S^1) \otimes C(S^1))$.

Consider also $k\colon A \otimes C(S^1) \otimes C(S^1) \to A$ the unital $*$-homomorphism such that $k(a \otimes f \otimes g) = af(1)\,g(1)$. With these notations we shall prove the following lemma.

9.8. LEMMA. *We have:*

$$(\alpha^{-1}\circ \beta)\,[\tau] = [\sigma] - k_*[\tau].$$

Proof. Of course this is equivalent to proving that $\alpha([\sigma] - k_*[\tau]) = \beta[\tau]$, which in turn can be seen as follows.

Let $R_-(a) \in C_{*s}(X, L(H))$ $(a \in A)$ be elements defining $(-[\tau])$. Then $\pi_{12*}([\sigma]-k_*[\tau])$ is defined modulo a trivial X-extension by $A \otimes C(S^1)$ by the elements $R(a) \oplus R_0(a) \oplus R_-(a)$, $(a \in A)$, and by unitary $I \oplus (-\exp(\pi i B_0)) \oplus I$. Putting together the first and third summands of these elements it is seen that $\pi_{12*}([\sigma] - k_*[\tau]) = 0$. Also modulo the same trivial extension, the unitary to be used in the construction of α is $U \oplus I$ and its extension to $X\times[0, 1]$ can be taken $(\widetilde{V}(\widetilde{V}\circ i_1\circ r)^{-1}) \oplus I$. But then

$$[((\widetilde{V}(\widetilde{V}\circ i_1\circ r)^{-1}) \oplus I),\ (R(a) \oplus R_0(a) \oplus R_-(a))\circ r] \in$$

$$\in C_n(X\times[0, 1],\ K(H \oplus H_0 \oplus H))$$

and

$$((\widetilde{V}(\widetilde{V}\circ i_1\circ r)^{-1} \oplus I)\,((W \oplus I)\circ r)\,((\widetilde{V}(\widetilde{V}\circ i_1\circ r)^{-1} \oplus I)^{-1} -$$

$$-(-\exp(\pi i(\widetilde{B} \oplus \widetilde{B}_0))) \oplus I \in C_n(X\times[0, 1], K(H \oplus H_0 \oplus H)).$$

In view of the fact that $\widetilde{B}(x, h) = (2h - 1)I$ and $\widetilde{B}_0(x, h) = B_0(x)$ it is now immediate that $\alpha([\sigma] - k_*[\tau]) = \beta[\tau]$. Q.E.D.

Note that it is a consequence of the preceding lemma that $\pi_{12*}([\sigma]-k_*[\tau]) = 0$, $\pi_{13*}([\sigma]-k_*[\tau]) = 0$ and $\pi_{23*}([\sigma]-k_*[\tau]) = 0$. Indeed this follows from the definition of the domain of α and from the fact that $\pi_{1*}\beta[\tau] = 0$ and $\pi_{2*}\beta[\tau] = 0$. In particular since $\pi_{23*}(k_*[\tau]) = 0$ it follows that $\pi_{23*}[\sigma] = 0$.

Thus in order to prove that $(\gamma \circ \alpha^{-1} \circ \beta)[\tau] = [\tau]$, remarking that $\gamma(k_*[\tau]) = [\tau]$, it follows that it will be sufficient to prove that $\gamma[\sigma] = 2[\tau]$.

Let us now recall the realization given in [12] for the projection $P' \in C(S^1) \otimes \otimes C(S^1) \otimes M_2$, corresponding to the pull-back of the Hopf line-bundle and which is used in the definition of γ. Consider $j\colon C(S^1) \otimes C(S^1) \otimes M_2 \to C([-1, 1]) \otimes \otimes C(S^1) \otimes M_2$ the $*$-homomorphism $j = j' \otimes \mathrm{id} \otimes \mathrm{id}$ where j' is induced by the map $[-1, 1] \ni t \to (-\exp(\pi i t)) \in S^1$. Then $j(P')$ can be described as the matrix-valued function on $[-1, 1] \times S^1$ given by

$$\begin{pmatrix} t^2 & -(t^+z + t^-)\sqrt{1-t^2} \\ -(t^+z^{-1} + t^-)\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$$

where $t^{\pm} = (1/2)(|t| \pm t)$.

In view of the description we have given of $[\sigma]$ (modulo a trivial extension which we shall no longer mention) we see that putting $\bar{B} = I \oplus B_0$ and $\bar{B}^{\pm} = (1/2)(|\bar{B}| \pm \bar{B})$ we have that $(\sigma \otimes \mathrm{id}_{M_2})(P)$ corresponds to

$$p\left(\begin{pmatrix} \bar{B}^2 & -\bar{B}^+(I-\bar{B}^2)^{1/2}U - \bar{B}^-(I-\bar{B}^2)^{1/2} \\ -\bar{B}^+(I-\bar{B}^2)^{1/2}U^{-1} - \bar{B}^-(I-\bar{B}^2)^{1/2} & I-\bar{B}^2 \end{pmatrix}\right).$$

Hence $\gamma[\sigma]$ will be given by restricting the elements

$$\begin{pmatrix} (\tau \oplus \tau_0)(a) & 0 \\ 0 & (\tau \oplus \tau_0)(a) \end{pmatrix}$$

to this projection.

Let now Q_0 denote the constant projection in $C_{*s}(X, L(H \oplus H_0))$ which projects $H \oplus H_0$ onto $0 \oplus H_0$. We shall prove that $\gamma[\sigma] = 2[\tau]$ by showing that the restrictions of

$$A \ni a \to \begin{pmatrix} (\tau \oplus \tau_0)(a) & 0 \\ 0 & (\tau \oplus \tau_0)(a) \end{pmatrix}$$

to $p(E_0)(\sigma \otimes \mathrm{id}_{M_2})(P)$ and $(I - p(E_0))(\sigma \otimes \mathrm{id}_{M_2})(P)$ where

$$E_0 = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_0 \end{pmatrix},$$

are both equivalent to $[\tau]$. Of course we must first check that $p(E_0)$ and $(\sigma \otimes \mathrm{id}_{M_2})(P)$ commute. This amounts to proving that $p(Q_0)$ commutes with $p(\bar{B}^2)$ (which is obvious) and $p(F^{\pm})\,p(U)$ where $F^{\pm} = -\bar{B}^{\pm}(I - \bar{B}^2)^{1/2}$.

Now, $[p(U), p(F^{\pm})] = 0$ and $F^{\pm}Q_0 = Q_0F^{\pm} = F^{\pm}$ which immediately give the desired commutation.

We have

$$(I - p(E_0))\,(\sigma \otimes \mathrm{id}_{M_2})\,(P) =$$

$$= \begin{pmatrix} I - p(Q_0) & 0 \\ 0 & 0 \end{pmatrix}$$

which shows that the restrictions to this projection of the elements

$$\begin{pmatrix} (\tau \oplus \tau_0)\,(a) & 0 \\ 0 & (\tau \oplus \tau_0)\,(a) \end{pmatrix}$$

defines an extension equivalent with $[\tau]$.

For the other restriction, to be shown equivalent, consider

$$D = \begin{pmatrix} U\bar{B}^{+} + B^{-} & 0 \\ -(I - \bar{B}^2)^{1/2} & 0 \end{pmatrix}.$$

Using the definitions of $U, \bar{B}^{+}, \bar{B}^{-}, \bar{B}$ and the relations connecting them it is not difficult to see that

$$p(D)^* p(D) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

$$p(D)\,p(D)^* = p(E_0)\,(\tau \otimes \mathrm{id}_{M_2})\,(P)$$

and that

$$\left[p(D), \begin{pmatrix} (\tau \oplus \tau_0)\,(a) & 0 \\ 0 & (\tau \oplus \tau_0)\,(a) \end{pmatrix}\right] = 0.$$

These relations give the desired equivalence of the restriction to $p(E_0)\,(\sigma \otimes \otimes\, \mathrm{id}_{M_2})\,(P)$ with $[\tau]$. This ends the proof of $\gamma[\sigma] = 2[\tau]$. Summing up our discussion and taking into account Theorem 9.1 we have proved the following proposition which is equivalent to the periodicity theorem.

9.9. PROPOSITION. *We have natural isomorphisms:*

$$\mathrm{Ext}(X, A) \xrightarrow{\beta} \{\eta \in \mathrm{Ext}(X \times S^1,\ A \otimes C(S^1))\ |\pi_{1*}\eta = 0,\ \pi_{2*}\eta = 0,\ t^*\eta = 0\} \to$$
$$\xrightarrow{\alpha^{-1}} \{\mu \in \mathrm{Ext}(X, A \otimes C(S^1) \otimes C(S^1))\ |\pi_{12*}\mu = 0,\ \pi_{13*}\mu = 0,\ \pi_{23*}\mu = 0\} \to$$
$$\xrightarrow{\gamma} \mathrm{Ext}(X, A)$$

and $\gamma \circ \alpha^{-1} \circ \beta$ *is the identity automorphism of* $\mathrm{Ext}(X, A)$.

Our next aim is to bring the periodicity theorem to a more familiar form.

For the A-variable, since A has a one-dimensional representation, Corollary 8.8 gives an isomorphism of $\operatorname{Ext}(X, \bar{S}^2 A)$ and

$$\{\mu \in \operatorname{Ext}(X, A \otimes C(S^1) \otimes C(S^1)) \mid \pi_{12*}\mu = 0, \ \pi_{13*}\mu = 0, \ \pi_{23*}\mu = 0\}.$$

Then γ, via this isomorphism, induces an isomorphism

$$\operatorname{Ext}(X, \bar{S}^2 A) \to \operatorname{Ext}(X, A)$$

which we shall denote by Per_*. This map coincides with the straightforward generalization of the periodicity map in [12]. The apparent difference in the constructions consisting in the "multiplying" by pull-back of the Hopf line-bundle instead of the Hopf line-bundle minus a trivial line-bundle is inessential, since replacing in the definition of γ the Hopf line-bundle by a trivial line-bundle yields a map which is zero on

$$\{\mu \in \operatorname{Ext}(X, A \otimes C(S^1) \otimes C(S^1)) \mid \pi_{12*}\mu = 0, \pi_{13*}\mu = 0, \ \pi_{23*}\mu = 0\}.$$

In particular when X is reduced to one point we get the same map as in [12]. We want also to remark that in the constructions we did, the projections to which we did restrict extensions did lift to projections, a fact which is not true in general.

For the periodicity in the X-variable it is better to consider a pointed space (X, x_0). In view of the naturality of β, we see that β gives an isomorphism:

$$\operatorname{Ext}(X, x_0; A) \xrightarrow{\beta} \{\eta \in \operatorname{Ext}(X \times S^1, A \otimes C(S^1)) \mid \pi_{1*}\eta = 0, \pi_{2*}\eta = 0, d^*\eta = 0\}$$

where d is the inclusion of $(X \times \{1\}) \cup (\{x_0\} \times S^1)$ into $X \times S^1$.

Using § 8 and Theorem 9.1 we see that $\alpha \circ \beta$ gives an isomorphism

$$\operatorname{Ext}(X, x_0; A) \to \operatorname{Ext}(\bar{S}^2 X, *; A)$$

(where $*$ denotes the basepoint).

This isomorphism will be denoted by Per^* and we shall show that it is the obvious generalization of the periodicity map in K-theory. Let $\theta_1: X \times S^1 \times S^1 \to X$ be the projection and by $\theta_2: X \times S^1 \times S^1 \to S^2$ the projection onto $S^1 \times S^1$ composed with the map $q: S^1 \times S^1 \to S^2$ we have already used. We want to prove that

$$\operatorname{Per}^*[\tau] = (\theta_2^*([L] - [1])) \cdot \theta_1^*[\tau],$$

where $\operatorname{Ext}(\bar{S}^2 X, *; A)$ has been identified in the usual way with a subgroup of $\operatorname{Ext}(X \times S^1 \times S^1; A)$. As usual, L is the Hopf line-bundle.

This can be seen as follows:

Let $f: X \times S^1 \times [0, 1] \to X$ denote the projection and let $[\eta], [\tau_0] \in \operatorname{Ext}(X, x_0; A)$ be such that $[\tau] + [\eta] = 0$, $[\tau_0] = 0$. Consider further $\tau' = f^*\tau$, $\eta' = f^*\eta$,

$\tau_0' = f^*\tau_0$ and $\widetilde{U} \in C_{*s}(X \times S^1 \times [0, 1], L(H \oplus H \oplus H))$ such that $\widetilde{U}(x, z, 1) = I_{H \otimes H \otimes H}$ and

$$\widetilde{U}(x, z, 0) = \begin{pmatrix} zI & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & z^{-1}I \end{pmatrix}.$$

Then the elements $p(\widetilde{U})((\tau' \oplus \eta' \oplus \tau_0')(a))p(\widetilde{U})^{-1}$ are the images of elements in

$$C_{*s}(X \times S^1 \times S^1, L(H \oplus H \oplus H))/C_n(X \times S^1 \times S^1, K(H \oplus H \oplus H))$$

which define, modulo a trivial extension, an extension equivalent with

$$\alpha(L_1 u^*[\tau] + L_0 u^*[\eta] + L_{-1} u^*[\tau_0]) =$$

$$= (L_1 u^*[\tau] - L_0 u^*[\tau]) = (\alpha \circ \beta)[\tau].$$

To prove our assertion remark that $\widetilde{U}$ can be chosen of the form $(a_{ij}(x, z, t))_{1 \leqslant i,j \leqslant 3} \otimes I_H$ where the unitary 3×3 matrix $V(x, z, t) = (a_{ij}(x, z, t))_{1 \leqslant i,j \leqslant 3}$ has scalar coefficients. Since the projections

$$V(x, z, t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^{-1}(x, z, t)$$

$$V(x, z, t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^{-1}(x, z, t)$$

$$V(x, z, t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1}(x, z, t)$$

are pullbacks of projections in $C(X \times S^1 \times S^1) \otimes M_3$ corresponding to line-bundles of class $\theta_1^*[L]$, $\theta_1^*[1]$, $\theta_1^*[L^{-1}]$ it is easily seen that:

$$(\alpha \circ \beta)[\tau] = \theta_1^*[L] \cdot \theta_2^*[\tau] +$$

$$+ \theta_1^*[1] \cdot \theta_2^*[\eta] + \theta_1^*[L^{-1}] \cdot \theta_2^*[\tau_0] =$$

$$= \theta_1^*[L] \cdot \theta_2^*[\tau] - \theta_1^*[1] \cdot \theta_2^*[\tau]$$

which is the desired fact.

Thus we can formulate the periodicity theorem as follows:

9.10. THEOREM. *Let (X, x_0) be a pointed finite-dimensional compact metrizable space and let A be a unital nuclear, g.q.d. C^*-algebra having a one-dimensional representation. Then the maps considered above*

$$\mathrm{Per}^*: \mathrm{Ext}(X, x_0; A) \to \mathrm{Ext}(\mathrm{S}^2X, *; A)$$

$$\mathrm{Per}_*: \mathrm{Ext}(X, x_0; \bar{\mathrm{S}}^2A) \to \mathrm{Ext}(X, x_0; A)$$

are natural isomorphisms and

$$\mathrm{Per}^* \circ \alpha^{-2} \circ \mathrm{Per}_* = \mathrm{id}_{\mathrm{Ext}(X, x; A)}.$$

To conclude this section we will give two corollaries of the periodicity theorem.

The first corollary is that we can now compute $\mathrm{Ext}(X, x_0; \mathbf{C} \oplus \mathbf{C})$ a fact which clarifies the problem of lifting projections from $C_{*s}(X, L(H))/C_n(X, K(H))$ to projections in $C_{*s}(X, L(H))$. Indeed, we have isomorphisms

$$\mathrm{Ext}(X, x_0; \mathbf{C} \oplus \mathbf{C}) \overset{\mathrm{Per}^*}{\tilde{\to}} \mathrm{Ext}(\bar{\mathrm{S}}^2X, *; \mathbf{C} \oplus \mathbf{C}) \overset{\alpha^{-1}}{\tilde{\to}}$$

$$\overset{\alpha^{-1}}{\tilde{\to}} \mathrm{Ext}(\bar{\mathrm{S}}^1X, *; C(S^1)) \simeq \tilde{\mathrm{K}}^0(\mathrm{S}X) \simeq \tilde{\mathrm{K}}^{-1}(X).$$

9.11. COROLLARY. *There is a natural isomorphism*

$$\mathrm{Ext}(X, x_0; \mathbf{C} \oplus \mathbf{C}) \simeq \mathrm{K}^{-1}(X).$$

The second corollary is related to the injectivity of the map β; it will be stated for the sake of simplicity only for the case when X is a point.

Let $\tau: A \to L(H)/K(H)$ be an injective unital $*$-homomorphism. Consider the group:

$$UC(\tau) = \{U \in L(H) \mid U \text{ unitary}, [p(U), \tau(A)] = \{0\}\}.$$

Clearly $UC(\tau)$ up to isomorphism depends only on $[\tau]$. On $UC(\tau)$ we shall consider two topologies: the norm-topology and a second topology we shall call the commutators-topology. The commutators-topology is defined as the weakest topology for which the following maps are continuous:

$$UC(\tau) \ni U \to U\xi \in H$$

$$UC(\tau) \ni U \to U^*\xi \in H$$

$$UC(\tau) \ni U \to [U, X] \in K(H),$$

where ξ runs over H, X runs over $p^{-1}(\tau(A))$ and H and $K(H)$ are given the norm-topologies.

Consider also the following loop denoted by φ:

$$S^1 \ni z \to zI \in UC(\tau).$$

9.12. Corollary. *Let A be a unital nuclear, g.q.d. C^*-algebra having a one-dimensional representation and let $\tau\colon A \to L(H)/K(H)$ be a unital $*$-monomorphism. Then the following assertions are equivalent:*

(i) $[\tau] = 0$

(ii) *φ is homotopic to zero for the norm-topology on $UC(\tau)$*

(iii) *φ is homotopic to zero for the commutators-topology on $UC(\tau)$.*

Proof. Clearly (ii) $\Rightarrow$ (iii).

That (i) $\Rightarrow$ (ii) can be seen as follows. Since $UC(\tau)$ depends only on the equivalence class of τ, we may assume $H = H_1 \otimes H_2$ (where H_1, H_2 are infinite-dimensional) and $\tau = p \circ (\rho \otimes I_{H_2})$ where ρ is a unital $*$-homomorphism. Then $\varphi(z) = I_{H_1} \otimes (zI_{H_2})$ and by Kuiper's theorem the loop $z \to I_{H_1} \otimes (zI_{H_2})$ is zero-homotopic in $I_{H_1} \otimes U(H_2)$ which is contained in $UC(\tau)$.

Next, we prove that (iii) $\Rightarrow$ (i). Let $\psi\colon S^1 \times [0, 1] \to UC(\tau)$ be continuous for the commutators-topology and assume $\psi(z, 0) = zI$ and $\psi(z, 1) = I$. Then ψ is a unitary element of $C_{*s}(S^1 \times [0, 1], L(H))$ such that $[(g^*\tau)(A), p(\psi)] = 0$ where $g\colon S^1 \times [0, 1] \to \{*\}$ and $[\tau]$ is viewed as an element of $\mathrm{Ext}(\{*\}, A)$. Then $g^*\tau$ and ψ determine a $(S^1 \times [0, 1])$-extension by $A \otimes C(S^1)$ which restricted to $S^1 \times \{0\}$ and $S^1 \times \{1\}$ gives extensions equivalent to $L_1 u^*[\tau]$ and $L_0 u^*[\tau]$, where $u\colon S^1 \to \{*\}$. By homotopy it follows that $L_1 u^*[\tau] = L_0 u^*[\tau]$ which means $\beta[\tau] = 0$. By the injectivity of β it follows that $[\tau] = 0$. Q.E.D.

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