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A WEIGHTED BILATERAL SHIFT WITH NO CYCLIC VECTOR

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Let us denote by $(e_n)_{n\in\mathbf{Z}}$ the canonical basis of $\ell^2(\mathbf{Z})$ and by T the weighted shift defined by $Te_n = w_n e_{n+1}$, $n \in \mathbf{Z}$, with

$$\begin{cases} w_n = 1 & n \geqslant 0 \\ w_n = 1/4 & n < 0. \end{cases}$$

This operator can also, of course, be considered as an operator on $L^2(\mathbf{T})$, which we denote also by T.

We are interested in the invariant subspaces for this operator; the following theorem proves that they are very numerous. In fact, no vector is cyclic.

THEOREM. *For all* $0 \neq f \in L^2(\mathbf{T})$, $\overline{\mathrm{span}}\,\{Tf, T^2f, \ldots, T^nf, \ldots\}$ *does not contain* f.

Proof. Assume, on the contrary, that we can find $f \neq 0$, $f \in \overline{\mathrm{span}}\{Tf, \ldots\}$. This means that, for all $\varepsilon > 0$, there exists a finite sequence of scalars (α_n) such that

$$\|f - \sum_{m\geqslant 1} \alpha_m T^m f\|_{L^2(\mathbf{T})} < \varepsilon. \tag{1}$$

Let $f = \sum_{-\infty}^{+\infty} a_k e^{ik\theta}$ be the Fourier decomposition. By computing the Fourier coefficients of $T^m f$, one can write (1) as

$$\sum_{k\geqslant 0} \left| a_k - \left(\sum_{1\leqslant m\leqslant k} \alpha_m a_{k-m} + \sum_{m>k} \alpha_m \frac{1}{4^{m-k}}\, a_{-m+k} \right) \right|^2 +$$

$$+ \sum_{k>0} \left| a_{-k} - \sum_{m\geqslant 1} \alpha_m \frac{1}{4^m}\, a_{-k-m} \right|^2 < \varepsilon^2.$$

By dividing each term of the second sum by the corresponding $1/4^k$, putting $b_k = a_k$ if $k \geqslant 0$, $b_k = \dfrac{1}{4^{|k|}}\, a_k$ if $k < 0$, one obtains

$$\sum_{k\in\mathbf{Z}} |b_k - \sum_{m\geqslant 1} \alpha_m b_{k-m}|^2 < \varepsilon^2. \tag{2}$$

This means that if U is the usual bilateral shift and if $g = \sum_{k\in\mathbf{Z}} b_k e^{ik\theta}$, then

$$\|g - \sum_{m\geqslant 1} \alpha_m U^m g\|_{L^2(\mathbf{T})} < \varepsilon.$$

Since a similar result holds for all $\varepsilon > 0$, it follows that $g \in \overline{\operatorname{span}}\,\{Ug, U^2g, \ldots\}$. From Szegö's theorem, we deduce that

$$\int_{-\pi}^{\pi} \log |g(\theta)|\, d\theta = -\infty.$$

But, by the definition of b_n,

$$\tilde{g}(z) = \sum_{k\in\mathbf{Z}} b_k z^k$$

is analytic in the domain $\left\{\frac{1}{4} < |z| < 1\right\}$, from which we obtain

$$\int_{-\infty}^{+\infty} \log |g(\theta)|\, d\theta > -\infty,$$

and the proof is complete.

REMARKS. 1. This operator is a typical example of a $C_1.$ contraction not satisfying the assumption of our paper [1] (the iterates $T^{-n}x$, $n \geqslant 0$, grow too quickly). One can wonder if the conclusion $x \notin \overline{\operatorname{span}}\,\{Tx, \ldots, T^n x, \ldots\}$ does not hold more generally in such a case.

2. On the contrary, since its spectrum is $\left\{\frac{1}{4} \leqslant |z| \leqslant 1\right\}$, it satisfies the assumption of the paper of Brown-Chevreau-Pearcy [2]. Since their method provides a point not belonging to the closed span of its iterates, it shows that in their paper no other description of the invariant subspaces was possible.

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