J. OPERATOR THEORY
4(1980), 297–306

FUNCTIONAL CALCULUS WITH SECTIONS OF AN ANALYTIC SPACE

MIHAI PUTINAR

The aim of this paper is to give an algebraic-topological approach to the representation theory of algebras of global sections of an analytic space, in connection with the functional calculus problem for finite commuting systems of linear continuous operators on a Fréchet space. The paper contains, as a particular case, the existence of the analytic functional calculus for commuting systems of regular operators on a Fréchet space as well as a spectral mapping theorem. The analytic space context yields a more refined functional calculus even for finite systems of commuting operators namely a functional calculus which takes not only the spectrum but also analytic relations satisfied by the operators into account. Moreover, the more general case of representations of algebras of global sections of a Stein space does not reduce, in general, to the case of finite systems of operators. Finally, working with sheaves of $\mathscr{C}^{\infty}$-functions we obtain an analogue of the theory which corresponds to generalized spectral systems of commuting operators.

Our technique is based on a relative homology theory (in the sense of [5, Cap. IX]) for topological algebras. Such a theory was developed for the proof of Grauert's coherence theorem (see for example [1, II]) and afterwards, independently, a homology theory (essentially the same) was elaborated and used by J. L. Taylor in the functional calculus theory.

Let us recall some notations and terminology. Let A be a nuclear Fréchet $\mathbf{C}$-algebra. In the relative homology theory which we shall use, the free A-modules are of the form $A \hat{\otimes} E$, where E is a Fréchet space and the admissible short exact sequences of Fréchet A-modules are that which are topologically $\mathbf{C}$-split. Each Fréchet A-module M has a natural free resolution (the Bar resolution):

$$0 \leftarrow M \leftarrow A \hat{\otimes} M \leftarrow A \hat{\otimes} A \hat{\otimes} M \leftarrow \dots .$$

The tensor product of two Fréchet A-modules M and N is the quotient space of $M \hat{\otimes} N$ by the subspace (not necessary closed) generated by the elements $am \otimes$

$\otimes n - m \otimes an$, $a \in A$, $m \in M$, $n \in N$. There are the derived functors of the tensor product, $\hat{\mathrm{Tor}}^A_\bullet(M, N)$, and they can be computed in a natural way by the Bar resolution:

$$\mathrm{B}^A_\bullet(M, N)\colon\ 0 \leftarrow M \hat{\otimes} N \leftarrow M \hat{\otimes} A \hat{\otimes} N \leftarrow M \hat{\otimes} A \hat{\otimes} A \hat{\otimes} N \leftarrow \dots$$

$$\hat{\mathrm{Tor}}^A_p(M, N) = \mathrm{H}_p\, \mathrm{B}^A_\bullet(M, N), \qquad p \in \mathbf{N}.$$

For details see [1, Exposé II] or [7].

Let us begin with some obvious correspondences. To give an operator $T \in L(\mathbf{C}^m)$ is equivalent to endowing $\mathbf{C}^m$ with the structure of a $\mathbf{C}[X]$-module (the action of X on $\mathbf{C}^m$ being identified with that of T, $\mathbf{C}[X]$ denoting the polynomial algebra in X). Let M be a Banach space and let T be a bounded linear operator on M. Then there is a topological $\mathcal{O}(\mathbf{C})$-structure on M associated with T in the following way:

$$f \cdot m = \sum_{n \geqslant 0} a_n T^n m, \quad \text{where} \quad f = \sum_{n \geqslant 0} a_n z^n \in \mathcal{O}(\mathbf{C}).$$

The continuity of the product $(f, m) \mapsto fm$ follows easily from the estimation:

$$\|f \cdot m\| = \|\sum_n a_n T^n m\| \leqslant \sum_n |a_n|\, \|T\|^n\, \|m\| = \|f\|_{\|T\|} \|m\|.$$

Conversely, each topological $\mathcal{O}(\mathbf{C})$-module structure on M determines an operator $T \in L(M)$. Similarly, n-tuples of commuting bounded operators on M correspond to topological $\mathcal{O}(\mathbf{C}^n)$-module structures on M.

The translation in this dictionary of the functional calculus problem for a system of commuting operators on the Banach space M is the following: for which open subsets $U \subset \mathbf{C}^n$ is there a topological $\mathcal{O}(U)$-structure on M, compatible with the $\mathcal{O}(\mathbf{C}^n)$-module structure?

We shall state this problem in a wider context and this will the central problem of the paper:

Let X be an analytic Stein space and M a Fréchet $\mathcal{O}(X)$-module. For which open subsets $U \subset X$, does the $\mathcal{O}(X)$-structure extend to a Fréchet $\mathcal{O}(U)$-module structure on M, compatible with the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$?

We shall use the following result which is a simplified version of a theorem in [1, Exposé II]:

(1) Let $X \xrightarrow{f} Y$ be a morphism of analytic Stein spaces and V on open Stein subset in Y. Then

$$\hat{\mathrm{Tor}}^{\mathcal{O}(Y)}_q(\mathcal{O}(X), \mathcal{O}(V)) \cong \mathcal{O}(f^{-1}V) \qquad \text{for } q = 0, \text{ and}$$
$$= 0 \qquad \text{for } q > 0,$$

the isomorphism being topological.

Here as well in rest of the paper the analytic spaces are separable. For standard notations in the theory of analytic spaces see for example [1] and [2].

LEMMA 1. *Let X be a Stein space and U an open Stein subspace of X. For each Fréchet $\mathcal{O}(U)$-modules M and N*

$$\hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(M, N) \cong \hat{\mathrm{Tor}}_p^{\mathcal{O}(U)}(M, N), \qquad p \in \mathbf{N},$$

the isomorphisms being topological and $\mathcal{O}(X)$-linear.

Proof. Let φ be the natural morphism of Bar resolutions

$$\varphi\colon \mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{O}(U), \mathcal{O}(U)) \to \mathrm{B}_\bullet^{\mathcal{O}(U)}(\mathcal{O}(U), \mathcal{O}(U)),$$

and $K_\bullet$ its cone. Then by (1), φ is a quasiisomorphism, hence the complex $K_\bullet$ is exact. But the components of $K_\bullet$ are free $\mathcal{O}(U)$-bimodules (i.e. of the form $\mathcal{O}(U) \hat{\otimes} \hat{\otimes} E \hat{\otimes} \mathcal{O}(U)$), so that $K'_\bullet = K_\bullet \hat{\otimes}_{\mathcal{O}(U)-\mathcal{O}(U)} (M \hat{\otimes} N)$ is still exact. But $K'_\bullet$ is the cone of the natural morphism

$$\varphi'\colon \mathrm{B}_\bullet^{\mathcal{O}(X)}(M, N) \to \mathrm{B}_\bullet^{\mathcal{O}(U)}(M, N)$$

therefore φ' is a quasiisomorphism:

$$\hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(M, N) \xrightarrow{\sim} \hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(U)}(M, N).$$

The bicontinuity of these isomorphisms is a simple observation:

(2) Each continuous quasiisomorphism of complexes of Fréchet spaces determines topological isomorphisms between their homology spaces (not necessarily separated), [2, Lemma 7.1.32].

COROLLARY 2. *Let $U \subset X$ be a Stein open subset in a Stein space X and M a Fréchet $\mathcal{O}(X)$-module.*

Then there is at most one Fréchet $\mathcal{O}(U)$-module structure on M, compatible with the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$.

Proof. Suppose M endowed with the structure of a Fréchet $\mathcal{O}(U)$-module, compatible with the $\mathcal{O}(X)$-structure. Then by Lemma 1, $M \cong \mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(X)} M$, topologically.

Let X be an analytic space and let M be a Fréchet $\mathcal{O}(X)$-module. The analogue $\sigma(X, M)$ of the spectrum will be defined as the smallest closed subset of X on the complement of which the complex of sheaves $\mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{O}, M)$ is exact, with $\mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{O}, M)$ denoting the sheaf associated to the presheaf $U \mapsto \mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{O}(U), M)$.

The components of the sheaf $\mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{O}, M)$ are of the form $\mathcal{O} \hat{\otimes} E$, where E is a Fréchet space and $\mathcal{O} \hat{\otimes} E$ is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U) \hat{\otimes} E$.

We assert that $U \mapsto \mathcal{O}(U) \hat{\otimes} E$ is a sheaf on the Stein open subsets of X. Indeed, let U be a Stein open subset of X and $\mathcal{U} = \{U_i\}$ a covering of U with Stein open sets. Then the Čech cochain complex

$$0 \to \mathcal{O}(U) \to \mathcal{C}^0(\mathcal{U}) \to \mathcal{C}^1(\mathcal{U}) \to \dots$$

is exact. But $\mathcal{C}^p(\mathcal{U})$ are Fréchet nuclear spaces, hence the complex

$$0 \to \mathcal{O}(U) \hat{\otimes} E \to \mathcal{C}^0(\mathcal{U}) \hat{\otimes} E \to \mathcal{C}^1(\mathcal{U}) \hat{\otimes} E \to \dots$$

is also exact. From the preceding exact sequence it follows also that $\Gamma(U, \mathcal{O} \hat{\otimes} E) = \mathcal{O}(U) \hat{\otimes} E$ and that $\mathcal{O} \hat{\otimes} E$ is acyclic on Stein open sets.

Let us remark that

$$\sigma(X, M) = \overline{\bigcup_{p \geqslant 0} \operatorname{Supp} \hat{\operatorname{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}, M)},$$

where $\hat{\operatorname{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}, M)$ are the sheaves associated to the presheaves $U \mapsto \hat{\operatorname{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(U), M)$.

PROPOSITION 3. *Let $U \subset X$ be an open Stein subset of a finite dimensional Stein space and let M be a Fréchet $\mathcal{O}(X)$-module. Then*

a) $U \cap \sigma(X, M) = \emptyset$ *if and only if* $\hat{\operatorname{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(U), M) = 0$, $p \geqslant 0$.

b) *If $U \supset \sigma(X, M)$, then there is a Fréchet $\mathcal{O}(U)$-module structure on M, compatible with the $\mathcal{O}(X)$-structure.*

c) *If there is a Fréchet $\mathcal{O}(U)$-module structure on M compatible with the $\mathcal{O}(X)$-structure, then $\bar{U} \supset \sigma(X, M)$.*

The existence condition in b) or c) means, by Lemma 1, $\hat{\operatorname{Tor}}_0^{\mathcal{O}(X)}(\mathcal{O}(U), M) \cong M$ and $\hat{\operatorname{Tor}}_q^{\mathcal{O}(X)}(\mathcal{O}(U), M) = 0$ for $q > 0$ and we shall denote this situation by $\mathcal{O}(U) \geqslant_{\mathcal{O}(X)} M$. Also the Tor-condition from a) will be denoted by $\mathcal{O}(U)\perp_{\mathcal{O}(X)} M$, [8].

Proof. a) Suppose $U \cap \sigma(X, M) = \emptyset$. Then the complex $B_\bullet^{\mathcal{O}(X)}(\mathcal{O}, M)$ is exact on U, it is bounded on the right and its components are acyclic on U, hence, because $\dim(X)$ is finite,

$$0 = H_q B_\bullet^{\mathcal{O}(X)}(\mathcal{O}(U), M) = \hat{\operatorname{Tor}}_q^{\mathcal{O}(X)}(\mathcal{O}(U), M), \qquad q \geqslant 0.$$

Conversely, suppose $\mathcal{O}(U) \perp M$, so that the complex $B_\bullet^{\mathcal{O}(X)}(\mathcal{O}(U), M)$ is exact. By applying to this complex the functor $\mathcal{O}(V) \hat{\otimes}_{\mathcal{O}(U)} \cdot$, where V is an open subset of U, the result is the exact complex $B_\bullet^{\mathcal{O}(X)}(\mathcal{O}(V), M)$, because the components of $B_\bullet^{\mathcal{O}(X)}(\mathcal{O}(U), M)$ are $\mathcal{O}(V) \hat{\otimes}_{\mathcal{O}(U)} \cdot$-acyclic.

b) Let $U \supset \sigma(X, M)$ be a Stein open subset of X. There is a covering $\mathscr{V}$ of $\complement U$ with Stein open subsets V_i, $V_i \cap \sigma(X, M) = \emptyset$, and with the nerve of finite dimension. Then $\mathscr{U} = \{U\} \cup \mathscr{V}$ is a covering of X and the associated Čech alternated cochains complex

$$(3) \qquad 0 \to \mathcal{O}(X) \to \mathcal{O}(U) \oplus \mathscr{C}'^0(\mathscr{U}) \to \mathscr{C}^1(\mathscr{U}) \to \ldots \to \mathscr{C}^N(\mathscr{U}) \to 0$$

is exact. Splitting (3) in short exact sequences, noticing that $\mathscr{C}'^0(\mathscr{U}), \mathscr{C}^1(\mathscr{U}), \ldots, \mathscr{C}^N(\mathscr{U})$ are $\perp$ $\mathcal{O}(U)$ and using (2), it follows that $\mathcal{O}(U) \geqslant M$.

c) Suppose $\mathcal{O}(U) \geqslant M$ and let V be an open Stein subset of X, $V \cap U = \emptyset$. Then $\mathcal{O}(U \cup V) = \mathcal{O}(U) \oplus \mathcal{O}(V)$ and M has a Fréchet $\mathcal{O}(U \cup V)$-module structure which is compatible with the $\mathcal{O}(X)$-structure, via the restriction map $\mathcal{O}(U \cup V) \to \to \mathcal{O}(U)$. By Lemma 1,

$$0 = \hat{\mathrm{Tor}}_p^{\mathcal{O}(U \cup V)}(\mathcal{O}(U \cup V), M) = \hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(U \cup V), M) =$$

$$= \hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(U), M) \oplus \hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(V), M),$$

hence $\hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{O}(V), M) = 0$ for $p > 0$. In the case $p = 0$,

$$M \cong M \oplus (\mathcal{O}(V) \hat{\otimes}_{\mathcal{O}(X)} M),$$

hence $\mathcal{O}(V) \hat{\otimes}_{\mathcal{O}(X)} M = 0$.

As a consequence of a), if $M \neq 0$, then $\sigma(X, M) \neq \emptyset$, because $\mathcal{O}(X)$ is not orthogonal on M.

This proposition contains a homological caracterisation of the spectrum and translates a part of our problem into homological conditions. The next theorem gives a more complete answer. First, a definition:

Let A be a Fréchet algebra and let $B_\bullet$, $C_\bullet$ be two complexes of Fréchet A-modules. Further let $K_\bullet$ be the simple complex associated to the complex of bicomplexes $\mathrm{B}_{\bullet}^A(B_\bullet, C_\bullet)$. We define

$$\hat{\mathrm{Tor}}_p^A(B_\bullet, C_\bullet) = \mathrm{H}_p K_\bullet, \quad p \in \mathbf{Z}.$$

Under nuclearity conditions $\hat{\mathrm{Tor}}_\bullet^A(*, C_\bullet)$ is a homological functor which extends our initial Tôr (the case when the complexes $B_\bullet$, $C_\bullet$ have only a component different from zero).

THEOREM 4. *Let X be a finite dimensional Stein space and let M be a Fréchet $\mathcal{O}(X)$-module.*

For each open set $U \supset \sigma(X, M)$ there is a Fréchet $\mathcal{O}(U)$-module structure on M (not necessarily unique) which extends the Fréchet $\mathcal{O}(X)$-module structure.

Proof. Let $\mathcal{U}$ be a covering of U with Stein open subsets of U and $\mathcal{V}$ a completion of $\mathcal{U}$ with Stein open sets V, $V \cap \sigma(X, M) = \emptyset$, to a covering of X with the nerve of finite dimension.

There is a short exact sequence of Čech complexes

$$(4) \qquad 0 \to \mathcal{D}_\bullet \to \mathcal{C}^\bullet(\mathcal{V}) \to \mathcal{C}^\bullet(\mathcal{U}) \to 0 .$$

The double complex $\mathrm{B}_\bullet^{\mathcal{O}(X)}(\mathcal{D}^\bullet, M)$ which gives $\hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(\mathcal{D}^\bullet, M)$ is bounded in the second argument, so that there is a bounded spectral sequence which converges to $\hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(\mathcal{D}^\bullet, M)$:

$$E^2_{pq} = \mathrm{H}^{-q}(\hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{D}, M)^\bullet), \qquad 0 \leqslant p, \; -N \leqslant q \leqslant 0.$$

But the components of $\mathcal{D}^\bullet$ are of the form $\prod_i \mathcal{O}(W_i)$, where W_i is Stein and disjoint from $\sigma(X, M)$, hence by the Proposition 3.a, $E^2_{pq} = 0$ for each p, q. Then $\mathcal{D}^\bullet \perp M$, i.e. $\hat{\mathrm{Tor}}_p^{\mathcal{O}(X)}(\mathcal{D}^\bullet, M) = 0$, $p \in \mathbf{Z}$.

The complex $\ldots 0 \to \mathcal{O}(X) \to 0$ is a free resolution of $\mathcal{C}^\bullet(\mathcal{V})$, therefore $\hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(\mathcal{C}^\bullet(\mathcal{V}), M) = \hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(\mathcal{O}(X), M)$ and we denote this by $\mathcal{C}^\bullet(\mathcal{V}) \geqslant M$.

By applying the functor $\hat{\mathrm{Tor}}_\bullet^{\mathcal{O}(X)}(*, M)$ to the exact sequence (4), it follows that $\mathcal{C}^\bullet(\mathcal{U}) \geqslant M$. There is a natural continuous morphism

$$(5) \qquad \mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(X)} M \to \hat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(\mathcal{C}^\bullet(\mathcal{U}), M) = M$$

which gives the desired $\mathcal{O}(U)$-structure.

The next simple example shows that this structure is not unique in general. Let $X = \mathbf{C}^2$ and $M = \mathbf{C}$ with the $\mathcal{O}(\mathbf{C}^2)$-structure given by the evaluation map at 0. If $U \ni 0$ is an open subset of $\mathbf{C}^2$ which is not a domain of holomorphy (i.e. Stein), there is in general not only one character on $\mathcal{O}(U)$ (i.e. a topological $\mathcal{O}(U)$-module structure on $\mathbf{C}$) which extends the evaluation in 0 by the restriction map $\mathcal{O}(\mathbf{C}^2) \to \mathcal{O}(U)$.

Let us remark that the map (5) is compatible with the restriction maps, thus the theorem says more, namely that there is a structure of Fréchet $\mathcal{O}(\sigma(X, M)) = \varinjlim_{U \supset \sigma(X, M)} \mathcal{O}(U)$ module on M.

THEOREM 5. *Let $f\colon X \to Y$ be a morphism of Stein spaces and let M be a Fréchet $\mathcal{O}(X)$-module. Then*

$$\overline{f\sigma(X, M)} = \sigma(Y, M)$$

where M has the $\mathcal{O}(Y)$-structure induced by the map

$$f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X).$$

Proof. Let V be an open Stein subset of Y. From (1) there are isomorphisms

$$\hat{\mathrm{Tor}}_{\bullet}^{\mathcal{O}(Y)}(\mathcal{O}(V), \mathcal{O}(X)) \cong \hat{\mathrm{Tor}}_{\bullet}^{\mathcal{O}(X)}(\mathcal{O}(f^{-1}V), \mathcal{O}(X)),$$

therefore the natural morphism of complexes

$$B_{\bullet}^{\mathcal{O}(Y)}(\mathcal{O}(V), \mathcal{O}(X)) \to B_{\bullet}^{\mathcal{O}(X)}(\mathcal{O}(f^{-1}V), \mathcal{O}(X)) \tag{6}$$

is a quasiisomorphism. Let $K_{\bullet}$ be its cone.

The components of $K_{\bullet}$ are $\cdot \hat{\otimes}_{\mathcal{O}(X)} M$-acyclic hence the complex $K_{\bullet} \hat{\otimes}_{\mathcal{O}(X)} M$ is also exact. But this is the cone of the morphism of complexes $\mathrm{B}_{\bullet}^{\mathcal{O}(Y)}(\mathcal{O}(V), M) \to \mathrm{B}_{\bullet}^{\mathcal{O}(X)}(\mathcal{O}(f^{-1}V), M)$ so that

$$\hat{\mathrm{Tor}}_{\bullet}^{\mathcal{O}(Y)}(\mathcal{O}(V), M) \xrightarrow{\sim} \hat{\mathrm{Tor}}_{\bullet}^{\mathcal{O}(X)}(\mathcal{O}(f^{-1}V), M). \tag{7}$$

Now suppose $V \cap \sigma(Y, M) = \emptyset$. Then $\mathcal{O}(V) \perp M$ over $\mathcal{O}(Y)$ and this is equivalent from (7) with $\mathcal{O}(f^{-1}V) \perp M$ over $\mathcal{O}(X)$, which in turn is equivalent by Proposition 3.a with

$$(f^{-1}V) \cap \sigma(X, M) = \emptyset, \text{ i.e. } V \cap f\sigma(X, M) = \emptyset\,.$$

Returning to the initial operatorial case, $X = \mathbf{C}^n$, M a Banach $\mathcal{O}(\mathbf{C}^n)$-module corresponding to a commuting n-tuple of operators $a = (a_1, a_2, \ldots, a_n)$, $\sigma(X, M)$ coincides, as proved in [8], with the J. L. Taylor's joint spectrum $\mathrm{Sp}(a, M)$. Thus Theorem 4 correspond to the analytic functional calculus theorem of J. L. Taylor [6] or [8] and Theorem 5 is a part of J. L. Taylor's spectral mapping theorem [6], namely the case of the mappings the domain of which is a domain of holomorphy (and contains the spectrum).

The following result is well known in the theory of analytic spaces but it can be proved also as a consequence of Theorem 5.

COROLLARY 6. *Each character of the Fréchet algebra $\mathcal{O}(X)$, where X is a finite dimensional Stein space, coincides with the evaluation at a point of X.*

Proof. Let φ be a character of $\mathcal{O}(X)$. This means a topological $\mathcal{O}(X)$-module structure on $\mathbf{C}$. For every morphism of analytic spaces $f: X \to \mathbf{C}$, the set $\overline{f\sigma(X, \mathbf{C})} = \sigma(\mathbf{C}, \mathbf{C})$ contains exactly one point. Indeed a topological structure on $\mathbf{C}$ is completely determined by the action of z on $\mathbf{C}$, hence by a complex number. But the set of morphisms $f: X \to \mathbf{C}$ separates the points of X, therefore $\sigma(X, \mathbf{C})$ reduces to a point $x \in X$. By Proposition 3.b, there is for each Stein open neighbourhood U of x a character φ_U of $\mathcal{O}(U)$ which extends φ with respect to the restriction $\mathcal{O}(X) \to \mathcal{O}(U)$. By passing to the inductive limit with respect to U, there is a character φ_x: $\mathcal{O}_x \to \mathbf{C}$ which extends φ. But φ_x is a morphism of analytic algebras so that φ_x coincides with the evaluation at x.

The same argument based on the separation of points of X by the elements of $\mathcal{O}(X)$ shows that every finite dimensional representation $\rho: \mathcal{O}(X) \to L(\mathbf{C}^m)$ has the support concentrated in a set with at most m elements; moreover ρ factorizes as a sum $\mathcal{O}_{x_1}/\mathfrak{m}_{x_1}^{\nu_1} \oplus \ldots \oplus \mathcal{O}_{x_p}/\mathfrak{m}_{x_p}^{\nu_p}$, $p \leqslant m$, $\nu_1 + \ldots + \nu_p = m$.

As a particular case, the joint spectrum of every commuting system of linear operators on $\mathbf{C}^m$ has at most m points.

For Stein spaces of finite Zariski dimension, the Theorem 4 say essentially the following fact (via the imbedding theorem):

COROLLARY 7. *Let $a = (a_1, \ldots, a_n)$ be a system of commuting operators on a Fréchet space M, which has a continuous $\mathcal{O}(\mathbf{C}^n)$-calculus.*

Suppose that there are entire series $f_i \in \mathcal{O}(\mathbf{C}^n)$ such that $f_i(a) = 0$, $1 \leqslant i \leqslant p$. Let X be the analytic space associated to the ideal I generated by f_i: $X = (V(I), \mathcal{O}/I\mathcal{O})$.

Then $\mathrm{Sp}(a, M) \subset X$ and the system a has a functional calculus with germs of sections of X in neighborhoods of the spectrum.

Proof. Because $\mathcal{O}(X) = \mathcal{O}(\mathbf{C}^n)/I$ and $I \subset \mathrm{Ann}(M)$, there is a structure of Fréchet $\mathcal{O}(X)$-module on M, induced by the $\mathcal{O}(\mathbf{C}^n)$-module structure. Let $j: X \to \mathbf{C}^n$ be the natural closed immersion. Then $j\sigma(X, M) = \mathrm{Sp}(a, M)$, hence $\mathrm{Sp}(a, M) \subset X$, and the rest follows from Theorem 4.

If the spectrum is contained in an analytic subset of $\mathbf{C}^n$, then it is not true in general that there is an analytic subspace X of $\mathbf{C}^n$ supported on this set, such that the preceding conclusion holds. For example, for a quasinilpotent operator which is not nilpotent, there is not a $\mathcal{O}(X) = \mathbf{C}[z]/(z^k)$-functional calculus, for every $k \in \mathbf{N}$.

In what follows we shall adapt our techniques to the case of separable manifolds and $\mathscr{C}^\infty$-functions on them.

PROPOSITION 8. *Let $f: X \to Y$ be a morphism of manifolds and $V \subset Y$ an open subset of Y. Then*

$$\hat{\mathrm{Tor}}_q^{\mathscr{E}(Y)}(\mathscr{E}(X), \mathscr{E}(V)) \cong \mathscr{E}(f^{-1}V) \qquad \text{for } q = 0, \text{ and}$$
$$= 0 \qquad \text{for } q > 0,$$

the isomorphisms being topological.

(We denote by $\mathscr{E}$ the sheaf of germs of $\mathscr{C}^\infty$-functions.)

Proof. First we suppose $Y = \mathbf{R}^n$.

The Koszul complex $K_\cdot(e_1, \ldots, e_n;\ \mathscr{E}(Y) \hat{\otimes} \mathscr{E}(V))$ corresponding to the system $e = (e_1, \ldots, e_n)$ of endomorphisms of $\mathscr{E}(Y) \hat{\otimes} \mathscr{E}(V)$:

$$e_i(f \otimes g) = (y_i f) \otimes g - f \otimes (y_i g)$$

where y_i are the coordinate functions on $\mathbf{R}^n$, is a free resolution of $\mathscr{E}(V)$. Indeed, for the exactness of the Koszul complex it suffices to prove that e is a regular sequence [5, VII.6.Ex.3]. Identifying $\mathscr{E}(Y) \hat{\otimes} \mathscr{E}(V)$ with $\mathscr{E}(Y \times V)$, we have

$$(e_i h)(y, v) = (y_i - v_i)\, h(x, v), \quad h \in \mathscr{E}(Y \times V),\ y \in Y,\ v \in V.$$

Then $\mathscr{E}(Y \times V)/(e_1, \ldots, e_k)\ \mathscr{E}(Y \times V) = \mathscr{E}(Y \times V \cap \bigcap_{i=1}^{n} (y_i = v_i))$ and $e_{k+1} = y_k - v_k$ is one to one on the last $\mathscr{E}(Y)$-module, for $1 \leqslant k < n$. The identification

$\mathscr{E}(Y \times V)/(e_1, \ldots, e_n)\, \mathscr{E}(Y \times V) \cong \mathscr{E}(Y \times V \cap \bigcap_{i=1}^{n} (y_i = v_i)) \cong \mathscr{E}(V)$ is obvious.

So

$$\hat{\mathrm{Tor}}_q^{\mathscr{E}(Y)}(\mathscr{E}(X), \mathscr{E}(V)) = \mathrm{H}_q(\mathscr{E}(X) \hat{\otimes}_{\mathscr{E}(Y)} K_{.}(e, \mathscr{E}(Y) \hat{\otimes} \mathscr{E}(V))) =$$

$$= \mathrm{H}_q K_{.}(f^*e, \mathscr{E}(X) \hat{\otimes} \mathscr{E}(V)),$$

the last groups being homology groups of the Koszul complex corresponding to the endomorphism $(f^*e)_i$ of $\mathscr{E}(X \times V)$:

$$((f^*e)_i h)(x, v) = (f_i(x) - v_i) h(x, v),$$

$h \in \mathscr{E}(X \times V)$, $x \in X$, $v \in V$, $1 \leqslant i \leqslant n$.

The system f^*e is also $\mathscr{E}(X \times V)$-regular and the zero homology space $\mathscr{E}(X \times V \cap \bigcap_{i=1}^{n} (f_i(x) = v_i))$ can be topologically identifyied with $\mathscr{E}(f^{-1} V)$.

In the general case we denote by $\mathscr{B}_{.}$ the sheaf associated to the presheaf

$$W \mapsto \mathrm{B}_{.}^{\mathscr{E}(W)}(\mathscr{E}(f^{-1}W), \ \mathscr{E}(W \cap V)).$$

The functor presheaf $\mapsto$ associated sheaf being exact, it follows from the preceding local computation that $\mathscr{B}_{.}$ is a resolution of the sheaf $W \mapsto \mathscr{E}(f^{-1}(W \cap V))$. The homology sheaves $\mathscr{H}_{.}$ of $\mathscr{B}_{.}$ are soft sheaves, hence

$$\Gamma(W, \mathscr{H}_q) = \mathrm{H}_q \mathrm{B}_{.}^{\mathscr{E}(W)}(\mathscr{E}(f^{-1}W), \mathscr{E}(W \cap V)), \ q \geqslant 0,$$

for each open subset W of Y.

Then $\mathscr{E}(X) \hat{\otimes}_{\mathscr{E}(Y)} \mathscr{E}(V) = \Gamma(Y, \mathscr{H}_0) = \mathscr{E}(f^{-1}V)$ and $\hat{\mathrm{Tor}}_q^{\mathscr{E}(Y)}(\mathscr{E}(X), \mathscr{E}(V)) = \Gamma(Y, \mathscr{H}_q) = 0$, for $q > 0$.

With this result, Proposition 3 and Theorems 4 and 5 are still valid and the proofs are the same, to Stein open subsets of a Stein space corresponding the open subsets of a manifold.

Let us observe that a structure of topological $\mathscr{E}(X)$-module on a Banach space M means a spectral distribution T: $\mathscr{E}(X) \to L(M)$, [9, 3.7.13], and conversely, the set $\sigma(X, M)$ being the support of T. A special case is $X = \mathbf{C}^n$, when Theorems 4 and 5 represents the $\mathscr{C}^\infty$-functional calculus and the spectral $\mathscr{C}^\infty$-mapping theorem for generalised spectral systems of commuting operators. Contrary to the analytic case, such a system does not caracterize a $\mathscr{E}(\mathbf{C}^n)$-structure.

Concluding, we mention that all the results hold in a category of ringed spaces with a sheaf of commutative nuclear Fréchet algebras as structure sheaf, category in which an analogue of Proposition 8 is valid.

One can prove with the techniques developed in this paper the following result, suggested to us by C. Bănică:

Let X a finite dimensional Stein space. Then $\mathcal{O}(X)$ has finite global homological dimension if and only if X is a complex manifolds. Moreover, in this case, the dimension of X coincides with the global homological dimension of $\mathcal{O}(X)$.

The global homological dimension (in our context) of a nuclear Fréchet algebra A is the smallest $n \leqslant \infty$ such that:

$$\hat{\mathrm{Tor}}_p^A(M, N) = 0$$

for each Fréchet A-modules M and N and $p > n$.

I would like to thank C. Bănică whose remarks have led to simplifications of the initial version of the present paper. I also thank F.-H. Vasilescu for his constant support and useful discussions.

REFERENCES

1. *Astérisque*, Rev. de la Soc. Math. de France, Séminaire de Géométrie Analytique, 1974.
2. Bănică, C.; Stănășilă, O., *Méthodes algebriques dans la théorie globale des espaces complexes*, Gauthier-Villars, 1977.
3. Frisch, J., *Introduction à la géométrie analytique*, Univ. de Pisa, 1971.
4. Jacobson, N., *Lectures in abstract algebra, Vol. II — Linear Algebra*, Van Nostrand, 1952.
5. Mac Lane, S., *Homology*, Springer, 1963.
6. Taylor, J. L., Analytic functional calculus for several commuting operators, *Acta Math.*, **125** (1970), 1–38.
7. Taylor, J. L., Homology and cohomology for topological algebras, *Advances in Math.*, **9** (1972), 131–182.
8. Taylor, J. L., A general framework for a multi-operator functional calculus, *Advances in Math.*, **9** (1972), 183–252.
9. Vasilescu, F. -H., *Multi-dimensional analytic functional calculus* (Romanian), Ed. Academiei, București, 1979.

MIHAI PUTINAR
Department of Mathematics
National Institute for Scientific and Technical Creation,
B-dul Păcii 220, 79622 Bucharest,
Romania.

Received April 10, 1980.