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POSITIVE MATRICES AND DIMENSION GROUPS AFFILIATED TO C^*-ALGEBRAS AND TOPOLOGICAL MARKOV CHAINS

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1. INTRODUCTION

In the very recent past, the study of C^*-algebras by means of partially ordered abelian groups has been carried out. For AF-C^*-algebras, "dimension groups" arise; these are direct limits (as ordered groups) of free finitely generated abelian groups equipped with the pointwise ordering. The maps between the free groups are determined by rectangular matrices with nonnegative integer entries.

Given a dimension group, it is of interest to know how to construct it explicitly, with the rectangular matrices given algorithmically (especially since dimension groups can be characterized abstractly). A convincing application of this is the recent embedding result of Pimsner and Voiculescu [13], which asserts that the irrational rotation algebra with angle $2\pi\alpha$, A_α, may be embedded in an AF-algebra C whose dimension group is $G_\alpha = \mathbf{Z} + \alpha\mathbf{Z}$ ordered as a subgroup of the reals. This depends on an explicit algorithm for obtaining G_α as an order limit of free groups of rank two, due to Effros and Shen [5]. This embedding has greatly increased the knowledge of the structure of A_α (see [14] for a discussion).

A major result of this article is to characterize those totally ordered (and somewhat more generally, simple) free abelian groups arising as the limit group with the same map repeated over and over. These admit an elementary algebraic characterization. The ordered groups that occur in an invariant (due to Krieger, [11], [12]) for irreducible subshifts of finite type (in the theory of topological Markov chains) are precisely these limit groups.

Up to order-isomorphism, these limits of stationary systems are classified by a triple, $(A, [\sigma\colon A \to \mathbf{R}], [I])$, where A is an integral order in a number field, $[\sigma]$ represents an equivalence class of real embeddings, and $[I]$ represents an ideal class.

If A is a matrix with strictly positive entries, then one of its eigenvalues is positive, real, and exceeds the absolute value of all other eigenvalues. Further, the corresponding eigenvectors (right and left) have strictly positive entries (up to multiplication by -1), and are the only eigenvectors with non-negative entries. This summarizes a portion of the standard Perron-Frobenius theory, and motivates to some extent our Theorem I, which is a partial (and in a sense the best possible) converse.

However, the immediate motivation lies in the theory of ordered abelian groups of a special kind (see [3], [4], [5], [6], [7], for example). If G is a partially ordered (and always abelian) group that is order-isomorphic to $\mathbf{Z}^n$ (with the coordinate-wise ordering), then G is called *simplicial*. If the partially ordered group G can be written as an order-direct limit of simplicial groups,

$$\mathbf{Z}^{n(1)} \xrightarrow[A_1]{} \mathbf{Z}^{n(2)} \xrightarrow[A_2]{} \mathbf{Z}^{n(3)} \xrightarrow[A_3]{} \cdots \tag{1}$$

where A_i are $n(i+1) \times n(i)$ matrices with integer entries, and $\mathbf{Z}^{n(i)}$ are thought of as groups of columns, then G is a countable *dimension group* cf. [3, Theorem 2.2]. If $n(i) = n$ and $A_i = A$ for all i, then the system in (1) is called a *stationary system*.

An element u of a partially ordered group G is called an *order unit* for G, if it belongs to G^+ and for all g in G, there exists n in $\mathbf{N}$ so that $g \leqslant nu$. A partially ordered group is *simple* (sometimes, o-simple) if every positive element other than zero, is an order unit.

A (normalized) state of a pair (G, u), u an order unit for G, is a group homomorphism,

$$f\colon (G, G^+, u) \to (\mathbf{R}, \mathbf{R}^+, 1).$$

We shall use unnormalized states (referring to them only as states, in contrast to other work in the subject), i.e. the requirement $f(u) = 1$ is relaxed to $f(u) \neq 0$. For simple dimension groups, [3, Theorem 1.4] asserts that

$$G^+ \setminus \{0\} = \{g \in G \mid f(g) \gneq 0 \text{ for all states } f \text{ of } (G, u)\};$$

that is, the states determine the ordering in a sense. When G is a simple dimension group arising from a stationary system, then some power of the matrix A has its entries strictly positive (of course in (1), A may be replaced by A^m), and G has (up to scalar multiple) just one state (this is due to Elliott, see [4, Proposition 2.2]).

Let r be the Perron eigenvalue of A (that is, the unique biggest one in any sense), and v the strictly positive left real eigenvector corresponding to r. Then v is a row, so acts naturally on the columns of $\mathbf{Z}^n$. Then we can define a compatible system of maps to $\mathbf{R}$,

$$(2)\qquad \begin{tikzcd} \mathbf{Z}^n \arrow[r, "A"] \arrow[dr, "v"'] & \mathbf{Z}^n \arrow[r, "A"] \arrow[d, "vr^{-1}"] & \cdots \arrow[dl, "vr^{-2}"] \\ & \mathbf{R} & \end{tikzcd} \qquad\qquad \begin{tikzcd} G \arrow[r, "\hat{A}"] \arrow[d, "v"'] & G \arrow[dl, "vr^{-1}"] \\ \mathbf{R} & \end{tikzcd}$$

Here $\hat{A}$ is the order-*automorphism* induced on G (even if $\det A = 0$!) by A, and $\hat{A}$ multiplies the state to vr.

If G is free, of finite rank, a simple dimension group with unique state, then we determine when it arises from a stationary system with A in $\mathrm{GL}(n, \mathbf{Z})$, and when (G, P), P a fixed automorphism can be obtained as $(G, \hat{A})$ for some A in $\mathrm{GL}(n, \mathbf{Z})$, as in (2).

Stationary systems arise in the study of subshifts of finite type, and classification by means of ordered groups has been initiated by Krieger and Cuntz & Krieger [11], [12], [2]. The pair $(G, \hat{A})$ is the invariant discussed there.

2. THEOREM I

Let A be a square matrix with real entries. We say A satisfies the *weak Perron property* if A has a real eigenvalue r of multiplicity one, such that for all other eigenvalues (in $\mathbf{C}$), a, of A,

$$r > |a|.$$

The large eigenvalue r is called the *weak Perron eigenvalue.*

A matrix (or row or column) is *strictly positive* if all of its entries are real and strictly greater than zero.

2.1. LEMMA. *Let B be a square matrix with real entries. Then some positive integer power of B has strictly positive entries if and only if:*

(i) *B has the weak Perron property;*

(ii) *the left and right eigenvectors corresponding to the weak Perron eigenvalue can be chosen strictly positive.*

Proof. If B or some power is strictly positive, the Perron-Frobenius theory applies to yield (i) and (ii).

Conversely, assume (i) and (ii); let r denote the weak Perron eigenvalue, with corresponding left and right strictly positive eigenvectors v, w respectively. Set $C = r^{-1}B$. Then C has 1 as an eigenvalue of multiplicity one, and all other eigenvalues have smaller modulus. It easily follows (use, for example, the Jordan normal form) that the sequence $\{C^m\}_{m\in\mathbf{N}}$ converges in the usual metric on $\mathbf{R}^{n\times n}$, to a rank one idempotent P. We shall show P has all of its entries strictly positive.

As v is a row, and w is a column, vw is a positive scalar, and wv is an n by n matrix with strictly positive entries, as is $Q = (vw)^{-1}wv$. Clearly Q is a rank one idempotent. Since $wvC = wv$, $wvC^m = wv$, so by continuity, $QP = Q$; similarly, $Q = PQ$. Thus Q and $P(I - Q)$ are orthogonal idempotents, so from

$$P = PQ + P(I - Q) = Q + P(I - Q),$$

we deduce $\operatorname{rank} P(I - Q) = \operatorname{rank} P - \operatorname{rank} Q = 0$, whence $P = PQ$, so $P = Q$. Thus P is strictly positive.

Since $\{C^m\}$ converges to a strictly positive matrix, some power of C must also be strictly positive, hence the same is true of B. ▨

It is convenient to use the notation $w = (w_1, w_2, \ldots, w_n)^{\mathrm{tr}}$ to denote a column; tr stands for *transpose*.

2.2. THEOREM I. *Let A be an n by n matrix with real entries such that:*

(i) *A has the weak Perron property;*

(ii) *If $w = (w_1, w_2, \ldots, w_n)^{\mathrm{tr}}$ is a right eigenvector corresponding to the weak Perron eigenvalue, then at least one of the ratios w_i/w_j, $w_j \neq 0$ is irrational.*

Then there exists P in $\mathrm{SL}(n, \mathbf{Z})$ such that some power of PAP^{-1} is strictly positive.

Proof. Let v, w be the left, right eigenvectors corresponding to the weak Perron eigenvalue r, chosen so that the scalar vw is non-negative. By using elementary row and column operations, we shall obtain P in $\mathrm{SL}(n, \mathbf{Z})$ such that both vP^{-1} and Pw are strictly positive. Then 2.1 applies to PAP^{-1}.

First we show vw is not zero. Suppose $vw = 0$. Define $Q' = wv$, an n by n matrix. Then $Q'^2 = 0$. Since the matrix $C = r^{-1}A$ has one eigenvalue 1, all the others of modulus less than 1, the sequence $\{C^m\}$ converges to a rank one idempotent T. Since $wvC = wv$, $wvC^m = wv$, whence $Q'T = Q'$, and similarly $Q' = TQ'$. As T is rank one, $Q' = TQ'T = sT$ for some real s. But $Q'^2 = 0$, so $s = 0$, and thus $Q' = 0$, which would imply that all the entries of v and of w are zero, a contradiction (cf. the proof of 2.1).

Hence $vw > 0$. We may assume v has a strictly positive entry (multiply both v and w by -1 if necessary), call it v_i.

Let us first note that if S is an elementary matrix whose action on the right ($v \mapsto vS$) adds m times the i-th column (of v, that is the i-th entry of v) to the j-th column, then the action of S^{-1} on the left ($w \mapsto S^{-1}w$) subtracts m times the j-th row of the column w, i.e. the j-th entry) from the i-th; observe that the roles of i, j are reversed, as is the sign. Both the inner product (the scalar, vw) and property (ii) are preserved by the simultaneous map $(v, w) \mapsto (vP, P^{-1}w)$ if P lies in $\mathrm{GL}(n, \mathbf{Z})$.

For ease of notation, after v, w have been transformed at each stage, we shall relabel the transformed vS, $S^{-1}w$ as v, w again.

Since v admits a strictly positive entry v_i, we may add enough positive integer multiples of v_i to all the other entries (that is, performing column operations on v) to make them strictly positive. We thus obtain S in $\mathrm{SL}(n, \mathbf{Z})$ (S is a product of at most $n-1$ elementary matrices) so that vS is strictly positive. Of course $S^{-1}w$ may be terrible. Relabelling, we have two cases, (a) and (b).

(a) *v is strictly positive, and w has a negative entry.*

Subtract enough multiples of the negative entry, w_j, from all the others to ensure that the other entries become strictly positive. The inverse operation applied to v *adds* positive multiples of the entries of v, so v remains strictly positive. An easy consequence of (ii) ensures that there is a w_k such that w_k/w_j is irrational. By subtracting another copy of w_j from any other entries (if they exist, i.e. if $n \geqslant 3$), we may assume that if $n \geqslant 3$, there is a distinct w_m with w_m/w_k also irrational. By normalizing both v and w, and permuting the entries (an elementary operation), we reduce to the following:

$$v = (1, a_2, a_3, \ldots)$$
$$w = (-1, b_2, b_3, \ldots)^{\mathrm{tr}}$$

with a's and b's strictly positive, where b_2 is irrational, and if $n \geqslant 3$, b_3/b_2 is irrational. We have two subcases, $n \geqslant 3$, and $n = 2$.

(a1) $n \geqslant 3$.

We have, vw is a positive scalar, say $vw = d > 0$. Hence $\sum a_i b_i = 1 + d$. As b_2/b_3 is irrational, we may apply the division algorithm to the pair (b_2, b_3) and continue until both remainders are arbitrarily small; e.g. if $b_2 < b_3$, subtract as many integral multiples of b_2 from b_3 so the remainder will be positive, and less than b_2; then apply the same process to reduce the b_2 entry. At each step the process is elementary and a subtraction of rows of w; hence the inverse operations applied to v simply add positive multiples of the second/third entries to the other, and so v remains strictly positive. Since b_2/b_3 is irrational, this process may be continued until the newly transformed b_2, b_3 are both less than d/n. Now, at this point, the ratio b_2/b_3 remains irrational, so that at least one of b_4/b_2, b_4/b_3 is irrational, and thus the division algorithm process can be applied to reduce b_4 to a positive number less than d/n. This may obviously be continued until all the b_i's have been ground down to positive numbers under d/n. Hence we are in the situation:

$$v = (1, \alpha_2, \alpha_3, \ldots) \quad \alpha_i > 0, \ \textstyle\sum \alpha_i \beta_i = 1 + d$$
$$w = (-1, \beta_2, \beta_3, \ldots)^{\mathrm{tr}} \quad 0 < \beta_i < d/n.$$

Now subtract enough multiples of 1, the first entry of v from each of the α_i, so that the remainder is strictly positive, but smaller than or equal one; the inverse operation adds multiples of β_i to the -1 entry of w. This yields:

$$v = (1, \alpha_2'', \alpha_3'', \ldots) \quad 0 < \alpha_i'' \leqslant 1$$
$$w = (c, \beta_2, \beta_3, \ldots)^{\mathrm{tr}}.$$

Now $\sum \beta_i \alpha_i'' < (d/n)(n-1) < d$, whence $c = vw - \sum \beta_i \alpha_i'' > d - d = 0$, so $c > 0$, and we have successfully transformed v, w to strictly positive vectors by products of elementary transformations.

(a2) $n = 2$.

Here we have

$$v = (1, a) \qquad ab > 1;\ a, b > 0$$

$$w = (-1, b)^{\mathrm{tr}} \qquad b \text{ irrational}.$$

Since $1/b$ is also irrational, given $\varepsilon < a - (1/b)$, we may find by the usual diophantine approximation methods, e.g. [10; Ch. 10], positive integers, p, q, r, s so that

$$a > \frac{p}{q} > \frac{1}{b} > \frac{r}{s} \qquad \text{(thus } q/p < b < s/r)$$

and $ps - qr = 1$.

Set $X = \begin{bmatrix} -p & -r \\ q & s \end{bmatrix}$. Then $\det X = -1$, so X belongs to GL(2, **Z**). Consider:

$$vX = (1, a)\begin{bmatrix} -p & -r \\ q & s \end{bmatrix} = (aq - p,\ as - r) = (q(a - (p/q)),\ s(a - (r/s)));$$

so vX is strictly positive, and

$$X^{-1}w = \begin{bmatrix} -s & -r \\ q & p \end{bmatrix}\begin{pmatrix} -1 \\ b \end{pmatrix} = \begin{pmatrix} s - br \\ pq - b \end{pmatrix} = \begin{pmatrix} r((s/r) - b) \\ p(b - (q/p)) \end{pmatrix},$$

again strictly positive.

(b) *v is strictly positive, but w has no negative entries.*

If w had just one nonzero entry, hypothesis (ii) would be violated. Hence there exist w_i, $w_j > 0$, $i \neq j$ as entries of w. Subtract enough multiples of w_i from w_j to force the latter to become negative. The inverse operation on v adds multiples of v_j to v_i, so of course the newly transformed v is also positive, and we have arrived at case (a).

Hence after the conclusion of case (a), we have obtained P in GL(n, **Z**) with vP and $P^{-1}w$ strictly positive. If $\det P = -1$, interchange any pair of positions; this preserves positivity, but changes the determinant of the transforming matrix by a factor of -1. Hence we can find this P in SL(n, **Z**). Now some power of $P^{-1}AP$ is strictly positive, by 2.1. ▨

If D is a subring of **R**, and the matrix A has entries from D, then one condition that will guarantee that (ii) holds (assuming (i) does), is that the weak Perron eigenvalue r not lie in the field of fractions of D. For example, if the characteristic polynomial of A is irreducible over the quotient field of D, then (ii) is automatic. The usual example for D is $\mathbf{Z}$; this will be applied in Sections 3 and 4.

On the other hand, even (especially ?) with $D = \mathbf{Z}$, if (ii) is dropped, the Theorem fails. Set $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$; although A is diagonable (via $GL(2, \mathbf{Z})$), A is not conjugate within $GL(2, \mathbf{Z})$ to a strictly positive matrix, nor to a matrix some power of which is strictly positive.

This type of example occurs when all the entries of both eigenvectors have only rational ratios (of course (ii) in Theorem I can be replaced by an irrational ratio in v, by simply applying the Theorem to A^{tr}, and transposing the matrices at the conclusion). In this case, we may assume all the entries are integers, and that the greatest common divisor of the nonzero entries in each of the vectors is 1.

Now if the matrix A were conjugate via $GL(n, \mathbf{Z})$ to a matrix some power of which is strictly positive, there will exist P in $GL(n, \mathbf{Z})$ such that both vP and $P^{-1}w$ are strictly positive. Since all the entries of v, w are integers, the same is true of vP and $P^{-1}w$; as $vw = vP \cdot P^{-1}w \geqslant n$, a necessary condition is that $vw \geqslant n$. This condition is possibly sufficient.

In any case, in the specific 2 by 2 example above,

$$v = (1,\ 3) \qquad w = (1,\ 0)^{tr}$$

so $vw = 1 < 2$, and thus A is not conjugate via $SL(n, \mathbf{Z})$ to a matrix some power of which is strictly positive.

On the other hand, to transform the pair to strictly positive vectors by means of $GL(n, \mathbf{Q})$ is completely straightforward, and yields Theorem I with (ii) dropped and $SL(n, \mathbf{Z})$ replaced by $SL(n, \mathbf{Q})$.

2.3. THEOREM. *If A is a square matrix of size n with real entries, and A has the weak Perron property, then there exists P in $SL(n, \mathbf{Q})$ such that some power of PAP^{-1} has strictly positive entries.*

Proof. This proof follows that of 2.2 (with $\mathbf{Z}$ replaced by $\mathbf{Q}$), until part (a) is reached; we may obviously assume all the entries of both v and w (the left and right eigenvectors corresponding to the Perron eigenvalue, r) are rational. Then we have

$$v = (a_1, a_2, \ldots, a_n) \qquad a_i, b_j \in \mathbf{N}$$

$$w = (-b_1, b_2, \ldots, b_n)^{tr} \qquad \sum_{i=2}^{n} a_i b_i = a_1 b_1 + d,\ d > 0.$$

Select $\varepsilon < d/(n-1)\max\{b_i | i \geqslant 2\}$. For $j \geqslant 2$, subtract a suitable *rational* multiple of a_1 from a_j, so that the remainder is positive but less than ε. The inverse operation adds rational multiples of b_j to $-b_1$. Then we are in the situation

$$v = (a_1, \alpha_2, \ldots, \alpha_n) \qquad 0 < \alpha_i < \varepsilon,\ 2 \leqslant i \leqslant n$$

$$w = (\beta_1, b_2, \ldots, b_n)^{\mathrm{tr}}.$$

Now

$$d = vw = a_1\beta_1 + \sum_{i=2}^{n} \alpha_i b_i < a_1\beta_1 + \varepsilon\cdot(n-1)\max\{b_i \mid i \geqslant 2\} < a_1\beta_1 + d.$$

So $\beta_1 > 0$, and 2.1 now applies as in the proof of 2.2. ▨

2.4. COROLLARY. *Let $S = \{r = r_1, r_2, \ldots, r_n\}$ be a set of (not necessarily distinct) complex numbers, closed under complex conjugation (counting multiplicity). If $r > |r_i|$ for all $i \geqslant 2$, then there exists a matrix of size n with real entries, A, whose set of eigenvalues (including multiplicities) is S, such that some power of A consists of strictly positive entries.*

Proof. If $\{s_j\}$ is an enumeration of the real members of S other than r, and $\{(t_{2k-1}, t_{2k}) = (a_k + ib_k, a_k - ib_k)\}$ is a list of the non-real ones (in both cases, keeping track of the multiplicities), define the n by n matrix

$$C = [r] \oplus \operatorname{diag}(s_j) \oplus \left(\oplus \begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix} \right).$$

Then C has S as its set of eigenvalues, and by 2.3, there exists P in $\mathrm{SL}(n, \mathbf{Q})$ so that $A = PCP^{-1}$ is ultimately strictly positive. ▨

To see explicitly the matrix A constructed in the proof of 2.4, observe that for C, the eigenvectors for r are

$$v = (1,0, \ldots, 0) = w^{\mathrm{tr}}.$$

Then, to obtain P, one simply follows the operations:

$$v \mapsto (1,1,1, \ldots, 1) \mapsto (1/n, 1, 1, \ldots, 1)$$

$$w \mapsto (1,0, \ldots, 0)^{\mathrm{tr}} \mapsto (1, 1/n, 1/n, \ldots, 1/n)^{\mathrm{tr}}.$$

We can use the method indicated above to construct examples illustrating the surprising behaviour of positive matrices.

Given $n \geqslant 2$ and $m \geqslant 2$, one can easily construct A in $M_n\mathbf{Z}$ so that A^{m-1} contains a negative entry, but no higher power does. It is more of a challenge to find such an A in $\mathrm{SL}(n,\mathbf{Z})$ (here $n \geqslant 3$ is required). We show how to do this with A having irreducible characteristic polynomial, $n = 3$, and $m = 2$ or 4. Additionally, A is conjugate (via $\mathrm{SL}(n, \mathbf{Z})$) to a square (4-th power) of a positive matrix, yet is not itself a square (4-th power) of a positive matrix, even in $M_3\mathbf{Q}$. It follows that A is shift equivalent to this square (lag 2), but I have not been able to establish strong shift equivalence.

Begin with the matrix

$$C = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

This has eigenvalues $r_i = 1 + z^i 2^{1/3} + z^{2i} 2^{2/3}$ $i = 0,1,2$ with z a primitive cube root of unity, has determinant 1, and its left and right Perron eigenvectors (corresponding to r_0) are

$$v = (1, 2^{1/3}, 2^{2/3}) \quad w = (2^{2/3}, 2^{1/3}, 1)^{\mathrm{tr}}.$$

Apply the first round of the division algorithm to the first two entries of v:

$$v \mapsto (1, 2^{1/3} - 1, 2^{2/3}) \mapsto (4 - 3 \cdot 2^{1/3}, 2^{1/3} - 1, 2^{2/3}).$$

This is implemented by the matrix $P_1 = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \oplus [1]$, so that if we set $E_1 = P_1^{-1} C P_1$, E_1 is ultimately strictly positive. In fact E_1 has two negative entries, but E_1^2, E_1^3 have none, hence the same is true for all higher powers; so with $A = E_1$, $m = 2$ in this case.

Set $B_1 = E_1^2$. It follows from the irreducibility of the characteristic polynomial of B that $\pm E_1$ are the only square roots of B, so B has no nonnegative square roots (even in $M_3\mathbf{Q}$). Being conjugate to C^2, B is shift equivalent to it (of lag 2).

If instead, we allow the division algorithm to continue another 1 1/2 rounds, we obtain $P_2 = \begin{pmatrix} 24 & -5 \\ -19 & 4 \end{pmatrix} \oplus [1]$. Setting $E_2 = P_2^{-1} C P_2$, E_2 is ultimately positive. A computer programme (arranged for me by Ronald Jubainville) yielded that E_2^n has negative entries for $n = 1,2,3$, and has only positive entries for $n = 4,5,6,7$ hence for all higher powers.

3. TOTALLY ORDERED GROUPS

Let G be a simple dimension group with unique state, and suppose G is free (as an abelian group of course), but not $\mathbf{Z}$. We shall determine when G arises from a stationary system, and if it does, when a power of a given automorphism is induced from the canonical automorphism of a stationary system.

We know G possesses a map, $f: G \to \mathbf{R}$, its state, such that $G^+ \setminus \{0\} = f^{-1}(\mathbf{R}^+ \setminus \{0\})$. Assume G is of finite rank (since otherwise it could not possibly arise from a stationary system), it is finitely generated (being free) so splits as

$$G \simeq f(G) \oplus \operatorname{Ker} f.$$

Now the relative ordering on $f(G)$ (as a subgroup of G) is equivalent to the total ordering as a subgoup of $\mathbf{R}$, so $f(G)$ is totally ordered. In particular, G is totally ordered if and only if f is one to one.

We shall investigate the totally ordered situation in this section, and the case with non-trivial kernel in the next. They could be done simultaneously, but the techniques are more transparent if the two types are done separately.

Now assume G is totally ordered, so $G \subset \mathbf{R}$. Let $E = \mathrm{End}_c G$ denote the ring of continuous endomorphisms of G (with respect to the relative topology); that is, $\mathrm{End}_c G$ consists of endomorphisms $\psi\colon G \to G$ that extend to real linear maps $\Psi\colon \mathbf{R} \to \mathbf{R}$,

$$\begin{array}{ccc} G & \subset & \mathbf{R} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ G & \subset & \mathbf{R} \end{array} .$$

The map $\psi \to \Psi(1)$ is easily seen to be an embedding of rings, $E \subset \mathbf{R}$, since Ψ is determined by its value at 1; of course the image of G is dense in $\mathbf{R}$.

We adopt the notation $M_n\mathbf{Z}$ to denote the ring of n by n matrices with integer coefficients, and $M_n\mathbf{Z}^{++}$ will indicate those with strictly positive entries.

The ring E acts in a natural and obvious way on G, so that G becomes an E-module. Let $K \subset \mathbf{R}$ be the field of fractions of E. Since $E \subset \mathrm{End}_{\mathbf{Z}} G \simeq M_n\mathbf{Z}$, and E is a commutative domain (from being embedded in $\mathbf{R}$), we have that $(E, +)$ is of finite rank, and thus $E \otimes \mathbf{Q} = K$. Hence $K \subset M_n\mathbf{Z} \otimes \mathbf{Q} = M_n\mathbf{Q}$, and it easily follows that K is a finite dimensional extension field of $\mathbf{Q}$, and that $[K\colon \mathbf{Q}]$ divides $n = \mathrm{rank} G$.

Next we show that if G had arisen from a stationary system,

$$G = 0\text{-}\varinjlim \mathbf{Z}^m \xrightarrow{A} \mathbf{Z}^m \xrightarrow{A} \mathbf{Z}^m \xrightarrow{A} \dots \qquad A \text{ in } M_n\mathbf{Z}^{++}$$

(with m possibly larger than n, but for all sufficiently high powers of t, $\mathrm{rank} A^t = n$), then $[K\colon \mathbf{Q}] = n$. (Of course the main result of this section is the converse, and the A can be chosen from $\mathrm{SL}(n, \mathbf{Z})$). Since $\mathrm{Ker} A^t$ is always a direct summand of $\mathbf{Z}^m$, by replacing A by a sufficiently high power, we may assume $\mathrm{Ker} A = \mathrm{Ker} A^2 = \dots$; observe that the limit ordered group G is unaffected. Set $T = \mathrm{Ker} A$; then $AT \subset T$, and so A induces an additive mapping on the quotient group $Y = \mathbf{Z}^m/T$, $\bar{A}\colon Y \to Y$. Of course Y is free and torsion-free of rank n (since $\bar{A}$ is one to one), and G as an abelian group (only) is isomorphic to the quotient limit

$$Y \xrightarrow{\bar{A}} Y \xrightarrow{\bar{A}} Y \xrightarrow{\bar{A}} \dots .$$

(It would make sense to impose a quotient ordering of sorts on Y, but this is not generally simplicial, and so we do not do so.)

Let v be a positive left Perron eigenvector for A. Then we obtain a (and therefore, up to scalar multiple, the only) state on G via (2) of Section 1, with n replaced by m. Since G is rank n and the state f on G is an embedding, $v(\mathbf{Z}^m)$ has rank n, and also $vT = (0)$. Hence v induces a one to one map $\bar{v}\colon Y \to \mathbf{R}$, and $\bar{v}\bar{A} = \bar{v}r$. If we choose a $\mathbf{Z}$-basis for Y, since

$$\operatorname{rank} Y = \operatorname{rank} G = \operatorname{rank} f(G) = \operatorname{rank} \bar{v}(Y),$$

and regard $\bar{v}$ as a row with real entries (depending upon the basis for Y), we see that $\bar{v}$ is one to one, so that its entries are linearly independent over $\mathbf{Q}$. On the other hand $\bar{v}$ is a left eigenvector for $\bar{A}$, and the latter has integer entries. It easily follows that the characteristic polynomial of $\bar{A}$ (as an endomorphism of $Y \simeq \mathbf{Z}^n$) is irreducible.

Now the endomorphism induced on G by A, equivalently by $\bar{A}$, is just multiplication by a real number r, so is continuous. Hence any polynomial in $\bar{A}$ induces a continuous endomorphism, and since $\{I, \bar{A}, \bar{A}^2, \ldots, \bar{A}^{n-1}\}$ are $\mathbf{Z}$-independent, we see that rank $(\mathrm{End}_c G, +) \geqslant n$, whence equality holds. As $K = E \otimes \mathbf{Q}$, $[K\colon \mathbf{Q}] = n$.

Before establishing Theorem II (3.3), we require a classical result about number fields and their units. If K is a finite dimensional extension field of $\mathbf{Q}$, let $\mathbf{Z}_K$ denote the ring of algebraic integers in K; $\mathbf{Z}_K^*$ will denote the collection of elements of $\mathbf{Z}_K$ which are invertible in $\mathbf{Z}_K$. The members of $\mathbf{Z}_K^*$ are known as the units of the number field K. As usual, r_1 is the number of real embeddings of K, and r_2 is one-half the number of non-real complex embeddings; also, $r_1 + 2r_2 = n = [K\colon \mathbf{Q}]$.

3.1. LEMMA (essentially [16; Corollary 5—3—8]). *Let K be a number field, and $\sigma_0\colon K \to \mathbf{R}$, a fixed embedding to the reals. Then $\mathbf{Z}_K^{*+} = \{u \in \mathbf{Z}_K^* \mid \sigma_0(u) > 0\}$ is a torsion-free abelian group, free of rank $r_1 + r_2 - 1$, and a free $\mathbf{Z}$-basis may be constructed of the form $\{u_i\}$, where*

$$\sigma_0(u_i) > 1 > |\sigma(u_i)| \qquad \textit{for all } i,$$

for all homomorphisms $\sigma\colon K \to \mathbf{C}$, $\sigma \neq \sigma_0$.

Proof. By [16; Corollary 5—3—8], there exists a unit u_1 such that $|\sigma_0(u_1)| > 1 > |\sigma(u_1)|$ for all other $\sigma\colon K \to \mathbf{C}$. By replacing u_1 by $-u_1$ if necessary, we may assume $\sigma_0(u_1) > 1$. We may also assume u_1 is not a power of anything (if $u_0^s = u_1$, replace u_1 by $\pm u_0$; as is well-known, $\mathbf{Z}_K^*$ is finitely generated, so this process terminates). Hence $\{u_1\}$ may be enlarged to a $\mathbf{Z}$-basis for the free abelian group $\mathbf{Z}_K^{*+}$, $\{u_1, u_2', u_3', \ldots, u_r'\}$, $r = r_1 + r_2 - 1$. Then for each $i > 1$, there exists a sufficiently high power of u_1 so that $u_i = u_1^m u_i'$ satisfies the desired inequalities; of course $\{u_i\}$ is also a $\mathbf{Z}$-basis for $\mathbf{Z}_K^{*+}$. ▨

I am indebted to Cam Stewart for pointing out the existence of [16; Corollary 5—3—8] to me.

Assuming that $[K: \mathbf{Q}] = \text{rank}G = n$ (in our notation), we have $E = \text{End}_c G \subset \mathbf{Z}_K$ (since $(E,+)$ is a subgroup of the finitely generated free abelian group $(M_n\mathbf{Z},+)$, and is thus finitely generated), and $\text{rank}(E,+) = \text{rank}(\mathbf{Z}_K,+) = n$. Hence there exists an integer m so that $m\mathbf{Z}_K \subset E$, whence $\mathbf{Z} + m\mathbf{Z}_K \subset E$. Now $\mathbf{Z}_K/m\mathbf{Z}_K$ is a finite ring, so its group of units is finite. Hence given a unit u in $\mathbf{Z}_K$, there exists an integer t so that $u^t - 1$ belongs to $m\mathbf{Z}_K$, whence $u^t \in \mathbf{Z} + m\mathbf{Z}_K \subset E$. Hence given a unit u in $\mathbf{Z}_K$, some positive power lies in E; in particular E^* has torsion-free rank equalling that of $\mathbf{Z}_K^*$.

Let

$$F = \{u \in E^* \mid \sigma_0(u) > |\sigma(u)| \quad \text{all } \sigma\colon K \to \mathbf{C},\ \sigma \neq \sigma_0\}.$$

By the preceding comment and Lemma 3.1, F generates a torsion-free abelian group of rank $r_1 + r_2 - 1$ inside E^*.

If $\mathbf{Q} \subset L \subset K$, and the torsion-free rank of $\mathbf{Z}_L^*$ equals that of $\mathbf{Z}_K^*$ (obviously it cannot exceed it), then elementary manipulations yield that either $L = K$ or K has no real embeddings; the latter does not arise here, so $K = L$. We shall employ this comment later.

Call an element y of K, *introspective* if $\mathbf{Q}(y) = \mathbf{Q}(y^m)$ for all m in $\mathbf{N}$. Since $[K: \mathbf{Q}] = n$, given z in K, z^{2^n} is introspective.

Let $M = \mathbf{Q}(u)$ be a subfield of K such that:

(i) u is introspective;

(ii) u belongs to F;

(iii) $[M: \mathbf{Q}]$ is maximal with respect to the above two properties.

We wish to show that $M = K$. Select x in F. Let S denote the *finite* set of fields L with $\mathbf{Q} \subset L \subset K$. For each prime number $p > 2$, define a function

$$f_p\colon \{1,2,\ldots,n\} \to S$$
$$f_p(i) = \mathbf{Q}((u^p x)^{2^i}) \in S.$$

Now $\{f_p\}$ is an infinite collection of functions between finite sets. Hence $f_p = f_q$ for some pair of distinct primes, $p < q$. Hence

$$\mathbf{Q}((u^p x)^{2^i}) = \mathbf{Q}((u^q x)^{2^i}) \quad 1 \leqslant i \leqslant n.$$

Then $(u^p x)^{2^n}$, $(u^q x)^{2^n}$ are both automatically introspective and since the fields they generate are equal, $u^{(q-p)2^n}$ belongs to $\mathbf{Q}((u^p x)^{2^n})$. As u is introspective, u thus also belongs, and hence x^{2^n} does too. In particular, M is contained in $\mathbf{Q}((u^p x)^{2^n})$, so by introspectivity of $(u^p x)^{2^n}$, and the maximality of M, $M = \mathbf{Q}((u^p x)^{2^n})$, whence x^{2n} belongs to M. Thus the torsion-free rank of $\mathbf{Z}_M^*$ is at least that of $\mathbf{Z}_K^*$, so $K = M$.

We have shown:

3.2. LEMMA. *Let K be a number field equipped with a fixed embedding, σ_0: $K \to \mathbf{R}$. There exists a unit u in $\mathbf{Z}_K$ such that $\sigma_0(u) > |\sigma(u)|$ for all embeddings σ: $K \to \mathbf{C}$ unequal to σ_0, and $K = \mathbf{Q}(u^i)$ for all $i = 1,2,\ldots$. If $\mathbf{Z}_K$ is replaced by a subring E which is also an order in K, u may be chosen to belong to E.*

(The final comment is a consequence of our earlier deduction that $\mathbf{Z}_K^*/E^*$ is finite).

The case $n = 2$ of the following result was established by Effros and Shen [5].

3.3. THEOREM II. *Let G be a totally ordered subgroup of $\mathbf{R}$, free of finite rank, n. The following are equivalent:*

(i) *G arises as the limit of a stationary system of the form*

$$\mathbf{Z}^m \xrightarrow{A} \mathbf{Z}^m \xrightarrow{A} \mathbf{Z}^m \xrightarrow{A} \cdots$$

for some A in $M_m\mathbf{Z}^{++}$ with $m \geqslant n$.

(ii) *G arises as the limit of a stationary system of the form*

$$\mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^n \xrightarrow{A} \cdots$$

for some A in $\mathrm{SL}(n, \mathbf{Z})^{++}$.

(iii) *There exists r in $\mathbf{R}$ such that $rG \otimes \mathbf{Q}$ (inside $\mathbf{R}$) is a field.*

(iv) ($\mathrm{End}_c G$, $+$) *is of additive rank n.*

(v) *If K is the field of fractions of* $\mathrm{End}_c G$, *then* $[K\colon \mathbf{Q}] = n$.

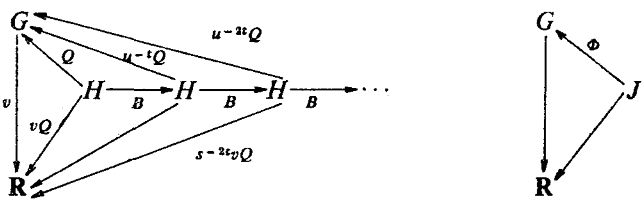
(vi) *For any nonzero element g of G,* $(\mathrm{End}_c G)(g)$ *is of rank n.*

Proof. A priori, (ii) implies (i); that (i) implies (v) follows from the discussion prior to 3.1, and (v) if and only if (iv) is trivial. The implications, (iv) $\Rightarrow$ (vi) $\Rightarrow$ (iii) are routine. Assume (iii) holds; we establish (iv). Since multiplication by r is an order-isomorphism, we may assume $r = 1$, so $G \subset L \subset \mathbf{R}$, where L is a field of dimension n over $\mathbf{Q}$. Select s in L. Then $sG \subset sL = L$, so, as G is of maximal rank in the finite rank additive group $(L,+)$, and sG is free, there exists a positive integer m such that $msG \subset G$. Hence ms belongs to $\mathrm{End}_c G$, so that $(\mathrm{End}_c G,+)$ is of the same rank as L, namely n (also $L = K$). Hence (iii) through (vi) are equivalent.

Now assume (v). By Lemma 3.2, we may find u in $E = \mathrm{End}_c G$, a unit such that $\sigma_0(u)$ (the real multiplier) exceeds all the other values for $|\sigma(u)|$. In particular, u is order-preserving (it corresponds to multiplication by the real number $\sigma_0(u)$, where we have chosen $\sigma_0 \colon K \to \mathbf{R}$ to be the fixed embedding obtained from $\psi \to \Psi(1)$).

Select a $\mathbf{Z}$-basis for G, and write a (the) state for G as a row, determined by this basis. Hence we have a row v of real numbers such that $G^+ = \{g \in G \mid vg > 0\} \cup \cup \{0\}$; here "$g$" is regarded as a column in terms of the fixed basis. Now $u \colon G \to G$ is order-preserving, so vu is also a state, and thus $vu = sv$ for some positive real number s; in fact $s = \sigma_0(u)$, the multiplier. If we call U the matrix corresponding to u, in terms of the fixed basis, then v is a left eigenvector for U, with respect to s, and it easily follows that s is a weak Perron eigenvalue for U (the eigenvalues are simply the values $\sigma(u)$, as $\sigma \colon K \to \mathbf{C}$ ranges over all such embeddings). Refer to G as a group with the fixed basis (and a corresponding simplicial ordering) as H.

Since U satisfies the hypothesis of Theorem I, we may find P in $\mathrm{Aut}(H) = = \mathrm{GL}(n, \mathbf{Z})$ so that some power of $A = PUP^{-1}$ has strictly positive entries (in the fixed basis). Then vP^{-1} is the left eigenvector for the Perron eigenvalue of A, hence up to multiplication by -1, is strictly positive. Suppose $B = A^t$, a power of A, is strictly positive, and $c = s^t$ is the corresponding eigenvalue. Set $Q = P^{-1}$; then $Q = U^{-t}QB$. Consider the diagram:



Here H is the underlying group structure of G, and we wish to show the compatible sequence of maps $U^{-mt}Q \colon H \to G$ induces an order-isomorphism of the limit of the stationary system, $J = \lim H \to H \to H \to \ldots$; here H is simplicially ordered by the fixed basis, with respect to which B is strictly positive.

First, note that the diagram commutes. Second, J is a limit of a stationary system, and since B is strictly positive, the limit is a simple dimension group, with unique state, which is determined as in (2) (Section 1), by the left eigenvector vQ. Next Q, U^{-mt}, are group isomorphisms, so the limit map from J to G, Φ, is a group isomorphism.

Select x in $J^+ \setminus \{0\}$. Then x may be replaced by $\underline{x}$ in H at some level. Now $s^{-kt}\, vQ\underline{x} > 0$ (as the state on J determines the ordering); but the image in G is positive at the state of G, v: $vU^{-kt}\, Q\underline{x} = s^{-kt}\, vQ\underline{x} > 0$; since G^+ is also determined by the state v, $\Phi(J^+) \subset G^+$, so Φ is order-preserving.

Conversely, if y lies in $G^+ \setminus \{0\}$, then $vy > 0$. There exists x in J with $\Phi x = y$. Hence we may find $\underline{x}$ in some level, H, with $\underline{x}$ representing x, so that $U^{-kt}\, Q\underline{x} = y$. As $vy > 0$, $vu^{-kt}Q\underline{x} > 0$, whence $s^{-kt}\, Q\underline{x} > 0$; as the state determines the ordering on J (see Section 1), $x > 0$. Hence $\Phi(J^+) = G^+$, so Φ is an order-isomorphism, and thus G is order-isomorphic to the limit of a stationary system. ▨

3.4. COROLLARY. *Let G be a totally ordered subgroup of* $\mathbf{R}$, *and suppose that G is free of prime rank. Then G is the limit of a stationary system as in* 3.3(ii), *if and only if* $\mathrm{End}_c G$ *is not trivial*, i.e., *G has a continuous endomorphism that is not multiplication by any integer.*

Proof. Since $\mathrm{rank}(\mathrm{End}_c G, +)$ divides $\mathrm{rank} G$, and the latter is prime, either the rank is 1 (in which case $\mathrm{End}_c G = \mathbf{Z}$, since $\mathrm{End}_c G$ is additively finitely generated) or the rank is that of G. By Theorem II, the result follows. ▨

Given a stationary system, with order limit J, the strictly positive matrix A induces an order-automorphism $\hat{A}$ of J, that resembles a shift:

$$\begin{array}{ccc} \begin{array}{ccccccc} \mathbf{Z}^n & \xrightarrow{A} & \mathbf{Z}^n & \xrightarrow{A} & \mathbf{Z}^n & \xrightarrow{A} & \cdots \\ {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & \\ \mathbf{Z}^n & \xrightarrow[A]{} & \mathbf{Z}^n & \xrightarrow[A]{} & \mathbf{Z}^n & \xrightarrow[A]{} & \cdots \end{array} & \qquad & \begin{array}{c} J \\ {\scriptstyle \hat{A}}\downarrow \\ J \end{array} \end{array} .$$

(Think of the direct limit as a quotient of a subgroup of the direct product; A acts as a backward shift. This resembles the original motivation for stationary systems, see for example [11], [12], [2]).

The method of proof of Theorem II actually yields more than is stated. Given a stationary system, with the generating matrix A strictly positive and in $\mathrm{GL}(n, \mathbf{Z})$, together with an order-automorphism ψ, the proof gives a necessary and sufficient criterion under which some power of ψ is of the form $\hat{B}$ for some B in $\mathrm{GL}(n, \mathbf{Z})^{++}$, generating a stationary system whose order limit is G, namely:

$$\begin{cases} \Psi(1) > |\text{all other eigenvalues of } \psi| \\ \text{the minimal polynomial of } \psi \text{ (as an element of } \mathrm{End}_{\mathbf{Z}} G \simeq M_n\mathbf{Z}\text{), is of degree } n = \mathrm{rank} G. \end{cases}$$

The latter condition is vacuous (in the presence of the first), if n is a prime number. The real number $\ln\Psi(1)$ is the entropy of the subshift associated to B.

Of course if a totally ordered group does arise from a stationary system generated by an n by n matrix A with $n = \mathrm{rank} G$, then A must have its characteristic polynomial irreducible, and this characterizes the totally ordered situation.

4. NON-TOTALLY ORDERED SIMPLE DIMENSION GROUPS WITH UNIQUE STATE

Let G be a simple dimension group with unique state, and say free of finite rank. Following Section 3, G can be written as a direct sum, $G = f(G) \oplus \mathrm{Ker} f$

$$G^+ = \{0\} \cup \{(a, k) \mid a > 0\},$$

where $f\colon G \to \mathbf{R}$ is the state. The order-automorphisms of G leave the state invariant (up to positive scalar multiple), so leave the kernel, Ker$f$, invariant; further, they induce order-automorphisms of the totally ordered subgroup, $f(G)$. Conversely, given any order-automorphism of $f(G)$, any group automorphism of Ker$f$, and any additive map whatever from G to Ker$f$, these can be put together in a unique way to yield an order-automorphism of G. This is a special case of the remark in the concluding paragraphs of Section 6 of [9]; a very special case is proved computationally in [15; Proposition 1.2].

If such a G arises as the limit of a stationary system, say generated by A in $M_m\mathbf{Z}^{++}$, with $m \geqslant n = \operatorname{rank} G$, then A induces an order-automorphism $\hat{A}$ of G, which in turn yields automorphisms of both $f(G)$ (as an ordered group) and Ker$f$ (as a group). If we write $\widetilde{A}$ for the induced order-automorphism on $f(G)$, then identical considerations as in Section 3, yield that $\widetilde{A}$ has irreducible characteristic polynomial (as a group endomorphism of $f(G)$). The matrix A has 0 as an eigenvalue of multiplicity $m - n$, and its Perron eigenvalue is just the real number that $\widetilde{A}$ multiplies $f(G)$ by.

Hence we deduce necessary conditions for an automorphism to be induced by the canonical one of a stationary system: and the converse holds with the appropriate modifications of 3.3.

The case that $n = 3$ and Ker$f$ be nonzero, of the following, was established by Shen [15; 2.1].

4.1. THEOREM. *Let G be a simple dimension group with unique state f, and suppose G is free of finite rank, n. Then G arises as the order limit of the stationary system*

$$\mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^n \xrightarrow{A} \ldots$$

for some A in SL $(n, \mathbf{Z})^{++}$ *if and only if $f(G)$, the totally ordered subgroup of* $\mathbf{R}$, *arises as the limit of a stationary system* (viz. 3.3).

Proof. The preceding paragraph indicated that if G is the limit of a stationary system induced by some B (a possibly larger size matrix), then the induced order-automorphism $\widetilde{B}$ of $f(G)$ has full rank, that is $\{I, \widetilde{B}, \widetilde{B}^2, \ldots, \widetilde{B}^{t-1}\}$ $(t = \operatorname{rank} f(G))$ is linearly independent over $\mathbf{Z}$, so $f(G)$ satisfies 3.3(iii).

Conversely, assume that $f(G)$ is the limit of a stationary system as in 3.3(ii), say with t by t matrix B $(t > 1)$. Then B necessarily has irreducible characteristic polynomial (as an endomorphism of $\mathbf{Z}^t$), and its Perron eigenvalue must exceed 1 (since $\det B = \pm 1$). Form an automorphism of G simply by defining $u = \hat{B} \oplus i$, where $i\colon \operatorname{Ker} f \to \operatorname{Ker} f$ is the identity. It is easily checked that u is an order-automorphism. Now the same process as in the proof of 3.3 (ii) can be carried through, since u as an element of $\mathrm{End}_{\mathbf{Z}}G \simeq M_n\mathbf{Z}$ satisfies the hypothesis of Theorem I. ▨

What about the non-free situation? Several complications arise owing to the group structure of limits of stationary systems, and also the failure of Theorem I, when the irrationality hypothesis is dropped. Provided the range of f is at least rank 2, the problem can always be resolved, but people might not find the solution to their taste.

(1) The rank of the image of the state is one.

If such a dimension group arises as a limit of a stationary system, then after normalization $f(G)$ must be $\mathbf{Z}[1/m]$ (the ring obtained by adjoining $1/m$ to $\mathbf{Z}$), where $m \in \mathbf{N}$ is the Perron eigenvalue of the generating matrix. Of course, if G has rank 1, then $G = \mathrm{Lim}\ \mathbf{Z} \xrightarrow{\times m} \mathbf{Z} \xrightarrow{\times m} \ldots$

More usually, the rank of G exceeds one. In that case, the kernel of the state is of rank $n - 1$, and need not be a direct summand. The group structure of G can then be very complicated and interferes with attempts to classify the outcome. In addition, the right and left eigenvectors of the matrix will have all rational ratios, so it will be difficult to tell when a specific group arises from a stationary system (see the discussion at the conclusion of Section 2).

(2) The rank of the image of the state exceeds one.

Here the totally ordered situation becomes interesting. If $G \subset \mathbf{R}$ does arise from a stationary system with $\mathrm{rank}G = n$, then we have A in $M_m\mathbf{Z}^{++}$ ($m \geqslant n$) generating it. Proceeding as before we obtain induced group homomorphisms $\bar{A}: \mathbf{Z}^n \to \mathbf{Z}^n$, whose limit is G. Then $\bar{A}$ satisfies the weak Perron property, and since its characteristic polynomial is irreducible (over $\mathbf{Q}$), Theorem I asserts that $\bar{A}$ is conjugate to a strictly positive matrix; it easily follows that there exists B in $M_n\mathbf{Z}^{++}$ such that G is order-isomorphic to the limit of the stationary system generated by B (as in the proof of 3.3). Hence we may have assumed $m = n$.

We also note that if $\hat{B}$ is the automorphism of G induced by B, then $\mathbf{Z}[\hat{B}, \hat{B}^{-1}] \subset \mathrm{End}_c G$, and the latter is a finitely generated module over the former. It follows that

(i) K, the field of fractions of $\mathrm{End}_c G$, is of dimension n over $\mathbf{Q}$;

(ii) There exists an integer m such that $m\mathbf{Z}_K[\hat{B}^{-1}] \subset \mathrm{End}_c G$;

(iii) It follows from (ii) that the units of $\mathrm{End}_c G$ have torsion-free rank at least $r_1 + r_2$ (strictly more than the rank of $\mathbf{Z}_K^*$), but are finitely generated.

So a necessary condition for a totally ordered (not necessarily free) subgroup of $\mathbf{R}$ to arise from a stationary system is

(iv) $(\mathrm{End}_c G, +)$ is of rank equalling that of G and

(v) The ring $\mathrm{End}_c G$ is finitely generated over a ring of the form $\mathbf{Z}[r, r^{-1}]$, for some r in $\mathbf{Z}_K$; alternately, there exists an element s in $\mathbf{Z}_K$ (K is the field of fractions of $\mathrm{End}_c G$) with $\mathbf{Z}_K[s^{-1}]$ a module of finite type over $\mathrm{End}_c G$; and ${}_E G$ is finitely generated.

In fact (iv) and (v) together are sufficient as well. The group structure underlying G now plays a role, and interferes with the attempt to apply directly the techniques suggested in the course of the proof of 3.3. This will be discussed in a subsequent paper, where a more general discussion of indecomposable torsion-free abelian groups and their endomorphism rings will take place.

5. CLASSIFICATION OF TOTALLY ORDERED GROUPS

In this section, we show the classification of free totally ordered subgroups which arise from stationary systems (equivalently, with "enough" continuous endomorphisms, 3.3(iii), (v), or (vi)) is roughly the same as that of certain ideal classes in integral orders in number fields, with a fixed real embedding.

This correspondence is similar to that obtained in Section 4 of [8], and here the techniques are easier to implement. So the details will only be sketched, and emphasis will be on the comparison with the results of [8; § 4].

Let G be a dense subgroup of $\mathbf{R}^m$; impose the relative topology on G, and define the ring $\mathrm{End}_c G = E$, as the ring of continuous endomorphisms of G; that is, $\mathrm{End}_c G$ consists of those endomorphisms ψ of G that extend to real linear maps,

$$\begin{array}{ccc} G & \subset & \mathbf{R}^m \\ \psi\downarrow & & \downarrow\Psi \\ G & \subset & \mathbf{R}^m \end{array} .$$

In case G is free of rank $m+1$, and $(\mathrm{End}_c G,+)$ is of rank $m+1$ (equivalently, $(\mathrm{End}_c G)(g)$ is of rank $m+1$ for any nonzero g in G), then G is classified by a triple [8; IV.6]:

$$(E = \mathrm{End}_c G,\ [D], [{}_E G]).$$

Here ${}_E G$ indicates G is being considered as an E-module; it turns out that as E-modules, ${}_E G$ is isomorphic to an ideal I of E such that $\mathrm{End}_E I \simeq E$; $[{}_E G]$ denotes the ideal class of this I. Also, $D\colon E \to \mathbf{R}$ is a distinguished real embedding; $[D] = [D']$ for another real embedding D', if there is an automorphism σ of E, such that $D\sigma = D'$. Conversely, every such triple is attainable [8; IV.5].

Now suppose $m = 1$, and G is a subgroup of $\mathbf{R}$, free of rank n. Then any continuous endomorphism u of G, has the property that u^2 and one of $\{u, -u\}$ are order-preserving. Hence, $(\mathrm{End}_c G)^*$ provides virtually complete information about $\mathrm{Aut}_0 G$, the group of order-automorphisms of G.

Assume $(\mathrm{End}_c G,+)$ has full rank n (so in particular, G is the limit of a stationary system). With $E = \mathrm{End}_c G$, G becomes an E-module of E-rank 1 (this means G is contained in a one-dimensional vector space over the field of fractions of E, K), and ${}_EG$ is obviously E-torsion-free, so is isomorphic to an ideal of E. Further, the module ${}_EG$ has the same reflexivity property as in the previous situation: the natural map

$$E \to \mathrm{End}_E G$$

is an isomorphism of rings. This may be proved exactly as in [8; IV.3].

The embedding $G \subset \mathbf{R}$, induces a distinguished real embedding $\sigma_0 \colon E \to \mathbf{R}$, $\psi \to \Psi(1)$. Declare two ring embeddings $\sigma_1, \sigma_2 \colon E \to \mathbf{R}$ equivalent if there is an automorphism σ of E (this is stronger than merely an automorphism of K) such that $\sigma_1\sigma = \sigma_2$; $[\sigma_1] = [\sigma_2]$.

Then totally ordered free subgroups of $\mathbf{R}$ with enough continuous endomorphisms are classified by the triple

$$(\mathrm{End}_c G, [\sigma_0], [{}_EG]).$$

This means, given two such groups G_1, G_2, they are topologically and order-isomorphic if and only if there is an isomorphism of rings $E_1 = \mathrm{End}_c G_1 \to E_2$, and an $E_1 - E_2$ semilinear module isomorphism, ${}_{E_1}G_1 \to {}_{E_2}G_2$, so that the σ_0's differ by at most an automorphism of E_2. The important thing is that the module isomorphism is just as modules.

Conversely, given an integral order E in a field K, together with a real embedding $\tau \colon K \to \mathbf{R}$, and an ideal I of E such that the natural map $E \to \mathrm{End}_E I$ ("semi-invertible" in [8]) is an isomorphism, there is a unique up to topological/order-isomorphism free subgroup of $\mathbf{R}$, G, whose triple $(\mathrm{End}_c G, [\sigma_0], [{}_EG])$ matches $(E, [\tau], [{}_EG])$. (Prescription: Take for G, $\tau(I) \subset \mathbf{R}$.)

The proofs are merely Bowdlerized versions of [8; IV.6] and [8; IV.5] respectively; all the nasty bits are removed because the vector space is one-dimensional.

In particular, certain of the ideal classes of these orders are the appropriate object of study. It is known that for fixed E, there are only finitely many [1; p. 543, 2.3].

In case G arises from a stationary system, but is not free, it still makes sense to consider G as an $\mathrm{End}_c G$-module; of course it is isomorphic to an ideal, and the reflexivity property still holds. It would be surprising if the analogous results dealing with isomorphism failed to hold.

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