J. OPERATOR THEORY
6(1981), 363–374

STONE-WEIERSTRASS THEOREMS FOR SEPARABLE C^*-ALGEBRAS

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1. INTRODUCTION

Suppose A is a C^*-algebra and B is a C^*-subalgebra of A. In this context, the classical commutative Stone-Weierstrass theorem asserts that if B separates the pure states of A, then $B = A$. The main results of this paper are as follows. Assume A is separable and unital.

1) If B separates the pure states of A and for each state f on B, $\pi_f(B)''$ contains a regular maximal abelian subalgebra, then $B = A$.
2) If B separates the factor states of A and each factor state on B extends to a factor state on A, then $B = A$.

See [1], [4], [7], [9], [10], and [12] for other Stone-Weierstrass theorems for noncommutative C^*-algebras.

If A is a nonunital C^*-algebra, let $\tilde{A}$ denote the C^*-algebra obtained by adjoining an identity to A. As noted in [12], if B separates the pure states of A and zero, then $C^*(B, 1)$ separates the pure states of $\tilde{A}$. We may thus assume that A has a unit. Moreover, if B separates the pure states of A, then B contains the unit of A [12, Lemma 1]. We therefore assume throughout that A is unital and $1 \in B \subseteq A$.

The paper is organized as follows. In Section 2 we collect some results that will be needed in the sequel. Section 3 contains the main results and in Section 4 we present some miscellaneous related facts.

If A is a C^*-algebra, then we use $\mathrm{S}(A)$, $\mathrm{P}(A)$, and $\mathrm{F}(A)$ to denote the states on A, the pure states on A, and the factor states on A, respectively. If $f \in \mathrm{S}(A)$, then $\pi_f, \mathscr{H}_f$, and 1_f denote the cyclic representation arising from f, the Hilbert space on which $\pi_f(A)$ acts and the cannonical cyclic vector. If S is a set of operators on a Hilbert space, then $W^*(S)$ denotes the von Neumann algebra generated by S and if η is a vector, then $[S\eta]$ denotes the closed subspace generated by elements of S acting on η. Finally maximal abelian subalgebras of von Neumann algebras are always assumed to be self-adjoint.

2. PRELIMINARY RESULTS

Probably the best partial solution to the Stone-Weierstrass problem for C^*-algebras is due to Sakai [13, 4.7.6] or [12]. Since it will be used repeatedly, we begin by recording it. Throughout A and B shall denote fixed unital *separable* C^*-algebras with $1 \in B \subset A$.

SAKAI'S STONE-WEIERSTRASS THEOREM. *Suppose B separates $\mathrm{P}(A)$, $\pi : A \to B(\mathscr{H})$ is a representation of A on a separable Hilbert space and $\mathscr{M}$ is a maximal abelian subalgebra of $\pi(A)'$. If Φ is a linear norm one map of $\pi(A)$ into $W^*(\pi(A), \mathscr{M})$ such that $\Phi(\pi(b)) = \pi(b)$ for all b in B, then $\Phi(\pi(a)) = \pi(a)$ for all a in A.*

We now present some corollaries to Sakai's theorem. Some have appeared elsewhere and, no doubt, the rest are known to many experts.

COROLLARY 1. *If B separates $\mathrm{P}(A)$, $\pi : A \to B(\mathscr{H})$ is a representation of A on a separable Hilbert space and $\mathscr{M}$ is a maximal abelian subalgebra of $\pi(A)'$, then $\mathscr{M}$ is a maximal abelian subalgebra of $\pi(B)'$.*

Proof. Fix a projection P in $\pi(B)' \cap \mathscr{M}'$ and define Φ on $\pi(A)$ by

$$\Phi(\pi(a)) = P\pi(a)P + P^{\perp}\pi(a)P^{\perp}$$

where $P^{\perp} = 1 - P$. As $\mathscr{M}$ is maximal abelian in $\pi(A)'$, $\mathscr{M}' = W^*(\pi(A), \mathscr{M})$ and so Φ maps $\pi(A)$ into $W^*(\pi(A), \mathscr{M})$. Since $P \in \pi(B)'$, $\Phi(\pi(b)) = \pi(b)$ for b in B. Clearly Φ has norm one, so Sakai's theorem applies and $\Phi(\pi(a)) = \pi(a)$ for a in A. It follows that $P \in \pi(A)'$ and so $\pi(B)' \cap \mathscr{M}' = \pi(A)' \cap \mathscr{M}' = \mathscr{M}$.

For a von Neumann algebra R, let $Z(R)$ denote the center of R.

COROLLARY 2. *If B separates $\mathrm{P}(A)$ and $\pi : A \to B(\mathscr{H})$ is a representation of A, then $\mathrm{Z}(\pi(B)'') \subseteq \mathrm{Z}(\pi(A)'')$.*

Proof. First suppose $\mathscr{H}$ is separable. If $z \in \mathrm{Z}(\pi(B)'') = \mathrm{Z}(\pi(B)')$, then z belongs to every maximal abelian subalgebra of $\pi(B)'$ and so by Corollary 1 z belongs to every maximal abelian subalgebra of $\pi(A)'$. Hence $z \in \mathrm{Z}(\pi(A)') = \mathrm{Z}(\pi(A)'')$. Now suppose π is an arbitrary representation and write $\pi = \sum \oplus \pi_\alpha$ where each π_α acts on a separable Hilbert space. If $z \in \mathrm{Z}(\pi(B)'')$, then $z = \sum \oplus z_\alpha$ where each $z_\alpha \in \mathrm{Z}(\pi_\alpha(B)'')$. By the first part of the proof, $z_\alpha \in \mathrm{Z}(\pi_\alpha(A)'')$ for each α and therefore $z \in \pi(A)'$, so that $z \in \mathrm{Z}(\pi(A)'')$.

Note that since A^{**} may be identified with $\pi_u(A)''$, where π_u denotes the universal representation of A and B^{**} may be identified with $\pi_u(B)''$, the corollary shows that if B separates $\mathrm{P}(A)$, then $\mathrm{Z}(B^{**}) \subseteq \mathrm{Z}(A^{**})$.

COROLLARY 3. *If B separates $\mathrm{P}(A)$ and $f \in \mathrm{F}(A)$, then $f | B \in \mathrm{F}(B)$.*

Proof. Suppose $f \in \mathrm{F}(A)$ so that $\pi_f(A)''$ is a factor. If we write $g=f|B$, then π_g is equivalent to a subrepresentation of $\pi_f|B$. By Corollary 2, $\pi_f(B)''$ is a factor and so [5, 5.3.4, 5.3.5] the weak closure of each subrepresentation of $\pi_f|B$ is a factor. Hence $g \in \mathrm{F}(B)$.

The following corollary is a special case of a result due to Effros [7, Theorem 11.1]. Let $r : \mathrm{S}(A) \to \mathrm{S}(B)$ denote the restriction map.

COROLLARY 4. *If B separates* $\mathrm{P}(A)$ *and there is an affine map $D : \mathrm{S}(B) \to \mathrm{S}(A)$ such that $r \circ D$ is the identity, then $B = A$.*

Proof. As D is affine on $\mathrm{S}(B)$, a standard argument (for example a slight variant of the proof of Lemma 6.7 in Chapter III of [15]) shows that D extends to a linear map (also denoted by D) of B^* into A^* such that D is continuous with $\|D\| \leqslant 2$. Since D maps $\mathrm{S}(B)$ into $\mathrm{S}(A)$, the adjoint map $D^* : A^{**} \to B^{**}$ is positive and self-adjoint. As $r \circ D$ is the identity, $D^*(b) = b$ for all b in B and since D^* is weak*-continuous, D^* is a projection of A^{**} onto B^{**} which has norm one by the Russo-Dye theorem (see [3, p. 211]). Fix a separable representation $\pi : A \to B(\mathscr{H})$ and let $\tilde{\pi}$ denote its unique normal extension to A^{**}. By [1, Lemma III.4] there is a projection q in $B^{**} \cap \mathrm{Z}(A^{**})$ such that qA^{**} is the ultra-weak closure of $\ker\pi$. Define $\Phi : \pi(A) \to \pi(B)''$ by $\Phi(\pi(a)) = \tilde{\pi}(D^*(a(\mathbb{1} - q)))$. It follows that Φ is a well-defined norm one map with $\Phi(\pi(b)) = \pi(b)$, for b in B. By Sakai's theorem, $\pi(A)'' = \pi(B)''$. Thus, every representation that is cyclic for A is cyclic for B. But this is impossible unless $B = A$ [1, proof of III.7].

3. THE MAIN RESULTS

In this section the theory of decomposition of states shall be used to obtain our main theorem (Theorem 5). This theory has a long history and important contributions have been made by many authors including Choquet, Ruelle, Sakai and Skau. For further details and references see [3, p. 451—454]. We shall employ the theory of orthogonal measures first exposed by Skau [14]. We shall also refer to the slightly different expositions found in [15, IV Section 6] and [3, Sections 4.1 and 4.2].

Throughout this section we continue to assume that A and B are separable unital C^*-algebras with $1 \in B \subseteq A$. We say that a Borel subset S of $\mathrm{S}(A)$ is a *set of agreement* for B if the restriction map r is injective on S and if $\pi_f(A)'' = \pi_f(B)''$ for each f in S. We now recall some facts and introduce some notation. If μ is a probability measure on the Borel subsets of $\mathrm{S}(A)$, let f_μ denote the resultant of μ given by the formula

$$f_\mu(a) = \int_{\mathrm{S}(A)} \hat{a}(f)\, \mathrm{d}\mu(f)$$

where $\hat{a}(f)=f(a)$ and let $\{\pi_\mu, \mathscr{H}_\mu, 1_\mu\}$ denote the cyclic representation of A that arises from f_μ. Also if $\varphi \in L^\infty(\mathrm{S}(A), \mu)$, then the formula

$$(K_\mu(\varphi)\,\pi_\mu(a)\,1_\mu, 1_\mu) = \int_{\mathrm{S}(A)} \varphi(f)\,\hat{a}(f)\,\mathrm{d}\mu(f)$$

defines an element $K_\mu(\varphi)$ in $\pi_\mu(A)'$. If μ is an orthogonal measure, then K_μ is a $*$-isomorphism of $L^\infty(\mathrm{S}(A), \mu)$ onto an abelian von Neumann algebra in $\pi_\mu(A)'$. The range of K_μ is denoted by $\mathscr{N}_\mu$. Conversely, if $f \in \mathrm{S}(A)$ and $\mathscr{N}$ is an abelian von Neumann algebra in $\pi_f(A)'$, then there is a unique orthogonal measure μ on $\mathrm{S}(A)$ such that $\mathscr{N} = \mathscr{N}_\mu$ and $f = f_\mu$.

THEOREM 5. *If $S \subseteq \mathrm{S}(A)$ is a set of agreement for B and ν is an orthogonal measure on $\mathrm{S}(B)$ with $\nu(r(S)) = 1$, then there is an orthogonal measure μ on $\mathrm{S}(A)$ such that*

i) *The resultant f_μ on A extends f_ν.*
ii) *If $\{\pi_\nu, \mathscr{H}_\nu, 1_\nu\}$ is regarded as a subrepresentation of $\{\pi_\mu|B, \mathscr{H}_\mu, 1_\mu\}$ so that $1_\nu = 1_\mu$ and $\mathscr{H}_\nu \subseteq \mathscr{H}_\mu$, then $\mathscr{H}_\nu = \mathscr{H}_\mu$ and $\pi_\mu|B = \pi_\nu$.*
iii) *$\mathscr{N}_\nu = \mathscr{N}_\mu$ and $\pi_\mu(A)'' \subseteq W^*(\pi_\mu(B), \mathscr{N}_\mu)$.*
iv) *If $\mathrm{P}(A) \subseteq S$, then the representation $\pi_\mu : A \to B(\mathscr{H}_\nu)$ is the unique representation of A such that π_μ extends π_ν and $\mathscr{N}_\mu \subseteq \pi_\mu(A)'$.*

Proof. Since A and B are separable, $\mathrm{S}(A)$ and $\mathrm{S}(B)$ are Polish spaces and since r is continuous it is a Borel mapping. As $r|S$ is injective, its range $T = r(S)$ is a Borel set and $r|S$ is a Borel isomorphism of S onto T [2, Theorem 3.3.2]. For a Borel subset E of $\mathrm{S}(A)$ write

$$\mu(E) = \nu(r(E \cap S)).$$

Since $r|S$ is a Borel isomorphism and $\nu(T) = 1$, this formula defines a Borel probability measure μ on $\mathrm{S}(A)$. Note that if φ is an integrable Borel function on $\mathrm{S}(A)$, then

$$(*) \qquad \int_S \varphi(f)\,\mathrm{d}\mu(f) = \int_T \varphi\circ r^{-1}(g)\,\mathrm{d}\nu(g).$$

In particular if b is in B, then

$$f_\mu(b) = \int_S \hat{b}(f)\,\mathrm{d}\mu(f) = \int_T \hat{b}(g)\,\mathrm{d}\nu(g) = f_\nu(b)$$

and f_μ extends f_ν. (We use the symbol $\hat{b}$ to denote both the function b induces on $\mathrm{S}(A)$ and the analogous function on $\mathrm{S}(B)$.) We next show μ is an orthogonal measure. Fix a Borel subset E in $\mathrm{S}(A)$ and let p_μ and q_μ denote the resultants of $\mu|E$ and $\mu|\mathrm{S}(A)\setminus E$, respectively. If p_ν and q_ν denote the resultants of $\nu|r(E)$ and $\nu|\mathrm{S}(B)\setminus r(E)$, then it follows from $(*)$ that $p_\nu = p_\mu|B$ and $q_\nu = q_\mu|B$. If t is a positive functional on A such that $t \leqslant p_\mu$ and $t \leqslant q_\mu$, then $0 \leqslant t|B \leqslant p_\nu, q_\nu$ and since ν is orthogonal $t|B = 0$. But $1 \in B$, so $t(1) = 0$ and therefore $t = 0$. Hence μ is an orthogonal measure. Since f_μ extends f_ν, we may regard π_ν as a subrepresentation of $\pi_\mu|B$ with $1_\nu = 1_\mu$ and $\mathscr{H}_\nu \subseteq \mathscr{H}_\mu$. Note that under this identification we have $\pi_\mu(b)1_\mu = \pi_\nu(b)1_\nu$ for all b in B. We next use direct integral theory to show $\pi_\mu(A)'' \subset W^*(\pi_\mu(B), \mathscr{N}_\mu)$. First note that by [15, IV 8.31 and its proof] there is a unitary operator U mapping $\mathscr{H}_\mu$ onto

$$\int_S^{\oplus} \mathscr{H}_f \, d\mu(f)$$

such that for a in A and φ in $L^\infty(S, \mu)$

$$U\pi_\mu(a)\, U^* = \int_S^{\oplus} \pi_f(a)\, d\mu(f) \quad \text{and} \quad UK_\mu(\varphi)\, U^* = \int_S^{\oplus} \varphi(f)\, d\mu(f).$$

Thus, for each a in A, $U\pi_\mu(a)\, U^*$ is a decomposable operator and $U\mathscr{N}_\mu U^*$ is the diagonal algebra associated with this direct integral decomposition. Fix a dense subsequence $\{b_n\}$ in B and let R denote the von Neumann algebra generated by the diagonal algebra and the decomposable operators of the form

$$U\pi_\mu(b_n)\, U^* = \int_S^{\oplus} \pi_f(b_n)\, d\mu(f).$$

Clearly $R = UW^*(\pi_\mu(B), \mathscr{N}_\mu)\, U^*$. By [6, Theorem 1 (ii), p. 171] R contains every decomposable operator

$$\int_S^{\oplus} x(f)\, d\mu(f)$$

such that $x(f) \in \{\pi_f(b_n) : n = 1, 2, \ldots\}'' = \pi_f(B)''$ for almost all f. Since S is a set of agreement for B

$$U\pi_\mu(a) U^* = \int_S^{\oplus} \pi_f(a)\, d\mu(f)$$

has this form for every a in A. Thus, $U\pi_\mu(A)\, U^* \subset UW^*(\pi_\mu(B), \mathscr{N}_\mu)\, U^*$.

We assert that $\mathscr{N}_\mu \mathscr{H}_\nu \subseteq \mathscr{H}_\nu$. Fix a projection P in $\mathscr{N}_\mu$ and let E denote the Borel subset of $S(A)$ such that $K_\mu(\chi_E) = P$, where χ_E denotes the characteristic function of E. If b, c are in B and we write $F = r(E)$, then

$$(P\pi_\mu(b)\, 1_\mu,\ \pi_\mu(c)\, 1_\mu) = \int_S \chi_E(f)\, \widehat{c^*b}(f)\, d\mu(f) =$$

$$= \int_T \chi_F(g)\, \widehat{c^*b}(g)\, d\nu(g) = (K_\nu(\chi_F)\pi_\nu(b)\, 1_\nu,\ \pi_\nu(c)\, 1_\nu) =$$

$$= (K_\nu(\chi_F)\, \pi_\mu(b)\, 1_\mu,\ \pi_\mu(c)\, 1_\mu).$$

Thus, if Q denotes the projection of $\mathscr{H}_\mu$ onto $\mathscr{H}_\nu$, then $QPQ|\mathscr{H}_\nu = K_\nu(\chi_F)$ and QPQ is a projection. It follows that $QPQ^\perp PQ = 0$ and Q commutes with P. Hence $P\mathscr{H}_\nu \subseteq \mathscr{H}_\nu$ and our assertion follows. Now note that

$$\mathscr{H}_\mu = [\pi_\mu(A)\,1_\mu] \subseteq [W^*(\pi_\mu(B), \mathscr{N}_\mu)\,1_\mu] = \mathscr{H}_\nu$$

so that $\mathscr{H}_\nu = \mathscr{H}_\mu$ and $Q = 1$. Moreover if E is a Borel subset of $\mathrm{S}(A)$ the calculation above shows that $K_\mu(\chi_E) = K_\nu(\chi_F)$. Since $r|S$ is a Borel isomorphism, $\mathscr{N}_\mu = \mathscr{N}_\nu$. It only remains to show iv). Suppose $\rho : A \to B(\mathscr{H}_\nu)$ is a representation of A such that ρ extends π_ν and $\mathscr{N}_\mu \subseteq \rho(A)'$. Select a maximal abelian subalgebra $\mathscr{M}$ of $\rho(A)'$ that contains $\mathscr{N}_\mu$ and define Φ on $\rho(A)$ by $\Phi(\rho(a)) = \pi_\mu(a)$. We have $\ker\rho \cap B = \ker\pi_\mu \cap B$ so by [1, Lemma III.4] $\ker\rho = \ker\pi_\mu$ and Φ is a well-defined $*$-isomorphism. By iii) we have

$$\pi_\mu(A) \subset W^*(\pi_\mu(B), \mathscr{N}_\mu) = W^*(\rho(B), \mathscr{N}_\mu) \subset W^*(\rho(B), \mathscr{M})$$

and so by Sakai's theorem $\rho = \pi_\mu$.

THEOREM 6. *If B separates* $\mathrm{P}(A)$, $f \in \mathrm{S}(B)$ *and $\mathscr{M}$ is a maximal abelian subalgebra of $\pi_f(B)'$, then there is a representation $\pi : A \to B(\mathscr{H}_f)$ such that*

i) *π extends π_f,*

ii) $\mathscr{M} \subseteq \pi(A)'$,

iii) $\pi(A)'' \subseteq W^*(\pi_f(B), \mathscr{M})$

and

iv) *π is the unique representation of A that satisfies* i) *and* ii).

Proof. Since A is separable, $\mathrm{P}(A)$ is a Borel subset of $\mathrm{S}(A)$ [11,4.3.2] and since B separates $\mathrm{P}(A)$, $\mathrm{P}(A)$ is a set of agreement for B and $r(\mathrm{P}(A)) = \mathrm{P}(B)$ [5,11.1.7]. Write ν for the orthogonal measure on $\mathrm{S}(B)$ with $\mathscr{N}_\nu = \mathscr{M}$. Since $\mathscr{M}$ is maximal abelian in $\pi_f(B)''$, ν is supported by $\mathrm{P}(B)$ [14]. Thus, Theorem 5 applies and the theorem follows.

COROLLARY 7. *If B separates* $\mathrm{P}(A)$, $f \in \mathrm{S}(B)$ *and $\pi : A \to B(\mathscr{H}_f)$ is a representation that extends π_f then π is the unique representation that extends π_f if and only if $\pi(A)'' = \pi_f(B)''$.*

Proof. If π is the unique representation that extends π_f, then by Theorem 6 $\pi(A)'$ contains every maximal abelian subalgebra of $\pi_f(B)'$ and so $\pi(A)'' = \pi_f(B)''$. If $\pi(A)'' = \pi_f(B)''$, $\rho : A \to B(\mathscr{H}_f)$ is a representation of A that extends π_f and $\mathscr{M}$ is a maximal abelian subalgebra of $\rho(A)'$, then by Corollary 1 $\mathscr{M}$ is maximal abelian in $\pi_f(B)' = \pi(A)'$ and so $\rho = \pi$ by part iv) of Theorem 6.

THEOREM 8. *If B separates* $\mathrm{P}(A)$ *and for each f in* $\mathrm{S}(B)$ *there is a unique representation $\hat{\pi}_f : A \to B(\mathscr{H}_f)$ that extends π_f, then $B = A$.*

Proof. It suffices to show there is a map $D : \mathrm{S}(B) \to \mathrm{S}(A)$ satisfying the hypotheses of Corollary 4. For f in $\mathrm{S}(B)$ define $D(f)$ on A by

$$D(f)(a) = (\hat{\pi}_f(a)1_f, 1_f).$$

As $\hat{\pi}_f$ extends π_f, $r(D(f)) = f$. We need to show that D is affine. Fix g and h in $\mathrm{S}(B)$, $0 < t < 1$, and write $f = tg + (1-t)h$. As $tg \leqslant f$, there is an element b' in $\pi_f(B)'$ such that $0 \leqslant b' \leqslant 1$ and $tg(b) = (\pi_f(b)b'1_f, b'1_f)$ for b in B. By Corollary 7, the range projection Q of b' lies in $\hat{\pi}_f(A)'$. We may then identify π_g with $Q\pi_f|Q\mathscr{H}_f$ so that $1_g = (1/\sqrt{t})b'1_f$. Since $Q \in \hat{\pi}_f(A)'$, $Q\hat{\pi}_f|Q\mathscr{H}_f$ is a representation of A that extends π_g, and therefore

$$tD(g)(a) = (\hat{\pi}_f(a)b'1_f, b'1_f).$$

A similar argument shows that

$$(1-t)D(h)(a) = (\hat{\pi}_f(a)b''1_f, b''1_f),$$

with $(b'')^2 + (b')^2 = 1$. Therefore,

$$tD(g) + (1-t)D(h) = D(f).$$

Recall that a maximal abelian subalgebra $\mathscr{M}$ of a von Neumann algebra R is said to be *regular* if R is generated by the unitaries U in R that normalize $\mathscr{M}$ in the sense that $U\mathscr{M}U^* = \mathscr{M}$.

PROPOSITION 9. *If B separates* $\mathrm{P}(A)$, *$f \in \mathrm{S}(B)$ and $\pi_f(B)'$ contains a regular maximal abelian subalgebra, then there is a unique representation $\pi : A \to B(\mathscr{H}_f)$ that extends π_f.*

Proof. Suppose $\mathscr{M}$ is a regular maximal abelian subalgebra of $\pi_f(B)'$ and select by Theorem 6 a representation $\pi : A \to B(\mathscr{H}_f)$ that extends π_f and such that $\pi(A)'$ contains $\mathscr{M}$. If U is a unitary in $\pi_f(B)'$ that normalizes $\mathscr{M}$ and we write $\rho(a) = U\pi(a)U^*$, then ρ is a representation of A that extends π_f. Moreover $\mathscr{M} = U\mathscr{M}U^* \subseteq \rho(A)'$, and so by part iv) of Theorem 6 $\rho = \pi$. Hence $U \in \pi(A)'$ and since $\pi_f(B)'$ is generated by such unitaries $\pi(A)'' = \pi_f(B)''$. By Corollary 7 π is the unique representation that extends π_f.

We say that a C^*-algebra C is *regular* if $\pi_f(C)''$ contains a regular maximal abelian subalgebra for each f in $\mathrm{S}(C)$.

THEOREM 10. *If B is a regular C^*-algebra and B separates* $\mathrm{P}(A)$, *then $B = A$.*

Proof. By Theorem 8 it suffices to show that for each f in $\mathrm{S}(B)$ there is a unique representation extending π_f. Fix f in $\mathrm{S}(B)$ and select an orthonormal basis $\{\eta_n\}$ for $\mathscr{H}_f$ with $\eta_1 = 1_f$. Let ω_n denote the vector state on $\pi_f(B)''$ defined by $\omega_n(X) =$

$=(X\eta_n, \eta_n)$ and write $g = \sum 2^{-n}\omega_n$. The state g induces a normal representation $\{\theta, \mathscr{H}_g, 1_g\}$ of $\pi_f(B)''$ such that 1_g is a cyclic and separating vector for $R=\theta(\pi_f(B)'')$. Moreover if we write $\rho = \theta \circ \pi_f$, then $\{\rho, \mathscr{H}_g, 1_g\}$ is the cyclic representation of B arising from $g \circ \pi_f$ and $\rho(B)'' = R$. Since B is regular R contains a regular maximal abelian subalgebra $\mathscr{M}_0$. By Tomita's theorem [11, 8.13.14] there is an isometric involution J such that $JRJ = R'$ and $\mathscr{M} = J\mathscr{M}_0 J$ is a regular maximal abelian subalgebra of R'. By Proposition 9 there is a representation $\hat{\rho} : A \to B(\mathscr{H}_g)$ that extends ρ and such that $\hat{\rho}(A)'' = R = \rho(B)''$. It follows that for every subrepresentation $\{\rho_0, \mathscr{H}_0\}$ of ρ there is a representation $\hat{\rho}_0 : A \to B(\mathscr{H}_0)$ that extends ρ_0 and such that $\hat{\rho}_0(A)'' = \rho_0(B)''$. Since $1/2 f \leqslant g \circ \pi_f$, π_f is unitarily equivalent to a subrepresentation of ρ and there is a representation $\pi : A \to B(\mathscr{H}_f)$ extending π_f and such that $\pi(A)'' = \pi_f(B)''$. By Corollary 7 π is the unique representation that extends π_f.

We do not know of an example of a von Neumann algebra that does not contain a regular maximal abelian subalgebra. Many von Neumann algebras do contain such algebras (see, for example [8, p. 332] or [16]). It is straightforward that every type I von Neumann algebra contains a regular maximal abelian subalgebra and therefore type I C^*-algebras are regular. We do not know of any other examples of regular C^*-algebras although it seems likely that many others exist.

We now turn to a consideration of the factorial Stone-Weierstrass problem. If A is abelian, then $\mathrm{F}(A) = \mathrm{P}(A)$. Thus, it is possible that the correct generalization of the Stone-Weierstrass theorem should require that B separate $\mathrm{F}(A)$. The content of our next theorem is that the only obstacle in the way of establishing this weaker version is showing that factor states extend to factor states.

PROPOSITION 11. *If B separates* $\mathrm{F}(A)$, *then* $\mathrm{F}(A)$ *is a set of agreement for B.*

Proof. As A is separable $\mathrm{F}(A)$ is a Borel subset of $\mathrm{S}(A)$ [11, 4.8.3] and since B separates $\mathrm{F}(A)$, the restriction map r is injective on $\mathrm{F}(A)$. Fix f in $\mathrm{F}(A)$, a unit vector η in $\mathscr{H}_f$ and a unitary U in $\pi_f(B)'$ and define f_1 and f_2 on A by

$$f_1(a) = (\pi_f(a)\eta, \eta), \quad f_2(a) = (\pi_f(a)U\eta, U\eta).$$

As η and $U\eta$ are unit vectors f_1 and f_2 are states on A and since $U \in \pi_f(B)'$, $r(f_1) = r(f_2)$. Moreover, f_1 and f_2 give rise to representations that are unitarily equivalent to subrepresentations of π_f and so f_1 and f_2 are factor states [5,5.3.4,5.3.5]. Thus, $f_1 = f_2$ and since η was arbitrary $U \in \pi_f(A)'$. Therefore $\pi_f(A)'' = \pi_f(B)''$ and $\mathrm{F}(A)$ is a set of agreement for B.

We remark that if B separates $\mathrm{F}(A)$ the proof given above also shows that if π is any representation of A with $\pi(A)''$ a factor, then $\pi(A)'' = \pi(B)''$. It can also be shown that disjoint factor representations of A restrict to disjoint factor representations of B. Thus, by analogy with [5,11.1.1] we may say that if B separates $\mathrm{F}(A)$, then B is *ultrarich* in A.

THEOREM 12. *If B separates $\mathrm{F}(A)$ and each factor state on B extends to a factor state on A, then $B = A$.*

Proof. By Proposition 11, $\mathrm{F}(A)$ is a set of agreement for B. Our additional assumption means that $r(\mathrm{F}(A)) = \mathrm{F}(B)$. Fix f in $\mathrm{S}(B)$ and let ν denote the central measure for f. If Z denotes the center of $\pi_f(B)''$, then by [15, IV 6.29,6.32] ν is the orthogonal measure associated with Z and ν is concentrated on $\mathrm{F}(B)$. Hence Theorem 5 applies and there is a representation $\pi : A \to B(\mathscr{H}_f)$ that extends $\pi_f = \pi_\nu$ and such that $\pi(A)'' \subseteq W^*(\pi_f(B), Z) = \pi_f(B)''$. The theorem now follows from Corollary 7 and Theorem 8.

Recall that a maximal abelian subalgebra $\mathscr{M}$ of a von Neumann algebra R is said to be *semiregular* if the unitaries in R that normalize $\mathscr{M}$ generate a factor.

PROPOSITION 13. *If B separates $\mathrm{P}(A)$, $f \in \mathrm{F}(B)$ and $\pi_f(B)'$ contains a maximal abelian subalgebra $\mathscr{M}$ such that either*

i) *$\mathscr{M}$ is a semiregular maximal abelian subalgebra of $\pi_f(B)'$,*

or

ii) *there is an injective factor R such that $\mathscr{M} \subseteq R \subseteq \pi_f(B)'$*

then f extends to a factor state on A.

Proof. Suppose $\mathscr{M}$ is a semiregular subalgebra of $\pi_f(B)'$ and let $\pi : A \to B(\mathscr{H}_f)$ denote a representation of A that extends π_f and is such that $\mathscr{M} \subseteq \pi(A)'$. As in the proof of Proposition 9, it follows that the unitaries that normalize $\mathscr{M}$ belong to $\pi(A)'$ and therefore there is a factor R such that $\mathscr{M} \subseteq R \subseteq \pi(A)'$. As $\mathscr{M}$ is maximal abelian the center of $\pi(A)'$ is contained in $\mathscr{M}$ and therefore lies in the center of R. As this latter algebra consists of the scalar multiples of the identity $\pi(A)''$ is a factor and f extends to a factor state on A. Now suppose $\mathscr{M} \subseteq R \subseteq \pi_f(B)'$, where R is an injective factor and let $\pi : A \to B(\mathscr{H}_f)$ be as before. We have then

$$\pi_f(B)'' \subseteq R' \subseteq \mathscr{M}' = W^*(\pi_f(B), \mathscr{M}).$$

Since R' is injective, there is a norm one projection mapping $B(\mathscr{H}_f)$ onto R'. This projection restricts to a map Φ on $\pi(A)$ that satisfies the hypotheses of Sakai's theorem. Thus

$$\pi(A)'' \subseteq R', \quad \mathscr{M} \subseteq R \subseteq \pi_f(A)'$$

and as above f extends to a factor state on A.

Again there does not seem to be an example known of a factor that does not contain a semiregular maximal abelian subalgebra. Regarding ii) of Proposition 13: One of Kadison's Baton Rouge problems was: Is each self-adjoint element in a II_1 factor contained in some hyperfinite subfactor ? In ii) we are asking whether every factor contains just one maximal abelian subalgebra contained in an injective factor. If so, and if B separates $\mathrm{F}(A)$, then $B = A$.

4. MISCELLANEOUS RELATED RESULTS

We continue to use A and B to denote separable unital C^*-algebras with $1 \in B \subset A$.

PROPOSITION 14. *If B separates* F(A) *and $\pi : A \to B(\mathscr{H})$ is a representation with $\mathscr{H}$ separable, then $\pi(A)'' = W^*(\pi(B), Z)$, where Z denotes the center of $\pi(A)''$.*

Proof. By the central decomposition of representations [5, Chapter 8] there is a measurable field $x \to \pi(x)$ of factor representations of A such that

$$\pi = \int^{\oplus} \pi(x)\, d\mu(x)$$

for some measure μ. Moreover the diagonal algebra for this direct integral decomposition is Z. By the proof of Proposition 11 we have that for each x, $\pi(x)(A)'' = \pi(x)(B)''$. The proof is now completed by arguing as in the proof of part iii) of Theorem 5.

COROLLARY 15. *Suppose B separates* F(A), $f \in$ S(B), *and $\pi_i : A \to B(\mathscr{H}_f)$ is a representation of A that extends π_f, $i = 1, 2$. If Z_i denotes the center of $\pi_i(A)''$, $i = 1, 2$ and Z_1 commutes with Z_2, then $\pi_1 = \pi_2$.*

Proof. Since Z_1 and Z_2 commute, there is a maximal abelian subalgebra $\mathscr{M}$ of $\pi_f(B)'$ that contains both Z_1 and Z_2. Let $\pi : A \to B(\mathscr{H}_f)$ denote a representation of A that extends π_f and is such that $\mathscr{M} \subseteq \pi(A)'$. By Proposition 14 we have that

$$\pi_i(A)'' = W^*(\pi_f(B), Z_i) \subseteq W^*(\pi(A), \mathscr{M}).$$

Define $\Phi_i : \pi(A) \to \pi_i(A)''$ by $\Phi_i(\pi(a)) = \pi_i(a)$, $i = 1,2$. As in the proof of part iv) of Theorem 5 we have that $\ker\pi = \ker\pi_i$, $i = 1,2$ and so Φ_i is a well-defined $*$-isomorphism for $i = 1,2$. Hence by Sakai's theorem $\pi_1 = \pi = \pi_2$.

PROPOSITION 16. *If B separates* P(A) *and each factor state on B has a unique state extension to A, then $B = A$.*

Proof. Fix f in F(B). By Theorem 6, there is a representation $\pi : A \to B(\mathscr{H}_f)$ that extends π_f. Let U be a unitary operator in $\pi_f(B)'$ and let x be a unit vector in $\mathscr{H}_f$. Then $b \to (\pi_f(b)x, x)$ is a factor state on B and has $a \to (\pi(a)x, x)$ and $a \to (\pi(a)Ux, Ux)$ as state extensions to A. Hence $(U^*\pi(a)Ux, x) = (\pi(a)x, x)$ for all x in $\mathscr{H}$ and a in A. Thus $U \in \pi(A)'$ and $\pi_f(B)'' = \pi(A)''$ and f extends to a factor state on A. By Theorem 12 $B = A$.

Proposition 17. *If B separates* F(A) *and if disjoint representations of A restrict to disjoint representations of B, then $B = A$.*

Proof. Fix f in F(B). By Theorem 6 there is a representation $\pi: A \to B(\mathscr{H}_f)$ that extends π_f. As in the proof of Proposition 16, it is enough to show $\pi(A)'' = \pi_f(B)''$. By Proposition 14 $\pi(A)'' = W^*(\pi_f(B), Z)$ where Z denotes the center of $\pi(A)''$ so it suffices to show Z is trivial. Suppose Z is not trivial so that π has disjoint subrepresentations ρ_1 and ρ_2. By our hypothesis, $\rho_1|B$ and $\rho_2|B$ are also disjoint. On the other hand these restrictions are subrepresentations of the factor representation π_1 and are therefore not disjoint [5,5.3.4,5.3.5]. Hence Z is trivial and $B = A$.

We conclude by mentioning a fact which is apparently unpublished folklore. Let K denote the compact convex set of linear functionals f on A such that $f = f^*$, $f(B) = 0$ and $\|f\| \leqslant 2$. If $B \neq A$, then by the Hahn-Banach theorem $K \neq \{0\}$ and so has a nonzero extreme point f. By [5, 12.3.4], there are unique positive functionals f^+ and f^- such that $f = f^+ - f^-$ and $\|f\| = \|f^+\| + \|f^-\|$. As f is extreme, $\|f\| = 2$ and so f^+ and f^- are states on A. Write π_+ and π_- for the representations of A that these states induce and $\pi = \pi_+ \oplus \pi_-$.

PROPOSITION 18. *With the notation as above, if B separates* P(A) *then $\pi(B)''$ is a factor.*

Proof. If z denotes the central support of π in B^{**}, then by [11, 3.8.13] we need only to prove that z is a minimal central projection in B^{**}. Now by Corollary 2 z is in the center of A^{**} and it therefore follows that if z were not minimal then f would not be an extreme point of K.

Both authors were supported in part by grants from the National Science Foundation.

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Received November 24, 1980.