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THE FAILURE OF THE SLICE MAP CRITERION

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1. INTRODUCTION

Let A, B be von Neumann algebras on Hilbert spaces $\mathscr{H}, \mathscr{K}$; $\mathscr{H} \otimes \mathscr{K}$ the Hilbert space tensor product, $A \odot B \subseteq L(\mathscr{H} \otimes \mathscr{K})$ the algebraic tensor product, $A \otimes B$ its norm closure and $A \mathbin{\overline{\otimes}} B$ its weak closure. For each $\psi \in B_*$, $\mathrm{id}_A \otimes \psi : a \otimes b \to \psi(b)a$ defines a σ-weakly continuous map $A \otimes B \to A$ with a continuous extension $L_\psi : A \mathbin{\overline{\otimes}} B \to A$. For $\varphi \in A^*$ the generalized right slice map $R_{\tilde{\varphi}} : A \mathbin{\overline{\otimes}} B \to B$ is then defined by $\langle R_{\tilde{\varphi}}(x), \psi \rangle = \langle L_\psi(x), \varphi \rangle$ $(x \in A \mathbin{\overline{\otimes}} B, \psi \in B_*)$ (see [1]). For $c \in A \mathbin{\overline{\otimes}} B$ the map $r_c : \varphi \to R_{\tilde{\varphi}}(c)$ is a map from A_1^*, the unit ball of A^* with the w*-topology, into B with the norm topology. If $c \in A \otimes B$ then r_c is continuous and if at least one of A and B is a finite type I algebra the converse holds. Tomiyama [2, § 2] has suggested that continuity of r_c might be a sufficient condition for c to lie in $A \otimes B$ in general and has shown that such a criterion would be extremely convenient. In this note we show that, unfortunately, the criterion is not generally valid. Specifically, we construct an element c of $L(\mathscr{H} \otimes \mathscr{H}) = L(\mathscr{H}) \mathbin{\overline{\otimes}} L(\mathscr{H})$, with $\mathscr{H} = \ell^2(\mathrm{N})$, for which r_c is continuous but such that $c \notin L(\mathscr{H}) \otimes L(\mathscr{H})$. This construction actually applies to any pair A, B of von Neumann algebras, neither of which is finite of type I.

2. THE CONSTRUCTION

(a) For $n \in \mathrm{N}$, let $\mathscr{H}_n$ be a Hilbert space of dimension n with orthonormal basis $\{\xi_1, \ldots, \xi_n\}$, and let $\{e_{ij} : 1 \leqslant i, j \leqslant n\}$ be matrix units in $L(\mathscr{H}_n)$ such that $e_{ij}\xi_k = \delta_{jk}\xi_i$ $(1 \leqslant i, j \leqslant n)$. Tr_n will denote the usual trace on $L(\mathscr{H}_n)$. Let

$$x_n = n^{-\frac{1}{2}} \sum_i e_{1i} \otimes e_{1i} \in L(\mathscr{H}_n \otimes \mathscr{H}_n).$$

Then $x_n x_n^* = e_{11} \otimes e_{11}$, so that $\|x_n\| = \|x_n x_n^*\|^{\frac{1}{2}} = 1$.

Let f_n be the linear functional on $L(\mathscr{H}_n \otimes \mathscr{H}_n) \cong L(\mathscr{H}_{n^2})$ given by $f_n(z) = \mathrm{Tr}_{n^2}(x_n^* z)$. Then

$$f_n(z) = \sum_{i,j} (z(\xi_i \otimes \xi_j) \mid (x_n(\xi_i \otimes \xi_j)) =$$

$$= (zn^{-\frac{1}{2}} \sum (\xi_k \otimes \xi_k) \mid \xi_1 \otimes \xi_1)$$

so that $\|f_n\| \leqslant 1$. As $f_n(x_n) = 1$, $\|f_n\| = 1$.

It is well-known (and easy to prove) that for each g in $L(\mathscr{H}_n)^*$, there is an $a \in L(\mathscr{H}_n)$ such that $g(y) = \mathrm{Tr}_n(ay)$ and $\|g\| = \mathrm{Tr}_n(|a|)$. It follows that the injective cross-norm λ on $L(\mathscr{H}_n) \odot L(\mathscr{H}_n)$ is given by

$$\lambda(z) = \sup\{|\langle g \otimes h, z\rangle| : g, h \in L(\mathscr{H}_n)_1^*\} =$$

$$= \sup\{|\mathrm{Tr}_{n^2}(z(a \otimes b))| : a, b \in L(\mathscr{H}_n), \mathrm{Tr}_n(|a|), \mathrm{Tr}_n(|b|) \leqslant 1\}.$$

If $a, b \in L(\mathscr{H}_n)$,

$$|\mathrm{Tr}_{n^2}((a \otimes b)x_n)| = n^{-\frac{1}{2}} |\sum_i \mathrm{Tr}_n(ae_{1i})\mathrm{Tr}_n(be_{1i})| =$$

$$= n^{-\frac{1}{2}} |\sum_j (a\xi_1 \mid \xi_j)(b\xi_1 \mid \xi_j)| \leqslant$$

(1)
$$\leqslant n^{-\frac{1}{2}} [\sum_j |(a\xi_1 \mid \xi_j)|^2]^{\frac{1}{2}} [\sum_j |(b\xi_1 \mid \xi_j)|^2]^{\frac{1}{2}} = \qquad \text{(by Cauchy-Schwartz)}$$

$$= n^{-\frac{1}{2}} \|a\xi_1\| \, \|b\xi_1\|.$$

If $\mathrm{Tr}_n(|a|)$, $\mathrm{Tr}_n(|b|) \leqslant 1$, (1) implies that

$$|\mathrm{Tr}_{n^2}((a \otimes b)x_n)| \leqslant n^{-\frac{1}{2}} \mathrm{Tr}_n(aa^*)^{\frac{1}{2}} \mathrm{Tr}_n(bb^*)^{\frac{1}{2}} \leqslant$$

$$\leqslant n^{-\frac{1}{2}} \mathrm{Tr}_n(|a|)\mathrm{Tr}_n(|b|),$$

and so $\lambda(x_n) \leqslant n^{-\frac{1}{2}}$.

On the other hand, if $a, b \in L(\mathscr{H}_n)$, by (1)

$$|f_n(a \otimes b)| = |\mathrm{Tr}_{n^2}((a^* \otimes b^*)x_n)| \leqslant$$

$$\leqslant n^{-\frac{1}{2}} \|a^*\xi_1\| \, \|b^*\xi_1\| \leqslant n^{-\frac{1}{2}} \|a\| \, \|b\|.$$

(b) Let $\mathscr{H} = \ell^2(\mathrm{N})$. $\mathscr{H}$ has a decomposition $\mathscr{H} = \bigoplus_{r=1}^{\infty} \mathscr{K}_r$, where $\mathscr{K}_r = \mathscr{H}_{r^4}$. If e_r is the projection onto $\mathscr{K}_r$, $z_r = x_{r^4}(e_r \otimes e_r) \in L(\mathscr{H}) \odot L(\mathscr{H})$ has unit norm,

$$z_r(\mathscr{K}_r \otimes \mathscr{K}_r) \subseteq \mathscr{K}_r \otimes \mathscr{K}_r,\ z_r \mid (\mathscr{K}_r \otimes \mathscr{K}_r)^{\perp} = 0,$$

and

$$\lambda(z_r) = \lambda(x_{r^4}) \leqslant \frac{1}{r^2}\cdot$$

Also, $\mathrm{Tr}(x_r x_s^*) = \delta_{rs}$, and if $a, b \in L(\mathscr{H})$,

$$\mathrm{Tr}(z_r^*(a \otimes b)) = f_{r^4}(e_r a e_r \otimes e_r b e_r) \leqslant$$

$$\leqslant \frac{1}{r^2}\, \|a\| \cdot \|b\| \to 0 \quad \text{as } r \to \infty.$$

Let $c = \sum_{r=1}^{\infty} z_r$, which exists as a weak limit. Then $\mathrm{Tr}(cz_r^*) = 1$ and $\|c\| = 1$. As the linear functional $y \mapsto \mathrm{Tr}(z_r^* y) = f_{r^4}((e_r \otimes e_r)y(e_r \otimes e_r))$ on $L(\mathscr{H} \otimes \mathscr{H})$ has norm 1 for each r, it follows that $\mathrm{dist}(c, L(\mathscr{H}) \odot L(\mathscr{H})) = 1$, so that $c \notin L(\mathscr{H}) \otimes L(\mathscr{H})$.

(c) We show, finally, that r_c is continuous.
A norm β on $L(\mathscr{H} \otimes \mathscr{H})$ is defined by

$$\beta(x) = \sup\{\|r_x(\varphi)\| : \varphi \in L(\mathscr{H})_1^*\} =$$

$$= \sup\{|\langle r_x(\varphi), \psi\rangle| : \varphi \in L(\mathscr{H})_1^*,\ \psi \in L(\mathscr{H})_{*1}\} =$$

$$= \sup\{|\langle \varphi, L_\psi(x)\rangle| : \varphi \in L(\mathscr{H})_1^*,\ \psi \in L(\mathscr{H})_{*1}\} =$$

$$= \sup\{|\langle \varphi \otimes \psi, x\rangle| : \varphi, \psi \in L(\mathscr{H})_{*1}\}\,.$$

Identifying x with r_x, β is just the norm of uniform convergence of $L(\mathscr{H})$-valued functions on the compact space $L(\mathscr{H})_1^*$; also, $\beta \mid L(\mathscr{H}) \odot L(\mathscr{H}) = \lambda$.

Let

$$c_n = \sum_1^n z_r \quad (n = 1, 2, \ldots).$$

If $\varphi, \psi \in L(\mathscr{H})_{*1}$,

$$|\langle \varphi \otimes \psi, c - c_n\rangle| = \left|\sum_{n+1}^{\infty} \langle \varphi \otimes \psi, z_r\rangle\right| \leqslant$$

$$\leqslant \sum_{n+1}^{\infty} \beta(z_r) \leqslant \sum_{n+1}^{\infty} r^{-2}.$$

Thus $\beta(c - c_n) \leqslant \sum_{n+1}^{\infty} r^{-2} \to 0$ as $n \to \infty$.

Each r_{c_n} is continuous, as $c_n \in L(\mathscr{H}) \odot L(\mathscr{H})$. Hence r_c, being the uniform limit of continuous functions, is continuous.

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