J. OPERATOR THEORY
9(1983), 297 - 317

© Copyright by INCREST, 1983

PERTURBATION THEORY FOR DEFINITIZABLE OPERATORS IN KREĬN SPACES

H. LANGER and B. NAJMAN

Definitizable selfadjoint operators in Kreĭn spaces have a spectral function with, possibly, certain critical points. Besides the real spectrum they can also have a finite number of nonreal eigenvalues, see [1]*), [3], [7]. It is the aim of this note to generalize some classical results on continuous perturbations of selfadjoint operators in Hilbert space, in particular the Theorem of F. Rellich about the strong convergence of the spectral functions and the Theorem of H. F. Trotter — T. Kato about the strong convergence of the corresponding unitary groups (see [5], [12], [17]) to the case of definitizable selfadjoint operators in Kreĭn spaces.

§ 2 contains some estimates for the resolvents of definitizable operators which are the main tools for the proofs in § 3.

The generalizations of F. Rellich's Theorem are given in § 3. We first consider a sequence of definitizable selfadjoint operators A_n in a Kreĭn space with definitizing polynomials of uniformly bounded degrees, which converges strongly (in resolvent sense) to a selfadjoint operator A. The Kreĭn space structure, that is the indefinite scalar product, is also allowed to depend on n. In Theorem 3.1 we give conditions which ensure that for certain complex sets Δ the spectral projections $E_n(\Delta)$ of A_n converge strongly to the spectral projection $E(\Delta)$ of A. Subsequently (Theorem 3.4) these results are specialized for the case of Pontrjagin spaces of (fixed) index $\varkappa$ (each selfadjoint operator in such a space has a definitizing polynomial of degree $\leqslant 2\varkappa$). In some situations the spectral projections, which correspond to nonreal eigenvalues, converge even in norm, while the operators themselves are only supposed to converge strongly (see Remark 4 after Theorem 3.4).

In § 4 we prove the above mentioned generalization of the Trotter — Kato theorem under the additional stipulation that the definitizing polynomials are of even degree.

An application to a second order differential equation in Hilbert space is given in § 5. However, the results of §§ 3,4 (in particular Theorem 3.5) allow the

*) In [1] definitizable operators are called positizable.

treatment of cases not covered by Theorem 5.1. We intend to consider these applications in a subsequent paper.

In § 6 we deal shortly with the case where the spaces (and not only the indefinite forms) may change. Then the main results of §§ 3,4 still hold true with obvious modifications.

Partial results of §§ 3,4 have been obtained earlier in [6], Satz 1.2, [8] and in particular, in [9], where corresponding results for Pontrjagin spaces have been obtained under stronger assumptions. We mention also the papers [2], [4], [13], [14], [15], which contain related results. As a rule, to each statement in § 3 for definitizable sefadjoint operators there corresponds a statement for definitizable unitary operators. Their formulations will be left to the reader.

1. PRELIMINARIES

Let $\mathscr{H}$ be a Hilbert space with the scalar product (,), J a regular (i.e. a bounded and boundedly invertible) selfadjoint operator in $\mathscr{H}$. If J is not definite, the space $\mathscr{H}$ equipped with the Hermitian form

$$[u, v] := (Ju, v) \qquad (u, v \in \mathscr{H}) \tag{1.1}$$

is a *Kreĭn space* ([1]) which will be denoted by $\mathscr{K} = (\mathscr{H}, J)$.

For a selfadjoint operator[*)] B in $\mathscr{H}$ with spectral function E_B we write $\operatorname{ind} B := \dim E_B((-\infty, 0)) \ (\leqslant \infty)$. If $\operatorname{ind} J = \varkappa < \infty$ the space $\mathscr{K} = (\mathscr{H}, J)$ is a $\pi_\varkappa$-space or *Pontrjagin space of index* $\varkappa$; in this case we shall also write $\operatorname{ind} \mathscr{K} = \varkappa$.

Operators which are selfadjoint or unitary with respect to the form (1.1) are called *J-selfadjoint* or *J-unitary*, respectively, or sometimes, as in [1], simply *selfadjoint* or *unitary* operators in the Kreĭn space $\mathscr{K}$.

The J-selfadjoint operator A is called *definitizable* if $\rho(A) \neq \emptyset$ and if there exists a polynomial p (the *definitizing polynomial*) such that

$$[p(A)u, u] \geqslant 0 \qquad (u \in \mathscr{D}(A^k))$$

where $k = \deg p$. Without loss of generality we shall always assume that p is real and normalized, i.e. $\|p\| = 1$; where for $p(z) = \sum_{j=0}^{n} p_j z^j$ we put

$$\|p\| := \sum_{j=0}^{n} |p_j|. \tag{1.2}$$

If $\operatorname{ind} J = \varkappa < \infty$ and A is J-selfadjoint, then A is definitizable with a definitizing polynomial of degree $\leqslant 2\varkappa$, see [1], Theorem IX.7.3. If A is J-selfadjoint and

[*)] Throughout the paper operators are always linear but not necessarily bounded.

$\operatorname{ind} J(A-\lambda I)=\varkappa<\infty$ for some $\lambda \in \mathbf{R}$, then A is definitizable with a definitizing polynomial of degree $\leqslant 2\varkappa+1$, see [7].

A definitizing polynomial p of A is called *minimal* if it is of minimal possible degree. Evidently, every definitizable operator has a minimal definitizing polynomial.

A definitizable J-selfadjoint operator has a unique spectral function, possibly with *critical points* on the real axis, see [1], [3], [7]. The spectral function of the definitizable operator $A\,(A_n)$ will always be denoted by $E(E_n$, resp.), the set of critical points of E, which are also called the *critical points of* A, by $c(A)$. We recall that $E(\Delta)$ is defined for all complex Borel sets Δ not having any critical point of E on the boundary, and $\lambda_0 \in \mathbf{R}$ is a critical point of A if for each interval Δ containing λ_0 and with endpoints not in $c(A)$ the subspace $E(\Delta)\mathscr{K}$ is indefinite, i.e. contains positive as well as negative elements.

We shall make use of the following results of [7]: If A is a definitizable operator with definitizing polynomial p, $z_0 \in \rho(A)$ and $r(z):=\dfrac{p(z)}{(z_0-z)^k(\bar{z}_0-z)^k}$ $(2k \geqslant$ $\geqslant \deg p)$, then for each pair $x, y \in \mathscr{K}$ there exists a real function $\sigma_{x,y}$ of bounded variation on the real axis such that

$$\int_{\mathbf{R}} |\mathrm{d}\sigma_{x,y}(t)| \leqslant [r(A)x, x]^{1/2}\,[r(A)y, y]^{1/2}, \tag{1.3}$$

$$[(zI-A)^{-1}x, y] = \frac{1}{r(z)} \int_{\mathbf{R}} \frac{\mathrm{d}\sigma_{x,y}(t)}{z-t} + \frac{1}{r(z)}[Q(z, A)x, y] \quad (z \in \rho(A),\ r(z) \neq 0), \tag{1.4}$$

where $Q(t, z):= \dfrac{r(t)-r(z)}{t-z}$, and

$$[E(\Delta)x, y] = \int \frac{\mathrm{d}\sigma_{x,y}(t)}{r(t)} \tag{1.5}$$

for each real interval Δ with the property $p(t) \neq 0$ if $t \in \ddot{\Delta} \cap \sigma(A)$.

We also recall that the nonreal spectrum $\sigma_0(A)$ of a definitizable J-selfadjoint operator A consists of a finite number of isolated points which are poles of the resolvent $R_z := (zI-A)^{-1}$. It is symmetric with respect to the real axis and coincides with the set of nonreal zeros of any minimal definitizing polynomial.

If Δ is a real interval and $a>0$, we put $\mathfrak{k}(\Delta;\,a):=\{z : \operatorname{Re} z \in \Delta,\ |\operatorname{Im} z|<a\}$, $\Gamma(\Delta;\,a)$ is the boundary of $\mathfrak{k}(\Delta;\,a)$ with positive orientation. If the nonreal part of $\Gamma(\Delta;\,a)$ is contained in the resolvent set of A and the endpoints of Δ are not critical points, we write $\tilde{E}(\Delta;a) = E(\mathfrak{k}(\Delta;a))$. In that case

$$\tilde{E}(\Delta;\,a) = E(\Delta) + \sum E(\{\lambda\})$$

where summation takes place over all $\lambda \in \mathfrak{k}(\Delta;a) \cap \sigma_0(A)$.

Finally, $\pi_0(A)$ denotes the set of all eigenvalues of A with a nonpositive eigenvector; evidently $\pi_0(A)$ contains $\sigma_0(A)$.

2. BASIC ESTIMATES

By A we denote again a definitizable J-selfadjoint operator with definitizing polynomial p, by E its spectral function. Further, let $z_0 \in \rho(A)$, $r(t) :=$ $:= \dfrac{p(t)}{(z_0 - t)^k (\bar{z}_0 - t)^k}$, where $2k \geqslant \deg p$, and $\gamma(J) := \|J\| \, \|J^{-1}\|$.

The following two lemmas are easy consequences of the relations (1.4) and (1.5).

LEMMA 2.1. *If Δ is a real interval, then*

$$\|E(\Delta)\| \leqslant \frac{1}{\min\limits_{t \in \bar{\Delta} \cap \sigma(A)} |r(t)|} \cdot \gamma(J) \, \|r(A)\| . \tag{2.1}$$

Proof. Note that (2.1) has sense only if $p(t) \neq 0$ on $\bar{\Delta} \cap \sigma(A)$. In that case we can use (1.5) and (1.3):

$$|(E(\Delta)x, Jy)| = \left| \int\limits_{\Delta \cap \sigma(A)} \frac{d\sigma_{x,y}(t)}{r(t)} \right| \leqslant (\min_{t \in \bar{\Delta} \cap \sigma(A)} |r(t)|)^{-1} [r(A)x, x]^{1/2} [r(A)y, y]^{1/2} \leqslant$$

$$\leqslant (\min_{t \in \bar{\Delta} \cap \sigma(A)} |r(t)|)^{-1} \|r(A)\| \, \|J\| \, \|x\| \, \|y\|,$$

and (2.1) follows.

In the following lemma we choose the function r as above with $k = \deg p$.

LEMMA 2.2. *Let $\mathfrak{k}$ be a closed set in the complex plane, $z_0, \bar{z}_0 \notin \mathfrak{k}$, k_0 a positive integer and $c > 0$. Then for any J-selfadjoint definitizable operator A with a definitizing polynomial p of degree $k \leqslant k_0$ and with the property $z_0 \in \rho(A)$, $\|(z_0 I - A)^{-1}\| \leqslant c$, we have*

$$\|(zI - A)^{-1}\| \leqslant |r(z)|^{-1} \gamma(J)^2 (\gamma_1 + \gamma_2 (\operatorname{dist}(z, \sigma(A) \cap \mathbf{R}))^{-1}) \quad (z \in \mathfrak{k} \cap \rho(A))$$

with constants γ_1, γ_2 independent of A, J, p and z (depending only on $\mathfrak{k}$, c and z_0).

Proof. First we observe that for arbitrary $l = 0, 1, \ldots, k$ it holds

$$\|R_{\bar{z}_0}^k\| = \|(R_{\bar{z}_0}^l)^*\| = \|(z_0 I - JAJ^{-1})^{-l}\| = \|JR_{z_0}^l J^{-1}\| \leqslant c^l \gamma(J),$$

$$\|A^l R_{z_0}^k\| = \|(-I + z_0 R_{z_0})^l R_{z_0}^{k-l}\| \leqslant (1 + |z_0| c)^{k_0} \max(1, c^{k_0}) =: d$$

and

$$\|A^l R_{\bar{z}_0}^k\| = \|JA^l R_{z_0}^k J^{-1}\| \leqslant \gamma(J) d.$$

Thus because of $\|p\| = 1$ we have

$$\|p(A)\, R^k_{z_0}\| \leqslant d, \quad \|p(A)\, R^k_{\bar{z}_0}\| \leqslant \gamma(J)d.$$

Moreover,

$$Q(A; z) = (A - zI)^{-1}(r(A) - r(z)) =$$

$$= -R_z(p(A) R^k_{z_0} R^k_{\bar{z}_0} - p(z)\,(z_0 - z)^{-k}\,(\bar{z}_0 - z)^{-k}) =$$

$$= -\,p(A)\, R_z R^k_{z_0}(R^k_{\bar{z}_0} - (z_0 - z)^{-k}) - R_z(p(A) - p(z)) R^k_{\bar{z}_0}(z_0 - z)^{-k} -$$

$$- R_z p(z)(z_0 - z)^{-k}(R^k_{\bar{z}_0} - (\bar{z}_0 - z)^{-k}) =$$

$$= p(A)\, R^k_{\bar{z}_0} \sum_{l=0}^{k-1} (z_0 - z)^{l-k}\, R^{l+1}_{z_0} + (z_0 - z)^{-k} \sum_{j=1}^{k} p_j \sum_{l=0}^{j-1} A^l z^{j-l-1} R^k_{z_0} +$$

$$+\, p(z)\,(z_0 - z)^{-k} \sum_{l=0}^{k-1} (\bar{z}_0 - z)^{l-k}\, R^{l+1}_{\bar{z}_0} =: \alpha_1 + \alpha_2 + \alpha_3.$$

Further, by the above estimates, we get

$$\|\alpha_1\| \leqslant \gamma(J) d \sum_{l=0}^{k-1} |z_0 - z|^{l-k} \|R_{z_0}\|^{l+1} \leqslant \gamma(J) d\, \delta_1,$$

$$\|\alpha_2\| \leqslant |z_0 - z|^{-k} \gamma(J) d \sum_{j=1}^{k} |p_j| \sum_{l=0}^{j-1} |z|^{j-l-1} = \gamma(J) d\, \delta_2,$$

$$\|\alpha_3\| \leqslant |p(z)|\, |z_0 - z|^{-k} \sum_{l=0}^{k-1} |\bar{z}_0 - z|^{l-k}\, c^{l+1} \gamma(J) \leqslant \gamma(J)\, \delta_3.$$

Here δ_1, δ_2, δ_3 depend only on c, k_0, z_0 and z and remain bounded if z runs through $\mathfrak{k}$. Thus it follows

$$\|Q(A; z)\| \leqslant \gamma(J)\gamma_1 \quad \text{for all } z \in \mathfrak{k} \cap \rho(A)$$

and for all operators A, J with the properties formulated in the lemma; γ_1 depends only on c, k_0, z_0 and $\mathfrak{k}$. Moreover,

$$\|r(A)\| = \|p(A) R^k_{z_0} R^k_{\bar{z}_0}\| \leqslant \gamma(J) c^k d,$$

and we get finally from (1.4) and (1.3) for $z \in \mathfrak{k} \cap \rho(A)$

$$\|R_z\| \leqslant |r(z)|^{-1}[\gamma(J)\, \|r(A)\|\, (\operatorname{dist}(z; \sigma(A) \cap \mathbf{R}))^{-1} + \gamma(J)\, \|Q(A; z)\|] \leqslant$$

$$\leqslant \gamma(J)^2\, |r(z)|^{-1}\, (\gamma_1 + \gamma_2 \operatorname{dist}(z, \sigma(A) \cap \mathbf{R})^{-1}).$$

The lemma is proved.

3. EXTENSION OF RELLICH'S THEOREM

3.1. We start with some simple observations about a sequence (p_n) of polynomials, $\|p_n\| = 1$, $\deg p_n \leqslant k$. Convergence of a sequence of polynomials to a polynomial means uniform convergence on compact subsets of the complex plane, which is equivalent to the convergence of the sequences of coefficients.

(i) *There exists a subsequence* (p_{n_j}) *of* (p_n), *converging to a polynomial* p *of degree* $\leqslant k$, $\|p\| = 1$.

(ii) *If* (p_n) *converges to* p *and* z_0 *is a zero of* p *of multiplicity* k_0 *then for sufficiently large* n *each* p_n *has zeros of total multiplicity* k_0 *near* z_0.

(iii) *If* $\deg p < \underline{\lim}_n \deg p_n$, *and* (p_n) *converges to* p, *then for sufficiently large* n *each* p_n *has zeros of total multiplicity* $\deg p_n - \deg p$ *near* ∞.

For a given sequence (p_n) of polynomials by $\pi[(p_n)]$ we denote the set of all limit points of the zeros of p_n: $\lambda_0 \in \pi[(p_n)]$ if for each neighbourhood $\mathfrak{u}_0$ of λ_0 there are infinitely many polynomials p_n having a zero in $\mathfrak{u}_0$.

In the sequel (A_n) is a sequence of definitizable J_n-selfadjoint operators in $\mathscr{H}$, A and J are operators in $\mathscr{H}$, such that the following conditions are satisfied:

$$\text{(j)} \qquad J_n \overset{w}{\to} J \quad (n \to \infty), \qquad \sup_n \|J_n^{-1}\| < \infty.$$

(a) There exists a $z_0 \in \rho(A) \cap \rho(A_n)$ for all n such that

$$\mathrm{R}_{z_0}^{(n)} := (z_0 I - A_n)^{-1} \to (z_0 I - A)^{-1} =: \mathrm{R}_{z_0} \quad (n \to \infty).$$

(p) There exists an integer k such that each A_n has a definitizing polynomial p_n such that $\deg p_n \leqslant k$.

Condition (a) implies $\|\mathrm{R}_{z_0}^{(n)}\| \leqslant C$ for some $C > 0$ and all $n = 1, 2, \ldots$. Moreover, using (j), (p) and Lemma 2.2 we conclude:

(iv) *For each compact set* $\mathfrak{k}$ *with* $\mathfrak{k} \cap \pi[(p_n)] = \emptyset$ *and* $z_0, \bar{z}_0 \notin \mathfrak{k}$ *there exist numbers* $C > 0$ *and* $n_0 > 0$ *such that*

$$\text{(3.1)} \qquad \|\mathrm{R}_z^{(n)}\| \leqslant \frac{C}{|\mathrm{Im}\, z|} \quad \textit{for all } n > n_0,\ z \in \mathfrak{k},\ z \neq \bar{z}.$$

(v) *For each compact set* $\mathfrak{k}$ *with* $\mathfrak{k} \cap \pi[(p_n)] = \emptyset$, $z_0, \bar{z}_0 \notin \mathfrak{k}$ *and* $\mathrm{dist}(\mathfrak{k}, \sigma(A_n) \cap \mathbf{R}) \geqslant \gamma$ *for some* $\gamma > 0$ *and all sufficiently large* n *there exist numbers* $C > 0$ *and* $n_0 > 0$ *such that* $\|\mathrm{R}_z^{(n)}\| < C$ *for all* $z \in \mathfrak{k}$, $n > n_0$.

In a similar way we can deduce from Lemma 2.1:

(vi) *If* Δ *is a bounded real interval,* $\bar{\Delta} \cap \pi[(p_n)] = \emptyset$, *then there exist numbers* $C > 0$, $n_0 > 0$ *such that*

$$\|E_n(\Delta)\| \leqslant C \qquad \textit{for all } n > n_0.$$

Here and in the sequel E_n denotes the spectral function of A_n.

REMARK 1. If the operators A_n and A are bounded, the condition (a) is evidently equivalent to $A_n \xrightarrow{s} A$ $(n \to \infty)$. If the operators are unbounded, the following two conditions (ã) and (b) are sufficient for (a):

(ã) There is a core $\mathscr{D}$ of A (see [5]) such that $u \in \mathscr{D}$ implies $u \in \mathscr{D}(A_n)$ for n large enough and $A_n u \to Au$ $(n \to \infty)$;

(b) There exists $z_0 \in \rho(A_n) \cap \rho(A)$, $n = 1,2, \ldots$ such that $\sup_n \|R_{z_0}^{(n)}\| < \infty$.

REMARK 2. If the conditions (j) and

$$(\varkappa) \qquad \varkappa := \operatorname{ind} J = \operatorname{ind} J_n < \infty, \quad n = 1,2, \ldots$$

hold, then (ã) implies (a).

This statement is contained in [9], Theorem 3.1.b).

3.2. THEOREM 3.1. *Assume that the sequence (A_n) of definitizable J_n-selfadjoint operators and the operator A satisfy* (j), (a) *and* (p). *Then the following conclusions hold:*

1) *The operator A is J-selfadjoint and definitizable.*

2) *If $\lambda_0 \in \sigma_0(A)$ and u_0 is an open neighbourhood of λ_0 such that $u_0 \cap \sigma(A) = \{\lambda_0\}$ then*

$$E_n(u_0) \xrightarrow{s} E(\{\lambda_0\}) \quad (n \to \infty). \tag{3.2}$$

3) *If Δ is a bounded real interval, $a > 0$ and*

$$(\bar{\Delta} \setminus \Delta) \cap \sigma_p(A) = \emptyset, \quad \Gamma(\Delta; a) \cap \pi[(p_n)] = \emptyset, \tag{3.3}$$

then

$$\tilde{E}_n(\Delta; a) \xrightarrow{s} \tilde{E}(\Delta; a) \quad (n \to \infty). \tag{3.4}$$

Proof. We can additionally suppose that the sequence (p_n) converges to a polynomial p. Indeed, by (i) above, for 1) this is evident, for 2) (and similarly for 3)) this follows from the fact that if each subsequence of $(E_n(u_0))$ contains a subsequence which converges to $E(\{\lambda_0\})$ then (3.2) holds.

1) It follows from (a) that $(R_{z_0}^{(n)})^k$ strongly converges for every k, so any polynomial in $R_{z_0}^{(n)}$ also strongly converges. This implies that the rational functions $p_n(A_n)(R_{z_0}^{(n)})^k$ strongly converge to $p(A)R_{z_0}^k$.

Denote $[x,y]_n := (J_n x, y)$. From (j) we have

$$[p(A)\, R_{z_0}^k x,\ R_{z_0}^k x] = \lim_{n \to \infty} [p_n(A_n)\,(R_{z_0}^{(n)})^k x, (R_{z_0}^{(n)})^k x]_n \geqslant 0 \quad (x \in \mathscr{H}).$$

Thus A is definitizable and p a definitizing polynomial.

2) Without loss of generality we assume $u_0 := \{z : |z - \lambda_0| < r\}$ to be such that it does not intersect the real axis. Let Γ_0 be the boundary of u_0, n_0 an integer such that $n \geqslant n_0$ implies $\Gamma_0 \subset \rho(A_n)$. From the statement (v) it follows that $\|R_z^{(n)}\|$ is

uniformly bounded on Γ_0. By [5], Theorem VIII.1.3 we conclude that $R_z^{(n)}$ strongly converges to R_z, uniformly on Γ_0. Therefore,

$$E_n(\{\lambda_0\})x = \frac{1}{2\pi i}\int_{\Gamma_0} R_z^{(n)} x \, dz \to \frac{1}{2\pi i}\int_{\Gamma_0} R_z \, x \, dz = E(\{\lambda_0\})\, x$$

as $n \to \infty$ for every $x \in \mathscr{H}$.

3) Let $\Delta = (\alpha, \beta)$. By assumption, p has no zeros on $\Gamma(\Delta; a)$, in particular α and β are not zeros of p.

Assume for the moment that p does not vanish on $\bar{\Delta}$. Choose $a > 0$ so that $\mathfrak{k}(\Delta; a)$ contains neither z_0 (from (a)) nor zeros of p. By (ii), we can assume that p_n, $n = 1, 2, \ldots$, has no zeros in $\mathfrak{k}(\Delta; a)$.

Statement (iv) implies that the functions $z \mapsto R_{z_0}^{(n)}(z-\alpha)(z-\beta)$ are bounded on $\Gamma(\Delta; a) \setminus \{\alpha, \beta\}$, uniformly in n. From [5], Theorem VIII.1.3 it follows that for each $z \in \Gamma(\Delta; a) \setminus \{\alpha, \beta\}$ we have $R_z^{(n)} \overset{s}{\to} R_z$ as $n \to \infty$. Therefore by the dominated convergence theorem we get for every $x \in \mathscr{H}$

$$\lim_{n\to\infty} E_n(\Delta)(A_n - \alpha I)(A_n - \beta I)(R_{z_0}^{(n)})^2 x =$$

$$(3.5) \qquad = \lim_{n\to\infty} (2\pi i)^{-1} \int_{\Gamma(\Delta; a)} \frac{(z-\alpha)(z-\beta)}{(z_0 - z)^2} R_z^{(n)} x \, dz =$$

$$= (2\pi i)^{-1} \int_{\Gamma(\Delta; a)} \frac{(z-\alpha)(z-\beta)}{(z_0 - z)^2} R_z x \, dz = E(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x.$$

Further,

$$\|E_n(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x - E(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x\| \leqslant$$

$$\leqslant \|E_n(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x - E_n(\Delta)(A_n - \alpha I)(A_n - \beta I)(R_{z_0}^{(n)})^2 x\| +$$

$$+ \|E_n(\Delta)(A_n - \alpha I)(A_n - \beta I)(R_{z_0}^{(n)})^2 x - E(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x\| \leqslant$$

$$\leqslant \|(A - \alpha I)(A - \beta I) R_{z_0}^2 x - (A_n - \alpha I)(A_n - \beta I)(R_{z_0}^{(n)})^2 x\| \sup_n \|E_n(\Delta)\| +$$

$$+ \|E_n(\Delta)(A_n - \alpha I)(A_n - \beta I)(R_{z_0}^{(n)})^2 x - E(\Delta)(A - \alpha I)(A - \beta I) R_{z_0}^2 x\|.$$

By (vi), (a) and (3.5) both terms on the right hand side converge to zero as $n \to \infty$. Using (vi) once again as well as the density of the range of $(A - \alpha I)(A - \beta I)$ we conclude $E_n(\Delta) \overset{s}{\to} E(\Delta)$, thus 3) is proved in this particular case.

Now let p have zeros in Δ. We can assume that t_0 is the only zero of p in Δ and that $0 \notin \Delta$. Let α', β' be such that $\alpha', \beta' \notin \sigma_p(A)$, $\alpha < \alpha' < t_0 < \beta' < \beta$ and set $\Delta_1 = (\alpha, \alpha')$, $\Delta_2 = (\beta', \beta)$, $\Delta_0 = (\alpha', \beta')$. Then

$$E(\Delta) = E(\Delta_1) + E(\Delta_2) + E(\Delta_0).$$

Choose $a > 0$ such that $\mathfrak{k}(\Delta_i; a)$ contains no zeros of p, $i = 1,2$, and that t_0 is the only zero of p in $\mathfrak{k}(\Delta_0; a)$. Then for $n \geqslant n_0$, $\mathfrak{k}(\Delta_i; a)$ contains no zeros of p_n, $i = 1,2$. For such n define

$$\hat{A}_n := A_n(I - E_n(\Delta_1 \cup \Delta_2)), \quad \hat{A} := A(I - E(\Delta_1 \cup \Delta_2)).$$

Then

$$[\hat{A}^2 p(\hat{A})x, x] = [Ap(A)(I - E(\Delta_1 \cup \Delta_2))x, \ A(I - E(\Delta_1 \cup \Delta_2))x] \geqslant 0$$

for every $x \in \mathscr{D}(A^{k+2})$; the corresponding statement for $\hat{A}_n$ also holds. Therefore the operators $\hat{A}_n$, $\hat{A}$ are definitizable with definitizing polynomials $z^2 p_n(z)$ and $z^2 p(z)$ respectively. Moreover, as $E_n(\Delta_i) \xrightarrow{s} E(\Delta_i)$, $i = 1,2$, by the first part of 3) in this proof we find $(z_0 I - \hat{A}_n)^{-1} \xrightarrow{s} (z_0 I - \hat{A})^{-1}$ as $n \to \infty$. The corresponding spectral projections obviously satisfy

$$\tilde{\hat{E}}_n(\Delta_0; a) = \tilde{E}_n(\Delta_0; a), \quad \tilde{\hat{E}}(\Delta_0; a) = \tilde{E}(\Delta_0; a),$$

so all we have to prove is the relation

$$\tilde{\hat{E}}_n(\Delta_0; a) \xrightarrow{s} \tilde{\hat{E}}(\Delta_0; a) \quad (n \to \infty). \tag{3.6}$$

Let Δ_0' be such that $\bar{\Delta}_0 \subset \Delta_0' \subset \Delta$. For sufficiently large n we have $\Gamma(\Delta_0'; a) \subset \subset \rho(\hat{A}_n)$, and (3.6) follows as above using the Riesz contour integral:

$$\tilde{\hat{E}}_n(\Delta_0; a) = \frac{1}{2\pi i} \int_{\Gamma(\Delta_0'; a)} (zI - \hat{A}_n)^{-1} \, dz \xrightarrow{s} \frac{1}{2\pi i} \int_{\Gamma(\Delta_0'; a)} (zI - \hat{A})^{-1} dz = \tilde{\hat{E}}(\Delta_0; a)$$

as $n \to \infty$.

The theorem is proved.

We extract a special case of 3) in Theorem 3.1 and note a semicontinuity property of the spectrum.

COROLLARY 3.2. *Let (A_n), (A) be as in Theorem 3.1.*

1) *If Δ is a real interval such that $\bar{\Delta}$ contains no points of $\pi[(p_n)]$ and that the endpoints of Δ are not eigenvalues of A, then $E_n(\Delta) \xrightarrow{s} E(\Delta)$ $(n \to \infty)$.*

2) *Let $\lambda \in \sigma(A)$ and let $\mathfrak{u}$ be a neighbourhood of λ. There exists n_0 such that $\mathfrak{u} \cap \sigma(A_n) \neq \emptyset$ if $n \geqslant n_0$.*

Proof. Only 2) has to be proved. The case of nonreal λ is contained in part 2) of Theorem 3.1. If λ is real, we can again assume (without loss of generality) that $p_n \to p$ $(n \to \infty)$, $\mathfrak{u} = \mathfrak{t}(\Delta; a)$ for some Δ and a, that the endpoints of Δ are not eigenvalues of A and that p has no zeros on $\Gamma(\Delta; a)$. It remains to apply part 3) of Theorem 3.1.

REMARK. The first condition in (3.3) is familiar from the classical Rellich theorem. In order to see that the second condition in (3.3) is essential, choose $\mathscr{H} := \ell^2$, $J_1 := -(\cdot, e_1)\, e_1 + \sum_{j=2}^{\infty} (\cdot, e_j)\, e_j$ where (e_j) is the canonical orthogonal system in ℓ^2, A_1 a bounded J_1-selfadjoint operator in ℓ^2 with a singular critical point at some given real λ and U the (right) shift operator. Then $A_n := U^n A_1 U^{*n}$ is J_n-selfadjoint, $J_n := I - U^n U^{*n} + U^n J_1 U^{*n}$, and we have $A_n \overset{s}{\to} 0$, $J_n \overset{s}{\to} I$ $(n \to \infty)$. Evidently, the conditions (j), (a), (p) are satisfied. However, $\tilde{E}_n(\Delta; a)$ is not even defined if λ is a boundary point of Δ.

We mention that a condition analogous to (j) was used in [10] in the Hilbert space case in order to treat variable scalar products.

It is easy to find examples where real or nonreal eigenvalues or parts of the continuous spectrum disappear in the limit. If, however $\operatorname{ind} J_n$ is finite and constant for all sufficiently large n, then, as J is supposed to be regular, it is preserved in the limit and the eigenvalues in $\pi_0(A)$ are "stable". The second condition in (3.3) can be considerably simplified in this case. This will be considered in the following section.

3.3. If $\lambda \in \pi_0(A)$ we denote by $\varkappa_A(\lambda)$ the dimension of a maximal J-nonpositive subspace of the algebraic eigenspace $\mathscr{L}_A(\lambda)$ of A corresponding to λ, and for a subset $\mathfrak{u}$ of the complex plane we put

$$\varkappa_A(\mathfrak{u}) := \sum \varkappa_A(\lambda)$$

where the summation runs over all $\lambda \in \pi_0(A) \cap \mathfrak{u}$ with $\operatorname{Im} \lambda \geqslant 0$.

The proof of the following simple lemma will be left to the reader.

LEMMA 3.3. *Assume (E_n) is a sequence of bounded J_n-selfadjoint operators such that $E_n \overset{s}{\to} E$, $J_n \overset{w}{\to} J$ $(n \to \infty)$. Then for sufficiently large n*

$$\operatorname{ind} JE \leqslant \operatorname{ind} J_n E_n.$$

THEOREM 3.4. *Let the sequence (A_n) of J_n-selfadjoint operators and the operator A satisfy the conditions* (j) *and* (a). *Moreover suppose*

$$\operatorname{ind} J = \operatorname{ind} J =: \varkappa < \infty \quad (n = 1,2, \ldots).$$

Then the following conclusions hold:

1) *$\pi_0(A_n)$ converges to $\pi_0(A)$ in the Hausdorff set metrics. Moreover, if $\lambda_0 \in \pi_0(A)$ and $\mathfrak{u}_0$ is a neighbourhood of λ_0 such that $\bar{\mathfrak{u}}_0 \cap \pi_0(A) = \{\lambda_0\}$ then there exists an n_0 such that*

$$\varkappa_{A_n}(\mathfrak{u}_0) = \varkappa_A(\lambda_0) \quad (n \geqslant n_0). \tag{3.7}$$

2) *If* $\lambda_0 \in \sigma_0(A)$ *and* $\mathfrak{u}_0$ *is an open neighbourhood of* λ_0 *such that* $\bar{\mathfrak{u}}_0 \cap \sigma(A) = \{\lambda_0\}$, *then*

$$E_n(\mathfrak{u}_0) \xrightarrow{s} E(\{\lambda_0\}) \quad (n \to \infty).$$

3) *If* Δ *is a real interval,* $a > 0$ *and* $\Gamma(\Delta; a) \cap \sigma_p(A) = \emptyset$, *then*

$$\tilde{E}_n(\Delta, a) \xrightarrow{s} \tilde{E}(\Delta, a) \quad (n \to \infty).$$

Proof. The condition ($\varkappa$) implies that (p) also holds [1]. The definitizing polynomials p_n can be chosen of minimal degree ($\leqslant 2n$) and we can suppose that they converge to a definitizing polynomial p of A (see the proof of Theorem 3.1).

Moreover, in order to find "enough" points on the real axis which do not belong to $\sigma_p(A)$, we suppose that the space $\mathscr{H}$ is separable. Otherwise it can be decomposed as $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ such that $\mathscr{H}_0$ is separable, reduces all the operators A_n, A, J_n, J and $J \mid \mathscr{H}_1$, $J_n \mid \mathscr{H}_1$ are uniformly positive. The space $\mathscr{H}_0$ can be chosen, e.g., as the closed linear span, generated by the vectors of $E_{J_n}((-\infty, 0))\mathscr{H}$, $E_J((-\infty, 0))\mathscr{H}$ and their images with respect to all the "powers" $B_{n_1}^{j_1} B_{n_2}^{j_2} \dots B_{n_k}^{j_k}$, where B_n is any of the operators A, J, A_n, J_n, $n = 1,2, \dots$, and $j_1, \dots, j_k$ are arbitrary nonnegative integers, $k = 1,2, \dots$. Then it is sufficient to consider the restrictions of all the operators to $\mathscr{H}_0$.

As the p_n are of minimal degree each zero λ of p_n belongs to $\pi_0(A_n)$ and $\varkappa_{A_n}(\lambda) > 0$. Consider an arbitrary neighbourhood $\mathfrak{u}_0$ of $\pi_0(A)$, which is a union of mutually disjoint rectangles $\mathfrak{k}_\mu := \mathfrak{k}(\Delta_\mu; a_\mu)$ with boundary points not in $\sigma_p(A)$, each containing one real point λ_μ and no other points of $\pi_0(A)$, $\mu = 1,2, \dots, m'$, and nonreal discs $\mathfrak{k}_\mu$, each containing one point λ_μ of $\pi_0(A)$, $\operatorname{Im} \lambda_\mu \neq 0$, $\mu = m' + 1, \dots, m$. Assume that there is a subsequence (A_{n_j}) of (A_n), such that each $\pi_0(A_{n_j})$ contains some point $\lambda_{n_j}^{(0)}$, not belonging to $\mathfrak{u}_0$. Then from Theorem 3.1 it follows

$$\tilde{E}_{n_j}(\Delta_\mu; a_\mu) \xrightarrow{s} E(\Delta_\mu; a_\mu), \quad \mu = 1,2, \dots, m' \quad (j \to \infty),$$

$$E_{n_j}(\mathfrak{k}_\mu) \xrightarrow{s} E(\{\lambda_\mu\}), \quad \mu = m' + 1, \dots, m \quad (j \to \infty).$$

Hence, by Lemma 3.3 we have for sufficiently large j

$$\varkappa = \sum_{\mu=1}^{m} \varkappa_A(\mathfrak{k}_\mu) \leqslant \sum_{\mu=1}^{m} \varkappa_{A_{n_j}}(\mathfrak{k}_\mu) \leqslant \varkappa - \varkappa_{A_{n_j}}(\lambda_{n_j}^{(0)}),$$

which is impossible as $\varkappa_{A_{n_j}}(\lambda_{n_j}^{(0)}) > 0$. This yields $\pi_0(A_n) \subset \mathfrak{u}_0$. By Lemma 3.3 each $\mathfrak{k}_\mu$ contains at least one point of $\pi_0(A_n)$. Hence $\pi_0(A_n)$ converges to $\pi_0(A)$ in the Hausdorff set metrics. Further, $\sum_{\mu=1}^{m} \varkappa_{A_{n_j}}(\mathfrak{t}_\mu)=\varkappa$ implies $\varkappa_A(\mathfrak{t}_\mu)=\varkappa_{A_{n_j}}(\mathfrak{t}_\mu)$ for all μ and all sufficiently large j. Thus 1) is proved. The statement 2) follows immediately from Theorem 3.1, 2). The statement 3) for bounded intervals Δ follows from Theorem 3.1, 3) since $\pi[(p_n)]$ is contained in $\sigma_p(A)$ by the proof of 1). The case of an unbounded interval can easily be treated by means of the statement for bounded intervals and corresponding results for the Hilbert space case, or by using a fractional linear mapping which transforms Δ into a bounded interval.

REMARK 1. We do not claim that (p_n) is convergent.

REMARK 2. Examples show that the projections $\tilde{E}_n(\Delta\,;a)$, $\tilde{E}(\Delta\,;a)$ in 3) can not in general be replaced by $E_n(\Delta)$, $E(\Delta)$ respectively.

REMARK 3. Under the conditions of Theorem 3.4 let (p_n) be a minimal definitizing polynomial of A_n and p a limit point of (p_n). Then we have

$$\pi[(p_n)]=\pi_0(A)=\{t:\ p(t)=0\}.$$

REMARK 4. The strong convergence in 2) can be often strengthened to norm convergence. This holds, e.g., if the weak convergence in condition (j) is replaced by strong convergence, in particular, if J_n is independent of n.

We shall show this, e.g., in the case $\dim E_n(\mathfrak{u}_0)=1$, $n=1,2,\ldots$. Writing $E_n(\mathfrak{u}_0)=(\cdot\,,e_n)f_n$, $E(\{\lambda_0\})=(\cdot\,,e)f$, the strong convergence in 2) is equivalent to $f_n \xrightarrow{s} f$, $e_n \xrightarrow{w} e$ $(n\to\infty)$. On the other hand, with $\mathfrak{u}_1:=\{\bar{z}:z\in\mathfrak{u}_0\}$

$$E_n(\mathfrak{u}_1)=J_n^{-1}E_n(\mathfrak{u}_0)^*J_n=(\cdot\,,J_nf_n)J_n^{-1}e_n,\quad n=1,2,\ldots,$$

and 2), applied to $\bar{\lambda}_0$, gives $J_n^{-1}e_n \xrightarrow{s} J^{-1}e$ $(n\to\infty)$. Thus from $J_n \xrightarrow{s} J$ we get $e_n \xrightarrow{s} e$ $(n\to\infty)$, and the statement follows.

3.4. In some applications the form $[u,u]$ has an infinite number of negative squares but the number of negative squares of the form $[Au,u]$ $(u\in\mathcal{D}(A))$ is finite. By $\pi_1(a)$ we denote the set of all eigenvalues of the definitizable selfadjoint operator A for which there exists a JA-nonpositive eigenvector x_0 (i.e. $[Ax_0,x_0]\leqslant 0$). If $\lambda\in\pi_1(A)$ we denote by $\varkappa_A^{(1)}(\lambda)$ the dimension of a maximal JA-nonpositive subspace of $\mathcal{L}_A(\lambda)$ and for a subset $\mathfrak{u}$ in the complex plane

$$\varkappa_A^{(1)}(\mathfrak{u}):=\sum_{\substack{\lambda\in\pi_1(A)\cap\mathfrak{u}\\ \operatorname{Im}\lambda\geqslant 0}}\varkappa_A^{(1)}(\lambda).$$

THEOREM 3.5. *Assume that the sequence (A_n) of J_n-selfadjoint operators and the operator A satisfy* (j), ($\tilde{\text{a}}$) *and* (b). *Moreover suppose*

(3.8) $$\operatorname{ind} J_nA_n=\operatorname{ind} JA=:\varkappa\,(<\infty),\quad n=1,2,\ldots.$$

Then the following conclusions hold:

1) $\pi_1(A_n)$ *converges to* $\pi_1(A)$ *in the Hausdorff set metrics. Moreover, if* $\lambda_0 \in \pi_1(A)$ *and* $\mathfrak{u}_0$ *is a neighbourhood of* λ_0 *such that* $\bar{\mathfrak{u}}_0 \cap \pi_1(A) = \{\lambda_0\}$, *then there exists* n_0 *such that*

$$\varkappa^{(1)}_{A_n}(\mathfrak{u}_0) = \varkappa^{(1)}_A(\lambda_0) \quad (n \geqslant n_0). \tag{3.9}$$

2) *If* $\lambda_0 \in \sigma_0(A)$ *and* $\mathfrak{u}_0$ *is an open neighbourhood of* λ_0 *such that* $\mathfrak{u}_0 \cap \sigma(A) = \{\lambda_0\}$, *then* $E_n(\mathfrak{u}_0) \xrightarrow{s} E(\{\lambda_0\})$ $(n \to \infty)$.

3) *If* Δ *is a bounded real interval,* $a > 0$ *and* $\Gamma(\Delta; a) \cap (\sigma_p(A) \cup \{0\}) = \emptyset$, *then*

$$\tilde{E}_n(\Delta\,;\,a) \xrightarrow{s} \tilde{E}(\Delta\,;\,a) \quad (n \to \infty).$$

Proof. The operators A_n and A are definitizable with definitizing polynomials $p_n(t) := t\hat{p}_n(t)$, $p(t) := t\hat{p}(t)$, where $\hat{p}_n$, $\hat{p}$ are polynomials of degree $\leqslant 2\varkappa$; $\hat{p}_n$ can be chosen so that its zeros coincide with points of $\pi_1(A_n)$.

Now the proof of (3.9) is similar to the proof of (3.7). Let $\mathfrak{v}$ be a neighbourhood of $\pi_1(A)$; assume (after passing to a subsequence) that p_n converges to p. Let $\mathfrak{v}_1 \subset \mathfrak{v}$ be a neighbourhood of $\pi_1(A)$ which is the union of mutually disjoint sets $\mathfrak{k}_\mu$ as in the proof of Theorem 3.4 and with $\Gamma(\Delta_j; a_j)$ not intersecting $\sigma_p(A)$ or zero. Note that $E_n(\mathfrak{v}_1)$ is J_n-selfadjoint, $E(\mathfrak{v}_1)$ is J-selfadjoint. From Theorem 3.1, 3) we again find $E_n(\mathfrak{v}_1) \xrightarrow{s} E(\mathfrak{v}_1)$ $(n \to \infty)$. This implies for $u \in \mathscr{D}$

$$\|A_n E_n(\mathfrak{v}_1)u - AE(\mathfrak{v}_1)u\| \leqslant \|E_n(\mathfrak{v}_1)(A_n - A)u\| + \|[E_n(\mathfrak{v}_1) - E(\mathfrak{v}_1)]Au\| \to 0 \tag{3.10}$$
$(n \to \infty)$.

As $\mathscr{D}$ is dense in $\mathscr{H}$, the numbers of negative squares of $[AE(\mathfrak{v}_1)u, u]$ on $\mathscr{H}$ and $\mathscr{D}$ coincide, so from (3.10) we conclude $\operatorname{ind} J_n A_n E_n(\mathfrak{v}_1) \geqslant \operatorname{ind} JAE(\mathfrak{v}_1) = \varkappa$. By (3.8) we have equality, thus $\pi_1(A_n) \subset \mathfrak{v}_1$. From this one can finish the proof in a similar way as in the proof of Theorem 3.4.

REMARK. We mention that the definitizing polynomial $p_n(p)$ of A_n (A resp.) can be chosen such that for any limit point p of (p_n) we have

$$\pi[(p_n)] = \pi_1(A) \cup \{0\} = \{t : p(t) = 0\}.$$

4. EXTENSIONS OF THE TROTTER—KATO THEOREM

If A is a definitizable J-selfadjoint operator in a separable Hilbert space[*)] and ∞ is not a singular critical point of A, then A generates a strongly continuous group

*) In this section the separability condition is imposed in order to ensure the existence of real points not belonging to $\sigma_p(A)$. It can be replaced by the separability of all the ranges of the positive (or negative) parts of J_n, J.

of (definitizable) J-unitary operators e^{itA}, $t\in\mathbf{R}$. They can be defined e.g. in the following way. Choose a bounded real interval $\Delta=(\alpha,\beta)$ and $a>0$ such that $\alpha,\beta\notin\sigma_p(A)$ and $\mathfrak{t}(\Delta;\,a)\supset\pi_0(A)$. Then

$$(4.1)\quad (\mathrm{e}^{itA}x,y)=\frac{1}{2\pi\mathrm{i}}\int'_{\Gamma(\Delta;\,a)}\mathrm{e}^{itz}(\mathrm{R}_z\,x,y)\,\mathrm{d}z+\int_{\mathbf{R}\setminus\Delta}\mathrm{e}^{it\lambda}\,\mathrm{d}(E(\lambda)\,x,\,y)\quad (x,y\in\mathscr{H},\ t\in\mathbf{R}).$$

Here $\int'_{\Gamma(\Delta;\,a)}$ stands for the integral in the principal value sense at α,β. The relation (4.1) implies

$$(4.2)\qquad \mathrm{e}^{itA}=\mathrm{e}^{itA}\tilde{E}(\Delta;\,a)+\int_{\mathbf{R}\setminus\Delta}\mathrm{e}^{it\lambda}\,\mathrm{d}E(\lambda)\qquad (t\in\mathbf{R}).$$

We mention that ∞ is not a critical point of A if A has a definitizing polynomial of even degree.

THEOREM 4.1. *Let $\mathscr{H}$ be a separable Hilbert space and let the sequence (A_n) of definitizable J_n-selfadjoint operators and the operator A satisfy the conditions* (j), (a) *and* (p). *Additionally suppose that the definitizing polynomials p_n are of even degree and that their zeros are contained in a compact subset of the complex plane. Denote* $w:=\max\{|\mathrm{Im}\ z|:z\in\pi[(p_n)]\}$. *Then for every $\varepsilon>0$ there exist $C>0$ and $n_0>0$ such that*

$$(4.3)\qquad \|\mathrm{e}^{itA_n}\|\leqslant C\,\mathrm{e}^{|t|(w+\varepsilon)}\qquad (t\in\mathbf{R},\,n\geqslant n_0).$$

Moreover, we have

$$(4.4)\qquad \mathrm{e}^{itA_n}\xrightarrow{s}\mathrm{e}^{itA}\qquad (n\to\infty,\quad t\in\mathbf{R}).$$

Proof. We can again suppose that the sequence (p_n) is convergent. Then its limit is a polynomial p of even degree, hence e^{itA_n}, e^{itA} are well defined. If we only show that (4.3) holds, then (4.4) follows from [5], Theorem IX. 2.16.

Let p be a definitizing polynomial of A, deg $p=2k$. Put $r(s)=\dfrac{p(s)}{(z_0-s)^k(\bar{z}_0-s)^k}$, where z_0 is the complex number in (a), and let Δ be chosen as in (4.1). Then (1.5) implies

$$(4.5)\qquad \begin{aligned}\Big|\int_{\mathbf{R}\setminus\Delta}\mathrm{e}^{it\lambda}\mathrm{d}[E(\lambda)x,y]\Big|&=\Big|\int_{\mathbf{R}\setminus\Delta}\mathrm{e}^{it\lambda}r(\lambda)^{-1}\mathrm{d}\sigma_{x,y}(\lambda)\Big|\leqslant\\ &\leqslant(\inf_{\lambda\in\mathbf{R}\setminus\Delta}|r(\lambda)|)^{-1}\cdot\|r(A)\|\,\|J\|\,\|x\|\,\|y\|.\end{aligned}$$

Now choose $a > w$ and $\Delta = (\alpha, \beta)$ such that $\pi[(p_n)] \subset \mathfrak{k}(\Delta; a)$ and $\alpha, \beta \notin \sigma_p(A)$. Then $\mathfrak{k}(\Delta; a)$ contains all the zeros of p_n for sufficiently large n.

From (4.2) we conclude for these n

$$\mathrm{e}^{itA_n} = \mathrm{e}^{itA_n}\tilde{E}_n(\Delta; a) + \int_{\mathbf{R}\setminus\Delta} \mathrm{e}^{it\lambda}\, \mathrm{d}E_n(\lambda) \quad (t \in \mathbf{R}). \tag{4.6}$$

The definition of Δ implies $\rho_0 := \underline{\lim}_n \inf_{s\in\mathbf{R}\setminus\Delta} |r_n(s)| > 0$. From the proof of Lemma 2.2 it follows that there exists $C > 0$ such that $\|r_n(A_n)\| \leqslant C$ for all sufficiently large n. By (4.5) and (j) the second term on the right hand side of (4.6) is uniformly bounded for all $t \in \mathbf{R}$ and sufficiently large n). In order to estimate the first term, we choosea (bounded interval Δ_1, $\bar{\Delta} \subset \Delta_1$. Then

$$\mathrm{e}^{itA_n}\tilde{E}_n(\Delta; a) = (2\pi \mathrm{i})^{-1} \int_{\Gamma(\Delta_1; a)} \mathrm{e}^{itz} \mathrm{R}_z^{(n)} \tilde{E}_n(\Delta; a)\, \mathrm{d}z \tag{4.7}$$

as soon as n is so large that all zeros of p_n are in $\mathfrak{k}(\Delta; a)$. Note that $\mathrm{R}_z^{(n)}\tilde{E}_n(\Delta; a)$ is norm-continuous in z on $\Gamma(\Delta_1; a)$.

The operators $\hat{A}_n := A_n \tilde{E}_n(\Delta; a)$, $\hat{A} := A\tilde{E}(\Delta; a)$ evidently satisfy the conditions (j) and (p). Moreover, the relation

$$\mathrm{R}_z^{(n)}\tilde{E}_n(\Delta; a) = (zI - \hat{A}_n)^{-1} + z^{-1}(I - \tilde{E}_n(\Delta; a))$$

and Theorem 3.1, 3) imply that (a) also holds. Therefore we conclude from Theorem 3.1, 3) and the statement (v) of § 3.1 that $\mathrm{R}_z^{(n)}\tilde{E}_n(\Delta; a)$ are uniformly bounded:

$$\overline{\lim_n} \sup_{z\in\Gamma(\Delta_1; a)} \|\mathrm{R}_z^{(n)}\tilde{E}_n(\Delta; a)\| < \infty.$$

Hence (4.7) yields

$$\|\mathrm{e}^{itA_n}\tilde{E}_n(\Delta; a)\| \leqslant C\,\mathrm{e}^{a|t|} \quad (t \in \mathbf{R}) \tag{4.8}$$

for some $C > 0$ and all sufficiently large n. Thus (4.3) follows and the theorem is proved.

REMARK 1. It follows from Theorem 4.1 that e^{itA} is of type $\max\{\operatorname{Im} z : z \in \sigma(A)\}$ (for the definition of the type of a semigroup see [5], IX. 1.4).

REMARK 2. If $(\varkappa)$, (j) and $(\tilde{a})$ are satisfied, then the conclusions of Theorem 4.1 hold true. Indeed, in this case (a) holds by Remarks 1 and 2 of § 3.1, $(\varkappa)$ implies (p). In this case it is possible to choose p_n of even degree, $\deg p_n = \deg p$ and the zeros of p_n accumulate only in points of $\pi_0(A)$ (see Theorem 3.4.).

REMARK 3. If the strong convergence of the projections, corresponding to nonreal spectral points of A, can be replaced by norm convergence (see Remark 4 after Theorem 3.4), then the projections of e^{itA_n} to the corresponding subspaces converge in norm, even in Hilbert-Schmidt norm. This fact can be useful for numerical approximation methods.

5. AN APPLICATION TO ABSTRACT DIFFERENTIAL EQUATIONS

Let $\mathscr{G}$ be a Hilbert space and K, H be selfadjoint operators in $\mathscr{G}$. We consider the differential equation

$$\ddot{u}(t) - iK\dot{u}(t) + Hu(t) = 0 \quad \text{on } \mathbf{R}^{*)} \tag{5.1}$$

where u is a function on $\mathbf{R}$ with values in $\mathscr{G}$ and $\dot{u}$, $\ddot{u}$ denote the derivatives of u. We always assume that the operator H (as well as the operators H_n below) has a bounded inverse H^{-1} (H_n^{-1} respectively).

The set $\mathscr{D}(|H|^{1/2})$ equipped with the norm $\| |H|^{1/2} u \|$ $(u \in \mathscr{D}(|H|^{1/2}))$ is a Hilbert space denoted by $\mathscr{G}_1$. We put

$$\mathscr{H} := \mathscr{G}_1 \oplus \mathscr{G}.$$

The form

$$[u, v] := (\operatorname{sgn} H |H|^{1/2} u_1, |H|^{1/2} v_1) + (u_2, v_2) \quad (u = u_1 \oplus u_2,\ v = v_1 \oplus v_2)$$

is continuous on $\mathscr{H}$, symmetric and nondegenerate. Since H^{-1} is bounded, $\mathscr{K} := (\mathscr{H}, [\cdot, \cdot])$ is a Kreĭn space. It is a $\pi_\varkappa$-space if and only if $\operatorname{ind} H = \varkappa$.

With the differential equation (5.1) we associate the operator

$$A := \begin{bmatrix} 0 & I \\ H & K \end{bmatrix}, \quad \mathscr{D}(A) := \mathscr{D}(H) \oplus (\mathscr{D}(K) \cap \mathscr{D}(|H|^{1/2})). \tag{5.2}$$

If K is an $|H|^{1/2}$-bounded operator, A is a closed operator in $\mathscr{H}$ with $\mathscr{D}(A) = \mathscr{D}(H) \oplus \mathscr{D}(|H|^{1/2})$. Moreover A is selfadjoint in the Kreĭn space $\mathscr{K}$.

We mention that this definition of A corresponds to the usual writing of (5.1) as a first order system:

$$\dot{v}(t) = iAv(t), \quad v(t) = u(t) \oplus [-i\,\dot{u}(t)].$$

Now suppose that besides (5.1) we are given a sequence of "approximating" equations in $\mathscr{G}$ with operators K, H replaced by K_n, H_n. The spaces $\mathscr{H}_n$, $\mathscr{K}_n$ and th

---

*) The term $-iK\dot{u}(t)$ is called *gyroscopic*. In what follows the letters u and v denote functions as well as elements of some Hilbert space.

operator A_n are defined in the same way. We assume that the following conditions are satisfied:

(I) $\mathscr{D}(|H|^{1/2}) = \mathscr{D}(|H_n|^{1/2})$ for all n*) and there exist $m, M > 0$ such that $m\| \, |H|^{1/2} u\| \leqslant \| \, |H_n|^{1/2} u\| \leqslant M \| \, |H|^{1/2} u\| \quad (u \in \mathscr{D}(|H|^{1/2}),\ n = 1,2, \ldots)$;

(II) $K_n|H|^{-1/2}$, $K|H|^{-1/2}$ are bounded;

(III) $K_n|H|^{-1/2} \xrightarrow{s} K|H|^{-1/2}$ in $\mathscr{G}$ $(n \to \infty)$;

(IV) There is a core $\mathscr{D}$ of H such that $u \in \mathscr{D}$ implies $u \in \mathscr{D}(H_n)$ for n large enough and $H_n u \to Hu$ $(n \to \infty)$.

The assumption (I) implies that $\mathscr{H}_n$ coincides with $\mathscr{H}$ (with equivalent norms). Therefore we can use the scalar product $(\cdot, \cdot)$ of $\mathscr{H}$ in $\mathscr{H}_n$. Then the spaces $\mathscr{K}_n$ and $\mathscr{K}$ are Kreĭn spaces and we have

$$[u, v]_n = (J_n u, v), \quad [u, v] = (Ju, v) \quad (u, v \in \mathscr{H})$$

with $J_n = \begin{bmatrix} |H|^{-1}H_n & 0 \\ 0 & I \end{bmatrix}$, $\quad J := \begin{bmatrix} \operatorname{sgn} H & 0 \\ 0 & I \end{bmatrix}$.

As a consequence of (II), A_n is J_n-selfadjoint, A is J-selfadjoint.

THEOREM 5.1. *Let K_n, H_n $(n = 1,2, \ldots)$, K and H be selfadjoint operators in $\mathscr{G}$ such that* (I) — (IV) *are satisfied. Moreover assume*

$$\varkappa := \operatorname{ind} H = \operatorname{ind} H_n < \infty \quad \textit{for all } n. \tag{5.3}$$

Then A_n, A satisfy the conditions (j), ($\tilde{a}$) *and* ($\varkappa$) *from* § 3. *Therefore the conclusions of Theorems* 3.1, 3.4 *and* 4.1 *hold for these operators.*

Proof. Condition (I) implies $\|J_n^{-1}\| \leqslant 1 + \| \, |H|^{1/2} H_n^{-1} |H|^{1/2}\| \leqslant C < \infty$ for all n. Let $u_1 \in \mathscr{D}$, $u_2 \in \mathscr{G}$, $u = u_1 \oplus u_2$. From (IV) we have $J_n u \to Ju$ $(n \to \infty)$ in $\mathscr{H}$. Since $\mathscr{D}$ is dense in $\mathscr{G}_1$ and the sequence (J_n) is bounded by (I), we conclude $J_n \xrightarrow{s} J$, thus (j) holds. As $\mathscr{D}' = \mathscr{D} \oplus \mathscr{D}(|H|^{1/2})$ is a core of A, the conditions (III) and (IV) imply ($\tilde{a}$). Finally (5.3) is exactly the condition ($\varkappa$) of § 3. Thus Theorems 3.1, 3.4 and 4.1 are applicable (note Remark 2 in § 3.1 and Remark 2 after Theorem 4.1).

COROLLARY 5.2. *Let the conditions of Theorem* 5.1 *be satisfied and suppose $u_{n,0} \in \mathscr{D}(H_n)$, $u_0 \in \mathscr{D}(H)$, $u_{n,1}$, $u_1 \in \mathscr{G}_1$ and $u_{n,0} \to u_0$ in $\mathscr{G}_1$, $u_{n,1} \to u_1$ in $\mathscr{G}$ $(n \to \infty)$. Denote by u_n, u the unique solutions of the initial problems*

$$\ddot{u}_n(t) - \mathrm{i}K\dot{u}_n(t) + H_n u_n(t) = 0, \quad u_n(0) = u_{n,0}, \quad \dot{u}_n(0) = u_{n,1}$$

$$\ddot{u}(t) - \mathrm{i}K\dot{u}(t) + Hu(t) = 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1$$

---

*) In this and subsequent conditions "for all n" can evidently be replaced by "for all sufficiently large n".

such that

$$u_n \in C^2(R, \mathscr{G}) \cap C^1(R, \mathscr{G}_1), \quad H_n u \in C(R, \mathscr{G}),$$

$$u \in C^2(R, \mathscr{G}) \cap C^1(R, \mathscr{G}_1), \quad Hu \in C(R, \mathscr{G}).$$

Then for any $t \in \mathbf{R}$ *we have* $u_n(t) \to u(t)$, $\dot{u}_n(t) \to \dot{u}(t)$ *in* $\mathscr{G}$ $(n \to \infty)$. *Moreover,*

$$\|\dot{u}(t)\|^2 + (Hu(t), u(t)) = \|u_1\|^2 + (Hu_0, u_0) \quad (t \in \mathbf{R}),$$

$$\|\dot{u}_n(t)\|^2 + (H_n u_n(t), u_n(t)) = \|u_{n,1}\|^2 + (H_n u_{n,0}, u_{n,0}) \quad (t \in \mathbf{R}, n \in \mathbf{N}).$$

REMARK 1. Note that the Remark 3 after Theorem 4.1 sometimes gives additional information about the convergence of u_n.

REMARK 2. We could weaken the assumptions on $u_{n,0}, u_0, u_{n,1}, u_1$ allowing $u_{n,0}$, $u_0 \in \mathscr{G}_1$, $u_{n,1}$, $u_1 \in \mathscr{G}$ for all n, by considering the weak solutions instead of the strong ones. We could weaken the assumptions even more, leaving the space $\mathscr{H}$ and using a larger space. In that case we should use Theorem 3.5 instead of Theorem 3.4. These questions will be considered in a subsequent paper.

REMARK 3. In many approximation schemes for the equation (5.1), the approximating equations (with K_n, H_n) act in different spaces $\mathscr{G}_n$ and $\mathscr{G}_n$ "approximates" $\mathscr{G}$. In that case also $\mathscr{H}_n$ is different from $\mathscr{H}$ but "approximates" it (see § 6). Note that even in case $\mathscr{G}_n = \mathscr{G}$ this situation may arise if (I) does not hold.

In connection with the results of [16] we formulate the following consequence of Theorem 5.1 and Remark 4 after Theorem 3.4 for the special case $K_n = K = 0$. It implies in particular the stability of the negative eigenvalues and associated eigenvectors of H under the imposed assumptions.

COROLLARY 5.3. *Let* H_n, H *be selfadjoint operators in the Hilbert space* $\mathscr{G}$ *such that* (I), (IV) *and* (5.3) *hold. Then*

$$\|E_n((-\infty, t]) - E((-\infty, t])\| \to 0 \quad (n \to \infty,\ t < 0,\ t \notin \sigma_p(H))$$

where E_n, E *are the spectral functions of* H_n, H *respectively.*

In order to see this, we only have to note that the eigenvalues of A in the upper half plane are the square roots of the negative eigenvalues of H.

6. VARIABLE SPACES

6.1. Let $\mathscr{H}$, $\mathscr{H}_n$ $(n = 1,2, \ldots)$ be Banach spaces, P_n (not necessarily linear) mappings from $\mathscr{H}$ into $\mathscr{H}_n$. We say that $\mathscr{H}_n$ *approximates* $\mathscr{H}$, (see [11], I; the verb "approximate" is used here for "discretely converge" in [11]), if the relations

(6.1) $\|P_n u\|_n \to \|u\|$ and $\|\alpha P_n u + \beta P_n v - P_n(\alpha u + \beta v)\|_n \to 0 \quad (n \to \infty)$

hold for all $u, v \in \mathscr{H}$.

If there exists a dense linear manifold Φ in $\mathscr{H}$ and mappings P_n from Φ into $\mathscr{H}_n$ with the properties (6.1) then P_n can be extended by continuity to all of $\mathscr{H}$ in a unique way and $\mathscr{H}_n$ approximates $\mathscr{H}$. If the mappings P_n are linear and continuous, they are uniformly bounded [12].

We shall say that the sequence (u_n), $u_n \in \mathscr{H}_n$ $(n = 1,2, \ldots)$ strongly approximates $u \in \mathscr{H}: u_n \xrightarrow{s} u$, if $\|u_n - P_n u\|_n \to 0$ $(n \to \infty)$. The strong approximation of operators is defined analogously: If S_n, S are bounded linear operators in $\mathscr{H}_n$, $\mathscr{H}$ respectively, then S_n *approximates* S *strongly* $(S_n \xrightarrow{s} S)$ if $S_n P_n u \xrightarrow{s} S_u$ holds for each $u \in \mathscr{H}$ $(n \to \infty)$.

If $\mathscr{H}_n, \mathscr{H}$ are Hilbert spaces, Φ is a dense linear manifold in $\mathscr{H}$ and P_n are mappings from Φ into $\mathscr{H}_n$ $(n = 1,2, \ldots)$, then $\mathscr{H}_n$ approximates $\mathscr{H}$ if and only if

$$(P_n u, P_n v)_n \to (u, v) \quad (n \to \infty;\ u, v \in \Phi).$$

We say that the sequence (u_n), $u_n \in \mathscr{H}_n$ $(n = 1,2, \ldots)$ *weakly approximates* $u \in \mathscr{H}: u_n \xrightarrow{w} u$, if $(u_n, P_n v)_n \to (u, v)$ $(n \to \infty)$ for each $v \in \mathscr{H}$. It is easy to see that $u_n \xrightarrow{s} u$ if and only if $u_n \xrightarrow{w} u$ and $\| u_n \|_n \to \| u \|$ $(n \to \infty)$.

If S_n, S are as above, then the sequence (S_n) *weakly approximates* S: $S_n \xrightarrow{w} S$, if

$$(S_n P_n u, P_n v)_n \to (Su, v) \quad (n \to \infty; u, v \in \mathscr{H}).$$

Generalizations of the Rellich and Trotter—Kato theorems to the case of approximating sequences of operators are given in [10] and [12] (see also [5], [17]).

6.2. Let $\mathscr{H}$, $\mathscr{H}_n (n = 1,2, \ldots)$ be Hilbert spaces, P_n be mappings from $\mathscr{H}$ into $\mathscr{H}_n$ such that $\mathscr{H}_n$ approximates $\mathscr{H}$. If J_n, J are selfadjoint operators in $\mathscr{H}_n$ and $\mathscr{H}$, respectively, such that $\mathscr{K}_n = (\mathscr{H}_n, J_n)$, $\mathscr{K} = (\mathscr{H}, J)$ are Kreĭn spaces and $J_n \xrightarrow{w} J$, then we say that $\mathscr{K}_n$ *approximates* $\mathscr{K}$. If, additionally, $\sup_n \| J_n^{-1} \|_n < \infty$, the approximation is called *stable*.

In other words, a sequence $\mathscr{K}_n$ of Kreĭn spaces with indefinite scalar products $[\cdot, \cdot]_n$ approximates the Kreĭn space $\mathscr{K}$ with indefinite scalar product $[\cdot, \cdot]$ if

$$[P_n u, P_n v]_n \to [u, v] \quad (n \to \infty;\ u, v \in \mathscr{K})$$

and if there exist bounded selfadjoint regular positive *) operators K_n, K in $\mathscr{K}_n$, $\mathscr{K}$, respectively, such that

$$[K_n P_n u, P_n v]_n \to [Ku, v] \quad (n \to \infty; u, v \in \mathscr{K}).$$

Here it is again sufficient that this convergence holds for all u, v from a dense linear manifold in $\mathscr{K}$.

---

*) This means, e.g, for K_n, $[K_n u, u]_n \geqslant \gamma_n \|u\|_n^2$ $(u \in \mathscr{K}_n)$ with some $\gamma_n > 0$.

The approximation is stable if there exists a $\gamma > 0$, independent of u and n, such that

$$[K_n^{-1} u, u]_n \geqslant \gamma |[u, u]_n| \quad (u \in \mathscr{K}_n;\ n = 1,2, \ldots).$$

As $J_n \xrightarrow{w} J$ implies $\sup_n \|J_n\|_n < \infty$ (see [11, I], p. 60), the approximation is stable if $\sup_n \gamma(J_n) < \infty$.

Now suppose that the sequence $(\mathscr{K}_n)$ of Kreĭn spaces stably approximates the Kreĭn space $\mathscr{K}$. Moreover, assume that A_n, A are definitizable selfadjoint operators in $\mathscr{K}_n$, $\mathscr{K}$, respectively, $n = 1,2, \ldots$, with definitizing polynomials p_n, p such that the following conditions are satisfied:

(p) $\sup_n \deg p_n < \infty$;

(a) there exists a $z_0 \in \rho(A_n) \cap \rho(A)$, $n = 1,2, \ldots$, such that $(z_0 I - A_n)^{-1}$ strongly approximates $(z_0 I - A)^{-1}$:

$$(z_0 I - A_n)^{-1} \xrightarrow{s} (z_0 I - A)^{-1} \quad (n \to \infty).$$

Then the conclusions of Theorems 3.1, 3.4 and 3.5 remain valid, if the strong convergence is replaced by strong approximation. If, additionally, the P_n are linear and continuous, then Theorem 4.1 remains also valid. This follows by repeating the proofs of §§ 3, 4, using Stummel's method [11, II] in order to obtain the convergence of the contour integrals.

In the special case of Pontrjagin spaces the stable approximation of $\mathscr{K}$ by the $\mathscr{K}_n$ can be obtained from the following statement:

Assume $\mathscr{K}_n$, $\mathscr{K}$ are Pontrjagin spaces, Φ is a dense linear manifold in $\mathscr{K}$ and P_n are mappings from Φ into $\mathscr{K}_n$ such that

(i) $\operatorname{ind} \mathscr{K}_n = \operatorname{ind} \mathscr{K} < \infty \quad (n = 1,2, \ldots).$

(ii) $\lim_{n\to\infty} [P_n u, P_n v]_n = [u, v] \quad (u, v \in \Phi).$

Then $\mathscr{K}_n$ stably approximates $\mathscr{K}$.

If $\Phi = \mathscr{K}$ and if the P_n are linear continuous surjections, the proof can be found in [9]. These two stipulations can easily be dropped by choosing a fundamental decomposition $\mathscr{K} = \mathscr{K}^- [\dot{+}] \mathscr{K}^+$ of $\mathscr{K}$ such that $\mathscr{K}^- \subset \Phi$.

The second author has been supported in part by SIZ VI SRH.

REFERENCES

1. Bognár, J., *Indefinite inner product spaces*, Berlin, 1974.
2. Jonas, P., Compact perturbations of definitizable operators. II, *J. Operator Theory*, **8**(1982), 3–18.
3. Jonas, P., On the functional calculus and the spectral function for definitizable operators in Kreĭn spaces, *Beiträge Anal.*, to appear.

4. JONAS, P.; LANGER, H., Compact perturbations of definitizable operators, *J. Operator Theory*, **2**(1979), 311--325.
5. KATO, T., *Perturbation theory for linear operators*, Berlin, 1966.
6. KREĬN, M. G.; LANGER, H., Uber die Q-Function eines π-hermiteschen Operators im Raume $\Pi_{\varkappa}$, *Acta Sci. Math. (Szeged)*, **34**(1973), 191--230.
7. LANGER, H., Spectral functions of definitizable operators in Kreĭn spaces, *Functional Analysis*, Proceedings, Dubrovnik 1981, Lecture Notes in Math., Springer-Verlag, Berlin, **948**(1982), 1--46.
8. LANGER, H., Zur Spektraltheorie polynomialer Scharen selbstadjungierter Operatoren, *Math. Nachr.*, **65**(1975), 301--319.
9. NAJMAN, B., Perturbation theory for selfadjoint operators in Pontrjagin spaces, *Glasnik Mat.*, **15**(35) (1980), 351--370.
10. RANNACHER, R., Zur asymptotischen Störungstheorie für Eigenwertaufgaben mit diskreten Teilspektren, *Math Z.*, **141**(1975), 219—233.
11. STUMMEL, F., Diskrete Konvergenz linearer Operatoren. I, *Math. Ann.*, **190**(1970), 45—92; II., *Math. Z.*, **120**(1971), 231—264.
12. USHIJIMA, T., Approximation theory for semigroups of linear operators and its applications to approximation of wave equations, *Japan. J. Math.*, **1**(1975), 185—224.
13. VESELIĆ, K., A spectral theorem for a class of J-normal operators, *Glasnik Mat.*, **5**(25) (1970), 97--102.
14. VESELIĆ, K., On the perturbation theory for a class of J-selfadjoint operators, *Glasnik Mat.*, **5**(25) (1970), 103—108.
15. VESELIĆ, K., On spectral properties of a class of J-selfadjoint operators. I, *Glasnik Mat.*, **7**(27) (1972), 229--248.
16. WEIDMANN, J., Continuity of the eigenvalues of selfadjoint operators with respect to the strong operator topology, *Integral Equations and Operator Theory*, **3**(1980), 138—142.
17. TROTTER, H. F., Approximation of semi-groups of operators, *Pacific J. Math.*, **8**(1958), 887—919.

HEINZ LANGER
Sektion Mathematik,
Technische Universität Dresden,
8027 Dresden,
DDR.

BRANKO NAJMAN
Department of Mathematics,
University of Zagreb, PO Box 187.
41000 Zagreb,
Yugoslavia.

Received November 12, 1981; revised July 8, 1982.