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ON MULTIDIMENSIONAL SINGULAR INTEGRAL OPERATORS. I: THE HALF-SPACE CASE

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INTRODUCTION

Fredholm properties of a multidimensional singular integral operator (equation)

$$A\varphi(x) \equiv a_0 \varphi(x) + \int_{\mathbf{R}^{n+}} \frac{\Omega(x-y)}{|x-y|^n} \varphi(y)\,\mathrm{d}y = f(x), \tag{0.1}$$

$x \in \mathbf{R}^{n+} = \mathbf{R}^+ \times \mathbf{R}^{n-1}$, $\mathbf{R}^+ = [0, \infty)$, is investigated when $\varphi, f \in L_p^N(\mathbf{R}^{n+})$ $(1 < p < \infty)$; in particular, criterion for the existence of left and right regularizers (of left and right inverses) are obtained (cf. Theorems 2.7—2.8).

Those results will be used in the second part of the paper for the investigation of systems of multidimensional singular integral equations on a compact manifold with boundary in vector Sobolev-Slobodeckiĭ spaces.

Most interesting in § 1, which deals with auxiliary propositions, is Theorem 1.4; it shows that Calderon-Zygmund (cf. [4,5]) and other similar theorems on the boundedness of the operator (0.1) in $L_p(\mathbf{R}^{n+})$ space is valid also for the space with weight $L_p(\mathbf{R}^{n+}, x_1^\alpha)$; the weight x_1^α is not pointwise here (as in Stein's Theorem; cf. [22]).

Besides the interest of its own singular integral equations play an important role in mathematical physics, mechanics and boundary value problems for the partial differential equations (cf. [8, 13, 18—20, 22, 24, 26, 30, 32]). It is impossible to observe here all results, obtained earlier on this subject and we refer to books [8, 13, 18, 20—22, 32]. We mention only a few results closely related to our investigations.

Classical theory of multidimensional singular integral equations on manifolds without boundary were developed by Tricomi, Michlin, Calderon, Zygmund, and others (cf. [22]); later Simonenko [31] treated the case of a half space $\mathbf{R}^{n+}$ and of a compact manifold M with boundary $\partial M \neq \emptyset$, but operators were investigated in the space $L_2^N(\mathbf{R}^{n+})$ and $L_2^N(M)$ (cf. also [13, 24, 26, 27]). Simultaneously with

Simonenko, Wišik and Eskin (cf. [13]) treated the case of Sobolev-Slobodeckiĭ spaces $H^{s2}(M)$, $\partial M \neq \emptyset$ and used obtained results for the investigation of boundary value problems for partial differential equations.

There was one more attempt to investigate operators (0.1) in $L_p^N(\mathbf{R}^{n+})$ and in Sobolev-Slobodeckiĭ $(H^{sp})^N(\mathbf{R}^{n+})$ spaces (cf. [29]); but the author made mistakes (proofs of Corollary 2.3 and Theorem 6.1 fail); nevertheless the reduction of the matrix-case $N > 1$ to the scalar one $N = 1$ can be carried out as in [29] (obtaining more precise asymptotics (2.7) instead of Corollary 2.3) and we follow this line here; instead of Theorem 6.1 from [29] we prove below Theorem 2.12.

Concerning Theorem 1.4, we do not need it here, but besides the interest of its own it can be used together with Theorem 2.12 for the investigation of the operator (0.1) in weighted spaces $L_p^N(\mathbf{R}^{n+}, x_1^\alpha)$. Besides Stein's Theorem, where the weight function $|x|^\alpha$ is considered, recently appeared papers (cf. [7,9]) dealing with much more general weight functions than $|x_1|^\alpha$ and $|x|^\alpha$; but singular integral operators have bounded and sufficiently smooth characteristics there (cf. also [25]).

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1. AUXILIARY PROPOSITIONS

1°. NOTATION.

▨ — end of the proof.

$\mathbf{R}^n := \mathbf{R} \times \ldots \times \mathbf{R}$, $\quad \mathbf{R} = (-\infty, \infty)$.

$\mathbf{R}^{n+} := \mathbf{R}^+ \times \mathbf{R}^{n-1}$, $\quad \mathbf{R}^+ = [0, \infty)$.

$[s]$ — integer part of a real number $s \in \mathbf{R}$.

$|\xi|_r := \left(\sum_{k=1}^{n} |\xi_k|^r\right)^{1/r}$, $\quad |\xi| \overset{\text{def}}{=} |\xi|_2 \quad (\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n)$.

$\xi' := (\xi_2, \ldots, \xi_n) \quad (\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n;\ \xi = (\xi_1, \xi'))$.

$S^{n-1} := \{\theta : \theta \in \mathbf{R}^n,\ |\theta| = 1\}$ — unit sphere in $\mathbf{R}^n$.

$$D_x^k := \frac{\partial^{|k|_1}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \quad (k = (k_1, \ldots, k_n);\ x \in \mathbf{R}^n).$$

$x^k := x_1^{k_1} \cdot x_2^{k_2} \ldots x_n^{k_n} \quad (x \in \mathbf{R}^n,\ k = (k_1, \ldots, k_n))$.

$\mathscr{F}$ and $\mathscr{F}^{-1}$ — Fourier transforms.

$$\mathscr{F}\varphi(t) = \int_{\mathbf{R}^n} e^{it\cdot\xi}\varphi(\xi)\, d\xi, \quad \mathscr{F}^{-1}\psi(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{-it\cdot\xi}\psi(t)\, dt,$$

$$t, \xi \in \mathbf{R}^n,\ t\cdot\xi = \sum_{k=1}^{n} t_k\xi_k.$$

$\mathscr{F}_{t_1 \to \xi_1}$ — partial Fourier transform.

$$\mathscr{F}_{t_1 \to \xi_1} \varphi(\xi) = \int_{-\infty}^{\infty} e^{i t_1 \cdot \xi_1} \varphi(t_1, \xi') \, dt_1 \quad (\xi = (\xi_1, \xi') \in \mathbf{R}^n).$$

$\mathscr{L}(X_1, X_2)$ — space of all linear bounded operators between the Banach spaces X_1 and X_2.

$\mathscr{L}(X) \stackrel{\text{def}}{=} \mathscr{L}(X, X)$.

$\mathfrak{S}(X_1, X_2)$ — space of all compact operators from $\mathscr{L}(X_1, X_2)$.

$\mathfrak{S}(X) \stackrel{\text{def}}{=} \mathfrak{S}(X, X)$.

$\operatorname{supp} a$ — closure of the set $\{\xi : a(\xi) \neq 0\}$ for a function $a(\xi)$, $\xi \in \mathbf{R}^n$ (support of the function a).

$C_0^\infty(\mathbf{R}^n)$ — algebra of all infinitely differentiable functions on $\mathbf{R}^n$ with compact supports.

P_{1+} — restricting operator $P_{1+}\varphi(t) = \varphi(t)|\mathbf{R}^{n+}$.

l_{1+} — extending operator (right inverse to P_{1+}) $l_{1+}\varphi(t) = \varphi(t)$ if $t \in \mathbf{R}^{n+}$ and $l_{1+}\varphi(t) = 0$ if $t \in \mathbf{R}^n \setminus \mathbf{R}^{n+}$ for $\varphi(t)$ defined on $\mathbf{R}^{n+}$.

$$\chi_{k\pm}(t) \equiv \chi_{k\pm}(t_k) = \frac{1}{2}(1 \pm \operatorname{sgn} t_k) \quad (t = (t_1 \ldots, t_n) \in \mathbf{R}^n, \ k = 1,2, \ldots n).$$

$H(\mathbf{R}^n)$ — algebra of homogeneous (of order 0) functions $a(\lambda\xi) \equiv a(\xi)$ $(\lambda > 0, \xi \in \mathbf{R}^n)$.

$HC^m(\mathbf{R}^n)$ — algebra of homogeneous (of order 0) functions, having continuous derivatives $D_\theta^k a(\theta)$ for all $|k|_1 \leqslant m$ on the unit sphere S^{n-1}.

$(HC^m)^{N \times N}(\mathbf{R}^n)$ — algebra of matrix-functions $a(\xi) = \|a_{jk}(\xi)\|_{j,k=1}^N$ with entries $a_{jk} \in HC^m(\mathbf{R}^n)$.

$\operatorname{diag}(a_1, \ldots, a_N) \stackrel{\text{def}}{=} \|a_j \delta_{jk}\|_{j,k=1}^N$ (δ_{jk} — symbol of Kroneker).

$\operatorname{diag} a^\sigma \stackrel{\text{def}}{=} \operatorname{diag}(a_1^{\sigma_1}, a_2^{\sigma_2}, \ldots, a_N^{\sigma_N})$ $(a = (a_1, \ldots, a_N),\ \sigma = (\sigma_1, \ldots, \sigma_N))$.

$L_p(M, \rho)$ — the space with norm $\|\varphi\|_{p\rho} = \left(\int_M |\rho\varphi|^p \, d\mu \right)^{1/p}$, where $d\mu$ is a certain measure on M.

$L_p^N(M, \rho)$ — the space of vector-functions $\varphi = (\varphi_1, \ldots, \varphi_n) \in L_p(M, \rho) \times \ldots \times L_p(M, \rho)$, $\|\varphi\|_{p\rho} = \left(\sum_{j=1}^N \|\varphi_j\|_{p\rho}^p \right)^{1/p}$.

$H^{sp}(\mathbf{R}^n)$ — Sobolev-Slobodeckiĭ space, defined as the closure of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|\varphi\|_{sp} = \left(\int_{\mathbf{R}^n} |\mathscr{F}^{-1}((1+|\xi|^2)^{s/2}(\mathscr{F}\varphi))(t)|^p \, dt\right)^{1/p}, \quad 1 < p < \infty, \ s \in \mathbf{R}.$$

$H_0^{sp}(\mathbf{R}^{n+})$ $(H_0^{sp}(\mathbf{R}^{n-}))$ — subspace of $H^{sp}(\mathbf{R}^n)$, obtained by the closure of the set of functions $\varphi \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp}\varphi \subset \mathbf{R}^{n+}$ $(\operatorname{supp}\varphi \subset \mathbf{R}^{n-} = \mathbf{R}^n \setminus \mathbf{R}^{n+}$, respectively).

$H^{sp}(\mathbf{R}^{n+})$ — the factor-space $H^{sp}(\mathbf{R}^{n+}) \stackrel{\text{def}}{=} H^{sp}(\mathbf{R}^n)/H_0^{sp}(\mathbf{R}^{n-})$.

For $s < 0$ the space $H^{s2}(\mathbf{R}^n)$ contains non-regular distributions, but for $s > \frac{n}{2}$ it is a Banach algebra (Peetre) and all functions there are continuous (Sobolev); the norm in the space $H^{s2}(\mathbf{R}^n)$ $(s > 0)$ can be defined also by the equality

$$\|\varphi\|_{sp} = \left(\int_{\mathbf{R}^n} |\varphi(t)|^2 \, dt + \sum_{|k|_1 = m} \int_{\mathbf{R}^n} dy \int_{\mathbf{R}^n} \frac{|D_x^k\varphi(x+y) - D_x^k\varphi(x)|^2}{|y|^{n+2\lambda}} \, dx\right)^{1/2},$$

where $s = m + \lambda$, $0 < \lambda < 1$ and m is an integer (cf. [12, 30]).

$H^{sp}(M)$ — Sobolev-Slobodeckiĭ space on a r-smooth n-dimensional compact manifold without boundary $(-\infty < s < \infty,\ 1 < p < \infty,\ r > |s|)$ defined as follows: we choose a finite covering $U_1, \ldots, U_m$ of M $\left(\bigcup_{j=1}^m U_j = M\right)$ and homeomorphisms $\varkappa_j \colon U_j^0 \to U_j$, where $U_j^0 \subset \mathbf{R}^n$ is an open set; let $\psi_1, \ldots, \psi_m$ be the partition of the unity on M, subjected to this covering (i.e. $\sum_{j=1}^m \psi_j(t) \equiv 1$, $\operatorname{supp}\psi_j \subset U_j$ and $\varkappa_j^*\psi(x) \equiv \psi_j(\varkappa_j(x)) \in C^r(U_j^0)$); suppose the transformation $\varkappa_{ij} = \varkappa_i^*\varkappa_j^{-1}$ $(\varkappa_{ij} \colon U_i^0 \cap U_j^0 \to U_i^0 \cap U_j^0$ if $U_i^0 \cap U_j^0 \neq \emptyset)$ belongs to $C^r(U_i^0 \cap U_j^0)$ and the Jakobian $|D\varkappa_{ij}(x)| \neq 0$; the norm in the space $H^{sp}(M)$ is defined by the equality

$$\|\varphi\|_{sp} = \left(\sum_{j=1}^m \|\varkappa_j^*(\psi_j\varphi)\|_{sp}^p\right)^{1/p};$$

the definition depends obviously on the partition of the unity, but it can be proved (cf. [1, 17, 20, 34]) that different norms defined by the different partitions are equivalent.

$H^{sp}(\tilde{M})$, $H_0^{sp}(\tilde{M})$ — Sobolev-Slobodeckiĭ spaces on a compact manifold $\tilde{M}$ with boundary $\partial\tilde{M} \neq \emptyset$, defined as in the previous case but with the help of spaces $H^{sp}(\mathbf{R}^{n+})$ and $H_0^{sp}(\mathbf{R}^{n+})$.

It is known that $H_0^{sp}(\tilde{M}) \subset H^{sp}(\tilde{M})$, $H_0^{0p}(\tilde{M}) = H^{0p}(\tilde{M}) = L_p(\tilde{M})$, $H^{0p}(M) = L_p(M)$ and $H_0^{sp}(\tilde{M}) = H^{sp}(\tilde{M})$ for $1/p - 1 < s < 1/p$; conjugate spaces are $(H_0^{sp}(\tilde{M}))^* = H^{-sp'}(\tilde{M})$ $(-\infty < s < \infty,\ p' = p/(p-1))$ and $(H^{sp}(\tilde{M}))^* = H_0^{-sp'}(\tilde{M})$, $(H^{sp}(M))^* = H^{-sp'}(M)$; for an integer m the space $H^{mp}(M)$ consists of all functions $\varphi(x)$ having p-summable partial derivatives $D_x^k\varphi(x)$ for all $|k|_1 \leqslant m$.

$C^mH^{sp}(X, M)$ — the space of functions $b(x, \xi)$ $(x \in X \subset \mathbf{R}^l,\ \xi \in M \subset \mathbf{R}^n,\ m = 0, 1, \ldots, \infty)$ which have the property: all derivatives $D_x^k b(x, \xi) = b_k(x, \xi) \in H^{sp}(M)$ and are uniformly continuous

$$\lim_{x \to x_0} \|b_k(x, \cdot) - b_k(x_0, \cdot)\|_{sp} = 0$$

for all $|k|_1 \leqslant m$ and $x_0 \in X$ $(x_0 \in X \cup \{\infty\}$ if X is not compact).

$CH^{sp}(X, M) \overset{\text{def}}{=} C^0H^{sp}(X, M)$.

2°. Singular integral operators. Consider the (multidimensional) singular integral operators of the type

$$A\varphi(x) = \int_{\mathbf{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n}\, \varphi(y)\, dy, \tag{1.1}$$

where $\Omega(x, \xi)$ are measurable functions $\Omega(x, \lambda\xi) \equiv \Omega(x, \xi)$ $(\lambda > 0$; i.e. $\Omega(x, \cdot) \in H(\mathbf{R}^n)$ for all $x \in \mathbf{R}^n)$; the function $\Omega(x, \xi)$ is called the *characteristic* of the operator (1.1).

Theorem 1.1. (cf. [22]). *Let $r > -1/2$ $(r > 0$ or $r > (n-1)(1/p' - 1/2))$; $1 < p < \infty$, $p' = p/(p-1)$ and:*

a) $$\int_{S^{n-1}} \Omega(x, \theta)\, d\theta \equiv 0,$$

b) $$c_{rs} = \sum_{|k|_1 \leqslant [|s|]+1} \operatorname*{ess\,sup}_{x \in \mathbf{R}^n} \|D_x^k\Omega(x, \cdot)\|_{H^{r2}(S^{n-1})} < \infty.$$

The operator (1.1) is bounded in the space $H^{sp}(\mathbf{R}^n)$ for $p = 2$ (for $2 < p < \infty$ and for $1 < p < 2$, respectively) and $\|A\|_{H^{sp}(\mathbf{R}^n)} \leqslant C \cdot C_{rs}$, where C is a constant, independent of Ω.

Theorem 1.2. (Caldéron-Zygmund; cf. [4, 22]). *Let $\Omega(x, \lambda\xi) \equiv \Omega(x, \xi)$ $(\lambda > 0;\ x, \xi \in \mathbf{R}^n)$ and:*

a) $$\int_{S^{n-1}} \Omega(x, \theta)\, d\theta = 0,$$

b) $$c_p = \sup_{x \in \mathbf{R}^n} \|\Omega(x, \cdot)\|_{L_{p'}(S^{n-1})} < \infty \quad \left(1 < p < \infty,\ p' = \frac{p}{p-1}\right).$$

The operator (1.1) *is bounded in the space* $L_p(\mathbf{R}^n)$ *and* $\|A\|_p \leqslant C \cdot C_p$, *where* C *is a constant, independent of* Ω.

THEOREM 1.3. (Caldéron-Zygmund; cf. [5, 22]). *Let* $\Omega(x, \xi) \equiv \Omega(\xi) \equiv \equiv \Omega(\lambda \cdot \xi)$ $(\lambda > 0, \xi \in \mathbf{R}^n)$ *and:*

a)
$$\int_{S^{n-1}} \Omega(\theta)\, d\theta = 0,$$

b)
$$\|\Omega\|_{L_1(S^{n-1})} < \infty \quad \|\tilde{\Omega} ln^+ \tilde{\Omega}\|_{L_1(S^{n-1})} < \infty,$$

where $\tilde{\Omega}(\theta) \equiv \Omega(\theta) + \Omega(-\theta)$.

The operator (1.1) *is bounded in the space* $L_p(\mathbf{R}^n)$ *for all* $1 < p < \infty$.

Now we prove the following

THEOREM 1.4. *Let* (1.1) *be a bounded operator in the space* $L_p(\mathbf{R}^n)$ $(1 < p < \infty)$ *and* $|\Omega(x, \cdot)| \in H(\mathbf{R}^n)$ *for all* $x \in \mathbf{R}^n$. *If*

$$K_p = \sup_{x \in \mathbf{R}^n} \left(\int_{S^{n-1}} |\Omega(x, \theta)|^{p'} d\theta \right)^{1/p'} < \infty,$$

then the operator (1.1) *is bounded in the space* $L_p(\mathbf{R}^n, |x_1|^\alpha)$ *for* $-1/p < \alpha < 1 - 1/p$ *and* $\|A\|_p \leqslant C(\|A\|_p + K_p)$.

If $\Omega(x, \xi) \equiv \Omega(\xi)$ *and*

$$K_1 = \int_{S^{n-1}} |\Omega(\theta)|\, d\theta < \infty,$$

then (1.1) *is a bounded operator in* $L_p(\mathbf{R}^n, |x_1|^\alpha)$ *and*[1] $\|A\|_{p\alpha} \leqslant C(\|A\|_p + K_1)$.

Proof. Assume first $\Omega(x, \xi) \equiv \Omega(\xi)$ and

$$(1.2) \qquad -\frac{1}{p} < \alpha < 0.$$

It obviously suffices to prove the boundedness of the operator

$$B\varphi(x) = \frac{1}{|x_1|^\beta} \int_{\mathbf{R}^n} \frac{(|y_1|^\beta - |x_1|^\beta)\, \Omega(x - y)}{|x - y|^n} \varphi(y)\, dy$$

in the space $L_p(\mathbf{R}^n)$, where $\beta = -\alpha$ (due to the boundedness of A in $L_p(\mathbf{R}^n)$).

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[1] This theorem is a multi-dimensional analog of Babenko-Chvedelidze theorem (cf. [6]).

Introducing a constant $0 < \delta < 1$ and using the Hölder inequality we get

$$|B\varphi(x)| \leqslant \frac{C_1}{|x_1|^\beta}\left(\int_{\mathbf{R}^n} \frac{|\varphi(y)|^p |\Omega(x-y)|\,|y_1|^{\delta p}\,\mathrm{d}y}{|x-y|^n\,|x_1-y_1|^{-\beta}}\right)^{1/p} \times$$

$$\times\left(\int_{\mathbf{R}^n} \frac{|\Omega(x-y)|\,\mathrm{d}y}{|x-y|^n\,|x_1-y_1|^{-\beta}\,|y_1|^{\delta p'}}\right)^{1/p'} =$$

$$= \frac{C_1}{|x_1|^\beta}\left(\int_{\mathbf{R}^n} \frac{|\varphi(y)|^p\,|\Omega(x-y)|\,|y_1|^{\delta p}\,\mathrm{d}y}{|x-y|^n\,|x_1-y_1|^{-\beta}}\right)^{1/p} \times$$

$$\times\left(\int_{-\infty}^{\infty} \frac{|y_1-x_1|^\beta\,\mathrm{d}y}{|y_1|^{\delta p'}} \int_{\mathbf{R}^{n-1}} \frac{|\Omega(x-y)|\,\mathrm{d}y'}{|x-y|^n}\right)^{1/p'} \leqslant \tag{1.3}$$

$$\leqslant \left(\frac{C_1}{|x_1|^\beta}\int_{\mathbf{R}^n} \frac{|\varphi(y)|^p \Omega(x-y)|\,|y_1|^{\delta p}\,\mathrm{d}y}{|x-y|^n\,|x_1-y_1|^{-\beta}}\right)^{1/p} \times$$

$$\times\left(\int_{-\infty}^{x_1} \frac{\mathrm{d}y_1}{|y_1-x_1|^{1-\beta}\,|y_1|^{\delta p'}} \int_{\mathbf{R}^{n-1}} \frac{|\Omega(1,\xi')|\,\mathrm{d}\xi'}{(1+|\xi'|^2)^{n/2}} + \right.$$

$$\left. + \int_{x_1}^{\infty} \frac{\mathrm{d}y_1}{|y_1-x_1|^{1-\beta}\,|y_1|^{\delta p'}} \int_{\mathbf{R}^{n-1}} \frac{|\Omega(-1,\xi')|\,\mathrm{d}\xi'}{(1+|\xi'|^2)^{n/2}}\right)^{1/p'}$$

because $\big|\,|x_1|^\beta - |y_1|^\beta\big| \leqslant |x_1-y_1|^\beta$ and we changed the variables $\xi' = (\xi_2, \ldots, \xi_n)$, $\xi_k = \dfrac{x_k - y_k}{x_1 - y_1}$ $(k = 2, \ldots, n$; due to homogeneity of $|\Omega(\xi)|$ we have $|\Omega(x_1 - y_1, x' - y')| = |\Omega\,(\mathrm{sgn}\,(x_1 - y_1), \xi')|)$.

Consider now the homeomorphism of the space $\mathbf{R}^{n-1}$ on the semi-sphere $S_\varepsilon^{n-1} = \{\theta = (\theta_1, \theta') \in S^{n-1} : \varepsilon\theta_1 \geqslant 0\}$ defined by the formulas

$$\theta_1 = \frac{\varepsilon}{r(\xi')}, \quad \theta_k = \frac{\xi_k}{r(\xi')} \quad \left(k = 2, \ldots, n; \quad r(\xi') = \sqrt{1 + \sum_{k=2}^n \xi_k^2}\right), \tag{1.4}$$

where $\varepsilon = \pm 1$ (inverse mapping is given by formulas $\xi_k = \theta_k/\theta_1$, $k = 2,3, \ldots, n$). We easily obtain from (1.4)

$$|\theta - \tilde{\theta}|^2 = \frac{|\xi' - \tilde{\xi}'|^2 - [r(\xi') - r(\tilde{\xi}')]^2}{r^2(\xi')\, r^2(\tilde{\xi}')} . \tag{1.5}$$

To change the variables in (1.3) we must find the relations between $d\xi'$ and $d\theta$; let's choose for this purpose the partition of $\mathbf{R}^{n-1}$ by the curves

$$\begin{cases} r(\xi') = \text{const}, & n-3 \text{ coordinates of } \xi' = (\xi_2, \ldots, \xi_n) \text{ are fixed;} \\ \xi_k = \gamma_k t & (k = 2, \ldots, n; \ -\infty < t < \infty). \end{cases}$$

Using (1.5) we calculate the ratio of distances between two points across these curves and their images on the sphere S^{n-1} when these points are converging to one; we get

$$\begin{cases} \lim\limits_{\tilde{\xi}' \to \xi'} \dfrac{|\theta - \tilde{\theta}|}{|\xi' - \tilde{\xi}'|} = \dfrac{1}{r(\xi')} & \text{for } r(\xi') = \text{const}, \\[2ex] \lim\limits_{\tilde{\xi} \to \xi} \dfrac{|\theta - \tilde{\theta}|}{|\xi' - \tilde{\xi}'|} = \dfrac{1}{r^2(\xi')} & \text{for } \xi_k = \gamma_k t; \end{cases}$$

hence

$$\frac{d\theta}{d\xi'} = \left[\frac{1}{r(\xi')}\right]^{n-2} \cdot \frac{1}{r^2(\xi')} = \frac{1}{r^n(\xi')} .$$

Using the obtained formula we get

$$\int\limits_{\mathbf{R}^{n-1}} \frac{|\Omega(\varepsilon, \xi')|\, d\xi'}{(1 + |\xi'|^2)^{n/2}} = \int\limits_{\mathbf{R}^{n-1}} \frac{|\Omega(\varepsilon, \xi')|\, d\xi'}{r^n(\xi')} =$$

$$= \int\limits_{S_\varepsilon^{n-1}} |\Omega(\varepsilon, |\theta_1|^{-1}\theta')|\, d\theta = \int\limits_{S_\varepsilon^{n-1}} |\Omega(\theta)|\, d\theta \leqslant \tag{1.6}$$

$$\leqslant \int\limits_{S^{n-1}} |\Omega(\theta)|\, d\theta = K_1.$$

Due to (1.3)

$$|B\varphi(x)| \leqslant \frac{C_1 K_1^{1/p'}}{|x_1|^\beta} \left(\int\limits_{\mathbf{R}^n} \frac{|\varphi(y)|^p |\Omega(x-y)|\, |y_1|^{\delta p}\, dy}{|x-y|^n\, |x_1 - y_1|^{-\beta}} \right)^{1/p} \times$$

$$\times \left(\int\limits_{-\infty}^{\infty} \frac{dy_1}{|y_1 - x_1|^{1-\beta}\, |y_1|^{\delta p'}} \right)^{1/p'} \leqslant \frac{C_2 K_1^{1/p'}}{|x_1|^{\beta/p + \delta}} \times$$

$$\times \left(\int\limits_{\mathbf{R}^n} \frac{|\varphi(y)|^p\, |\Omega(x-y)|\, |y_1|^{\delta p}\, dy}{|x-y|^n\, |x_1 - y_1|^{-\beta}} \right)^{1/p},$$

because

$$(1.7)\qquad \int_{-\infty}^{\infty} \frac{dy_1}{|y_1 - x_1|^{1-\beta}\,|y_1|^{\delta p'}} = C_3|x_1|^{\beta-\delta p'}, \quad \beta - \frac{\beta}{p'} = \frac{\beta}{p},$$

if $1 - \beta + \delta p' > 1$. Hence

$$\int_{\mathbf{R}^n} |B\varphi(x)|^p \, dx \leqslant C_2 K_1^{p/p'} \int_{\mathbf{R}^n} |y_1|^{\delta p}\, |\varphi(y)|^p \, dy \, \times$$

$$\times \int_{-\infty}^{\infty} \frac{|x_1 - y_1|^{\beta}\, dx_1}{|x_1|^{\beta+\delta p}} \int_{\mathbf{R}^n} \frac{|\Omega(x-y)|\, dy'}{|x-y|^n} \leqslant$$

$$\leqslant C_3 K_1^{p/p'} \int_{\mathbf{R}^n} |y_1|^{\delta p}\, |\varphi(y)|^p dy \int_{-\infty}^{\infty} \frac{dx_1}{|x_1|^{\beta+\delta p}\, |x_1 - y_1|^{1-\beta}} \, \times$$

$$\times \int_{S^{n-1}} |\Omega(\theta)|\, d\theta = C_4 K_1^{p/p'+1} \int_{\mathbf{R}^n} |\varphi(y)|^p \, dy,$$

because

$$(1.8)\qquad \int_{-\infty}^{\infty} \frac{dx_1}{|x_1|^{\beta+\delta p}\, |x_1 - y_1|^{1-\beta}} = C_5 |x_1|^{-\delta p}$$

if $\beta + \delta p < 1$; we used also the inequality (1.6).

We will be done with the case (1.2) if we prove the compatibility of inequalities (cf. (1.7)—(1.8))

$$0 < \delta < 1, \quad 1 - \beta + \delta p' > 1, \quad \beta + \delta p < 1;$$

but they are compatible, because $\beta = -\alpha$ and $0 < \beta < 1/p$ holds (cf. (1.2)).

Let now

$$0 < \alpha < 1 - \frac{1}{p} = \frac{1}{p'} \quad \left(-\frac{1}{p'} < -\alpha < 0\right);$$

consider the conjugate operator

$$A^*\psi(x) = \int_{\mathbf{R}^n} \frac{\Omega^*(x-y)}{|x-y|^n}\, \psi(y)\, dy, \quad \Omega^*(\xi) = \overline{\Omega(-\xi)}$$

in the conjugate space $L_{p'}(\mathbf{R}^n, |x_1|^{-\alpha})$; in virtue of the proved part of the theorem A^* is bounded and hence, A is bounded in $L_p(\mathbf{R}^n, |x_1|^\alpha)$.

There remains to prove only the first part of the theorem (case $\Omega(x, \xi) \not\equiv \not\equiv \Omega(\xi)$).

Assuming again (1.2), instead of (1.3) we write

$$|B\varphi(x_1)| \leqslant \frac{1}{|x|^\beta} \left(\int_{\mathbf{R}^n} \frac{|\varphi(y)|^p |y_1|^{\delta p} \, dy}{|x - y|^n |x_1 - y_1|^{-\beta}} \right)^{1/p} \times$$

$$\times \left(\int_{\mathbf{R}^n} \frac{|\Omega(x - y)|^{p'} \, dy}{|x - y|^n |x_1 - y_1|^{-\beta} |y_1|^{\delta p'}} \right)^{1/p'} \leqslant$$

$$\leqslant \frac{1}{|x_1|^\beta} \left(\int_{\mathbf{R}^n} \frac{|\varphi(y)|^p |y_1|^{\delta p} \, dy}{|x - y|^n |x_1 - y_1|^{-\beta}} \right)^{1/p} \times$$

$$\times \left(\int_{\infty}^{\infty} \frac{|y_1 - x_1|^{\beta - 1} \, dy_1}{|y_1|^{\delta p'}} \sup_{\eta \in \mathbf{R}^n} \int_{\mathbf{R}^{n-1}} \frac{|\Omega(\eta, 1, \xi')|^{p'} \, d\xi'}{(1 + |\xi'|^2)^{n/2}} \right)^{1/p'};$$

the remainder is the same as in the considered case. ▨

From Theorems 1.2—1.4 it immediately follows:

COROLLARY 1.5. *If conditions of Theorem* 1.2 *(of Theorem* 1.3*) hold and* $-1/p < < \alpha < 1 - 1/p$, *the operator* (1.1) *is bounded in the space* $L_p(\mathbf{R}^n, |x_1|^\alpha)$.

3°. INTEGRAL CONVOLUTION OPERATORS. Let $a(\xi) \in L_\infty(\mathbf{R}^n)$ and

$$W_a^0 \varphi = \mathscr{F}^{-1} a \mathscr{F} \varphi \quad (\varphi \in C_0^\infty(\mathbf{R}^n)); \tag{1.9}$$

by $M_p(\mathbf{R}^n)$ denote the algebra of all (*multipliers*) $a(\xi)$ for which W_a^0 admits the continuous extension to the space $L_p(\mathbf{R}^n)$ $(1 < p < \infty)$. By $m_p(\mathbf{R}^n)$ denote closure of the set $\bigcup_{r \notin (p, p')} M_r(\mathbf{R}^n)$ with the norm $\|a\|_p^0 = \|W_a^0\|_p$.

$$Hm_p(\mathbf{R}^n) = m_p(\mathbf{R}^n) \cap H(\mathbf{R}^n).$$

If $a(x, \xi) \in m_p(\mathbf{R}^n)$ depends as well on the variable $x \in N$, the operator will be written as $W_{a(x, \cdot)}^0$; if $\sup_x \|a(x, \cdot)\|_p^0 = \sup_x \|W_{a(x, \cdot)}^0\|_p < \infty$, we write $a(x, \xi) \in L_\infty m_p(N, \mathbf{R}^n)$ $(x \in N)$.

The operator (1.1) represents the example of the operator $W^0_{a(x,\cdot)}$ where

(1.10) $$a(x,\xi)=\int_{\mathbf{R}^n}\frac{e^{i\xi t}\Omega(x,t)}{|t|^n}\,dt$$

(cf. [22, 31]) and $a(x,\xi)\in HM_p(\mathbf{R}^n)$ if (1.1) is bounded in $L_p(\mathbf{R}^n)$; $a(x,\xi)$ is called the *symbol* of (1.1).

THEOREM 1.6. (cf. [29]). *Let $\Omega(x,\xi)\equiv\Omega(\xi)$; the symbol $a(\xi)\in H^{s2}(S^{n-1})$ (cf. (1.10)) if and only if the characteristic $\Omega(\xi)\in H^{s-n/2,2}(S^{n-1})$.*

COROLLARY 1.7. *Let $a(x,\cdot)\in H(\mathbf{R}^n)$ for all $x\in\mathbf{R}^n$,*

$$\max_{|\varkappa|_1\leqslant m}\ \sup_{x\in\mathbf{R}^n}\|D^k_x a(x,\cdot)\|_{H^{r2}(S^{n-1})}<\infty$$

and $r>(n-1)/2$, $r>n/2$, $r>(n-1)/p'+1/2$ for $p=2$, for $2<p<\infty$ and for $1<p<2$, respectively ($p'=p/(p-1)$); then $W^0_{a(x,\cdot)}$ is a bounded operator in $H^{sp}(\mathbf{R}^n)$ for all $|s|\leqslant m$.

Theorem 1.1 and Corollary 1.7 yield:

COROLLARY 1.8. *Let $a(\xi)\in H^{s2}(S^{n-1})\cap H(R^n)$ and $s>n/2$; then $a(\xi)\in Hm_p(\mathbf{R}^n)$ for $1<p<\infty$.*

If $s>n/2$, then $HC^s(\mathbf{R}^n)\subset Hm_p(\mathbf{R}^n)$.

Consider the operator

(1.11) $$W^1_{a(x,\cdot)}\stackrel{\text{def}}{=}P_{1+}W^0_{a(x,\cdot)}l_{1+};$$

if $a(x,\xi)\in L_\infty m_p(\mathbf{R}^{n+},\mathbf{R}^n)$, $W^1_{a(x,\cdot)}$ is a bounded operator in $L_p(\mathbf{R}^{n+})$.

Obviously

$$\lambda W^k_a+\mu W^k_b=W^k_{\lambda a+\mu b}\quad(k=0,1).$$

It is also easy to prove the following:

PROPOSITION 1.9. *If $a(\xi),b(\xi)\in M_p(\mathbf{R}^n)$, then $W^0_aW^0_b=W^0_{ab}$.*

If, additionally, $a(\xi)$ has an analytic extension in the half-plane $\operatorname{Im}\xi_1<0$ *for all $\xi'\in\mathbf{R}^{n-1}$ ($\xi=(\xi_1,\xi')\in\mathbf{R}^n$) or $b(\xi)$ — in the half-plane* $\operatorname{Im}\xi_1>0$ *(for all $\xi'\in\mathbf{R}^{n-1}$), then $W^1_aW^1_b=W^1_{ab}$.*

We need several well-known results, which we formulate below.

THEOREM 1.10 (Michlin-Hörmander; cf. [16, 21, 28]). *If*

$$\sum_{|k|_1 \leqslant [n/2]+1} \sup_{R>0} \int_{R<2|x|<4R} |x^k D_x^k a(x)|^2 \frac{dx}{R^n} < \infty, \tag{1.12}$$

then $a \in m_p(\mathbf{R}^n)$.

Let M be r-smooth manifold with boundary $\partial M \neq \emptyset$ and $\mathscr{L}_{pr}$, $\mathfrak{S}_{pr}$ denote the space of all bounded, all compact, operators $\mathscr{L}_{pr} = \mathscr{L}(L_{pr}^0, L_{pr})$, $\mathfrak{S}_{pr} := \mathfrak{S}(L_{pr}^0, L_{pr})$ where either

a) $L_{pr}^0 := H_0^{rp}(M)$, $L_{pr} := H^{rp}(M)$ $(1 < p < \infty,\ -\infty < r < \infty)$ or

b) $L_{pr}^0 = L_{pr} := H^{rp}(\mathbf{R}^n)$ $(1 < p < \infty,\ -\infty < r < \infty)$.

THEOREM 1.11. *Let* $A \in \mathscr{L}_{p_0 r_0} \cap \mathscr{L}_{p_1 r_1}$; *then* $A \in \mathscr{L}_{pr}$ *for*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad r = r_0(1-\theta) + r_1\theta \quad (0 < \theta < 1)$$

and

$$\|A\|_{pr} \leqslant C\|A\|_{p_0 r_0}^{1-\theta}\|A\|_{p_1 r_1}^{\theta},$$

where $\|A\|_{rp}$ *denotes the norm in* $\mathscr{L}_{pr}$.

If, additionally, $A \in \mathfrak{S}_{p_0 r_0}$ *then* $A \in \mathfrak{S}_{pr}$ $(0 < \theta < 1)$.

THEOREM 1.12. *Let* $a \in m_p(\mathbf{R}^n)$ $(1 < p < \infty)$, $b(\xi) \in C^r(\mathbf{R}^n)$ *and* $\lim_{|\xi|\to\infty} a(\xi) = \lim_{|\xi|\to\infty} b(\xi) = 0$; *then* bW_a^0, $W_a^0 b \in \mathfrak{S}(H^{sp}(\mathbf{R}^n)$ $(\in \mathfrak{S}(L_p(\mathbf{R}^n)$ *and* bW_a^1, $W_a^1 b \in \mathfrak{S}(H_0^{sp}(\mathbf{R}^{n+}), H^{sp}(\mathbf{R}^{n+}))$ *for* $|s| \leqslant r$ $(\in \mathfrak{S}(L_p(\mathbf{R}^{n+})))$.

For $s = 0$, $a, b \in C_0^\infty(\mathbf{R}^n)$ the theorem is well-known; the general case is treated with the help of Theorem 1.11 as the case $n = 0$ in [10], Lemma 7.1.

4°. ON THE TENSOR PRODUCT OF OPERATORS. If K is a finite dimensional operator

$$K\varphi(t) = \sum_{j=1}^{m} \varphi_j(t) \int_0^\infty \overline{\psi_j(\tau)}\varphi(\tau)\, d\tau \quad (\varphi_j \in L_p(\mathbf{R}^+),\ \psi_j \in L_{p'}(\mathbf{R}^+))$$

in the space $L_p(\mathbf{R}^+)$ $(1 < p < \infty)$ and $A' \in \mathscr{L}(L_p(\mathbf{R}^{n-1}))$, the *tensor product* $A = K \otimes A'$ is defined as

$$A\varphi(t) = (K \otimes A')\,\varphi(t) = \sum_{j=1}^{m} \varphi_j(t_1) \int_0^\infty \overline{\psi_j(\tau)}\,(A'\varphi)(\tau, t')\, d\tau \quad (t = (t_1, t'));$$

obviously $A \in \mathscr{L}(L_p(\mathbf{R}^{n+}))$. The closure of the set of such operators is denoted by $\mathfrak{S}_p^{(1)} \stackrel{\text{def}}{=} \mathfrak{S}^{(1)}(L_p(\mathbf{R}^{n+}))$; $\mathfrak{S}_p^{(1)}$ is an ideal in the algebra $\mathscr{L}_p = \mathscr{L}(L_p(\mathbf{R}^{n+}))$ and, obviously, $\mathfrak{S}_p = \mathfrak{S}(L_p(\mathbf{R}^{n+})) \subset \mathfrak{S}_p^{(1)}$.

LEMMA 1.13. (cf. [11], Lemma 2.1). *If* $T \in \mathfrak{S}_p^{(1)}$ *and* $B_j = C_j \otimes I$, *where* $\lim_{j\to\infty} \|C_j\psi\|_p = 0$ *for any* $\psi \in L_p(\mathbf{R}^+)$, *then* $\lim_{j\to\infty} \|B_jT\|_p = 0$.

Define the operator

$$V_\lambda\varphi(t) = \varphi(\lambda t) \quad (0 < \lambda < \infty);$$

obviously $V_\lambda^{-1} = V_{1/\lambda}$ and $V_\lambda W_a^k = W_a^k V_\lambda$ for any $a \in HM_p(\mathbf{R}^n)$ $(k = 0, 1)$.

LEMMA 1.14. *If* $T \in \mathfrak{S}_p^{(1)}$ *and* $V_\lambda T = TV_\lambda$, *then* $T = 0$.

Proof. Let

$$B_j\varphi(t) = v_j(t)\,\varphi(t), \qquad \varphi \in L_p(\mathbf{R}^{n+}),$$

where $v_j(t) = 1$ for $0 < t_1 \leqslant j^{-1}$ and $v_j(t) \equiv 0$ if $j^{-1} < t_1$; in virtue of Lemma 1.13

$$\lim_{j\to\infty} \|B_jT\|_p = 0. \tag{1.13}$$

For any $\varepsilon > 0$ there exists $\varphi_\varepsilon(t) \in L_p(\mathbf{R}^{n+})$ such that $\|\varphi_\varepsilon\|_p = 1$, $\|T\varphi_\varepsilon\|_p \geqslant \|T\|_p - \varepsilon$; if $\varphi_{\varepsilon,\lambda}(t) = \lambda^{n/p}\varphi_\varepsilon(\lambda t) = \lambda^{n/p}V_\lambda\varphi_\varepsilon(t)$, then $\|\varphi_{\varepsilon,\lambda}\|_p = 1$ and

$$\lim_{\lambda\to\infty} \|B_jT\varphi_{\varepsilon,\lambda}\|_p = \lim_{\lambda\to\infty} \|\lambda^{n/p}V_\lambda\, V_{\lambda^{-1}}v_j]\, T\varphi_\varepsilon\|_p =$$

$$= \lim_{\lambda\to\infty} \|(V_{\lambda^{-1}}v_j)\, T\varphi_\varepsilon\|_p = \|T\varphi_\varepsilon\|_p \geqslant \|T\|_p - \varepsilon\,;$$

hence $\|B_jT\|_p \geqslant \|T\|_p$. The converse inequality $\|B_jT\|_p \leqslant \|T\|_p$ is obvious and, therefore, $\|B_jT\|_p = \|T\|_p$; conclusion follows now from (1.13). ▨

LEMMA 1.15. *Let* $a \in HM_p(\mathbf{R}^n)$ *and* U *be any neighbourhood of the point* $y = (0, y_2, \ldots, y_n)$; *then*

$$\|W_a^k\|_p = \inf_{T\in\mathfrak{S}_p^{(1)}} \|\chi_u W_a^k + T\|_p \quad (k = 0, 1),$$

where $\chi_u(\xi)$ *is the characteristic function of* U.

Proof. Inequality $c_p = \inf_{T\in\mathfrak{S}_p^{(1)}} \|\chi_u W_a^1 + T\|_p \leqslant \|W_a^1\|_p$ is obvious.

Let for definiteness $k = 0$ (the case $k = 1$ is similar).

Assume now $\|W_a^0\|_p - c_p = 4\varepsilon > 0$ and $T_\varepsilon \in \mathfrak{S}_p^{(1)}$ be such that $c_p + \varepsilon > \|\chi_u W_a^0 + T_\varepsilon\|_p \geqslant c_p$; without the loss of generality we can suppose that $(0, 0, \ldots, 0) \in U$; otherwise we can use the shift operator

$$B_{-y}\varphi(x) = \varphi(x - y) = \varphi(x_1, x_2 - y_2, \ldots, x_u - y_n)$$

$$(\|B_{-y}\|_p = 1;\ B_{-y}W_a^0 = W_a^0 B_{-y};\ B_{-y}^{-1} = B_y).$$

Let $\chi_\delta(\xi)$ be the characteristic function of the set $[-\delta, \delta] \times \ldots \times [-\delta, \delta]$ $(\delta > 0)$; then $\|\chi_\delta T_\varepsilon\|_p < \varepsilon$ for some small δ (cf. Lemma 1.13) and $\chi_\delta \cdot \chi_u \equiv \chi_\delta$.

Let $\varphi(\xi)$ be such, that $\|\varphi\|_p = 1$ and $\|W_a^0\|_p < \|W_a^0\varphi\|_p + \varepsilon$; if $\tilde{V}_\lambda\varphi(\xi) \equiv \varphi_\lambda(\xi) = \lambda^{-n/p}\varphi(\lambda^{-1}\xi)$, then $\|\varphi_\lambda\|_p = 1$ and with the help of the equality

$$\lim_{\lambda\to 0}\|(V_\lambda\chi_\delta)\,\psi\|_p = \|\psi\|_p$$

we obtain

$$\|W_a^0\|_p < \|W_a^0\varphi\|_p + \varepsilon = \lim_{\lambda\to 0}\|(V_\lambda\chi_\delta)\,W_a^0\varphi\|_p + \varepsilon =$$

$$= \lim_{\lambda\to 0}\|\tilde{V}_\lambda(V_\lambda\chi_\delta)\,W_a^0\varphi\|_p + \varepsilon = \lim_{\lambda\to 0}\|\chi_\delta W_a^0\varphi_\lambda\|_p + \varepsilon \leqslant$$

$$\leqslant \|\chi_\delta W_a^0\|_p + \varepsilon \leqslant \|\chi_\delta(\chi_u W_a^0 + T_\varepsilon)\|_p + 2\varepsilon \leqslant$$

$$\leqslant \|\chi_u W_a^0 + T_\varepsilon\|_p + 2\varepsilon < c_p + 3\varepsilon = \|W_0^p\|_p - \varepsilon;$$

the obtained contradiction proves the lemma. ▨

LEMMA 1.16. *Let $a, b \in HM_p(\mathbf{R}^n)$ and $\chi_k(\xi)$ be the characteristic function of the set u_k $(k = 1, 2, 3)$; let $(c_{k1}, d_{k1}) \times \ldots \times (c_{kn}, d_{kn}) \subset u_k$ for some $c_{kj} < d_{kj}$ $(j = 1, 2, \ldots, n)$. Then*

$$\sup_\xi |a(\xi)\,b(\xi)| = \|W_{ab}^0\|_2 \leqslant |||W_{ab}^0|||_p = \|W_{ab}^1\|_p = \tag{1.14}$$
$$= |||\chi_1 W_a^0 \chi_2 W_b^0 \chi_3|||_p = |||W_a^1 W_b^1|||_p,$$

where

$$|||A|||_p = \inf_{T \in \mathfrak{S}(L_p(X))} \|A + T\|_p.$$

All relations in (1.14) except the first one are proved similarly to Lemma 1.5; the first relation $\|W_a^0\|_2 \leqslant \|W_a^k\|_p$ is well known (cf. [10, 16]).

We need one more inequality, which is also well-known (cf. [16]): if $a \in M_r(\mathbf{R}^n)$ and $p \in (r, r')$, $r' = r/(r - 1)$, then

$$\|W_a^k\|_p \leqslant \|W_a^k\|_r^{1-\gamma}\cdot \sup|a(\xi)|^\gamma, \quad \gamma = \frac{2(r - p)}{p(r - 2)},\ r \neq 2,\ k = 0, 1. \tag{1.15}$$

5°. LOCAL PRINCIPLE. Here some necessary information from [14], Chapter XII, § 1 will be given.

Let Σ be Banach algebra with the unit element e; a set $\Lambda \subset \Sigma$ is called a *localizing class* if $0 \notin \Lambda$ and for any $a, b \in \Lambda$ there exists $c \in \Lambda$ such that $ac = bc = ca = cb = c$.

Elements $x, y \in \Sigma$ are called *Λ-equivalent* if

$$\inf_{a\in\Lambda} \|(x-y)a\| = \inf_{a\in\Lambda} \|a(x-y)\| = 0,$$

and the notation $x \overset{\Lambda}{\sim} y$ is used.

An element $x \in \Sigma$ is called *left* (*right*) *Λ-invertible* if there exist $z \in \Sigma$ and $a \in \Lambda$ such that $zxa = a$ ($axz = a$).

A system of localizing classes $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called *covering* if from each choice of elements $\{a_\omega\}_{\omega\in\Omega}$ ($a_\omega \in \Lambda_\omega$) one can find a finite number whose sum is invertible in Σ.

THEOREM 1.17. *Let $\{\Lambda_\omega\}_{\omega\in\Sigma}$ be a covering system of localizing classes, and let $x \overset{\Lambda_\omega}{\sim} y_\omega$ ($x, y_\omega \in \Sigma$, $\omega \in \Omega$). If x commutes with all $a \in \bigcup_{\omega\in\Omega} \Lambda_\omega$, then it is left (right) invertible in Σ if and only if y_ω are left (right) Λ_ω-invertible for all $\omega \in \Omega$.*

2. SINGULAR INTEGRAL OPERATORS ON THE HALF-SPACE

1°. ON THE FACTORIZATION AND PARTIAL INDICES OF DISCONTINUOUS MATRIX-FUNCTIONS. Let $a(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$ ($m > n/2$) be elliptic (nondegenerate) $\inf_{\xi\in S^{n-1}} |\det a(\xi)| > 0$; consider the constant matrix

$$a_0 = a^{-1}(-1, 0, \ldots, 0)\, a(+1, 0, \ldots, 0) \tag{2.1}$$

and let $\lambda_1, \ldots, \lambda_l$ be the eigenvalues of a_0 with multiplicities $r_1, \ldots, r_l$ $\left(\sum_{j=1}^{l} r_j = N\right)$.

Then the representation

$$a_0 = gB(1)g^{-1}, \quad \det g \neq 0$$

$$B(1) = \operatorname{diag}[\lambda_1 B^{r_1}(1), \ldots, \lambda_l B^{r_l}(1)],$$

$$B^r(\alpha) = \|b_{\nu k}(\alpha)\|_{\nu,k=1}^{r}, \quad b_{\nu k} = \begin{cases} 0, & \nu < k, \\ 1, & \nu = k, \\ \dfrac{\alpha^{\nu-k}}{(\nu-k)!}, & \nu > k \end{cases}$$

is possible (a_0 and $B(1)$ have the same eigenvalues and therefore same Jordan form).

The matrix-functions $B^r(\alpha)$ have the property

$$B^r(\alpha+\beta) = B^r(\alpha)\cdot B^r(\beta); \quad B^r(0) = I;$$

in particular

$$B^r(-\alpha) = [B^r(\alpha)]^{-1}.$$

If now $\alpha_\pm(t) = (2\pi i)^{-1}\ln(t \pm i)$, we get $\lim\limits_{t\to-\infty}[\alpha_+(t)-\alpha_-(t)] = 0$, $\lim\limits_{t\to+\infty}[\alpha_+(t) - \alpha_-(t)] = 1$; hence

$$\lim_{t\to-\infty} B^r(\alpha_-(t))\,[B^r(\alpha_+(t))]^{-1} = I,$$

$$\lim_{t\to\infty} B^r(\alpha_-(t))\,[B^r(\alpha_+(t))]^{-1} = [B^r(1)]^{-1}.$$

The matrix-functions

(2.2) $$B_\pm(t) = \operatorname{diag}\,[B^{r_1}(\alpha_\pm(t)), \ldots, B^{r_l}(\alpha_\pm(t))]$$

are holomorphic in $\pm\operatorname{Im} t > 0$.

Let now

(2.3) $$\delta_j(a) = \delta_j = \frac{\ln\lambda_j}{2\pi i},\ \frac{1}{p} - 1 < \operatorname{Re}\delta_j \leqslant \frac{1}{p} \quad (j=1,2,\ldots,l;\ 1<p<\infty);$$

such a choice of δ_j is obviously possible and they are defined uniquely. Clearly

(2.4) $$\delta_0 < 1 - \operatorname{Re}[\delta_j - \delta_r] \quad (1 \leqslant j \leqslant l)$$

for some $\delta_0 > 0$.

Let

(2.5) $$\delta = (\delta_1', \ldots, \delta_N'), \quad \delta_j' = \delta_k \text{ for } \sum_{\nu=1}^{k-1} r_\nu < j < \sum_{\nu=1}^{k} r_\nu, \quad j = 1, 2, \ldots, N,$$

$$(t \pm i)^\delta = \operatorname{diag}[(t\pm i)^{\delta_1'}, \ldots, (t \pm i)^{\delta_N'}];$$

clearly $(t \pm i)^\delta B_\pm(t) = B_\pm(t)(t\pm i)^\delta$ since $(t \pm i)^\delta$ is a diagonal matrix-function having the same element inside of block of $B_\pm(t)$. We set

(2.6) $$a_*(\xi) = (\xi_1 - i)^{-\delta} B_-^{-1}(\xi_1)\, g^{-1} a^{-1}(-1, 0, \ldots, 0)\, a(\xi) g B_+(\xi_1)\,(\xi_1 + i)^\delta$$
$$(\xi = (\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{n-1}).$$

Let us notice that if $l = N$ (i.e. $r_1 = \ldots = r_N = 1$) then $B_\pm(t) \equiv I$ (I is the identity matrix).

LEMMA 2.1. $a_*(\xi_1, \xi') \in C^{m+2}(\mathbf{R})$ *for all* $\xi' \in \mathbf{R}^{n-1}$ (cf. (2.6)) *and* (cf. (2.4))

$$(2.7)\quad \begin{aligned} &D^k_{\xi_1}[a^*(\xi_1, \xi') - I]_{j\nu} = O(|\xi_1|^{-k-\mathrm{Re}\,\delta_j+\mathrm{Re}\,\delta_\nu'-1}) = O(|\xi_1|^{-k-\delta_0}),\\ &\xi_1 \to \pm\infty, \quad j, \nu = 1,2,\ldots,N, \quad k = 0,1,\ldots,m+2. \end{aligned}$$

Proof. In [29], Appendix, it is proved that

$$D^\nu_{\xi_1} b(\xi_1) = O(|\xi_1|^{-k-1}), \qquad k = 0,1,\ldots,m+2$$

$$\left(b(\xi_1) = B_-(\xi_1) g^{-1} a^{-1}(-1, 0, \ldots, 0)\, a(\xi)\, B_+(\xi_1) - \left(\frac{\xi_1 - \mathrm{i}}{\xi_1 + \mathrm{i}}\right)^\delta\right);$$

(2.7) is now obvious since

$$[a_*(\xi_1, \xi') - I]_{j\nu} = [(\xi_1 - \mathrm{i})^{-\delta} b(\xi_1)(\xi_1 + \mathrm{i})^\delta]_{j\nu} = (\xi_1 - \mathrm{i})^{-\delta_j} b_{j\nu}(\xi_1)(\xi_1 + \mathrm{i})^{\delta_\nu}. \quad ▨$$

By $W^r(\mathbf{R})$ ($r = 0,1,2,\ldots$) denote the subalgebra of the Wiener-algebra

$$W(\mathbf{R}) \stackrel{\mathrm{def}}{=} \{f(t) = c + \mathscr{F}g(t) : g \in L_1(\mathbf{R})\},$$

consisting of functions $f(t)$ with the property

$$(1 - \mathrm{i}t)^k D^k_t f(t) \in W(\mathbf{R}) \quad (k = 0, 1, \ldots, r).$$

LEMMA 2.2. (cf. [29]). $W^r(\mathbf{R})$ *is a Banach algebra with the norm*

$$\|f\|^{(r)} = \|f\|_W + \sum_{k=1}^{r} \|(1 - \mathrm{i}t)^k D^k_t f(t)\|_W =$$

$$= |c| + \int_{-\infty}^{\infty} |g(\lambda)|\, d\lambda + \sum_{k=1}^{r} \int_{-\infty}^{\infty} |(D_\lambda + 1)^k \lambda^k g(\lambda)|\, d\lambda,$$

where $f = c + \mathscr{F}g$ ($g \in L_1$) *and* $(D_\lambda + 1)^k = \mathscr{F}^{-1}(1 - \mathrm{i}t)^k \mathscr{F}$.

The singular integral operator

$$S_{\mathbf{R}}\varphi(t) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\varphi(\tau)\, d\tau}{\tau - t}$$

is bounded in $W^r(\mathbf{R})$ (it is decomposable) and the set of all rational functions, vanishing at infinity and having poles off $\mathbf{R}$ are dense in $W^r(\mathbf{R})$ (it is rationally dense).

LEMMA 2.3 (cf. [29]). *Let $r = 0, 1, 2, \ldots$ and the function $b(t) \in C^{r+1}(\mathbf{R})$ has the property*

$$D_t^k b(t) = O(|t|^{-k-\nu}), \quad k = 0,1, \ldots, r+1.$$

Then $b(t) \in W^r(\mathbf{R})$.

COROLLARY 2.4 (cf. [29]). $a_*(t, \theta') \in (W^{m+1})^{N\times N}(\mathbf{R})$ *for all* $\theta' \in S^{n-2}$ *(cf. (2.6)).*

Lemma 2.2, Corollary 2.4 and main Theorem from [2] yield:

THEOREM 2.5 (cf. [29]). *The matrix-function $a_*(t, \theta')$ (cf. (2.6); $t \in \mathbf{R}$; $\theta' \in S^{n-2}$) can be factored on the form*

$$a_*(t, \theta') = (a_*^-)^{-1}(t, \theta') \ \operatorname{diag}\left[\left(\frac{t-i}{t+i}\right)^{\varkappa(\theta')}\right] a_*^+(t, \theta'), \tag{2.8}$$

where $(a_*^-)^{\pm 1}(t, \theta')$, $(a_*^+)^{\pm 1}(t, \theta') \in (W^{m+1})^{N\times N}(\mathbf{R})$ have analytic extensions in lower $\operatorname{Im} t < 0$ and upper $\operatorname{Im} t > 0$ half-planes respectively for all $\theta' \in S^{n-2}$. $\varkappa(\theta') = (\varkappa_1(\theta'), \ldots, \varkappa_N(\theta'))$ is uniquelly determined $\varkappa_1(\theta') \geqslant \ldots \geqslant \varkappa_N(\theta')$; the integer

$$\varkappa(\theta') = \sum_{j=1}^{N} \varkappa_j(\theta') = \int_{-\infty}^{\infty} dt[\arg\det a_*(t, \theta)] \tag{2.9}$$

is continuous and partial sums

$$\sum_{j=1}^{r} \varkappa_j(\theta') \quad (1 \leqslant r \leqslant N)$$

are upper semi-continuous (i.e. do not increase under small perturbations) with respect to $\theta' \in S^{n-2}$.

Integers $\varkappa_1(\theta'), \ldots, \varkappa_N(\theta')$ will be called *partial p-indices* of $a(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$ (they depend on p obviously; cf. (2.3)).

REMARK 2.6. The index $\varkappa(\theta')$, defined by (2.9), does not depend on θ' for $n > 2$ (i.e. when S^{n-2} is connected set), because it is a continuous function admitting only integer values; for $n = 2$ $\varkappa(\theta') = \varkappa(\pm 1)$ and these two integers can differ $\varkappa(-1) \neq \varkappa(+1)$.

In the scalar case $\varkappa_1(\theta') \equiv \varkappa(\theta')$ and $\varkappa_1(\theta')$ is calculated by the formula (2.9); the integer $\varkappa_1(\theta')$ and the number $\varkappa_1(a)$ (cf. (2.3)) are the integer parts of the real numbers

$$\delta(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\{\arg a(\lambda, 1, \dots, 1)\} - \operatorname{Re}\delta, \quad \text{for } n > 2$$

$$\delta_{\pm}(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\{\arg a(\lambda, \pm 1)\} - \operatorname{Re}\delta, \quad \text{for } n = 2.$$

2°. STATEMENT OF THEOREMS

THEOREM 2.7. *Let* $a(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$ *be elliptic,* $\inf|\det a(\theta)| > 0$ *(*$\theta \in S^{n-1}$*) and* $\varkappa_N(\theta') \leqslant \dots \leqslant \varkappa_1(\theta')$ *be the partial p-indices of* $a(\xi)$ *(*$1 < p < \infty$*).*

*If (*cf. (2.3)*)*

(2.10) $$\operatorname{Re}\delta_j \neq 1/p \quad \text{for all } j = 1,2,\dots,l$$

and

(2.11) $$\varkappa_1(\theta') \equiv \dots \equiv \varkappa_N(\theta') \equiv 0 \quad (\theta' \in S^{n-2}),$$

the operator W_a^1 *is invertible in* $L_p^N(\mathbf{R}^{n+})$.

If (2.10) *holds,* $\varkappa_N(\theta') \geqslant 0$ *but* (2.11) *does not hold the operator* W_a^1 *is left-invertible in* $L_p^N(\mathbf{R}^{n+})$ *and* $\dim\operatorname{Coker} W_a^1 = \infty$.

If (2.10) *holds,* $\varkappa_1(\theta') \leqslant 0$ *but* (2.11) *does not hold the operator* W_a^1 *is right-invertible and* $\dim\operatorname{Ker} W_a^1 = \infty$.

*In all other cases (*i.e. (2.10) *does not hold or* (2.10) *holds but* $\varkappa_N(\omega) < 0$ *and* $\varkappa_1(\tilde\omega) > 0$ *for some* $\omega, \tilde\omega \in S^{n-2}$*) the operator* W_a^1 *has no left and no right regularizers in* $L_p^N(\mathbf{R}^{n+})$.

THEOREM 2.8. *Let* $a(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$; *if* $\inf|\det a(\theta)| = 0$ *(*$\theta \in S^{n-1}$*) the operator* W_a^1 *has no left and no right regularizers in* $L_p^N(\mathbf{R}^n)$.

COROLLARY 2.9. *Let conditions of Theorem* 2.7 *hold;* W_a^1 *is a Fredholm operator in* $L_p(\mathbf{R}^{n+})$ *if and only if* (2.10) *and* (2.11) *hold.*

COROLLARY 2.10. *Let conditions of Theorem* 2.7, *including* (2.10), *hold and*

$$a_\varkappa(\xi) = \left(\frac{\xi_1 - i|\xi'|}{\xi_1 + i|\xi'|}\right)^{\varkappa} a(\xi) \quad (\varkappa = 0, \pm 1, \pm 2, \dots)$$

there exist integers $\varkappa' \leqslant \varkappa''$ *such that* W_a^1 *is left invertible in* $L_p^N(\mathbf{R}^{n+})$ *for all* $\varkappa \geqslant \varkappa''$ *and is right invertible for all* $\varkappa \leqslant \varkappa'$.

Theorems will be proved below; Corollary 2.9 is obvious and we explain here only Corollary 2.10 (cf. [29]): partial p-indices of $a_\varkappa(\xi)$ are $\varkappa + \varkappa_j(\theta')$ ($j = 1,2, \dots, N$), where $\varkappa_j(\theta')$ are partial p-indices of $a(\xi)$; using semi-continuity of partial sums of p-indices (cf. Theorem 2.5) we conclude

$$-\infty < -\varkappa'' = \inf \varkappa_N(\theta') \leqslant \inf \varkappa_j(\theta') \leqslant \sup \varkappa_j(\theta') \leqslant$$
$$\leqslant \sup \varkappa_1(\theta) \leqslant -\varkappa' < \infty \quad (\theta' \in S^{n-2}).$$

REMARK 2.11. Conditions of Theorem 2.7 on $a(\xi) \in HC^{m+2}(\mathbf{R}^n)$ can be weakened (we can demand less smoothness of $a(\xi)$ with respect to $\xi' \in \mathbf{R}^{n-1}$); especially can be weakened the condition in Theorem 2.8 (we can demand there $a(\xi) \in Hm_p(\mathbf{R}^n) \cap C(S^{n-1})$).

Condition (2.10) seems to be in fact necessary and sufficient for W_a^1 being normally solvable (i.e. to have a closed range).

3°. ON THE INVERTIBILITY OF CERTAIN OPERATORS. We prove here a theorem which is a particular case of Theorem 2.7, but deals also with weighted spaces; this theorem will be used in proving Theorem 2.7 in § 2.4°.

THEOREM 2.12. *Let*

$$g_\gamma(\xi) = \left(\frac{i\xi_1 + |\xi'|}{i\xi_1 - |\xi'|}\right)^\gamma, \quad \xi = (\xi_1, \xi') \in \mathbf{R}^{n+},$$

where

$$\alpha + \frac{1}{p} - 1 < \operatorname{Re}\gamma < \alpha + \frac{1}{p}, \quad -\frac{1}{p} < \alpha < 1 - \frac{1}{p}, \quad 1 < p < \infty. \tag{2.12}$$

The operator $W_{g_\gamma}^1$ *is invertible in the space* $L_p(\mathbf{R}^{n+}, x_1^\alpha)$.

Proof. Consider first the case

$$\alpha + \frac{1}{p} - 1 < \operatorname{Re}\gamma < 0 \tag{2.13}$$

and factor function $g(\xi)$

$$g_\gamma(\xi) = g_-(\xi)\, g_+(\xi),$$
$$g_-(\xi) = (i\xi_1 + |\xi'|)^\gamma \quad g_+(\xi) = (i\xi_1 - |\xi'|)^{-\gamma}. \tag{2.14}$$

The operator

$$(W^1_{g_\gamma})^{-1} = W^1_{g_+^{-1}} W^1_{g_-^{-1}} = \chi_{1+} W^0_{g_+^{-1}} \chi_{1+} W^0_{g_-^{-1}} \chi_{1+} =$$

$$= \chi_{1+} W^0_{g_+^{-1}} W^0_{g_-^{-1}} \chi_{1+} - \chi_{1+} W^0_{g_+^{-1}} \chi_{1-} W^0_{g_-^{-1}} \chi_{1+} = W^1_{g_\gamma^{-1}} - A_1,$$

where

$$A_1 = \chi_{1+} W^0_{g_+^{-1}} \chi_{1-} W^0_{g_-^{-1}} \chi_{1+},$$

is inverse to $W^1_{g_\gamma}$ on the set $C_0^\infty(\mathbf{R}^{n+})$: $W^1_{g_\gamma}(W^1_{g_\gamma})^{-1}\varphi = (W^1_{g_\gamma})^{-1} W^1_{g_\gamma}\varphi = \varphi$ ($\varphi \in C_0^\infty(\mathbf{R}^{n+})$) and it remains to prove only the boundedness of the operator $(W^1_{g_\gamma})^{-1}$ in the space $L_p(\mathbf{R}^{n+}, x_1^\alpha)$; obviously it is sufficient to prove the boundedness of $A = x_1^\alpha A_1 x_1^{-\alpha} I$ in $L_p(\mathbf{R}^{n+})$.

We have (cf. [16])

$$W^0_{g_\pm^{-1}} \varphi(t) = \int_{\mathbf{R}^n} m_\pm(t-\xi)\, \varphi(\xi)\, d\xi \tag{2.15}$$

where $\varphi \in C_0^\infty(\mathbf{R}^n)$ and $m_\pm(\eta)$ exist as distributions[1)] $m_\pm = \mathscr{F}^{-1}(g_\pm^{-1})$;

$$A\varphi(t) = \int_{\mathbf{R}^{n+}} m(t_1, \tau_1;\ t' - \tau')\, \varphi(\tau)\, d\tau, \quad \tau = (\tau_1, \tau'),\ t = (t_1, t') \in \mathbf{R}^{n+},$$

$$m(t_1, \tau_1;\ t') = \left(\frac{t_1}{\tau_1}\right)^\alpha \int_0^\infty \int_{\mathbf{R}^{n-1}} m_+(t_1 + y;\ t' - \xi')\, m_-(-\tau_1 - y;\ \xi')\, dy\, d\xi' =$$

$$= (\mathscr{F}^{-1}_{\xi' \to t'} \widetilde{m})(t_1, \tau_1;\ t'),$$

$$\widetilde{m}(t_1, \tau_1;\ \xi') = \left(\frac{t_1}{\tau_1}\right)^\alpha \int_0^\infty \widetilde{m}_+(t_1 + y;\ \xi')\, \widetilde{m}_-(-\tau_1 - y;\ \xi')\, dy,$$

$$\widetilde{m}_\pm(t_1, \xi') = (\mathscr{F}^{-1}_{\xi_1 \to t_1} g_\pm^{-1})(t_1, \xi'),$$

where $\widetilde{m}_\pm$ exist as distributions again.

---

1) The convolutions (2.15) of distributional kernels $m_\pm$ with functions $\varphi \in C_0^\infty(\mathbf{R}^n)$ exist as regular functions.

With the help of formulas (cf. [15], 3.382)

$$\int_{-\infty}^{\infty} (\beta + \mathrm{i}x)^{-\nu} \mathrm{e}^{-\mathrm{i}\lambda x} \mathrm{d}x = -\frac{2\pi(-\lambda)^{\nu-1}}{\Gamma(\nu)} \mathrm{e}^{\beta\lambda} \chi_-(\lambda),$$

(2.17)
$$\int_{-\infty}^{\infty} (\beta - \mathrm{i}x)^{-\nu} \mathrm{e}^{-\mathrm{i}\lambda x} \mathrm{d}x = \frac{2\pi\lambda^{\nu-1}\mathrm{e}^{-\beta\lambda}}{\Gamma(\nu)} \chi_+(\lambda),$$

$$\chi_\pm(\lambda) = \frac{1}{2}(1 \pm \operatorname{sgn}\lambda), \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0,$$

we obtain (cf. (2.13) — (2.14))

$$m_+(t_1, \xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}t_1\xi_1} \mathrm{d}\xi_1}{(\mathrm{i}\xi_1 - |\xi'|)^{-\gamma}} = \frac{(-1)^\gamma}{\Gamma(-\gamma)} t_1^{-\gamma-1} \mathrm{e}^{-|\xi'|t_1} \chi_{1+}(t_1),$$

$$m_-(t_1, \xi') = -\left(\frac{\partial}{\partial t_1} - |\xi'|\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}t_1\xi_1} \mathrm{d}\xi_1}{(\mathrm{i}\xi_1 + |\xi'|)^{\gamma+1}} =$$

$$= -\left(\frac{\partial}{\partial t_1} - |\xi'|\right) \left[\frac{(-t_1)^\gamma}{\Gamma(1+\gamma)} \mathrm{e}^{|\xi'|t_1} \chi_{1-}(t_1)\right] =$$

$$= -\frac{(-1)^\gamma}{\Gamma(\gamma)} t_1^{\gamma-1} \mathrm{e}^{|\xi'|t_1} \chi_{1-}(t_1);$$

hence

$$m(t_1, \tau_1, t') = \mathscr{F}^{-1}_{\xi' \to t'} m(t_1, \tau_1, t') =$$

$$= \left(\frac{t_1}{\tau_1}\right)^\alpha \frac{(-1)^\gamma}{(2\pi)^{n-1}} \int_{\mathbf{R}^{n-1}} \mathrm{e}^{-t'\xi'} \mathrm{d}\xi' \int_0^\infty \frac{(t_1 + y)^{-\gamma-1}(\tau_1 + y)^{\gamma-1}}{\Gamma(-\gamma)\,\Gamma(\gamma)} \mathrm{e}^{-|\xi'|(t_1+\tau_1+2y)} \mathrm{d}y =$$

$$= \left(\frac{t_1}{\tau_1}\right)^\alpha \frac{(-1)^\gamma}{(2\pi)^{n-1}\Gamma(-\gamma)\,\Gamma(\gamma)} \int_0^\infty (t_1 + y)^{-\gamma-1}(\tau_1 + y)^{\gamma-1} \mathrm{d}y \times$$

$$\times \int_{\mathbf{R}^{n-1}} \mathrm{e}^{-[\mathrm{i}t'\cdot\xi' + (t_1+\tau_1+2y)|\xi'|]} \mathrm{d}\xi';$$

to calculate the last factor we notice that it is the Fourier transform of the function, depending only on the distance $|\xi'|$; it is easy to prove then that the transform also depends only on the distance $|t'|$ and one can take (without loss of generality) $t_2 = |t'| = x$, $t_3 = \dots = t_n = 0$; using then spherical coordinates $d\xi_3 \dots d\xi_n = r^{n-3} d\sigma dr$ $\left(r = \sum_{k=3}^{n} |\xi_k|^2, d\sigma \text{ — element of the sphere} \right)$ we easily get ($t_1 + \tau_1 + 2y = u$ for brevity)

$$\int_{R^{n-1}} e^{-(it'\xi' + u|\xi'|)} d\xi' = \int_{R^{n-1}} e^{-i(x\xi_2 + u|\xi'|)} d\xi' =$$

$$= \int_{-\infty}^{\infty} e^{-ix\xi_2} d\xi_2 \int_0^{\infty} r^{n-3} e^{-u\sqrt{\xi_2^2 + r^2}} dr \int_{S^{n-3}} d\sigma =$$

$$= 2c_0 \int_0^{\infty} r^{n-3} dr \int_0^{\infty} e^{-u\sqrt{\xi_2^2 + r^2}} \cos(x\xi_2) d\xi_2 =$$

$$= \frac{2c_0 u}{\sqrt{u^2 + x^2}} \int_0^{\infty} r^{n-2} K_1(r\sqrt{u^2 + x^2}) dr =$$

$$= \frac{2c_0 u}{(\sqrt{u^2 + x^2})^n} \int_0^{\infty} r^{n-2} K_1(r) dr = \frac{2c_1 u}{(\sqrt{u^2 + |t'|^2})^n},$$

where c_0, c_1 are constants and $K_1(\tau)$ is the modified Bessel function of the third kind.

Returning back to $m(t_1, \tau_1, t')$ we get

$$m(t_1, \tau_1, t') = \left(\frac{t_1}{\tau_1}\right)^{\alpha} \frac{(-1)^{\gamma} 2c_0}{(2\pi)^{n-1} \Gamma(-\gamma) \Gamma(\gamma)} \times$$

$$\times \int_0^{\infty} \frac{(t_1 + y)^{-\gamma-1} (\tau_1 + y)^{-1} (t_1 + \tau_1 + 2y)}{(\sqrt{(t_1 + \tau_1 + 2y)^2 + |t'|^2})^n} dy.$$

From (2.16) we have

$$|A\varphi(t)| \leqslant \left(\int_{R^{n+}} |m(t_1, \tau_1, t' - \tau')| \tau_1^{\delta p} |\varphi(\tau)|^p d\tau \right)^{1/p} \times$$

(2.18)

$$\times \left(\int_{R^{n+}} |m(t_1, \tau_1, t' - \tau')| \tau_1^{-\delta q} d\tau \right)^{1/p},$$

where

$$0 < \delta < \min\left(\frac{1}{p}, \frac{1}{q}\right), \quad q = \frac{p}{p-1}. \tag{2.19}$$

Consider last factor in (2.18) (notation: $\beta = \operatorname{Re}\gamma$)

$$J = \int_{\mathbf{R}^{n+}} |m(t_1, \tau_1, t' - \tau')| \, \tau_1^{-\delta q} \, d\tau \leqslant$$

$$\leqslant C_1 \int_0^\infty \frac{t_1^\alpha \, dy}{(y + t_1)^{1+\beta}} \int_0^\infty \frac{\tau_1^{-\delta q} \, d\tau_1}{(y + \tau_1)^{1-\beta}} \int_{\mathbf{R}^{n-1}} \frac{(t_1 + \tau_1 + 2y) \, d\tau'}{(\sqrt{(t_1 + \tau_1 + 2y)^2 + |\tau' - t'|^2})^n} =$$

(2.20)

$$= C_1 \int_0^\infty \frac{t_1^\alpha \, dy}{(y + t_1)^{1+\beta}} \int_0^\infty \frac{\tau_1^{-\delta q - \alpha} \, d\tau_1}{(y + \tau_1)^{1-\beta}} \int_{\mathbf{R}^{n-1}} \frac{d\tau'}{(\sqrt{1 + |\tau'|^2})^n} =$$

$$= C_2 t_1^\alpha \int_0^\infty \frac{y^{\beta - \delta q - \alpha} \, dy}{(y + t_1)^{1+\beta}} \int_0^\infty \frac{dz}{z^{\delta q + \alpha}(1 + z)^{1-\beta}};$$

here and in what follows constants $C_1, C_2, \ldots$ depend only on γ and p. If

$$0 < \delta q - \beta + \alpha < 1, \tag{2.21}$$

(2.20) yield

$$J \leqslant C_3 t_1^\alpha \int_0^\infty \frac{y^{\beta - \delta q - \alpha} \, dy}{(t_1 + y)^{1+\beta}} = C_3 t_1^{-\delta q + \alpha} \int_0^\infty \frac{dz}{(1 + z)^{1+\beta} z^{\delta q + \alpha - \beta}} = C_4 t_1^{\alpha - \delta q}.$$

From (2.18) we obtain now

$$\int_{\mathbf{R}^{n+}} |A\varphi(t)|^p \, dt \leqslant \int_{\mathbf{R}^{n+}} t_1^{\alpha\delta - p} \, dt \int_{\mathbf{R}^{n+}} \tau_1^{\beta p} |m(t_1, \tau_1, t' - \tau')| \, |\varphi(\tau)|^p \, d\tau \leqslant$$

$$\leqslant C_5 \int_{\mathbf{R}^{n+}} \tau_1^{\beta p - \alpha} |\varphi(\tau)|^p \, d\tau \int_0^\infty \frac{dy}{(\tau_1 + y)^{1-\beta}} \int_0^\infty \frac{dt_1}{t_1^{\delta p - \alpha}(t_1 + y)^{1+\beta}} \times$$

$$\times \int_{\mathbf{R}^{n-1}} \frac{(t_1 + \tau_1 + 2y) \, dt'}{(\sqrt{(t_1 + \tau_1 + 2y)^2 + |t' - \tau'|^2})^n} =$$

$$= C_6 \int_{\mathbf{R}^{n+}} \tau_1^{\beta p - \alpha} |\varphi(\tau)|^p \, d\tau \int_0^\infty \frac{y^{\alpha - \delta p - \beta} \, dy}{(\tau_1 + y)^{1-\beta}} \int_0^\infty \frac{dz}{z^{\delta p - \alpha}(1 + z)^{1+\beta}};$$

if

(2.22) $$0 < \beta + \delta p - \alpha < 1,$$

integrals exist and

$$\int_{\mathbf{R}^{n+}} |A\varphi(t)|^p \, dt \leqslant C_7 \int_{\mathbf{R}^{n+}} |\varphi(\tau)|^p \, d\tau \int_0^\infty \frac{dz}{(z+1)^{1-\beta} z^{\beta+\delta p-\alpha}} = C_8 \int_{\mathbf{R}^{n+}} |\varphi(\tau)|^p d\tau.$$

Finally $\|A\varphi\|_p \leqslant C_9 \|\varphi\|_p$ if (2.19) — (2.22) are valid; these inequalities are compatible, because $\beta = \operatorname{Re} \gamma$ satisfies (2.13).

Let now

(2.23) $$0 < \operatorname{Re} \gamma < \alpha + \frac{1}{p};$$

the conjugate operator $W^1_{\bar{g}_\gamma}$ to $W^1_{g_\gamma}$ in the conjugate space $L_q(\mathbf{R}^{n+}, x_1^{-\alpha})$ is invertible, because

$$\overline{g_\gamma(\xi)} = \left(\frac{i\xi_1 + |\xi'|_1}{i\xi_1 - |\xi'|_1} \right)^{-\gamma} = g_{-\gamma}(\xi), \quad q = \frac{p}{p-1}$$

and (2.23) can be rewritten as

$$-\alpha + \frac{1}{q} - 1 < \operatorname{Re}(-\gamma) < 0;$$

invertibility of $W^1_{\bar{g}}$ in $L_q(\mathbf{R}^{n+}, x_1^{-\alpha})$ yields the invertibility of $W^1_{g_\gamma}$ in $L_p(\mathbf{R}^{n+}, x_1^\alpha)$.

Consider now the last case: $\operatorname{Re} \gamma = 0$ but $\gamma \neq 0$; the inverse operator $(W^1_g)^{-1} = W^1_{g_+^{-1}} W^1_{g_-^{-1}}$ is bounded in $L_p(\mathbf{R}^{n+})$ since $g_\pm(\xi) = (i\xi_1 \pm |\xi'|_1)^{\pm\gamma} \in m_p(\mathbf{R}^n)$ (due to Theorem 1.10). The functions $g_\pm^{-1}(\xi)$ are homogeneous of order $\pm\gamma$ and $\operatorname{Re} \gamma = 0$. The operators $W^1_{g_\pm^{-1}}$ have then the form (cf. [16]):

$$W_{g_\pm^{-1}} \varphi(x) = \int_{\mathbf{R}^{n+}} \frac{\Omega_\pm(x-y)\, \varphi(y)\, dy}{|y-x|^n},$$

$$g_\pm^{-1}(\xi) = \mathscr{F}_{t \to \xi} [\Omega_\pm(t) |t|^{-n}](\xi)$$

and therefore (cf. (2.17))

$$\Omega_\pm(t) = C_\pm \cdot t_1^{\pm\gamma},$$

where $C_\pm$ are constants. Hence $|\Omega_\pm(t)| \equiv |C_\pm|$ and in virtue of Theorem 1.4 the operators $W^1_{g_\pm^{-1}}$ (and therefore operator $(W^1_{g_\gamma})^{-1}$) are bounded in $L_p(\mathbf{R}^{n+}, x_1^\alpha)$. ▨

4°. PROOF OF THEOREM 2.7. Let $\sigma\colon \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$, $|\sigma\xi| = |\xi|$ be a rotation of $\mathbf{R}^{n-1}$ about the origin and

$$\sigma_*\varphi(\xi) = \varphi(\sigma\xi).$$

The operator

$$W^0_{\chi_{2+}}\varphi(t') = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(\tau, t'')}{\tau - t_2}\,d\tau, \qquad t' = (t_2, t''),$$

$$\chi_{2+}(\xi') \equiv \chi_{2+}(\xi_2) = \frac{1}{2}(1 + \operatorname{sgn}\xi_2)$$

is bounded in $L_p(\mathbf{R}^{n-1})$ and therefore $\chi_{2+}(\xi) \in m_p(\mathbf{R}^{n-1})$. But then $\sigma_*\chi_{2+} \in m_p(\mathbf{R}^{n-1})$ (cf. [16]) and

$$\|W^0_{\sigma_* d}\|_p = \|\sigma_* W^0_d \sigma'_*\|_p = \|W^0_d\|_p \quad (d \in m_p(\mathbf{R}^{n-1})),$$

since σ_* and conjugate σ'_* are isomorphisms in $L_p(\mathbf{R}^{n-1})$.

Let $\Sigma = \mathscr{L}(L_p(\mathbf{R}^{n+}))$; $\Lambda \subset \Sigma$ denote the set of all operators W^1_χ, where

$$\chi(\xi) \equiv \chi(\xi') = \prod_{k=1}^{n-1} \sigma_*^{(k)}\chi_{1+}(\xi')$$

and $\sigma^{(1)}, \ldots, \sigma^{(n-1)}$ are rotations; obviously $W^1_\chi = I \otimes W^0_\chi$, $\chi \in H(\mathbf{R}^n)$ and $\|W^1_\chi\|_p < \infty$.

By Λ_ω denote the subset of Λ, consisting of all operators $W^1_\chi \in \Lambda$ with $\chi(\theta) \equiv \equiv \chi(\theta') = 1$ in some neighbourhood of $\omega \in S^{n-2}$. Clearly Λ_ω is the localizing class (cf. § 1.6°) and due to the independence of $\chi(\xi)$ on ξ_1, $AW^1_d = W^1_d A$ for any $d \in m_p(\mathbf{R}^n)$, $A \in \Lambda$; it is easy also to show that the system $\{\Lambda_\omega\}_{\omega \in S^{n-2}}$ is a covering in Σ $\left(\text{if } \bigcup_{k=1}^{m} \operatorname{supp}\chi^{(k)} \cap S^{n-2} = S^{n-2},\right.$ then $g = \sum_{k=1}^{m} \chi^{(k)}$ is invertible in $m_p(\mathbf{R}^n)$ and $\left.\sum_{k=1}^{m} W^1_{\chi^{(k)}} = W^1_g,\ (W^1_g)^{-1} = W^1_{g^{-1}};\ \text{cf. Proposition 1.9}\right)$.

Equivalence (cf. § 1.6°)

$$\text{(2.24)} \qquad W^1_a \overset{\Lambda_\omega}{\sim} W^1_{a_\omega} \quad (\omega \in S^{n-2}), \qquad a_\omega(\xi) = a(\xi_1|\xi'|^{-1}, \omega)$$

is the simple consequence of the inequality (1.17) and of homogeneity of $a(\xi)$ $(a(\xi) \equiv a(\xi_1|\xi'|^{-1}, \theta'),\ \theta' = |\xi'|^{-1}\xi' \in S^{n-2})$.

In virtue of Theorem 1.17 the operator W^1_a will be left (right) invertible in $L^N_p(\mathbf{R}^{n+})$ if all $W^1_{a_\omega}$ $(\omega \in S^{n-2})$ are left (right) invertible in $L^N_p(\mathbf{R}^{n+})$.

(2.5) — (2.8) yield

$$a(t,\omega) \equiv a_-\tilde{a}_-^{-1}(t,\omega)\left(\frac{it+1}{it-1}\right)^{\varkappa(\omega)+\delta}\tilde{a}_+(t,\omega), \quad a_- = a(-1,0,\ldots,0),$$

(2.25) $$\tilde{a}_\pm(t,\omega) = (t\pm \mathrm{i})^\delta a_\pm^{\pm}(t,\omega)(t\pm \mathrm{i})^{-\delta}B_\pm^{\mp 1}(t)g^{-1},$$

$$\left(\frac{it+1}{it-1}\right)^{\varkappa(\omega)+\delta} = \operatorname{diag}\left[\left(\frac{it+1}{it-1}\right)^{\varkappa_1(\omega)+\delta_1'},\ldots,\left(\frac{it+1}{it-1}\right)^{\varkappa_N(\omega)+\delta_N'}\right].$$

Hence (cf. (2.22) — (2.23))

$$a_\omega(\xi) = a_{\omega-}(\xi)g_{\varkappa(\omega)+\delta}(\xi)a_{\omega+}(\xi),$$

(2.26) $$g_\gamma = \operatorname{diag}[g_{\gamma_1},\ldots,g_{\gamma_N}], \quad g_\mu(\xi) = \left(\frac{\mathrm{i}\xi_1+|\xi'|}{\mathrm{i}\xi_1-|\xi'|}\right)^\mu,$$

$$a_{\omega-}(\overline{\xi}) = a_-\tilde{a}_-^{-1}(\xi_1|\xi'|^{-1},\omega), \quad a_{\omega+}(\xi) = \tilde{a}_+(\xi_1|\xi'|^{-1},\omega).$$

Our next step is to prove the inclusions

(2.27) $$a_{\omega\pm}(\xi),\, a_{\omega\pm}^{-1}(\xi) \in m_p^{N\times N}(\mathbf{R}^n);$$

it suffices to obtain the asymptotics

(2.28) $$\mathrm{D}_t^q(a_{\omega\pm}^{\pm 1})_{jr}(t,\omega) = \mathrm{O}(|t|^{-q}), \quad t\to\pm\infty \quad j,r = 1,2\ldots N \quad q = 0,1,2,\ldots m,$$

because then

$$\sum_{|k|_1\leqslant m}\sup_\xi |\xi^k \mathrm{D}_\xi^k(a_{\omega\pm}^{\pm 1})_{jr}(\xi)| < \infty$$

and due to Theorem 1.10 (2.27) holds.

We will prove that

$$\tilde{a}_\pm^{\pm 1}(t,\omega) \in (W^m)^{N\times N}(\mathbf{R}),$$

which by the definition of the algebra $W^m(\mathbf{R})$ yields (2.28).

Consider for the definiteness $\tilde{a}_+(t) \equiv \tilde{a}_+(t,\omega)$ (we omit $\omega \in S^{n-2}$ for brevity); others are similar. A typical entry of $\tilde{a}_+$ is

(2.29) $$(\tilde{a}_+)_{jr}(t) = (t+\mathrm{i})^{\delta_j'-\delta_r'}\sum_{q\geqslant r}(a_*^+)_{jq}(t)\,[\ln(t+\mathrm{i})]^{\nu_{qr}}, \quad \nu_{rr}=0, \quad \delta_r' = \delta_q'.$$

If $\operatorname{Re}(\delta_j' - \delta_r') < 0$, then $(\tilde{a}_+)_{jr} \in W^{m+1}(\mathbf{R})$; that follows from the inclusions $(t+\mathrm{i})^{\delta_j' - \delta_r'}[\ln(t+\mathrm{i})]^{\nu} \in W^{m+1}(\mathbf{R})$ (cf. Lemma 2.3), $(a_*^+)_{jq} \in W^{m+1}(\mathbf{R})$ (cf. Theorem 2.5), and algebraic properties of $W^{m+1}(\mathbf{R})$.

Let $j = r$; then $(a_+)_{rr} \in W^m(\mathbf{R})$, if $(a_*^+)_{jq}[\ln(t+\mathrm{i})]^{\nu_{jq}} \in W^{m+1}(\mathbf{R})$ for $j \neq q$ (we remind that $\nu_{rr} = 0$); but we will prove more (and that will include the last case): if $\operatorname{Re}(\delta_j' - \delta_r') \geqslant 0$, $j \neq r$, then

$$(2.30)\qquad (t+\mathrm{i})^{\delta_j' - \delta_r'}(a_*^+)_{jr}(t)\,[\ln(t+\mathrm{i})]^{\nu} \in W^m(\mathbf{R}),$$

for any $-\infty < \nu < \infty$ (we remind that $\delta_r' = \delta_q'$ in (2.29)).

The factorization (2.8) of a_* easily yields

$$a_*^+(t) - a_*^-(t) = \left[\left(\frac{t+\mathrm{i}}{t-\mathrm{i}}\right)^{k} - 1\right] a^-(t) +$$

$$(2.31)\qquad + \left(\frac{t+\mathrm{i}}{t-\mathrm{i}}\right)^{\varkappa} a_*^-(t)\,[a_*(t) - 1] = b_1(t) + b_2(t)\,[a_*(t) - 1],$$

$$b_1(t) = \left[\left(\frac{t+\mathrm{i}}{t-\mathrm{i}}\right)^{\varkappa} - 1\right] a_*^-(t), \quad b_2(t) = \left(\frac{t+\mathrm{i}}{t-\mathrm{i}}\right)^{\varkappa} a_*(t).$$

Since $a_*^- \in (W^{m+1})^{N\times N}(\mathbf{R})$ clearly

$$(2.32)\qquad \mathrm{D}_t^q b_1(t) = \mathrm{O}(|t|^{-q-1}), \quad t \to \pm\infty, \quad q = 0,1,\ldots,m+1.$$

By the same reason $b_2 \in (W_2^{m+1})^{N\times N}(\mathbf{R})$ and making use of (2.7) we easily obtain

$$\mathrm{D}_t^q[b_2(a_* - 1)]_{jr}(t) = \mathrm{D}_t^q \sum_{s=1}^{N} (b_2)_{js}(t)(a_* - 1)_{sr}(t) =$$

$$(2.33)\qquad = \sum_{s=1}^{N} \mathrm{O}(|t|^{\operatorname{Re}[\delta_r' - \delta_j' + (\delta_j' - \delta_s' - 1)] + \varepsilon - q}) = \mathrm{O}(|t|^{\operatorname{Re}(\delta_r' - \delta_j') - q - \delta_0}),$$

$q = 0,1,\ldots,m+1$, where δ_0 is defined by (2.4); (2.31) — (2.33) yield

$$\mathrm{D}_t^q(a_*^+ - a_*^-)_{jr}(t) = \mathrm{O}(|t|^{\operatorname{Re}(\delta_r' - \delta_j') - \delta_0 - q}) \quad (q = 0,1,\ldots,m+1).$$

If now $\varphi_k(t) = t^k D_t^k \varphi(t) = O(|t|^{-\nu})$ $(0 < \nu < 1,\ k = 0,1, \ldots, m+1)$ then clearly $\left(t = i\frac{1+z}{1-z},\quad -\infty < t < \infty,\ |z| = 1\right)$

$$t^k D_t^k S_{\mathbf{R}}\varphi(t) = \frac{t^k}{\pi i} D_t^k \int_{-\infty}^{\infty} \frac{\varphi(\tau)\,d\tau}{\tau - t} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi_k(\tau)\,d\tau}{\tau - t} =$$

$$= \frac{1}{\pi i}\int_{|\zeta|=1} \frac{1-z}{1-\zeta}\,\frac{\varphi_k\left(i\frac{1+\zeta}{1-\zeta}\right)d\zeta}{\zeta - z} = \frac{1}{\pi i}\int_{|\zeta|=1} \frac{\varphi_k\left(i\frac{1+\zeta}{1-\zeta}\right)}{\zeta - z}\,d\zeta -$$

$$- \frac{1}{\pi i}\int_{|\zeta|=1} \frac{\varphi_k\left(i\frac{1+\zeta}{1-\zeta}\right)}{\zeta - 1}\,d\zeta \qquad \left(\varphi_k\left(i\frac{1+\zeta}{1-\zeta}\right) \in H_\nu(\Gamma),\ \varphi_k(1) = 0\right)$$

and we easily conclude

$$t^k D_t^k S_{\mathbf{R}}\varphi(t) = O(|t|^{-\nu}) \qquad (|t| \to \infty,\ k = 0,1, \ldots, m).$$

The operator $P^+ = \frac{1}{2}(I + S_{\mathrm{R}})$ has the property $P^+ a_*^+ = a_*^+$, $P^+ a_*^- = 0$; hence

$$(a_*^+)_{jr}(t) = P^+(a_*^+ \cdot a_*^-)_{jr}(t) = O(|t|^{\operatorname{Re}(\delta_r' - \delta_j') - \nu - q})$$

$$(|t| \to \infty,\ 0 < \nu < \delta_0,\ q = 0,1, \ldots, m);$$

(2.30) follows now with the help of Lemma 2.3.

Thus, the inclusion (2.27) is proved and the operators $W^1_{a_{\omega\pm}}$ are invertible (cf. Proposition 1.9)

(2.34) $$W^1_{a_{\omega\pm}} W^1_{a^{-1}_{\omega\pm}} = W^1_{a^{-1}_{\omega\pm}} W^1_{a_{\omega\pm}} = I.$$

Let now (2.10) and (2.11) hold; then $\varkappa_j(\omega) = 0$ and in virtue of Theorem 2.12 the operator

$$W^1_{g_\delta} = W^1_{g_{\delta_1'}} \oplus W^1_{g_{\delta_2'}} \oplus \ldots \oplus W^1_{g_{\delta_N'}}$$

(cf. (2.26)) is invertible in $L_p^N(\mathbf{R}^{n+})$. Since (cf. Proposition 1.9)

$$W^1_{a_\omega} = W^1_{a_\omega} W^1_{g_\delta} W^1_{a_{\omega+}},$$

the operator $W^1_{a_\omega}$ is invertible for all $\omega \in S^{n-2}$ (cf. (2.34)); but then (cf. (2.24)) W^1_a is invertible in $L^N_p(\mathbf{R}^{n+})$.

Let now (2.10) holds, $\varkappa_1(\omega) \leqslant 0$ and (2.11) does not hold. Since (cf. Proposition 1.9)

$$W^1_{a_\omega} = W^1_{a_{\omega-}} W^1_{g_{\varkappa(\omega)}} W^1_{g_\delta} W^1_{a_{\omega+}}, \tag{2.35}$$

where all operators, except $W^1_{g_{\varkappa(\omega)}}$, are invertible and the operator $W^1_{g_{\varkappa(\omega)}}$ is right--invertible (we remind that $\varkappa_N(\omega) \leqslant \ldots \leqslant \varkappa_1(\omega) \leqslant 0$)

$$W^1_{g_{\varkappa(\omega)}} W^1_{g_{-\varkappa(\omega)}} = I,$$

the operator $W^{-1}_{a_\omega}$ will be right-invertible as well. Due to Theorem 1.17 and (2.24) W^1_a is right-invertible in $L^N_p(\mathbf{R}^{n+})$.

We will finish with the case $\varkappa_1(\omega) \leqslant 0$ if as soon as $\varkappa_N(\omega) < 0$ for some $\omega \in S^{n-2}$ the operator W^1_a can not have a right regularizer in $L^N_p(\mathbf{R}^{n+})$.

Because (2.10) holds and $\varkappa_N(\omega) < 0$ for some $\omega \in S^{n-2}$, it is easy to notice that

$$W^1_{g_{\varkappa_N(\omega)}} \varphi = 0,$$

$$\varphi(t) = \sum_{k=1}^{\varkappa_N(\omega)} |t'|^{k-1+1/p} t_1^{k-1} e^{-|t'|t_1} \varphi_k(t'), \quad t = (t_1, t') \in \mathbf{R}^{n+},$$

where $\varphi_k(t')$ are arbitrary functions from $L_p(\mathbf{R}^{n-1})$. Hence (cf. (2.35))

$$W^1_{a_\omega} \psi = 0, \quad \psi = W^1_{a^{-1}_{\omega+}} (W^1_{g_\delta})^{-1} \tilde{\varphi}, \quad \tilde{\varphi} = (\varphi, 0, \ldots, 0)$$

and therefore $\dim \operatorname{Ker} W^1_{a_\omega} = \infty$. That means, among others, that $W^1_{a_\omega}$ cannot have a right regularizer in $L^N_p(\mathbf{R}^{n+})$.

Assume now W^1_a has the right regularizer in $L^N_p(\mathbf{R}^{n+})$; due to (2.24) and to Theorem 1.17 [1]

$$W^1_\chi W^1_{a_\omega} R = W^1_\chi + T, \tag{2.36}$$

where $W^1_\chi \in \Lambda_\omega$, $T \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+}))$; (2.36) remains valid for all $W^1_{\tilde{\chi}}$ with $\operatorname{supp} \tilde{\chi} \cap S^{n-2} \subset \operatorname{supp} \chi \cap S^{n-2}$, because $W^1_\chi W^1_{\tilde{\chi}} = W^1_{\tilde{\chi}} W^1_\chi = W^1_{\chi\tilde{\chi}} = W^1_{\tilde{\chi}}$. Using this property

[1] To be more exact we must consider the factor-algebra $\hat{\Sigma} = \Sigma/\mathfrak{S}(L^N_p(\mathbf{R}^n))$; since the existence of the right regularizer in Σ means right invertibility of the corresponding class in Σ, due to Theorem 1.17 and (2.27) (which remain valid for classes) we get (2.36).

and also the property $\sigma_* W^1_{a_\omega} = W^1_{a_\omega}\sigma_*$ for any rotation [1] σ, we easily conclude: there exist the operators $W^1_{\chi_1}, \ldots, W^1_{\chi_q} \in \Lambda$, $T_1, \ldots, T_q \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+}))$ and $R_1 \ldots, R_q \in \mathscr{L}(L^N_p((\mathbf{R}^n))$ such that [2]

(2.37)
$$W^1_{\chi_j} W^1_{a_\omega} R_j = W^1_{\chi_j} + T_j \quad (j = 1,2, \ldots q),$$
$$\sum_{j=1}^{q} \chi_j(\xi) = 1.$$

Due to

$$W^1_{\chi_j} W^1_{a_\omega} = W^1_{a_\omega} W^1_{\chi_j}$$

from (2.37) it follows

(2.38)
$$W^1_{a_\omega} R = I + T,$$

where

$$R = \sum_{j=1}^{q} W^1_{\chi_j} R_j, \quad T = \sum_{j=1}^{q} T_j.$$

(2.38) is a contradiction, because $W^1_{a_\omega}$ cannot have the right regularizer in $L^N_p(\mathbf{R}^{n+})$.

Let now (2.10) holds, $\varkappa_N(\theta') \geqslant 0$ and (2.11) does not hold; the conjugate operator $W^1_{\bar{a}}$ to the operator W^1_a is then right invertible in the conjugate space $L^N_{p'}(\mathbf{R}^{n+})$ $(p' = p/(p-1))$ and $\dim \operatorname{Ker} W^1_{\bar{a}} = \infty$; but then W^1_a is left invertible and $\dim \operatorname{Coker} W^1_a = \infty$.

Let now (2.10) holds but $\varkappa_N(\omega) < 0$ and $\varkappa_1(\tilde{\omega}) > 0$ for some $\omega, \tilde{\omega} \in S^{n-2}$. As we already proved W^1_a cannot have a right regularizer (because $\varkappa_N(\omega) < 0$) and the conjugate operator $W^1_{\bar{a}}$ cannot have right regularizer as well (because $\varkappa_1(\tilde{\omega}) > 0$). Therefore W^1_a has no left and no right regularizer in $L^N_p(\mathbf{R}^{n+})$.

There remains to consider only the case when (2.10) does not hold ($j = 1$ for definiteness).

As we already proved, existence of a local regularizer of $W^1_{a_\omega}$ (i.e. (2.36) is valid) immediately yields (2.38) (existence of a global regularizer); hence due to Theorem 1.17, if $W^1_{a_\omega}$ has no left (or no right) regularizer for any $\omega \in S^{n-2}$ the same holds for W^1_a.

Thus it suffices to make sure that $W^1_{a_\omega}$ has no left and no right regularizer.

Due to the representation (2.26)

(2.39)
$$W^1_{a_\omega} = W^1_{a_{\omega-}} W^1_{g_{\varkappa(\omega)+\delta}} W^1_{a_{\omega+}},$$

---

[1] This property is due to the dependence of $a_\omega(\xi)$ only on $|\xi'|$, because then $\sigma_* a_\omega(\xi) \equiv a_\omega(\xi)$.

[2] Obviously $R_j = \sigma_* R \sigma'_*$ for some rotation σ.

where the middle operator is diagonal one and others in the right part of (2.39) are invertible (cf. Proposition 1.9 and (2.34)). Hence it suffices to consider only the operator $W^1_{g_{r+1/p}}$ in the space $L_p(\mathbf{R}^{n+})$ ($r = \varkappa_1(\omega)$ for brevity and $\delta_1' = 1/p$ by assumption).

Let first $r = 0$ and $A_0 = W^1_{g_{1/p}}$ have a left (or a right) regularizer; due to (1.15) and stability theorems the operators $A_{\pm\varepsilon} = W^1_{g_{1/p\pm\varepsilon}}$ also have left (or right) regularizers for sufficiently small $\varepsilon > 0$; but the operator $A_{-\varepsilon}$ is invertible and $A_{+\varepsilon}$ is left-invertible, $\dim\operatorname{Coker} A_{+\varepsilon} = \infty$, we obtain the contradiction: if $A_0 = W^1_{g_{1/p}}$ is invertible (has two-side regularizer) the same must be $A_{+\varepsilon}$; if A_0 has a left regularizer and $\dim\operatorname{Coker} A_0 = \infty$ (has a right regularizer and $\dim\operatorname{Ker} A_0 = \infty$) the same must be $A_{-\varepsilon}$.

Let now the integer $r \neq 0$ and for definiteness $r > 0$; if $B_0 = W^1_{g_{r+1/p}}$ has a regularizer, it must be left regularizer since $B_{\pm\varepsilon} = W^1_{g_{r+1/p\pm\varepsilon}}$ have only left regularizers and are close to the operator B_0 (cf. (1.15)).

As it is already proved, the operator $A_{-\varepsilon} = W^1_{g_{1/p-\varepsilon}}$ is invertible. Consider the factor-algebra

$$\hat{\Sigma} = \mathscr{L}(L_p(\mathbf{R}^{n+}))/\mathfrak{S}(L_p(\mathbf{R}^{n+})).$$

The class $\hat{B}_0 \in \hat{\Sigma}$, which contains the operator $B_0 = W^1_{g_{r+1/p}}$, is left-invertible in $\hat{\Sigma}$ (since B_0 has a left-regularizer in Σ); classes $\hat{B}_{-\varepsilon}$ are also left-invertible and since

$$\lim_{\varepsilon\to 0}\|\hat{B}_0 - \hat{B}_{-\varepsilon}\|_p \leqslant \lim_{\varepsilon\to 0}\|B_0 - B_{-\varepsilon}\|_p = 0$$

inverses $(\hat{B}_{-\varepsilon})_l^{-1}$ must be simultaneously bounded

$$\sup_{\varepsilon>0}\|(\hat{B}_{-\varepsilon})_l^{-1}\|_p \leqslant M < \infty.$$

But then classes $\hat{C}_\varepsilon = (\hat{B}_{-\varepsilon})^{-1}\hat{W}^1_{g_r}$ are also simultaneously bounded and therefore convergent

$$\|\hat{C}_\varepsilon - \hat{C}_\gamma\|_p \leqslant \|\hat{C}_\varepsilon\|_p\|\hat{C}_\gamma\|_p\|B_{-\gamma} - B_{-\varepsilon}\|_p\|W^1_{g_r}\|_p$$

$$\lim_{\varepsilon,\gamma\to 0}\|\hat{C}_\varepsilon - \hat{C}_\gamma\|_p = 0.$$

The limit

$$\hat{C}_0 = \lim_{\varepsilon\to 0}\hat{C}_\varepsilon$$

is inverse to the class

$$\hat{W}^1_{g_{1/p}} = \hat{W}^1_{g_{1/p+r}g_{-r}} = \hat{W}^1_{g_{-r}}\hat{W}^1_{g_{1/p+r}} = \lim_{\varepsilon\to 0}\hat{W}^1_{g_{-r}}\hat{B}_{-\varepsilon}.$$

We obtain the contradiction because $W^1_{g_{1/p}}$ has no left and no right regularizer. ▨

5°. PROOF OF THEOREM 2.8. Suppose $\det a(\theta_0) = 0$ for some $\theta_0 \in S^{n-1}$ but W^1_a has the left regularizer R_l in $L^N_p(\mathbf{R}^{n+})$.

$R_l W^1_a = I + T$, $T \in (L^N_p(\mathbf{R}^{n+}))$; then the class $\tilde{W}^1_a$ from the factor-algebra $\Sigma^{(1)} = \mathscr{L}(L^N_p(\mathbf{R}^{n+}))/\mathfrak{S}^{(1)}(L^N_p(\mathbf{R}^{n+}))$ (cf. § 1.4°) is left invertible $\tilde{R}_l \tilde{W}^1_a = \tilde{I}$ (since $T \in \mathfrak{S}(L^N_p) \subset \mathfrak{S}^{(1)}(L^N_p)$). Without loss of generality, we can suppose that the first column of the matrix $a(\theta_0)$ is zero (otherwise we can consider the operator $W^1_{ag} = W^1_a gI$, where g is the constant matrix such that $\det g \neq 0$ and the first column in $a(\theta_0)g$ dissapears; the operator $g^{-1}R_l$ will be the left-regularizer for W^1_{ag}).

Let now $V_\varepsilon(\xi) \in (C^\infty_0)^{N\times N}(\mathbf{R}^n)$,

$$V_\varepsilon(\xi) \equiv \operatorname{diag}[\tilde{V}_\varepsilon(\xi), 0, \ldots, 0], \quad \tilde{V}_\varepsilon(\xi) = \prod_{k=1}^{n} V_{\varepsilon k}(\xi_k),$$

$$\tilde{V}_{\varepsilon k}(\xi_k) \in C^\infty_0(\mathbf{R}), \quad \tilde{V}_{\varepsilon k}(\xi_k) = \begin{cases} 0, & |\xi_k - \theta^0_k| > 2\varepsilon, \\ 1, & |\xi_k - \theta^0_k| < \varepsilon, \end{cases}$$

$$\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n, \quad \theta_0 = (\theta^0_1, \ldots, \theta^0_n) \in S^{n-1};$$

then

$$T_\varepsilon = W^1_a W^1_{v_\varepsilon} - W^1_{av_\varepsilon} = \chi_{1+} W^0_a \chi_{1-} W^0_{v_\varepsilon} \chi_{1+} \in \mathfrak{S}^{(1)}(L^N_p(\mathbf{R}^{n+})), \tag{2.40}$$

since

$$\chi_{1-} W^0_{v_\varepsilon} \chi_{1+} = \operatorname{diag}[\chi_- W^0_{\tilde{v}_{\varepsilon 1}} \chi_+ \otimes W^0_{\tilde{v}_{\varepsilon 2}} \otimes \cdots \otimes W^0_{\tilde{v}_{\varepsilon n}}, 0, \ldots, 0]$$

$$\chi_\pm(\xi_1) = \frac{1}{2}(1 \pm \operatorname{sign}\xi_1)$$

and (cf., for example [10], Lemma 7.3)

$$\chi_{1-} W^0_{\tilde{v}_{\varepsilon 1}} \chi_{1+} \in \mathfrak{S}(L_p(\mathbf{R})).$$

By assumption the first column of $a(\theta_0)$ dissapears; hence due to (1.15)

$$\lim_{\varepsilon\to 0} \| W^1_{av_\varepsilon} \|_p = 0. \tag{2.41}$$

Let now prove

$$\| \tilde{W}^1_b \|_p = \inf_{T \in \mathfrak{S}^{(1)}_p} \| W^1_b + T \|_p = \| W^1_b \|_p,$$

$$b \in M_p(\mathbf{R}^n), \ \mathfrak{S}^{(1)}_p = \mathfrak{S}^{(1)}(L^N_p(\mathbf{R}^{n+})). \tag{2.42}$$

Obviously

$$\| W_b^1 \|_p \overset{\text{def}}{=} \| \chi_{1+} W_b^0 \chi_{1+} \|_p = \| S_\lambda \chi_{1+} W_b^0 \chi_{1+} \|_p =$$

$$= \| \chi_\lambda W_b^0 \chi_\lambda \|_p \geqslant \| \chi_\lambda W_b^1 \|_p \geqslant \| W_b^1 \|_p,$$

where $S_\lambda W_b^0 = W_b^0 S_\lambda$,

$$S_\lambda \psi(t) = \psi(t_1 - \lambda, t_2, \ldots, t_n), \quad \lambda > 0,$$

$$\chi_\lambda = S_\lambda \chi_{1+} \equiv \frac{1}{2}[1 + \operatorname{sgn}(t_1 - \lambda)];$$

due to Lemma 1.13

$$\lim_{\lambda \to \infty} \| \chi_\lambda T \|_p = 0 \quad \text{for all } T \in \mathfrak{S}_p^{(1)}.$$

Hence

$$\| W_b^1 \|_p = \lim_{\lambda \to \infty} \| \chi_\lambda W_b^1 + \chi_\lambda T \|_p \leqslant \| W_b^1 + T \|_p$$

and therefore

$$\| W_b^1 \|_p \leqslant \| \tilde{W}_b^1 \|_p;$$

the inverse inequality in (2.42) is trivial.

From (2.40) – (2.42) we get (cf. (1.14))

$$1 = \lim_{\varepsilon \to 0} \sup_{\xi} |\tilde{V}_\varepsilon(\xi)| \leqslant \lim_{\varepsilon \to 0} \| \tilde{W}_{v_\varepsilon}^1 \|_p =$$

$$= \lim_{\varepsilon \to 0} \| \tilde{R}_l \tilde{W}_a^1 \tilde{W}_{v_\varepsilon}^1 \|_p \leqslant \| \tilde{R}_l \|_p \lim_{\varepsilon \to 0} \| \tilde{W}_{av_\varepsilon}^1 \|_p =$$

$$= \| \tilde{R}_l \|_p \lim_{\varepsilon \to 0} \| W_{av_\varepsilon}^1 \| = 0;$$

obtained contradiction proves that W_a^1 with $a(\theta_0) = 0$ cannot have the left regularizer.

Similar contradiction will be obtained if W_a^1 with $a(\theta_0) = 0$ has the right regularizer. ▨

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