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REFLEXIVE ALGEBRAS WITH COMPLETELY DISTRIBUTIVE SUBSPACE LATTICES

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The structure of a reflexive operator algebra is determined by the properties of its subspace lattice. One of the nicest properties that a lattice can possess, distributivity, is automatic for commutative subspace lattices. Consequently, CSL algebras (reflexive algebras whose subspace lattices are commutative) form a tractable and interesting class of operator algebras. The strongest possible distributive law permits distribution of the lattice operations over families of arbitrary cardinality. Complete lattices which satisfy this property are said to be completely distributive (see Section 1 for a precise definition). Every complete lattice which is totally ordered is completely distributive; on the other hand, a complete Boolean algebra is completely distributive if, and only if, it is atomic. Thus the CSL algebras with completely distributive subspace lattices form a proper subclass of the CSL algebras which contains all nest algebras.

A CSL algebra may possess many compact operators (for a nest algebra the linear span of the rank one operators in the algebra is dense in the ultraweak topology) or it may possess none (for example, a maximal abelian von Neumann algebra with no minimal projections). Complete distributivity of the subspace lattice is intimately related to the presence of compact operators in a CSL algebra. It is already known, by results of Longstaff [10,11], that if a reflexive algebra has a completely distributive subspace lattice, then the rank one operators in the algebra determine the subspace lattice and that if the linear span of the rank one operators in a reflexive algebra is dense in the algebra, then the subspace lattice is completely distributive. In this paper we shall show that for a CSL algebra, the Hilbert-Schmidt operators in the algebra are ultraweakly dense in the algebra if, and only if, the subspace lattice is completely distributive.

One immediate corollary of this result is that any completely distributive commutative subspace lattice is synthetic (as defined in [1]). A much deeper consequence is that, for completely distributive commutative subspace lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, the tensor product formula $\mathrm{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathrm{Alg}\,\mathscr{L}_1 \otimes \mathrm{Alg}\,\mathscr{L}_2$ is always valid. This formula has been verified for several other classes of reflexive algebras. For von Neumann algebras, which are precisely the self-adjoint reflexive algebras, this is a very deep theorem: the Tomita tensor product commutation theorem. For nest algebras, the tensor product formula was proven in [4]. K. Harrison has verified this formula for the tensor product of a nest algebra with a CSL algebra whose subspace lattice has finite width [5]. J. Kraus has proven the tensor product formula for CSL-subalgebras of von Neumann algebras when the commutative subspace lattices generate totally atomic von Neumann algebras [6]. For general reflexive algebras, or even general CSL algebras, this problem remains a deep open question.

Section 1 of this paper is devoted to notation and preliminaries. In Section 2 we prove that the Hilbert-Schmidt operators in a CSL algebra are ultraweakly dense in the algebra if, and only if, the subspace lattice of the algebra is completely distributive. This is accomplished via a measure theoretic condition on the spectrum of the lattice which, by a result in [9], is equivalent to the ultraweak density of the Hilbert-Schmidt operators. As corollaries we obtain the facts that every completely distributive commutative subspace lattice $\mathscr{L}$ is synthetic and that $\mathrm{Alg}\,\mathscr{L} + \mathscr{K}$ is norm closed, where $\mathscr{K}$ is the set of compact operators.

Section 3 is devoted to tensor products of completely distributive commutative subspace lattices. Using Theorem 7 of Section 2, we show that the tensor product of two such lattices is again completely distributive. The main result of Section 3 is the verification of the tensor product formula $\mathrm{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathrm{Alg}\,\mathscr{L}_1 \otimes \otimes\, \mathrm{Alg}\,\mathscr{L}_2$ for completely distributive commutative subspace lattices.

In Section 4 we present, for the convenience of the reader, several examples of CSL algebras. One of these examples has the (perhaps unexpected) property that it possesses Hilbert-Schmidt operators but no finite rank operators.

1. NOTATION AND PRELIMINARIES

Throughout this paper all Hilbert spaces are separable. For any family $\mathscr{L}$ of orthogonal projections acting on a Hilbert space $\mathscr{H}$, let $\mathrm{Alg}\,\mathscr{L}$ denote the set of all bounded linear operators on $\mathscr{H}$ which leave invariant each projection in $\mathscr{L}$. $\mathrm{Alg}\,\mathscr{L}$ is, in fact, a weakly closed subalgebra of $\mathscr{B}(\mathscr{H})$. For any family $\mathscr{A}$ of operators acting on $\mathscr{H}$, let $\mathrm{Lat}\,\mathscr{A}$ denote the set of all orthogonal projections which are invariant under each operator in $\mathscr{A}$. It is easy to see that $\mathrm{Lat}\,\mathscr{A}$ is a strongly closed lattice of projections which contains 0 and I; such a lattice is called a *subspace lattice*. Since $\mathrm{Alg}\,\mathscr{L}$ is unchanged if $\mathscr{L}$ is replaced by the subspace lattice which it generates and $\mathrm{Lat}\,\mathscr{A}$ is unchanged if $\mathscr{A}$ is replaced by the weakly closed algebra which it generates, we shall generally consider only subspace lattices and weakly closed algebras.

A weakly closed algebra $\mathscr{A}$ is said to be *reflexive* if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. The class of reflexive algebras is exactly the class of algebras of the form $\operatorname{Alg}\mathscr{L}$. A subspace lattice which consists of mutually commuting projections is called a *commutative subspace lattice*; the associated reflexive algebra is called a *CSL algebra*. Reflexive lattices are defined analogously. Since all commutative subspace lattices are reflexive [1] and all subspace lattices in this paper will be commutative, reflexivity of lattices will not be an issue for us. A totally ordered subspace lattice is called a *nest* and the associated reflexive algebra is a *nest algebra*.

We shall need to make substantial use of Arveson's "spectral" representation theorem for commutative subspace lattices. This theorem states that the following scheme for constructing examples of commutative subspace lattices yields, up to unitary equivalence, all commutative subspace lattices. Let X be a compact metric space, let $\leqslant$ be a reflexive and transitive relation on X whose graph G is a closed subset of $X \times X$, and let m be a finite Borel measure on X. (The relation $\leqslant$ will be referred to as an order even though it need not be anti-symmetric.) A Borel subset $S \subseteq X$ is said to be *increasing* if $x \in S$ and $x \leqslant y$ imply $y \in S$. For each Borel subset S of X let P_S denote the corresponding orthogonal projection acting on the Hilbert space $\mathscr{H} = L^2(X, m)$, i.e., P_S is the multiplication operator obtained from the characteristic function of S. Let $\mathscr{L}(X, \leqslant, m) = \{P_S \mid S$ is an increasing Borel subset of $X\}$. Arveson's theorem [1, Theorem 1.3.1] asserts that every commutative subspace lattice acting on a separable Hilbert space is unitarily equivalent to some $\mathscr{L}(X, \leqslant, m)$.

For Borel subsets of X and the corresponding projections in $\mathscr{L}(X, \leqslant, m)$ we shall use a notation which, although a bit unusual, proves to be very convenient. In general, Borel subsets of X will be denoted by "hatted" etters, e.g. $\hat{N}$, with the "unhatted" letter, N, denoting the corresponding projection. Of course, $\hat{N}$ determines N uniquely, but not vice versa; N determines $\hat{N}$ only up to a null set. The following subsets and projections will be of particular use. For each $x \in X$, let $\hat{I}_x = \{y \in X \mid x \leqslant y\}$ and $\hat{D}_x = \{y \in X \mid y \leqslant x\}$. These sets are easily seen to be Borel sets; I_x and D_x denote the corresponding projections. For all x, $I_x \in \mathscr{L}$, while each D_x is a projection in the von Neumann algebra generated by $\mathscr{L}$. It is possible, though, that $I_x = 0$ for all $x \in X$ (and similarly for the D_x's).

Uncountable unions and intersections frequently cause difficulty in measure theoretic arguments; the problems which arise in the context under study can be skirted by use of the following proposition. (This result, which is due to Arveson, is stated in greater generality, viz. in the context of standard partially ordered spaces, in [1].)

PROPOSITION 1. *Let X be a compact metric space and $\leqslant$ an order whose graph is closed. Then there is a sequence $\hat{E}_1, \hat{E}_2, \ldots$ of increasing Borel subsets of X such that $x \leqslant y$ if, and only if, $\chi_{\hat{E}_n}(x) \leqslant \chi_{\hat{E}_n}(y)$, for all n.*

An abstract complete lattice is said to be *completely distributive* if the following identity and its dual hold for arbitrary index sets:

$$\bigwedge_{\alpha \in A} \Big(\bigvee_{\beta \in B_\alpha} E_{\alpha\beta} \Big) = \bigvee_{\varphi \in \prod_{\alpha \in A} B_\alpha} \Big(\bigwedge_{\alpha \in A} E_{\alpha\varphi(\alpha)} \Big).$$

It is known that every complete chain is completely distributive [2, p. 232] and that a complete Boolean algebra is completely distributive if, and only if, it is atomic. When working with subspace lattices, the identity above is often difficult to check. (It is never necessary to verify both the identity and its dual statement, since the two are always equivalent.) Fortunately, there is a more tractable characterization of complete distributivity, due to Longstaff. We need the following definitions:

$$M_- = \bigvee \{N \mid M \nleqslant N,\ N \in \mathscr{L}\}, \quad \text{for all } M \in \mathscr{L};$$

$$L_* = \bigwedge \{M_- \mid M \nleqslant L,\ M \in \mathscr{L}\}, \quad \text{for all } L \in \mathscr{L}.$$

(We use the conventions $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$, where necessary. We always assume that each lattice has a 0 and a I. This yields $0_- = 0$ and $I_* = I$.) It is easy to see that $L \leqslant L_*$, for all $L \in \mathscr{L}$. Longstaff has proven in [11] that $\mathscr{L}$ is completely distributive if, and only if, $L_* = L$, for all $L \in \mathscr{L}$. For an extensive discussion of complete distributivity, see [7].

In this paper, attention is restricted to subspace lattices. We should remark, first, that the notation M_- is compatible with its conventional meaning in the context of nests: M_- is the immediate predecessor of M if M has an immediate predecessor; otherwise $M_- = M$. Also, note that Ringrose's lemma on rank one operators in a nest algebra extends to general reflexive algebras:

LEMMA 2. (Longstaff, [10]). *The rank one operator $x \otimes y$ belongs to* Alg $\mathscr{L}$ *if, and only if, there is a projection M in $\mathscr{L}$ such that $y \in M$ and $x \in M_-^{\perp}$.*

Thus, we see that if $L = L_*$ for some projection $L \neq I$ in $\mathscr{L}$, then Alg $\mathscr{L}$ contains rank one operators. If $L = L_*$ for all $L \in \mathscr{L}$, i.e. if $\mathscr{L}$ is completely distributive, then, as Longstaff has shown [10], $\mathscr{L} = \operatorname{Lat} \mathscr{R}$, where $\mathscr{R}$ is the set of rank one operators in Alg $\mathscr{L}$. He has also shown [11] that if the linear span of $\mathscr{R}$ (which is automatically a subalgebra) is dense in Alg $\mathscr{L}$ (in any of the usual topologies), then $\mathscr{L}$ is completely distributive. In the next section we will prove that, for commutative subspace lattices, complete distributivity is equivalent to density of the Hilbert-Schmidt operators in Alg $\mathscr{L}$ (in any of the weak, strong, ultraweak or ultrastrong topologies).

In Section 3 it will be convenient to use the notion of a slice algebra, or Fubini algebra, as introduced by Tomiyama in [12]. For any Hilbert space $\mathscr{H}$, $\mathscr{B}(\mathscr{H})_*$ denotes the predual of $\mathscr{B}(\mathscr{H})$, i.e. the space of ultraweakly continuous linear functionals on $\mathscr{B}(\mathscr{H})$. If $\varphi \in \mathscr{B}(\mathscr{H})_*$ and $T \in \mathscr{B}(\mathscr{H} \otimes \mathscr{K})$, we can define a linear functional $f_{\varphi,T}$ on $\mathscr{B}(\mathscr{K})_*$ by

$$f_{\varphi,T}(\psi) = (\varphi \otimes \psi)(T), \quad \text{for all } \psi \in \mathscr{B}(\mathscr{K})_*.$$

This is a bounded linear functional on $\mathscr{B}(\mathscr{K})_*$, so there is an operator in $\mathscr{B}(\mathscr{K})$, which we call $R_\varphi(T)$, such that

$$f_{\varphi,T}(\psi) = \psi(R_\varphi(T)), \quad \text{for all } \psi \in \mathscr{B}(\mathscr{K})_*.$$

Similarly, for each ψ in $\mathscr{B}(\mathscr{K})_*$ we can define an element $L_\psi(T)$ of $\mathscr{B}(\mathscr{H})$ such that $\varphi(L_\psi(T)) = (\varphi \otimes \psi)(T)$, for all $\varphi \in \mathscr{B}(\mathscr{H})_*$. Then $\{R_\varphi \mid \varphi \in \mathscr{B}(\mathscr{H})_*\}$ and $\{L_\psi \mid \psi \in \mathscr{B}(\mathscr{K})_*\}$ are families of ultraweakly continuous maps from $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ to $\mathscr{B}(\mathscr{K})$ and $\mathscr{B}(\mathscr{H})$ respectively, and

$$R_\varphi\left(\sum_{i=1}^{n} A_i \otimes B_i\right) = \sum_{i=1}^{n} \varphi(A_i) B_i,$$

$$L_\psi\left(\sum_{i=1}^{n} A_i \otimes B_i\right) = \sum_{i=1}^{n} \psi(B_i) A_i.$$

If $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ and $\mathscr{B} \subseteq \mathscr{B}(\mathscr{K})$ are ultraweakly closed algebras containing the identity I, define the Fubini algebra $\mathscr{F}(\mathscr{A}, \mathscr{B})$ by

$$\mathscr{F}(\mathscr{A}, \mathscr{B}) = \{T \in \mathscr{B}(\mathscr{H} \otimes \mathscr{K}) \mid R_\varphi(T) \in \mathscr{B} \text{ and } L_\psi(T) \in \mathscr{A} \text{ for all } \varphi \text{ in } \mathscr{B}(\mathscr{H})_* \text{ and } \psi \text{ in } \mathscr{B}(\mathscr{K})_*\}.$$

Tomiyama introduced Fubini algebras for the purpose of extending the Tomita tensor product commutation theorem to contexts broader than that of von Neumann algebras. For reflexive algebras, Tomiyama's conjecture, $\mathscr{F}(\mathscr{A}, \mathscr{B}) = \mathscr{A} \otimes \mathscr{B}$, agrees with the conjecture natural to reflexive algebras, $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \operatorname{Alg} \mathscr{L}_1 \otimes \operatorname{Alg} \mathscr{L}_2$. The authors would like to thank Jon Kraus for drawing Fubini algebras to their attention.

We conclude this section by stating formally the (obvious) definitions for the tensor product of weakly closed algebras and of subspace lattices. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are weakly closed algebras of operators acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, then $\mathscr{A}_1 \otimes \mathscr{A}_2$ is the weakly closed algebra acting on $\mathscr{H}_1 \otimes \mathscr{H}_2$ generated by all elementary tensors $T_1 \otimes T_2$, where $T_1 \in \mathscr{A}_1$ and $T_2 \in \mathscr{A}_2$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are subspace lattices acting on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectivey, then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is the subspace lattice acting on $\mathscr{H}_1 \otimes \mathscr{H}_2$ generated by all elementary tensors of the form $P_1 \otimes P_2$, where $P_1 \in \mathscr{L}_1$ and $P_2 \in \mathscr{L}_2$.

2. DENSITY OF THE HILBERT-SCHMIDT OPERATORS

By virtue of Arveson's spectral representation theorem, we may assume that each commutative subspace lattice is of the form $\mathscr{L} = \mathscr{L}(X, \leqslant, m)$. In this section we shall prove that complete distributivity of $\mathscr{L}$ is equivalent to a measure theoretic statement involving the triple $(X, \leqslant, m)$, which in turn is equivalent to the ultraweak density of the Hilbert-Schmidt operators in Alg $\mathscr{L}$. First, however, it will be useful to obtain a set theoretic identification of N_-, where $N \in \mathscr{L}$. It would be desirable to obtain such an identification for N_* also; however, some of the examples given in Section 4 indicate that this may prove difficult to find.

In the following lemma and hereafter, $\hat{S}^c$ denotes the complement of $\hat{S}$ in X. Also, the sets $\hat{E}_n$ are the separating family of Borel sets given by Proposition 1.

LEMMA 3. *Let $N \in \mathscr{L} = \mathscr{L}(X, \leqslant, m)$. Let $\hat{N}$ be an increasing Borel subset of X which corresponds to the projection N. Let*

$$\hat{B} = \{x \mid m(\hat{N} \cap \hat{I}_x^c) > 0\} = \{x \mid N \nleqslant I_x\} = \cup \{\hat{E}_n \mid m(\hat{E}_n^c \cap \hat{N}) \neq 0\}.$$

Then $\hat{B}$ is an increasing measurable subset of X and the projection to which it corresponds is N_-.

Proof. Since $\hat{N} \times \hat{N} \cap G$ is a Borel subset of $X \times X$, every section of $\hat{N} \times \hat{N} \cap G$ is measurable. An x-section of $\hat{N} \times \hat{N} \cap G$, defined to be $\{y \mid (x, y) \in \hat{N} \times \hat{N} \cap G\}$, is clearly equal to $\hat{N} \cap \hat{I}_x$. The function $x \to m(N \cap \hat{I}_x)$ is a measurable function, whence $\hat{B}$ is a measurable set. Let B denote the projection corresponding to $\hat{B}$. To prove that $N_- \leqslant B$ it suffices to show that $M \leqslant B$ whenever $M \in \mathscr{L}$ and $N \nleqslant M$. (Reason: $N_- = \vee \{M \mid N \nleqslant M\}$.) Let $\hat{M}$ be an increasing Borel set corresponding to M. Let $x \in \hat{M}$. Then $\hat{I}_x \subseteq \hat{M}$. Since $m(\hat{N} \cap \hat{M}^c) > 0$, we must have $m(\hat{N} \cap I^c) > 0$. Thus $x \in \hat{B}$, i.e. $\hat{M} \subseteq \hat{B}$. Hence $M \leqslant B$.

It remains to prove that $B \leqslant N_-$. It is easy to see that, for any x, $\hat{I}_x = \cap \{\hat{E}_n \mid x \in \hat{E}_n\}$, whence $\hat{I}_x^c = \cup \{\hat{E}_n^c \mid x \in \hat{E}_n\}$. Consequently

$$\hat{B} = \{x \mid m(\hat{N} \cap \hat{I}_x^c) > 0\} =$$

$$= \{x \mid m(\hat{N} \cap (\cup \{\hat{E}_n^c \mid x \in \hat{E}_n\})) > 0\} =$$

$$= \{x \mid m(\cup \{\hat{N} \cap \hat{E}_n^c \mid x \in \hat{E}_n\}) > 0\} =$$

$$= \cup \{\hat{E}_n \mid m(\hat{N} \cap \hat{E}_n^c) > 0\}.$$

It is routine to verify that a countable union of sets corresponds to the join of the corresponding projections, hence

$$B = \vee \{E_n \mid m(\hat{N} \cap \hat{E}_n^c) > 0\} = \vee \{E_n \mid N \nleqslant E_n\}.$$

But if $N \nleqslant E_n$ then $E_n \leqslant N_-$, whence $B \leqslant N_-$.

REMARK. If $m\{x \mid m(\hat{I}_x) = 0\} > 0$ then $\mathscr{L}$ is not completely distributive. Indeed, let $\hat{C} = \{x \mid m(\hat{I}_x) = 0\}$ and let C be the projection corresponding to $\hat{C}$. Since $\hat{C}$ is measurable and increasing, $C \in \mathscr{L}$. Further, from Lemma 3, we have $C \leqslant M_-$, for all $M \neq 0$. If $L \in \mathscr{L}$, then $L_* = \wedge\{M_- \mid M \nleqslant L\}$; so $C \leqslant L_*$, for all $L \in \mathscr{L}$. Thus it suffices to find one projection $L \in \mathscr{L}$ such that $C \nleqslant L$ to show that $\mathscr{L}$ is not completely distributive. Let $\hat{E}_n$ be a separating sequence of increasing Borel sets as in Proposition 1. Suppose that, for all $n \in N$, either $m(\hat{C} \cap \hat{E}_n) = 0$ or $m(\hat{C} \cap \hat{E}_n^c) = 0$. Since $m(\hat{C}) > 0$, it can never occur that both conditions hold. Let $N_1 = \{n \mid m(\hat{C} \cap \hat{E}_n) = 0\}$ and $N_2 = \{n \mid m(\hat{C} \cap \hat{E}_n^c) = 0\}$. Then $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = \mathbf{N}$. Let $\hat{C}_1 = \bigcup_{n \in N_1} \hat{C} \cap \hat{E}_n$ and $\hat{C}_2 = \bigcup_{n \in N_2} \hat{C} \cap \hat{E}_n^c$. Both $m(\hat{C}_1) = 0$ and $m(\hat{C}_2) = 0$. Let $\hat{C}_0 = \hat{C} \setminus (\hat{C}_1 \cup \hat{C}_2)$. Then $m(\hat{C}_0) = m(\hat{C}) > 0$. For each $n \in \mathbf{N}$, either $\hat{C}_0 \subseteq \hat{E}_n$ or $\hat{C}_0 \subseteq \hat{E}_n^c$. Hence, if $x, y \in \hat{C}_0$ then $\chi_{\hat{E}_n}(x) = \chi_{\hat{E}_n}(y)$, for all n. Therefore $x \leqslant y$. In other words, $\hat{C}_0 \subseteq \hat{I}_x$, for each $x \in \hat{C}_0$. This yields $m(\hat{C}_0) \leqslant m(\hat{I}_x) = 0$, a contradiction. Therefore, there exists $n \in \mathbf{N}$ such that $m(\hat{C} \cap \hat{E}_n) > 0$ and $m(\hat{C} \cap \hat{E}_n^c) > 0$. In other words, $E_n \neq 0$ and $C \nleqslant E_n$. Since $C \leqslant E_{n*}$, we have $E_n \neq E_{n*}$ and $\mathscr{L}$ is not completely distributive.

It appears to be much more difficult to characterize L_* "set theoretically" than it is to characterize L_-. We do have the following tidbit:

LEMMA 4. *Let $L \in \mathscr{L}$ and let $\hat{L}$ be a corresponding increasing set. Let $\hat{L}_\smile = \{x \mid m(\hat{I}_x \cap \hat{L}^c) = 0\} = \{x \mid I_x \leqslant L\}$. Then $\hat{L}_\smile$ is an increasing measurable set and the corresponding projection, $L_\smile$, is a subprojection of L_*.*

Proof. It is clear that $\hat{L}_\smile$ is increasing and measurable. Now suppose that $N \in \mathscr{L}$ and $N \nleqslant L$. Let $\hat{N}$ correspond to N. By Lemma 3, N_- corresponds to $\{x \mid m(\hat{N} \cap \hat{I}_x^c) > 0\} = \{x \mid N \nleqslant I_x\}$. For any $x \in \hat{L}_\smile$, $I_x \leqslant L$, whence $N \nleqslant I_x$. Thus $\hat{L}_\smile \subseteq \{x \mid N \nleqslant I_x\}$ and so $L_\smile \leqslant N_-$. Hence $L_\smile \leqslant \wedge\{N_- \mid N \nleqslant L\} = L_*$.

REMARK. It is not necessarily the case that $L_\smile = L_*$. An example is given in Section 4 for which $L_\smile < L_*$ for some $L \in \mathscr{L}$. One can define $\hat{L}_{\approx} = \{x \mid I_x \leqslant L_\smile\}$ and speculate on the relation between $L_{\approx}$ and L_*. Examples will be given in Section 4 in which $L_* = L_{\approx}$ and in which $L_* < L_{\approx}$.

The set theoretic characterization of N_- leads to the following necessary and sufficient condition for Alg $\mathscr{L}$ to contain a rank one operator.

LEMMA 5. *Alg $\mathscr{L}$ contains a rank one operator if, and only if, there exist measurable sets $\hat{A}, \hat{B} \subseteq X$ such that $m(\hat{A}) > 0$, $m(\hat{B}) > 0$ and $x \leqslant y$ whenever $x \in \hat{A}$ and $y \in \hat{B}$.*

REMARK. The latter condition is equivalent to saying that the graph G contains a non-trivial measurable rectangle, viz. $\hat{A}\times\hat{B}$.

Proof. If Alg $\mathscr{L}$ contains a rank one operator then there exists a non-zero projection $N\in\mathscr{L}$ with $N_-^\perp\neq 0$. Let $\hat{E}_n$ be a sequence of increasing sets which determines the order, as in Proposition 1. Let $J=\{n\mid N\nleqslant E_n\}$. From the proof of Lemma 3, N_- corresponds to $\cup\{\hat{E}_n\mid n\in J\}$. Let $\hat{A}=X\setminus\bigcup_{n\in J}\hat{E}_n$. Then $m(\hat{A})>0$, since $\hat{A}$ corresponds to $N_-^\perp$. Let $\hat{B}=\hat{N}\setminus\cup\{N\cap\hat{E}_n^c\mid n\notin J\}$. When $n\notin J$, $N\leqslant E_n$, hence $m(\hat{N}\cap\hat{E}_n^c)=0$. Since $\mathbf{N}\setminus J$ is countable, $m(\hat{B})=m(\hat{N})>0$. Now let $x\in\hat{A}$ and $y\in\hat{B}$ be arbitrary. If $n\in J$, then $x\notin\hat{E}_n$ and $\chi_{\hat{E}_n}(x)=0\leqslant\chi_{\hat{E}_n}(y)$. If $n\notin J$, then $y\in\hat{E}_n$ and $\chi_{\hat{E}_n}(x)\leqslant 1=\chi_{\hat{E}_n}(y)$. Thus $x\leqslant y$, as desired.

For the converse, suppose sets $\hat{A}$ and $\hat{B}$ are given with $m(\hat{A})>0$, $m(\hat{B})>0$ and $\hat{A}\times\hat{B}\subseteq G$. Let A and B be the projections corresponding to $\hat{A}$ and $\hat{B}$. Then $A\neq 0$ and $B\neq 0$. It is sufficient to show that any operator T satisfying $T=BTA$ must lie in Alg$\mathscr{L}$. Let $L\in\mathscr{L}$ and suppose $T=BTA$. If $AL=0$ then T clearly leaves L invariant. If $AL\neq 0$ and if $\hat{L}$ is an increasing set corresponding to L, then $\hat{L}\cap\hat{A}\neq\emptyset$. Since $\hat{L}$ is increasing, it follows that $\hat{B}\subseteq\hat{L}$. Thus $B\leqslant L$ and T leaves L invariant. Thus we see that $T\in$ Alg $\mathscr{L}$ and the proof is completed by taking any rank one operator T which satisfies $T=BTA$.

The following lemma will also prove useful.

LEMMA 6. *For every* $x\in X$, $D_x\wedge I_{x^-}=0$.

Proof. Since I_{x^-} corresponds to $\hat{B}=\{y\mid m(\hat{I}_x\cap\hat{I}_y^c)>0\}=\{y\mid I_x\nleqslant I_y\}$, it suffices to show that $\hat{D}_x\cap\hat{B}=\emptyset$. But, if $y\in\hat{D}_x$ then $y\leqslant x$, whence $\hat{I}_x\subseteq\hat{I}_y$, so $\hat{D}_x\cap\hat{B}=\emptyset$.

We now turn to the Hilbert-Schmidt operators in Alg$\mathscr{L}$, where, as usual, $\mathscr{L}=\mathscr{L}(X,\leqslant,m)$ acts on the Hilbert space $\mathscr{H}=L^2(X,m)$. Every Hilbert-Schmidt operator T_f in $\mathscr{B}(\mathscr{H})$ can be expressed as an integral operator with respect to a kernel function f which is an element of $L^2(X\times X,\ m\times m)$. There is a simple necessary and sufficient condition for T_f to lie in Alg$\mathscr{L}$: f must have support on the graph G of the order $\leqslant$ ([1, Proposition 1.6.0]). From this it follows that there exists a Hilbert-Schmidt operator in Alg$\mathscr{L}$ if, and only if, $m\times m(G)>0$. The main result of this section is the following lattice theoretic characterization of the CSL algebras in which the Hilbert-Schmidt operators are ultraweakly dense.

THEOREM 7. *Let* $\mathscr{L}=\mathscr{L}(X,\leqslant,m)$ *be a commutative subspace lattice. The following are equivalent:*

(i) *The Hilbert-Schmidt operators in* Alg$\mathscr{L}$ *are ultraweakly dense in* Alg$\mathscr{L}$.

(ii) *If* $\hat{A}$ *is any Borel subset of* X *for which* $m(\hat{A}) > 0$ *then* $m\times m(\hat{A}\times\hat{A}\cap G) > 0$.
(iii) $\mathscr{L}$ *is completely distributive.*

Proof. The equivalence of (i) and (ii) has been proven in [9]. In fact, a slightly stronger result is proven: (ii) may be replaced by the same statement in which it is further assumed that $\hat{A}$ corresponds to a projection of the form $P - Q$, where $P, Q \in \mathscr{L}$ and $Q < P$. (Such a projection is called an interval.) We shall use this variation in proving the equivalence of (ii) and (iii).

To prove that (ii) implies (iii), assume that $\mathscr{L}$ is not completely distributive. Then there is a projection $M \in \mathscr{L}$ such that $M \neq M_*$. Consequently, $M < M_*$. Let $\hat{M}$ and $\hat{M}_*$ be increasing Borel subsets of X which correspond to M and M_* and which satisfy $\hat{M} \subset \hat{M}_*$. Let $\hat{A} = \hat{M}_* \cap \hat{M}^c$. The hypothesis assures that $m(\hat{A}) > 0$. Let $\hat{A}_1 = \{x \in \hat{A} \mid I_x \leqslant M\}$ and $\hat{A}_2 = \hat{A}\setminus\hat{A}_1$. Then at least one of $\hat{A}_1$ and $\hat{A}_2$ has positive measure; the proof that (ii) implies (iii) will be complete if we show that $m\times m(\hat{A}_i\times\hat{A}_i \cap G) = 0$ for both $i = 1$ and $i = 2$.

For any measurable subset $\hat{B}\subseteq X$, $m\times m(\hat{B}\times\hat{B}\cap G)$ can, by Fubini's theorem, be obtained by integrating the measure of the sections of $\hat{B}\times\hat{B}\cap G$. The x and y sections of $\hat{B}\times\hat{B}\cap G$ are given by

$$\{y \mid (x, y) \in \hat{B}\times\hat{B}\cap G\} = \hat{B}\cap\hat{I}_x, \quad \text{for } x \in \hat{B}$$

and

$$\{x \mid (x, y) \in \hat{B}\times\hat{B}\cap G\} = \hat{B}\cap\hat{D}_y, \quad \text{for } y \in \hat{B}.$$

So,

$$m\times m(\hat{B}\times\hat{B}\cap G) = \int_{\hat{B}} m(\hat{B}\cap\hat{I}_x)\mathrm{d}m(x) =$$

$$= \int_{\hat{B}} m(\hat{B}\cap\hat{D}_y)\mathrm{d}m(y).$$

In particular,

$$m\times m(\hat{A}_1\times\hat{A}_1\cap G) = \int_{\hat{A}_1} m(\hat{A}_1\cap\hat{I}_x)\mathrm{d}m(x).$$

But, for each $x \in \hat{A}_1$, $I_x \leqslant M$, hence $m(\hat{I}_x\cap\hat{M}^c) = 0$. Since $\hat{A}_1\subseteq\hat{M}^c$, $m(\hat{A}_1\cap\hat{I}_x) = 0$ for all $x \in \hat{A}_1$, and therefore $m\times m(\hat{A}_1\times\hat{A}_1\cap G) = 0$. Similarly,

$$m\times m(\hat{A}_2\times\hat{A}_2\cap G) = \int_{\hat{A}_2} m(\hat{A}_2\cap\hat{D}_y)\mathrm{d}m(y).$$

Now, for $y \in \hat{A}_2$, $I_y \nleq M$. Since $M_* = \bigwedge\{N_- \mid N \nleq M\}$ we have $M_* \leqslant I_{y}-$. But $A_2 \leqslant M_*$, so that $A_2 \leqslant I_y-$ and hence $A_2 \wedge D_y \leqslant (I_y-) \wedge D_y = 0$, by Lemma 6. Therefore $m(\hat{A}_2 \cap \hat{D}_y) = 0$ for all $y \in \hat{A}_2$ and $m \times m(\hat{A}_2 \times \hat{A}_2 \cap G) = 0$.

It remains to prove that (iii) implies (ii). Assume that there exists a Borel set $\hat{A} \subseteq X$ such that $m(\hat{A}) > 0$ and $m \times m(\hat{A} \times \hat{A} \cap G) = 0$. We may, by the remarks in the first paragraph of the proof, assume further that $\hat{A}$ corresponds to an interval $P - Q$, where $P, Q \in \mathscr{L}$ and $Q < P$. Let $\hat{P}$ and $\hat{Q}$ be increasing Borel sets corresponding to P and Q such that $\hat{Q} \subset \hat{P}$. If necessary, adjust $\hat{A}$ by a set of measure 0 so that $\hat{A} = \hat{P} \setminus \hat{Q}$. Let $A_0 = \{x \in \hat{A} \mid m(\hat{A} \cap \hat{I}_x) = 0\}$. Since $0 = m \times m(\hat{A} \times \hat{A} \cap G) = \int_{\hat{A}} m(\hat{A} \cap \hat{I}_x)\,\mathrm{d}m(x)$, we see that $m(\hat{A} \setminus \hat{A}_0) = 0$. Thus $\hat{A}_0$ also corresponds to $P - Q$. If $x \in \hat{A}_0$ then $x \in \hat{P}$ and so $\hat{I}_x \subseteq \hat{P}$, since $\hat{P}$ is increasing. But $0 = m(\hat{I}_x \cap \hat{A}_0) = m(\hat{I}_x \cap \hat{Q}^c)$. By Lemma 4, $P - Q \leqslant Q_*$, from which it follows that $P \leqslant Q_*$. Since $Q < P$, we have $Q \neq Q_*$ and $\mathscr{L}$ is not completely distributive. Thus (iii) implies (ii).

REMARK. Trivial modifications of the proof of Theorem 3.2 in [9] show that if the Hilbert-Schmidt operators in Alg $\mathscr{L}$ are dense in Alg $\mathscr{L}$ in any of the usual topologies, then condition (ii) in Theorem 7 is satisfied. Thus if the Hilbert-Schmidt operators are dense in any of the usual topologies, they are ultraweakly dense and $\mathscr{L}$ is completely distributive. On the other hand, if the Hilbert-Schmidt operators in Alg $\mathscr{L}$ are dense in Alg $\mathscr{L}$ in the ultraweak topology they are also dense in the ultrastrong topology. (Convex sets have the same closures in the ultraweak and ultrastrong topologies.) From this, of course, it follows that they are also dense in the strong and the weak operator topologies. Thus we see that density of the Hilbert-Schmidt operators in any one of these topologies is equivalent to density in any other.

REMARK. M. S. Lambrou has recently obtained an interesting result about the rank one operators in Alg $\mathscr{L}$ where $\mathscr{L}$ is a completely distributive subspace lattice on an arbitrary normed linear space X. He proves in [8] that complete distributivity of $\mathscr{L}$ is equivalent to the following condition: for each $x \in X$, $A \in \mathrm{Alg}\,\mathscr{L}$ and $\varepsilon > 0$ there is an operator F equal to a finite sum of rank one operators of Alg $\mathscr{L}$ such that $\|Ax - Fx\| < \varepsilon$. This condition is somewhat weaker than the strong density of the linear span of the rank one operators of Alg $\mathscr{L}$. Lambrou also gives an example of a (non-commutative) subspace lattice $\mathscr{L}$ which is not completely distributive but for which the finite rank operators in Alg $\mathscr{L}$ are strongly dense in Alg $\mathscr{L}$. This example shows, of course, that conditions (i) and (iii) in Theorem 7 are not equivalent in general.

It is pointed out in [9] that if condition (ii) is satisfied then $\mathrm{Alg}\mathscr{L} + \mathscr{K}$ is norm closed, where $\mathscr{K}$ is the set of compact operators on $L^2(X, m)$. Hence we have the following corollary.

COROLLARY 8. *If $\mathscr{L}$ is a completely distributive commutative subspace lattice then* $\mathrm{Alg}\mathscr{L} + \mathscr{K}$ *is norm closed.*

This should be compared with § 5 in [4], where it is proven that $\mathrm{Alg}\mathscr{L} + \mathscr{K}$ is closed whenever $\mathscr{L}$ has finite width.

A reflexive lattice $\mathscr{L}$ is defined to be *synthetic* (in [1]) if the only ultraweakly closed algebra $\mathscr{A}$ satisfying $\mathscr{A} \cap \mathscr{A}^* = (\mathrm{Lat}\,\mathscr{A})'$ and $\mathrm{Lat}\,\mathscr{A} = \mathscr{L}$ is the algebra $\mathrm{Alg}\mathscr{L}$. (Here, $\mathscr{B}'$ denotes the commutant of B.) Theorem 7 also yields the following corollary.

COROLLARY 9. *If $\mathscr{L}$ is a completely distributive commutative subspace lattice, then $\mathscr{L}$ is synthetic.*

Proof. Theorem 2.2.11 in [1] asserts that $\mathscr{L}$ is synthetic if, and only if, the pseudo-integral operators in $\mathrm{Alg}\mathscr{L}$ are ultraweakly dense in $\mathrm{Alg}\mathscr{L}$. (The reader is referred to [1] for the definition of pseudo-integral operators.) When $\mathscr{L}$ is completely distributive, the Hilbert-Schmidt operators in $\mathrm{Alg}\mathscr{L}$ are ultraweakly dense in $\mathrm{Alg}\mathscr{L}$. Not every Hilbert-Schmidt operator is a pseudo-integral operator; but every Hilbert-Schmidt operator with a bounded kernel is pseudo-integral. Since the Hilbert-Schmidt operators with bounded kernel in $\mathrm{Alg}\,\mathscr{L}$ are ultraweakly dense in $\mathrm{Alg}\mathscr{L}$ whenever the full set of Hilbert-Schmidt operators in $\mathrm{Alg}\mathscr{L}$ is dense, we see that the pseudo-integral operators are ultraweakly dense and $\mathscr{L}$ is synthetic.

3. TENSOR PRODUCTS

The main result in this section is the verification of the formula $\mathrm{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathrm{Alg}\mathscr{L}_1 \otimes \mathrm{Alg}\mathscr{L}_2$ whenever $\mathscr{L}_1$ and $\mathscr{L}_2$ are completely distributive commutative subspace lattices. The first step in this direction is a proof that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are completely distributive, then so is $\mathscr{L}_1 \otimes \mathscr{L}_2$. A related theorem has been proven by K. Harrison [5] — if $\mathscr{N}$ is a nest and $\mathscr{L}$ is a (not necessarily commutative) completely distributive subspace lattice, then $\mathscr{N} \otimes \mathscr{L}$ is completely distributive. Both Harrison's theorem and our theorem generalize a result in [4] which states that the tensor product of a finite number of nests is completely distributive.

THEOREM 10. *If $\mathscr{L}_1$ and $\mathscr{L}_2$ are completely distributive commutative subspace lattices, then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is completely distributive.*

Proof. As usual, represent $\mathscr{L}_1$ as $\mathscr{L}(X_1, \leqslant, m_1)$ and $\mathscr{L}_2$ as $\mathscr{L}(X_2, \leqslant, m_2)$. It is proven in [4] that $\mathscr{L}_1 \otimes \mathscr{L}_2$ is represented by $\mathscr{L}(X_1 \times X_2, \leqslant, m_1 \times m_2)$, where $\leqslant$ is the product ordering, viz. $(x_1, x_2) \leqslant (y_1, y_2)$ if, and only if, $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$.

Let G_i denote the graph of $(X_i, \leqslant)$ and G denote the graph of the product order. Let $X = X_1 \times X_2$ and $m = m_1 \times m_2$.

Choose an arbitrary Borel subset $A \subseteq X$. (We drop the ^-symbol, since it is no longer useful.) Let $E = A \times A \cap G$ and assume that $m \times m(E) = 0$. By Theorem 7, we need only show that $m(A) = 0$.

Let $\tilde{X} = X_2 \times X_1$ (with the measure $\tilde{m} = m_2 \times m_1$) and view $X \times X$ as $X_1 \times \tilde{X} \times X_2$. For $\tilde{x} = (x_2, x_1)$ let $E_{\tilde{x}}$ denote the $\tilde{x}$-section of E, viz.

$$E_{\tilde{x}} = \{(z_1, z_2) \in X_1 \times X_2 \mid (z_1, \tilde{x}, z_2) \in E\} =$$

$$= \{(z_1, z_2) \in X_1 \times X_2 \mid (z_1, x_2) \in A,\ (x_1, z_2) \in A \text{ and } (z_1, x_2) \leqslant (x_1, z_2)\} =$$

$$= \{z_1 \in X_1 \mid (z_1, x_2) \in A,\ z_1 \leqslant x_1\} \times \{z_2 \in A \mid (x_1, z_2) \in A,\ x_2 \leqslant z_2\} =$$

$$= [A_{x_2} \cap D_{x_1}] \times [A_{x_1} \cap I_{x_2}].$$

(A_{x_i} denotes the x_i-section of A. As earlier, $D_{x_1} = \{y \in X_1 \mid y \leqslant x_1\}$ and $I_{x_2} = \{y \in X_2 \mid x_2 \leqslant y\}$.) Fubini's theorem implies that $m_1 \times m_2(E_{\tilde{x}}) = 0$, for $\tilde{m}$-almost all $\tilde{x}$ in $\tilde{X}$. Hence either $m_1(A_{x_2} \cap D_{x_1}) = 0$, or $m_2(A_{x_1} \cap I_{x_2}) = 0$, for almost all $(x_1, x_2) \in X$.

Set

$$B = \{(x_1, x_2) \mid m_1 (A_{x_2} \cap D_{x_1}) = 0\}$$

and

$$C = \{(x_1, x_2) \mid m_2(A_{x_1} \cap I_{x_2}) = 0\}.$$

Then $X_1 \times X_2 \setminus (B \cup C)$ has m-measure zero. We shall show that $m(A \cap B) = 0$ and $m(A \cap C) = 0$; this immediately yields $m(A) = 0$ and thus will complete the proof.

If $x \in (A \cap B)_{x_2}$, then $(x, x_2) \in A \cap B \subseteq B$ and $m_1 (A_{x_2} \cap D_x) = 0$. Consequently, $m_1((A \cap B)_{x_2} \cap D_x) = 0$, for all $x \in (A \cap B)_{x_2}$. Hence,

$$m_1 \times m_1((A \cap B)_{x_2} \times (A \cap B)_{x_2} \cap G_1) = \int_{(A \cap B)_{x_2}} m_1((A \cap B)_{x_2} \cap D_x) \mathrm{d}m_1(x) = 0.$$

The complete distributivity of $\mathscr{L}_1$ implies that $m_1((A \cap B)_{x_2}) = 0$, for all $x_2 \in X_2$ (Theorem 7). Thus $m(A \cap B) = m_1 \times m_2(A \cap B) = 0$; $m(A \cap C) = 0$ can be shown in a similar way.

We now turn attention to the tensor product formula: $\mathrm{Alg}\,(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathrm{Alg}\,\mathscr{L}_1 \otimes \mathrm{Alg}\,\mathscr{L}_2$. We have defined the tensor product of two weakly closed algebras to be the weak closure of the algebraic tensor product. In one of the propositions below, some of the algebras involved are assumed to be ultraweakly closed, rather than weakly closed. In this context, the tensor products should be interpreted as ultraweak closures, rather than weak closures. In fact, a close reading of the proofs below reveals that one could define $\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathrm{Alg}\,\mathscr{L}_2$ to be the ultraweak closure of the algebraic tensor product and still obtain the formula $\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathrm{Alg}\,\mathscr{L}_2 = \mathrm{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$. In particular, for completely distributive lattices, the ultraweak closure of the algebraic tensor product of $\mathrm{Alg}\,\mathscr{L}_1$ and $\mathrm{Alg}\,\mathscr{L}_2$ is automatically weakly closed. We do not know if this happens for all (commutative) subspace lattices or if this refinement will prove useful.

Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces. Let $\{e_i\}_{i=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ be orthonormal bases for $\mathscr{H}$ and $\mathscr{K}$ respectively. Define $E_{ij} \in \mathscr{B}(\mathscr{H})$ as the rank one operator $E_{ij}h = (h, e_j)e_i$ and $F_{ij} \in \mathscr{B}(\mathscr{K})$ by $F_{ij}k = (k, f_j)f_i$. Any operator $T \in \mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ has the infinite matrix representation

$$T = \sum_{i,j=1}^{\infty} T_{ij} \otimes F_{ij} \quad \left(\text{or } T = \sum_{i,j=1}^{\infty} E_{ij} \otimes S_{ij}\right),$$

with convergence in the ultraweak topology. From this observation it is straightforward to check that, for any subspace lattice $\mathscr{L}_1$ on $\mathscr{H}$, $\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathscr{B}(\mathscr{K}) = \mathrm{Alg}(\mathscr{L}_1 \otimes I_{\mathscr{K}})$. ($\mathscr{L}_1 \otimes I_{\mathscr{K}}$ is the subspace lattice on $\mathscr{H} \otimes \mathscr{K}$ generated by $\{P \otimes I_{\mathscr{K}} \mid P \in \mathscr{L}_1\}$.) Similarly, for any subspace lattice $\mathscr{L}_2$ on $\mathscr{K}$, $\mathscr{B}(\mathscr{H}) \otimes \mathrm{Alg}\,\mathscr{L}_2 = \mathrm{Alg}(I_{\mathscr{H}} \otimes \mathscr{L}_2)$. Putting these facts together we have

$$[\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathscr{B}(\mathscr{K})] \cap [\mathscr{B}(\mathscr{H}) \otimes \mathrm{Alg}\,\mathscr{L}_2] = \mathrm{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2).$$

Thus the tensor product formula may be reduced to the equation:

$$\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathrm{Alg}\,\mathscr{L}_2 = [\mathrm{Alg}\,\mathscr{L}_1 \otimes \mathscr{B}(\mathscr{K})] \cap [\mathscr{B}(\mathscr{K}) \otimes \mathrm{Alg}\,\mathscr{L}_2].$$

In the next proposition we show the validity of the analogous statement with operator algebras replaced by Hilbert spaces. This is then exploited to yield similar results for the Hilbert-Schmidt operators in CSL algebras. For the special case when the Hilbert-Schmidt operators are ultraweakly dense, viz. when the lattices are completely distributive, this yields the tensor product formula.

PROPOSITION 11. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Let $\mathscr{U} \subseteq \mathscr{H}$ and $\mathscr{V} \subseteq \mathscr{K}$ be subspaces. Then $\mathscr{U} \otimes \mathscr{K} \cap \mathscr{H} \otimes \mathscr{V} = \mathscr{U} \otimes \mathscr{V}$.*

Proof. Let $\{e_i\}_{i \in I}$ be a basis for $\mathscr{U}$ and $\{e_i'\}_{i \in I'}$ be a basis for $\mathscr{U}^{\perp}$. Let $\{f_j\}_{j \in J}$ be a basis for $\mathscr{V}$ and $\{f_j'\}_{j \in J'}$ be a basis for $\mathscr{V}^{\perp}$. Then $\{e_i \otimes f_j, e_i' \otimes f_j, e_i \otimes f_j', e_i' \otimes f_j'\}$ forms a basis for $\mathscr{H} \otimes \mathscr{K}$ while $\{e_i \otimes f_j\}$ forms a basis for $\mathscr{U} \otimes \mathscr{V}$. It is obvious that $\mathscr{U} \otimes \mathscr{V}$ is contained in $\mathscr{U} \otimes \mathscr{K} \cap \mathscr{H} \otimes \mathscr{V}$. The reverse containment is almost equally clear: for each $x \in \mathscr{U} \otimes \mathscr{K} \cap \mathscr{H} \otimes \mathscr{V}$, the representation of x in terms of basis vectors involves only the basis vectors $e_i \otimes f_j$. This proves the proposition.

For the remainder of this section we shall employ the notation of a Fubini algebra and the notation described in Section 2. In particular, for ultraweakly closed algebras $\mathscr{A}\subseteq\mathscr{B}(\mathscr{H})$ and $\mathscr{B}\subseteq\mathscr{B}(\mathscr{K})$,

$$\mathscr{F}(\mathscr{A},\mathscr{B})=\{T\in\mathscr{B}(\mathscr{H}\otimes\mathscr{K})\mid R_\varphi(T)\in\mathscr{B}, L_\psi(T)\in\mathscr{A}, \text{for all } \varphi\in\mathscr{B}(\mathscr{H})_*, \psi\in\mathscr{B}(\mathscr{K})_*\}.$$

The next proposition, combined with the comments preceding Proposition 11, shows that, for reflexive algebras $\mathscr{A}$ and $\mathscr{B}$, the tensor product formula has the equivalent formulation: $\mathscr{F}(\mathscr{A},\mathscr{B})=\mathscr{A}\otimes\mathscr{B}$.

PROPOSITION 12. *Let $\mathscr{A}\subseteq\mathscr{B}(\mathscr{H})$ and $\mathscr{B}\subseteq\mathscr{B}(\mathscr{K})$ be ultraweakly closed algebras which contain I. Then $\mathscr{F}(\mathscr{A},\mathscr{B})=[\mathscr{A}\otimes\mathscr{B}(\mathscr{K})]\cap[\mathscr{B}(\mathscr{H})\otimes\mathscr{B}]$.*

Proof. Since the R_φ and L_ψ are ultraweakly continuous, it is clear from the definitions that $[\mathscr{A}\otimes\mathscr{B}(\mathscr{K})]\cap[\mathscr{B}(\mathscr{H})\otimes\mathscr{B}]\subseteq\mathscr{F}(\mathscr{A},\mathscr{B})$. It is also clear from the definition that $\mathscr{F}(\mathscr{A},\mathscr{B})=\mathscr{F}(\mathscr{A},\mathscr{B}(\mathscr{K}))\cap\mathscr{F}(\mathscr{B}(\mathscr{H}),\mathscr{B})$. Let $T=\sum_{i,j=1}^{\infty}T_{ij}\otimes F_{ij}$ be in $\mathscr{F}(\mathscr{A},\mathscr{B}(\mathscr{K}))$. Fix k,l and let $\psi\in\mathscr{B}(\mathscr{K})_*$ be defined by $\psi(X)=\operatorname{tr}(XF_{kl})$, where tr is the usual trace on $\mathscr{B}(\mathscr{K})$. Then $L_\psi(T)=T_{lk}$. Hence $T_{lk}\in\mathscr{A}$, for all k,l and therefore $T\in\mathscr{A}\otimes\mathscr{B}(\mathscr{K})$. Thus $\mathscr{F}(\mathscr{A},\mathscr{B}(\mathscr{K}))\subseteq\mathscr{A}\otimes\mathscr{B}(\mathscr{K})$; $\mathscr{F}(\mathscr{B}(\mathscr{H}),\mathscr{B})\subseteq$ $\subseteq\mathscr{B}(\mathscr{H})\otimes\mathscr{B}$ is shown similarly.

We now recall some properties of the algebra of Hilbert-Schmidt operators on a Hilbert space. For a separable Hilbert space $\mathscr{H}$, let $\mathscr{C}_2(\mathscr{H})$ denote the Hilbert-Schmidt operators on $\mathscr{H}$. For $A,B\in\mathscr{C}_2(\mathscr{H})$, define $\langle A,B\rangle=\operatorname{tr}(B^*A)$. With the inner product $\langle\cdot,\cdot\rangle$, $\mathscr{C}_2(\mathscr{H})$ is a Hilbert space. For $A\in\mathscr{C}_2(\mathscr{H})$, we will write $\|A\|_2$ for $\langle A,A\rangle^{1/2}$.

We shall make frequent use of the fact that convergence in the $\|\cdot\|_2$ norm implies operator norm convergence, which in turn implies ultraweak convergence. For example, if $\mathscr{A}$ is an ultraweakly closed subalgebra of $\mathscr{B}(\mathscr{H})$, then $\mathscr{A}\cap\mathscr{C}_2(\mathscr{H})$ is closed in the $\|\cdot\|_2$ norm. Thus we can consider $\mathscr{A}\cap\mathscr{C}_2(\mathscr{H})$ as a closed subspace of the Hilbert space $\mathscr{C}_2(\mathscr{H})$. In the next proposition, $\mathscr{C}_2(\mathscr{H})\overline{\otimes}\mathscr{C}_2(\mathscr{K})$ denotes the subalgebra of $\mathscr{B}(\mathscr{H}\otimes\mathscr{K})$ which is generated by elementary tensors $S\otimes T$ with $S\in\mathscr{C}_2(\mathscr{H})$ and $T\in\mathscr{C}_2(\mathscr{K})$ and is closed in the $\|\cdot\|_2$ norm on $\mathscr{C}_2(\mathscr{H}\otimes\mathscr{K})$. In essence, $\mathscr{C}_2(\mathscr{H})\otimes\mathscr{C}_2(\mathscr{K})$ is the Hilbert space tensor product of $\mathscr{C}_2(\mathscr{H})$ and $\mathscr{C}_2(\mathscr{K})$.

PROPOSITION 13. $\mathscr{C}_2(\mathscr{H}\otimes\mathscr{K})=\mathscr{C}_2(\mathscr{H})\otimes\mathscr{C}_2(\mathscr{K})$.

Proof. Let $\{e_i\}$ and $\{f_j\}$ be orthonormal bases for $\mathscr{H}$ and $\mathscr{K}$, respectively. Let $E_{ij}\in\mathscr{B}(\mathscr{H})$ and $F_{ij}\in\mathscr{B}(\mathscr{K})$ be matrix units as before. Then $\{E_{ij}\}$ is a basis for $\mathscr{C}_2(\mathscr{H})$ and $\{F_{ij}\}$ is a basis for $\mathscr{C}_2(\mathscr{K})$. Hence $\{E_{ij}\otimes F_{kl}\}$ is a basis for $\mathscr{C}_2(\mathscr{H})\otimes$ $\otimes\mathscr{C}_2(\mathscr{K})$. On the other hand, the orthonormal basis of $\mathscr{C}_2(\mathscr{H}\otimes\mathscr{K})$ consisting of the matrix units which correspond to the basis $\{e_i\otimes f_j\}$ of $\mathscr{H}\otimes\mathscr{K}$ is also $\{E_{ij}\otimes F_{kl}\}$. The proposition is now immediate.

LEMMA 14. *Let* $T \in \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$, $\varphi \in \mathscr{B}(\mathscr{H})_*$ *and* $\psi \in \mathscr{B}(\mathscr{K})_*$. *Then* $R_\varphi(T) \in \mathscr{C}_2(\mathscr{K})$ *and* $L_\psi(T) \in \mathscr{C}_2(\mathscr{H})$.

Proof. Let $T = \sum_{i,j=1}^{\infty} T_{ij} \otimes F_{ij}$ and let $T_n = \sum_{i,j=1}^{n} T_{ij} \otimes F_{ij}$. Since $\|T\|_2^2 = \sum_{i,j=1}^{\infty} \|T_{ij}\|_2^2$, we see that $T \in \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$ implies that each $T_{ij} \in \mathscr{C}_2(\mathscr{H})$ and also that $\|T_n - T\|_2 \to 0$ as $n \to \infty$. Thus $R_\varphi(T_n) \to R_\varphi(T)$ ultraweakly. Now $R_\varphi(T_n) = \sum_{i,j=1}^{n} \varphi(T_{ij}) F_{ij}$ is in $\mathscr{C}_2(\mathscr{K})$. Hence to show that $R_\varphi(T) \in \mathscr{C}_2(\mathscr{K})$, we need only show that $\{R_\varphi(T_n)\}$ is Cauchy in the Hilbert-Schmidt norm. For this it suffices to show that the sequence of numbers $\{|\varphi(T_{ij})|\}$ is square summable. Since φ is a bounded linear functional,

$$\sum_{i,j=1}^{\infty} |\varphi(T_{ij})|^2 \leqslant \|\varphi\|^2 \sum_{i,j=1}^{\infty} \|T_{ij}\|^2 \leqslant \|\varphi\|^2 \sum_{i,j=1}^{\infty} \|T_{ij}\|_2^2 = \|\varphi\|^2 \, \|T\|_2^2 < \infty.$$

A similar argument shows that $L_\psi(T) \in \mathscr{C}_2(\mathscr{H})$.

Let $\mathscr{A}_2 = \mathscr{A} \cap \mathscr{C}_2(\mathscr{H})$ and $\mathscr{B}_2 = \mathscr{B} \cap \mathscr{C}_2(\mathscr{K})$. As before, $\overline{\otimes}$ denotes the $\|\cdot\|_2$ norm closure in $\mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$ of the algebraic tensor product, i.e. the Hilbert space tensor product.

LEMMA 15.

$$[\mathscr{A} \otimes \mathscr{B}(\mathscr{K})] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = \mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})$$

$$[\mathscr{B}(\mathscr{H}) \otimes \mathscr{B}] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = \mathscr{C}_2(\mathscr{H}) \overline{\otimes} \mathscr{B}_2.$$

Proof. It is clear that $\mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K}) \subseteq [\mathscr{A} \otimes \mathscr{B}(\mathscr{K})] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$ since Hilbert--Schmidt convergence implies ultraweak convergence.

Let $T = \sum_{i,j=1}^{\infty} T_{ij} \otimes F_{ij}$ be in $[\mathscr{A} \otimes \mathscr{B}(\mathscr{K})] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$ and let $T_n = \sum_{i,j=1}^{n} T_{ij} \otimes F_{ij}$. Just as before, $T_n \to T$ in Hilbert-Schmidt norm and each $T_{ij} \in \mathscr{A}_2$. Consequently, each T_n is a sum of elementary tensors in $\mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})$. Since $T_n \to T$ in Hilbert-Schmidt norm, $T \in \mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})$. Thus $[\mathscr{A} \otimes \mathscr{B}(\mathscr{K})] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = \mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})$; the other inequality is verified in the same way.

THEOREM 16. $\mathscr{F}(\mathscr{A}, \mathscr{B}) \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = [\mathscr{A} \otimes \mathscr{B}] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$.

Proof. Since $\mathscr{F}(\mathscr{A}, \mathscr{B}) = [\mathscr{A} \otimes \mathscr{B}(\mathscr{K})] \cap [\mathscr{B}(\mathscr{H}) \otimes \mathscr{B})]$, Lemma 15 yields

$$\mathscr{F}(\mathscr{A}, \mathscr{B}) \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = [\mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})] \cap [\mathscr{C}_2(\mathscr{H}) \overline{\otimes} \mathscr{B}_2].$$

Let $\mathscr{A}_2 \subseteq \mathscr{C}_2(\mathscr{H})$ and $\mathscr{B}_2 \subseteq \mathscr{C}_2(\mathscr{K})$ be considered as closed subspaces of the Hilbert spaces $\mathscr{C}_2(\mathscr{H})$ and $\mathscr{C}_2(\mathscr{K})$ respectively. By Proposition 11 we have,

$$[\mathscr{A}_2 \overline{\otimes} \mathscr{C}_2(\mathscr{K})] \cap [\mathscr{C}_2(\mathscr{H}) \overline{\otimes} \mathscr{B}_2] = \mathscr{A}_2 \overline{\otimes} \mathscr{B}_2,$$

where each side is considered as a closed subspace of the Hilbert space $\mathscr{C}_2(\mathscr{H}) \overline{\otimes} \mathscr{C}_2(\mathscr{K}) = \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$. Since

$$\mathscr{A}_2 \overline{\otimes} \mathscr{B}_2 = [\mathscr{A} \cap \mathscr{C}_2(\mathscr{H})] \otimes [\mathscr{B} \cap \mathscr{C}_2(\mathscr{K})] \subseteq [\mathscr{A} \otimes \mathscr{B}] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}),$$

we obtain $\mathscr{F}(\mathscr{A}, \mathscr{B}) \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) \subseteq [\mathscr{A} \otimes \mathscr{B}] \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$. The reverse inequality is obvious.

THEOREM 17. *Let $\mathscr{L}_1 \subseteq \mathscr{B}(\mathscr{H})$ and $\mathscr{L}_2 \subseteq \mathscr{B}(\mathscr{K})$ be completely distributive commutative subspace lattices. Then*

$$\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \operatorname{Alg} \mathscr{L}_1 \otimes \operatorname{Alg} \mathscr{L}_2.$$

Proof. First note that $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathscr{F}(\operatorname{Alg} \mathscr{L}_1, \operatorname{Alg} \mathscr{L}_2)$, since both are equal to $[\operatorname{Alg} \mathscr{L}_1 \otimes \mathscr{B}(\mathscr{K})] \cap [\mathscr{B}(\mathscr{H}) \otimes \operatorname{Alg} \mathscr{L}_2]$. Therefore Theorem 16 yields

$$\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) = (\operatorname{Alg} \mathscr{L}_1 \otimes \operatorname{Alg} \mathscr{L}_2) \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}).$$

By Theorem 10, $\mathscr{L}_1 \otimes \mathscr{L}_2$ is completely distributive, hence $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) \cap \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K})$ is ultraweakly dense in $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$. But we have $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) \cap \cap \mathscr{C}_2(\mathscr{H} \otimes \mathscr{K}) \subseteq \operatorname{Alg} \mathscr{L}_1 \otimes \operatorname{Alg} \mathscr{L}_2$, so $\operatorname{Alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) \subseteq \operatorname{Alg} \mathscr{L}_1 \otimes \operatorname{Alg} \mathscr{L}_2$. The reverse containment is automatic and the theorem is proven.

4. EXAMPLES

In this section we point out several examples of commutative subspace lattices. The first three examples, already familiar in the literature, are presented to illustrate the concept of complete distributivity. The fourth example is the most interesting: it yields a CSL algebra which possesses Hilbert-Schmidt operators but no finite rank operators. The fifth example is a variation on the fourth which illustrates the difficulty of finding a set theoretic characterization of L_*. Each example is presented by specifying the ingredients $X, \leqslant, m$ in the spectral representation $\mathscr{L} = \mathscr{L}(X, \leqslant, m)$.

EXAMPLE 1. Let $X = [0,1]$, let m be Lebesgue measure, and let $\leqslant$ be the trivial order: $x \leqslant y$ if and only if $x = y$. Since $\mathscr{L}(X, \leqslant, m)$ is a non-atomic Boolean algebra, it is not completely distributive. For each $x \in X$, $I_x = 0$. From Lemma 3 it is clear that $L_- = I$ for all non-zero $L \in \mathscr{L}(X, \leqslant, m)$. This, in turn, implies that $L_* = I$ for all $L \neq 0$. Also note that $L_\sim = I$, for all $L \neq 0$ (Lemma 4).

EXAMPLE 2. Let $X = [0,1]\times[0,1]$, let m be Lebesgue measure, and let $\leqslant$ be the product order: $(a, b) \leqslant (c, d)$ if, and only if, $a \leqslant c$ and $b \leqslant d$. Since $(X, \leqslant, m)$ is the tensor product of two nests, it is completely distributive. (The two nests are the same: each is the nest constructed from [0,1] with the usual order and Lebesgue measure.) Each projection I_x is of the form $P\otimes Q$. With the aid of Lemma 3 it is easy to check that $(P\otimes Q)_- = (P^{\perp}\otimes Q^{\perp})^{\perp}$.

EXAMPLE 3. This example is taken from [9]. Let $X = [0,1]\times[0,1]$, let $\leqslant$ be the product order as in Example 2, and let m be the sum of two dimensional Lebesgue measure on the square and one dimensional Lebesgue measure on the diagonal $\{(x, y) \in X \mid x + y = 1\}$. Let L be the projection in $\mathscr{L}(X, \leqslant, m)$ which corresponds to $\{(x, y) \in X \mid x + y > 1\}$. From the definition of $L_{\sim}$ given in Lemma 4, it is clear that $L_{\sim}$ corresponds to $\{(x, y) \in X \mid x + y \geqslant 1\}$. Thus $L < L_{\sim} \leqslant L_*$ and $\mathscr{L}(X, \leqslant, m)$ is not completely distributive. It is not hard to see that $L_{\sim} = L_*$ in this example.

The authors would like to thank Tavan Trent for providing the following example:

EXAMPLE 4. Let $X = [0,1]$ and let m be Lebesgue measure on X. Let $R = \{(x, y) \mid 0 \leqslant x \leqslant 1/2,\ 1/2 \leqslant y \leqslant 1 \text{ and } y - x \in \mathbf{Q}\}$. R is a subset of $[0,1/2]\times[1/2,1]$ with measure 0. Let K be a compact subset of $[0,1/2]\times[1/2,1]\setminus R$ such that $m\times m(K) > 0$. A standard construction yields the set K. First, enumerate the rationals in [1/2,1]. For each rational r_n delete from $[0,1/2]\times[1/2,1]$ all points whose distance from the line $y = x + r_n$ is less than $\varepsilon/2^n$. For suitable ε, the set that remains is compact and has positive measure. In particular, K has the property that, for each $x \in [0,1/2]$, both $\{y \mid (x, y) \in K\} \cap [1/2, 3/4]$ and $\{y \mid (x, y) \in K\} \cap [3/4,1]$ have positive measure. Let $D = \{(x, x) \mid x \in [0,1]\}$ be the diagonal. Let $G = K \cup D$. Since K is a subset of $[0,1/2]\times[1/2,1]$, G is the graph of a transitiveand reflexive relation. Let $\leqslant$ be the relation so obtained.

If A and B are any measurable subsets of [0,1] with $m(A) > 0$ and $m(B) > 0$ then $A\times B$ is not a subset of G. Indeed, if $A\times B\subseteq G$ then clearly $A\times B\subseteq K$. But this means that $b - a \notin \mathbf{Q}$ for all $(a, b) \in A \times B$, i.e. $(B - A) \cap \mathbf{Q} = \emptyset$. But if A and B both have positive measure, then $B - A$ contains an interval. This shows that Alg$\mathscr{L}$ contains no rank one operator (Lemma 5). By using a result which will appear elsewhere, one can actually see that Alg$\mathscr{L}$ contains no finite rank operators. But $m\times m(G) > 0$, so Alg$\mathscr{L}$ does contain Hilbert-Schmidt operators.

By Lemma 2, $L_- = I$ for all $L \neq 0$ in $\mathscr{L}(X, \leqslant, m)$. Consequently, $L_* = I$ for all $L \neq 0$ and $\mathscr{L}(X, \leqslant, m)$ is not completely distributive. Now fix L equal to the projection which corresponds to [3/4,1]. Then $L_{\sim}$ corresponds to $\{x \mid m(\hat{I}_x \wedge \hat{L}^c) = 0\} = (1/2,1]$. Thus $L < L_{\sim} < L_*$ in this example. It is easy to check that $L_{\approx} = L_*$ in this example (see remark after Lemma 4).

EXAMPLE 5. $(X, \leqslant, m)$ is as in Example 4 except that the graph G is taken to be $G = K \cup D \cup ([0,1/4] \times [3/4,1])$. Let L be the projection corresponding to $[7/8,1]$. One can show that $L_{\sim}$ corresponds to $[1/2,1]$, L_{*} corresponds to $[1/4,1]$ and $L_{\approx} = I$. Thus $L < L_{\sim} < L_{*} < L_{\approx}$. We omit the details.

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Added in proof: Laurie and Longstaff [*Proc. Amer. Math. Soc.*, **89**(1983), 293–297] have shown that the statements (i), (ii), (iii) of Theorem 7 are all equivalent to:

(iv) *The algebra generated by the rank one operators in* Alg $\mathscr{L}$ *is ultraweakly dense in* Alg $\mathscr{L}$.