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REMARKS ON LEBESGUE-TYPE DECOMPOSITION OF POSITIVE OPERATORS

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0. INTRODUCTION

The purpose of the paper is to give alternative proofs to Ando's results, [1], based on a different technique, which will be explained below. Also, this technique is shown to provide quite a natural viewpoint to the subject matter.

Let a, b be positive (bounded) operators on a Hilbert space. As a generalization of absolute continuity in measure theory, we say that b is *a-absolutely continuous* if there exists a sequence $\{b_n\}$ of positive operators such that $b_n \uparrow b$ (strongly) as $n \uparrow \infty$ and $b_n \leqslant l_n a$ for some positive number l_n. We also say that b is *a-singular* if $c = 0$ follows whenever an operator c satisfies $0 \leqslant c \leqslant b$ and $0 \leqslant c \leqslant a$. In the paper, we consider a "Lebesgue decomposition" $b = b_1 + b_2$ with an a-absolutely continuous operator b_1 and an a-singular operator b_2.

In [1], Ando introduced the positive operator $[a]b$ $(\leqslant b)$ as the strong limit of a certain sequence of positive operators (see § 1). Among other results, he showed that

(i) $[a]b$ (resp. $b - [a]b$) is a-absolutely continuous (resp. a-singular),

(ii) $[a]b$ is maximal in the sense that $b' \leqslant [a]b$ whenever $0 \leqslant b' \leqslant b$ and b' is a-absolutely continuous.

In the paper, we further assume that a is non-singular ($\mathrm{Ker}\, a = \{0\}$). (Applications of this subject to the theory of operator algebras will appear in subsequent papers. And, in this context, the case when $\mathrm{Ker}\, a = \{0\}$ is most interesting.) We show that the subject matter is closely related (almost "equivalent") to recent theories [4], [7] of decompositions of unbounded operators and quadratic forms (into their closable parts and singular parts). Use of unbounded operators (and forms) actually gives quite a powerful tool although involved arguments are simple. In fact, based on this technique we obtain certain simple and (more importantly) explicit expressions of $[a]b$.

1. PRELIMINARIES

In this section, we collect some basic definitions as well as results. We fix positive (bounded) operators a, b on a Hilbert space $\mathscr{H}$ throughout, and further assume that a is non-singular.

DEFINITION 1. We set $T = b^{-1/2}a^{-1/2}$ so that T is a densely defined $(\mathscr{D}(T) = \mathscr{R}(a^{1/2}))$ operator on $\mathscr{H}$.

The operator T may or may not be closable. The operator $b^{1/2}$ being bounded, we have $T^* = a^{-1/2}b^{1/2}$ and

$$\mathscr{D}(T^*) = \{\xi \in \mathscr{H} \,;\, b^{1/2}\xi \in \mathscr{R}(a^{1/2})\}.$$

Due to the well known relation

$$\mathscr{D}(T^*)^{\perp} = \{\xi \in \mathscr{H}; (0, \xi) \in \Gamma(T)^{-}\},$$

where $\Gamma(T)$ denotes the graph of T, we have the following lemma:

LEMMA 2. *Let p be the projection onto the closure of $\{\xi \in \mathscr{H}; b^{1/2}\xi \in \mathscr{R}(a^{1/2})\}$. Then $1 - p$ is the projection onto $\{\xi \in \mathscr{H}; (0, \xi) \in \Gamma(T)^-\}$.*

The next result is implicit in [1] and indicates that [4] and [7] are closely related to our subject.

LEMMA 3. *The following three conditions are equivalent:*
(a) *b is a-absolutely continuous,*
(b) *T is closable,*
(c) *$p = 1$.*

Proof. (b) $\Leftrightarrow$ (c) is precisely Lemma 2.

(a) $\Rightarrow$ (b) Here (and in the proof of Theorem 8), the space $\mathscr{H}$ equipped with the topology induced by the new norm: $\xi \mapsto \|a^{1/2}\xi\|$ is denoted by $\mathscr{H}_a$. By the assumption, b can be written as $b_n \uparrow b$, $b_n \leqslant l_n a$. For each n, we have $\|b_n^{1/2}\xi\| \leqslant \leqslant l_n^{1/2}\|a^{1/2}\xi\|$ so that the map: $\xi \in \mathscr{H}_a \mapsto \|b_n^{1/2}\xi\| \in \mathbf{R}_+$ is continuous. Being the supremum of continuous functions, the map: $\xi \in \mathscr{H}_a \mapsto \|b^{1/2}\xi\| = \sup_n \|b_n^{1/2}\xi\| \in \mathbf{R}_+$ is thus lower semi-continuous. To show the closability of T, we now assume $\zeta_n = a^{1/2}\xi_n \to 0$ and $T\zeta_n = b^{1/2}\xi_n \to \eta$ in $\mathscr{H}$ as $n \to \infty$. For each $\varepsilon > 0$, we pick up a positive integer $N = N_\varepsilon$ such that

$$\|T\zeta_m - T\zeta_n\| = \|b^{1/2}(\xi_m - \xi_n)\| \leqslant \varepsilon \quad \text{for } n, m \geqslant N.$$

The sequence $\{\zeta_n\}$ tending to 0 in $\mathscr{H}$, for each fixed m, $\xi_m - \xi_n$ tends to ξ_m in

$\mathscr{H}_a$ as $n \to \infty$. The above mentioned lower semi-continuity thus implies

$$\|T\zeta_m\| = \|b^{1/2}\xi_m\| \leqslant \liminf_{n\to\infty} \|b^{1/2}(\xi_m - \xi_n)\| \leqslant \varepsilon$$

for each fixed $m \geqslant N$. Letting $m \to \infty$ and noting the arbitrariness of $\varepsilon > 0$, we conclude $\eta = 0$ as desired.

(b) $\Rightarrow$ (a) We always have $b^{1/2} = Ta^{1/2}$ so that we also have $b^{1/2} = \bar{T}a^{1/2}$ whenever T is closable. Let e_n be the spectral projection of $|\bar{T}|^2 = T^*\bar{T}$ corresponding to the interval $[0, n]$. It is straightforward to check that $a^{1/2}e_n|\bar{T}|^2a^{1/2} = a^{1/2}e_n|\bar{T}|^2e_na^{1/2} \leqslant na$ and $(a^{1/2}e_n|\bar{T}|^2a^{1/2}\xi \mid \xi) \uparrow (b\xi \mid \xi)$ for each $\xi \in \mathscr{H}$. Q.E.D.

DEFINITION 4. Let I be the imbedding of $\mathscr{D}(T)$ into the graph $\Gamma(T)$ ($\subseteq \mathscr{H} \oplus \mathscr{H}$) given by $I(\xi) = (\xi, T\xi)$. Also, let i be the restriction to $\Gamma(T)^-$ of the projection: $(\xi_1, \xi_2) \in \mathscr{H} \oplus \mathscr{H} \mapsto \xi_1 \in \mathscr{H}$ so that i is a contraction from $\Gamma(T)^-$ to $\mathscr{H}$.

It is easy to check that $1 \oplus p$ (or equivalently $0 \oplus (1 - p)$) leaves $\Gamma(T)^-$ invariant. We obviously have

$$\operatorname{Ker} i = \{(0, \xi) \in \Gamma(T)^-\} = \{0 \oplus (1 - p)\}\Gamma(T)^-,$$

which is the "obstruction" for the closability of T.

Finally we recall the notion of a parallel sum ([3]). For positive operators c, d, their parallel sum $c : d$ is defined as the positive operator determined by

$$((c : d)\xi \mid \xi) = \inf\{(c\xi_1 \mid \xi_1) + (d\xi_2 \mid \xi_2);\ \ \xi = \xi_1 + \xi_2,\ \ \xi_i \in \mathscr{H}\}.$$

It follows immediately that $(na) : b \leqslant na$, $(na) : b \leqslant b$, and $\{(na) : b\}_{n=1,2,\ldots}$ is an increasing sequence of positive operators. Ando, [1], set $[a]b = \text{s-}\lim_{n\to\infty} (na) : b$ ($= \sup_n (na) : b$) and showed (i) and (ii) in § 0 (see [1], and also [5] for further properties).

2. LEBESGUE DECOMPOSITION

In this section, we obtain a Lebesgue decomposition of b with respect to a. More precisely, we show that $b^{1/2}pb^{1/2}$ is a-absolutely continuous and exactly $[a]b$ in the sense of Ando. Actually, we present two proofs (for the latter result) for the following reasons: (a) to understand a relation between his approach and our approach, (b) to show strength of simple arguments in [4], [7], (c) to make the paper self-contained. Namely, in Theorem 6 we show $[a]b = b^{1/2}pb^{1/2}$ based on the definition of $[a]b$ and the maximality of $[a]b$ ((ii) in § 0), while in Theorem 7 we directly prove the maximality of $b^{1/2}pb^{1/2}$ based on arguments in [4], [7].

We begin with introducing

$$T_c = pT, \text{ the closable part of } T \ (\mathscr{D}(T_c) = \mathscr{D}(T)),$$

$$T_s = (1-p)T, \text{ the singular part of } T \ (\mathscr{D}(T_s) = \mathscr{D}(T)).$$

Among other results, Jorgensen, [4], showed:

(iii) T_c is closable as its name indicates,

(iv) If the projection from $\mathscr{H} \oplus \mathscr{H}$ onto its closed subspace $\Gamma(T)^-$ is $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ (called the characteristic matrix for T, see [6] for details), then that corresponding to $\Gamma(T_c)^- = \Gamma(\bar{T}_c)$ is $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & pp_{22} \end{bmatrix}$. Proofs of (iii) and (iv) are quite easy and are found in [4], pp. 285–286. We also note that $\Gamma(\bar{T}_c)$ is exactly the "operator part" of $\Gamma(T)^-$ in the sense of [2].

LEMMA 5. *The operator $b^{1/2}pb^{1/2}$ is a-absolutely continuous.*

Proof. Notice that $pb^{1/2} = T_c a^{1/2} = \bar{T}_c a^{1/2}$. Thus, arguments in (b) $\Rightarrow$ (a) of Lemma 3 show the a-absolute continuity of $b^{1/2}pb^{1/2}$. Q.E.D.

THEOREM 6. *We have $b^{1/2}pb^{1/2} = [a]b$.*

Proof. Due to the previous lemma, $b^{1/2}pb^{1/2} \leqslant b$, and the maximality of $[a]b$ ((ii) in § 0), we have $b^{1/2}pb^{1/2} \leqslant [a]b$. Since $(na):b \uparrow [a]b$, it suffices to show $(na):b \leqslant b^{1/2}pb^{1/2} = a^{1/2}|\bar{T}_c|^2 a^{1/2}$ for each n.

However, it actually suffices to show just $a:b \leqslant a^{1/2}|\bar{T}_c|^2 a^{1/2}$. In fact, if we replace a by $a' = na$ here, then we have

$$a'^{1/2} = n^{1/2}a^{1/2}$$

$$T' = b^{1/2}a'^{-1/2} = n^{-1/2}b^{1/2}a^{-1/2} = n^{-1/2}T$$

$$T'_c = n^{-1/2}T_c .$$

Here, the last equality follows from the fact that the range of $a^{1/2}$ is the same as that of $a'^{1/2}$. Hence we have

$$a'^{1/2}|\bar{T}'_c|^2 a'^{1/2} = a^{1/2}|\bar{T}_c|^2 a^{1/2} = b^{1/2}pb^{1/2}.$$

For a vector $\xi \in \mathscr{H}$, we have

$$((a:b)\xi|\xi) = \inf\{\|a^{1/2}\xi - a^{1/2}\zeta\|^2 + \|b^{1/2}\zeta\|^2;\ \zeta \in \mathscr{H}\} =$$

$$= (\text{Distance}\{(a^{1/2}\xi, 0), \Gamma(T)\})^2 =$$

$$= (\text{Distance}\{(a^{1/2}\xi, 0), \Gamma(T)^-\})^2$$

because $\{(a^{1/2}\zeta, b^{1/2}\zeta) \in \mathscr{H} \oplus \mathscr{H};\ \zeta \in \mathscr{H}\}$ is precisely the graph of T. The equality

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} a^{1/2}\xi \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & pp_{22} \end{bmatrix} \begin{bmatrix} a^{1/2}\zeta \\ 0 \end{bmatrix}$$

says that the nearest point on $\Gamma(T)^-$ from $(a^{1/2}\xi, 0)$ is the same as the nearest point on $\Gamma(T_c)^- = \Gamma(\overline{T}_c)$ from $(a^{1/2}\xi, 0)$. (Recall (iv)). We thus estimate

$$((a:b)\xi \mid \xi) = (\text{Distance}\{(a^{1/2}\xi, 0),\ \Gamma(T_c)^-\})^2 =$$

$$= (\text{Distance}\{(a^{1/2}\xi, 0),\ \Gamma(T_c)\})^2 =$$

$$= \inf\{\|a^{1/2}\xi - a^{1/2}\zeta\|^2 + \|T_c a^{1/2}\zeta\|^2;\quad \zeta \in \mathscr{H}\} \leqslant$$

$$\leqslant \|T_c a^{1/2}\xi\|^2 = (b^{1/2}pb^{1/2}\xi \mid \xi).$$

Q.E.D.

THEOREM 7. *If a positive operator b' is a-absolute continuous and $b' \leqslant b$, then $b' \leqslant b^{1/2}pb^{1/2}$.*

(Direct proof based on arguments in [7].) Starting from $T' = b'^{1/2}a^{-1/2}$ ($\mathscr{D}(T') = \mathscr{D}(T) = \mathscr{R}(a^{1/2})$), we consider I' and i' associated with T' as in Definition 4. The a-absolute continuity of b' implies that T' is closable (Lemma 3) so that $\operatorname{Ker} i' = \{0\}$. (see the paragraph after Definition 4).

We now define $j: \Gamma(T) \to \Gamma(T')\ (\subseteq \Gamma(T')^- = \Gamma(\overline{T}'))$ by $j(\xi, T\xi) = (\xi, T'\xi)$. It follows from $b' \leqslant b$ that $\|(\xi, T'\xi)\| \leqslant \|(\xi, T\xi)\|$. Thus, j extends uniquely to a contraction (still denoted by j) from $\Gamma(T)^-$ to $\Gamma(\overline{T}')$. We also note

$$j \circ I = I'$$

$$i' \circ j = i.$$

In fact, the both sides of the second equality are contractions from $\Gamma(T)^-$ to $\mathscr{H}$, and they give the same result when applied to $(\xi, T\xi) \in \Gamma(T)$.

The above second equality and the injectivity i' imply that $\operatorname{Ker} i = \operatorname{Ker} j$. The operator $(0 \oplus (1-p))\,|_{\Gamma(T)^-}$ being the projection from $\Gamma(T)^-$ onto $\operatorname{Ker} i$, we then have

$$j \circ ((0 \oplus (1-p))\,|_{\Gamma(T)^-}) \circ I(\xi) = 0 \quad \text{for} \quad \xi \in \mathscr{R}(a^{1/2}),$$

or equivalently,

$$j \circ ((1 \oplus p)\,|_{\Gamma(T)^-}) \circ I(\xi) = i \circ I(\xi) = I'(\xi).$$

Thus, for $a^{1/2}\zeta \in \mathscr{R}(a^{1/2})$, we estimate

$$\|a^{1/2}\zeta\|^2 + \|T'a^{1/2}\zeta\|^2 = \|I'(a^{1/2}\zeta)\|^2$$

$$= \|j\cdot((1\oplus p)|_{\Gamma(T)^-})\, I(a^{1/2}\zeta)\|^2 \leqslant$$

$$\leqslant \|(1\oplus p)|_{\Gamma(T)^-}\, I(a^{1/2}\zeta)\|^2 \quad \text{(since } j \text{ is a contraction)}$$

$$= \|(a^{1/2}\zeta,\ pTa^{1/2}\zeta)\|^2 = \|a^{1/2}\zeta\|^2 + \|pTa^{1/2}\zeta\|^2.$$

This means $\|T'a^{1/2}\zeta\|^2 \leqslant \|pTa^{1/2}\zeta\|^2$, that is, $(b'\zeta|\zeta) \leqslant (b^{1/2}pb^{1/2}\zeta|\zeta)$. Q.E.D.

3. AN EXPRESSION OF $b^{1/2}pb^{1/2}$

In the previous section, we showed that $[a]b$ is just $b^{1/2}pb^{1/2}$. In this section, we further eliminate the projection p and obtain a certain expression of $[a]b = b^{1/2}pb^{1/2}$ involving only simpler quantities. The expression and its proof are closely related to the well-known recent characterization of closed forms in terms of lower semi-continuity (see [8] for example).

THEOREM 8. *For a vector* $\xi \in \mathscr{H}$, *we have* $(b^{1/2}pb^{1/2}\xi|\xi) = \inf\{\liminf_{n\to\infty}(b\xi_n|\xi_n);$ $\{\xi_n\}$ *in* $\mathscr{H}$ *satisfying* $\lim_{n\to\infty}(a(\xi-\xi_n)|\xi-\xi_n)=0\} = \inf\{\lim_{n\to\infty}(b\xi_n|\xi_n);\ \{\xi_n\}$ *in* $\mathscr{H}$ *satisfying* (a) $\lim_{n\to\infty}(a(\xi-\xi_n)|\xi-\xi_n)=0$, (b) $\lim_{n\to\infty}(b\xi_n|\xi_n)$ *exists*$\}$.

Proof. We denote the first and second infima by α and β respectively so that we have $\alpha \leqslant \beta$.

At first we note that $\lim_{n\to\infty}(a(\xi-\xi_n)|\xi-\xi_n)=0$ is same as $\xi_n \to \xi$ in $\mathscr{H}_a$ (see the proof of (a) $\Rightarrow$ (b) in Lemma 3). Since $b^{1/2}pb^{1/2}$ is a-absolutely continuous (Lemma 5), the map $\mathscr{H}_a \ni \zeta \mapsto (b^{1/2}pb^{1/2}\zeta|\zeta) \in \mathbf{R}_+$ is lower semi-continuous. Thus, whenever $\xi_n \to \xi$ in $\mathscr{H}_a$, we must have

$$(b^{1/2}pb^{1/2}\xi|\xi) \leqslant \liminf_{n\to\infty}(b^{1/2}pb^{1/2}\xi_n|\xi_n) \leqslant \liminf_{n\to\infty}(b\xi_n|\xi_n).$$

Hence we have shown $(b^{1/2}pb^{1/2}\xi|\xi) \leqslant \alpha$.

Recall that $0\oplus(1-p)$ sends $\Gamma(T)^-$ onto $\Gamma(T)^- \cap (0\oplus\mathscr{H})$. Hence, for $a^{1/2}\xi \in \mathscr{D}(T)$ and $Ta^{1/2}\xi = b^{1/2}\xi$, the vector $(0\oplus(1-p))(a^{1/2}\xi,\ b^{1/2}\xi) = (0, (1-p)b^{1/2}\xi)$ in $\mathscr{H}\oplus\mathscr{H}$ must belong to $\Gamma(T)^- \cap (0\oplus\mathscr{H})$. In particular, we can pick up a sequence $\{a^{1/2}\zeta_n\}$ in $\mathscr{R}(a^{1/2})$ such that $a^{1/2}\zeta_n \to 0$ and $Ta^{1/2}\zeta_n = b^{1/2}\zeta_n \to (p-1)b^{1/2}\xi$. Setting $\xi_n = \zeta_n + \xi$, we notice that $a^{1/2}\xi_n \to a^{1/2}\xi$ and $b^{1/2}\xi_n \to (p-1)b^{1/2}\xi + b^{1/2}\xi = pb^{1/2}\xi$. We thus have $\beta \leqslant (b^{1/2}pb^{1/2}\xi|\xi)$. Q.E.D.

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