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ORBITS OF THE UNIT SPHERE OF $\mathscr{L}(\mathscr{H}, \mathscr{K})$ UNDER SYMPLECTIC TRANSFORMATIONS

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1. INTRODUCTION

The set of all biholomorphic transformations of the open unit disc of the complex plane onto itself comprises the mappings

$$\varphi(z) = \lambda \frac{z - \alpha}{1 - \bar{\alpha} z}$$

where $|\alpha| < 1$ and $|\lambda| = 1$. These mappings play an important role in the multiplicative theory of analytic functions, and it is not surprising that their analogues are prominent in the extension of this theory to more general domains. One domain for which there is a rich theory is the unit ball Δ of the space $\mathscr{L}(\mathscr{H}, \mathscr{K})$ of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, with the operator norm, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces. We say that $\Psi: \Delta \to \Delta$ is biholomorphic if it is Fréchet differentiable on Δ and has a Fréchet differentiable inverse $\Psi^{-1}: \Delta \to \Delta$. It then transpires (see [4]) that every biholomorphic mapping of Δ onto itself is of the form $L \circ \Phi$ where L is a linear isometry on $\mathscr{L}(\mathscr{H}, \mathscr{K})$ and Φ is a *symplectic transformation* — that is, a mapping of the form

$$\Phi(X) = (AX + B)(CX + D)^{-1} \tag{1}$$

where $A \in \mathscr{L}(\mathscr{K})$, $B \in \mathscr{L}(\mathscr{H}, \mathscr{K})$, $C \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, $D \in \mathscr{L}(\mathscr{H})$ (we write $\mathscr{L}(\mathscr{H})$ for $\mathscr{L}(\mathscr{H}, \mathscr{H})$) and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^* & -C^* \\ -B^* & D^* \end{bmatrix}. \tag{2}$$

These 2×2 matrices of operators represent elements of $\mathscr{L}(\mathscr{K} \oplus \mathscr{H})$ in an obvious way. The relation (2) is precisely what is needed to ensure that the linear fractional transformation Φ be defined throughout Δ and map Δ into itself: see [10]. The set of all transformations Φ defined by (1) and (2) is a group. It can also be described as the connected component of the identity in the group of all biholomorphic transformations of the identity under a certain natural topology [10].

Symplectic transformations arise in many other ways too. A seminal paper by C. L. Siegel [12] inaugurated the study in a number-theoretic spirit of the geometry of the unit ball of the symmetric $n \times n$ matrices under this group. They occur naturally in the theory of spaces with indefinite inner product (Kreĭn spaces: see [8]) and hence also in sundry interpolation and approximation problems which are closely connected with Kreĭn spaces (see [1, 2]). They turn up in the theory of electrical circuits: specifically, they correspond to the insertion of lossless matching circuits between a power source and a load [9]. J. W. Helton [6] has given a beautiful solution using symplectic geometry of the problem of matching impedances over a wide bandwidth. V. Pták and the author [11] used symplectic transformations of shifts to exhibit extremals for a certain maximum problem for matrices.

In studying the action of symplectic transformations on canonical models of contractions and intertwining dilations, I have encountered the problem of orbits of the symplectic transformations. Now this group acts transitively on Δ; Siegel [12] showed this for the symmetric $n \times n$ matrices, and his proof extends directly to the most general domains [5]. However, it is a consequence of condition (2) that the formula (1) for $\Phi(X)$ makes sense when $\|X\| = 1$, and so we may regard the symplectic transformations as acting on the closed unit ball $\bar{\Delta}$ of $\mathscr{L}(\mathscr{H}, \mathscr{K})$, and indeed much interest attaches to the orbits of operators which lie irredeemably on the unit sphere — notably shifts and projections. It is the purpose of this paper to characterize the orbits of $\bar{\Delta}$ under the symplectic group.

There are two identities which play a vital role: if Φ is defined by (1) and (2) then (see [5])

$$I - \Phi(X)^*\Phi(X) = (CX + D)^{*-1}(I - X^*X)(CX + D)^{-1},$$
(3)
$$I - \Phi(X)\Phi(X)^* = (XB^* + A^*)^{-1}(I - XX^*)(XB^* + A^*)^{*-1}.$$

In order that X and Y lie in the same orbit it is thus necessary that $I - X^*X, I - XX^*$ be congruent to $I - Y^*Y, I - YY^*$ respectively (we say that Hermitian operators M, N on $\mathscr{H}$ are congruent if there exists P invertible in $\mathscr{L}(\mathscr{H})$ such that $M = P^*NP$). One can hardly resist guessing that these conditions are also sufficient, and this is very nearly true. There is just one recalcitrant class of operators for which it fails: we shall call $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ *essentially unitary* if both $I - X^*X$ and $I - XX^*$ are compact operators. The orbit of an essentially unitary operator X is characterized by the congruence class of $I - X^*X$ and $I - XX^*$ *and* by the Fredholm index ind X; for any other operator the two congruence classes alone characterize the orbit.

We assume throughout that $\mathscr{H}$ and $\mathscr{K}$ are separable (but not necessarily infinite-dimensional) Hilbert spaces.

In Section 2 we perform a little algebraic manipulation to write symplectic transformations in a different way. The characterization of orbits, Theorem 1, and most of its proof is in Section 3: there is just one delicate point left over to Section 4. We need to answer the question: given $T \in \mathscr{L}(\mathscr{H})$, $\|T\| \leqslant 1$, does there exist a unitary U such that $U - T$ is invertible? This too has a neat answer. Yes, unless T is a compact perturbation of a non-unitary isometry or co-isometry (Theorem 3). Section 5 gives an example which shows that the index really is needed for essentially unitary operators. Finally, Section 6 characterizes those operators which lie in the orbit of a partial isometry.

We shall call an element of $\mathscr{L}(\mathscr{H}, \mathscr{K})$ of norm no greater than 1 a *contraction*.

2. REARRANGEMENT OF SYMPLECTIC TRANSFORMATIONS

In manipulating scalar linear fractional transformations it is sometimes convenient to rewrite them:

$$\frac{az+b}{cz+d} = \frac{b}{d} + \frac{ad-bc}{d}\,\frac{z}{cz+d}.$$

A similar thing can be done for the symplectic transformation (1). Since D is invertible, we can write for any contraction $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$,

$$\Phi(X) = (AX+B)(CX+D)^{-1} =$$

$$= BD^{-1} - (BD^{-1}C - A)X(I + D^{-1}CX)^{-1}D^{-1}.$$

Let us introduce the notation

$$\Psi_\rho(X) = F - EX(I + GX)^{-1}H \tag{4}$$

where ρ is the 2×2 matrix of operators

$$\rho = \begin{bmatrix} E & F \\ G & H \end{bmatrix}. \tag{5}$$

Here $E \in \mathscr{L}(\mathscr{K})$, $F \in \mathscr{L}(\mathscr{H}, \mathscr{K})$, $G \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ and $H \in \mathscr{L}(\mathscr{H})$, so that ρ is the matrix of an operator on $\mathscr{K} \oplus \mathscr{H}$. Then $\Phi = \Psi_\rho$ when

$$\rho = \begin{bmatrix} BD^{-1}C - A & BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix}. \tag{6}$$

A rather messy calculation shows that condition (2) implies that ρ is unitary (this fact is widely known — see for example [2, 6]; it seems to be traditional to leave the calculation to the reader). Conversely, if ρ in (5) is unitary and H is invertible then Ψ_ρ is a symplectic transformation. Indeed, if we let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} FH^{-1}G - E & FH^{-1} \\ H^{-1}G & H^{-1} \end{bmatrix} \tag{7}$$

then it may be verified that (2) is satisfied, so that $\Phi(X) = (AX + B)(CX + D)^{-1}$ is symplectic, and that $\Phi = \Psi_\rho$.

If ρ is unitary but H is not invertible then the transformation Ψ_ρ defined by (4) is clearly not symplectic. $\Psi_\rho(X)$ is still defined as long as $\|X\| < 1$, but it may not be when $\|X\| = 1$; that is, $I + GX$ need not be invertible. $\Psi_\rho(X)$ can still be given a natural interpretation in the finite-dimensional case, but in general it cannot: see [7].

Let us establish the congruence relations (3) in the Ψ notation.

LEMMA 1. *Let*

$$\rho = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

be the matrix of an operator on $\mathscr{K} \oplus \mathscr{H}$ and let $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ be such that $I + GX$ is invertible in $\mathscr{L}(\mathscr{H})$.

(i) *If $\rho^*\rho = I$ then*

$$I - \Psi_\rho(X)^*\Psi_\rho(X) = H^*(I + X^*G^*)^{-1}(I - X^*X)(I + GX)^{-1}H;$$

(ii) *if $\rho\rho^* = I$ then*

$$I - \Psi_\rho(X)\Psi_\rho(X)^* = E(I + XG)^{-1}(I - XX^*)(I + G^*X^*)^{-1}E^*.$$

Proof (ii). We have, since $\rho\rho^* = I$,

$$EE^* + FF^* = I,$$

$$GG^* + HH^* = I,$$

$$EG^* = -FH^*, \quad GE^* = -HF^*.$$

Hence

$$I - \Psi_\rho(X)\Psi_\rho(X)^* = I - \{F - EX(I+GX)^{-1}H\}\{F^* - H^*(I+X^*G^*)^{-1}X^*E^*\} =$$

$$= I - FF^* + EX(I+GX)^{-1}HF^* + FH^*(I+X^*G^*)^{-1}X^*E^* -$$

$$- EX(I+GX)^{-1}HH^*(I+X^*G^*)^{-1}X^*E^* =$$

$$= EE^* - EG^*(I+X^*G^*)^{-1}X^*E^* - EX(I+GX)^{-1}GE^* -$$

$$- E(I+XG)^{-1}XHH^*X^*(I+G^*X^*)^{-1}E^* =$$

$$= E(I+XG)^{-1}[(I+XG)(I+G^*X^*) - (I+XG)G^*X^* -$$

$$- XG(I+G^*X^*) - XHH^*X^*](I+G^*X^*)^{-1}E^* =$$

$$= E(I+XG)^{-1}[1 + X(-GG^* - HH^*)X^*](I+G^*X^*)^{-1}E^* =$$

$$= E(I+XG)^{-1}(I - XX^*)(I+G^*X^*)^{-1}E^*.$$

To prove (i) note that $\Psi_\rho(X)^* = \Psi_\nu(X^*)$ where

$$\nu = \begin{bmatrix} H^* & F^* \\ G^* & E^* \end{bmatrix}.$$

If $\rho^*\rho = I$ then $\nu\nu^* = I$. On applying (ii) with ρ replaced by ν and X replaced by X^* we obtain (i).

3. CHARACTERIZATION OF ORBITS

In view of the foregoing discussion we can re-state our problem as follows: given contractions $X, Y \in \mathscr{L}(\mathscr{H},\mathscr{K})$, find a unitary ρ on $\mathscr{K}\oplus\mathscr{H}$, with invertible (2,2) entry (i.e. compression of ρ to $\mathscr{H}$), such that $\Psi_\rho(X) = Y$. It turns out that deriving a suitable ρ entails solving a sort of quadratic equation in operators, and this can be achieved by the traditional process of "completing the square".

THEOREM 1. *Let $X, Y \in \mathscr{L}(\mathscr{H},\mathscr{K})$ be contractions. There exists a symplectic transformation Φ such that $\Phi(X) = Y$ if and only if*

(i) *$I - X^*X$, $I - XX^*$ are congruent to $I - Y^*Y$, $I - YY^*$ respectively, and*

(ii) *if X is essentially unitary then*

$$\operatorname{ind} X = \operatorname{ind} Y.$$

Note that if X is essentially unitary then, by (i), Y is also, hence X and Y are both Fredholm, so that $\operatorname{ind} X$ and $\operatorname{ind} Y$ are defined.

It will be helpful to distinguish various sorts of operators typographically; accordingly we shall continue to denote operators from $\mathscr{H}$ or $\mathscr{K}$ to $\mathscr{H}$ or $\mathscr{K}$ by Roman capitals and elements of $\mathscr{L}(\mathscr{K}\oplus\mathscr{H})$ by small Greek letters, while operators from $\mathscr{K}\oplus\mathscr{H}$ to $\mathscr{H}$ or $\mathscr{K}$ will be denoted by small Roman letters. For consistency we denote the identity operator on $\mathscr{K}\oplus\mathscr{H}$ by ι henceforth.

LEMMA 2. *Let $X, Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ be contractions and let $P \in \mathscr{L}(\mathscr{H})$ be invertible and satisfy*

$$I - Y^*Y = P^*(I - X^*X)P. \tag{8}$$

Define operators ρ, E, F, G and H by

$$\rho = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \rho_1\rho_2 \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}) \tag{9}$$

where

$$\rho_1 = [v^* \ \ b^*M] \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}), \tag{10}$$

$$\rho_2 = \begin{bmatrix} (I + XPP^*X^*)^{-1/2} & XPM \\ -MP^*X^* & M \end{bmatrix} \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}), \tag{11}$$

$$b = [Y^* \ \ P^*] : \mathscr{K}\oplus\mathscr{H} \to \mathscr{H},$$

$$M = (bb^*)^{-1/2} = (Y^*Y + P^*P)^{-1/2} = (I + P^*X^*XP)^{-1/2} \in \mathscr{L}(\mathscr{H})$$

and $v \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}, \mathscr{K})$ is a partial isometry with initial space Ker$\,b$.

Then $\rho\rho^* = \iota$ and

$$\begin{aligned} Y &= F - EXP, \\ H &= (I + GX)P. \end{aligned} \tag{12}$$

Note that if $I + GX$ is invertible then the last two equations yield

$$Y = F - EX(I + GX)^{-1}H,$$

that is, $\Psi_\rho(X) = Y$. Note also that, since P is invertible, Ker$\,b$ and $\mathscr{K}$ have the same dimension, so that there does exist a partial isometry $v \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}, \mathscr{K})$ with initial space Ker$\,b$ (that is, an operator which is isometric on Ker$\,b$ and zero on $(\text{Ker}\,b)^\perp$).

We denote by σ the Hermitian projection in $\mathscr{L}(\mathscr{K}\oplus\mathscr{H})$ with range Ker$\,b$. It is elementary that, for $v \in \mathscr{L}(\mathscr{K}\oplus\mathscr{H}, \mathscr{K})$, v is a partial isometry with initial space Ker$\,b$ if and only if $v^*v = \sigma$.

If there is a unitary ρ such that $\Psi_\rho(X) = Y$ then, by Lemma 1 (i) we have

$$I - Y^*Y = H^*(I + X^*G^*)^{-1}(I - X^*X)(I + GX)^{-1}H.$$

Comparison with Lemma 1 suggests we try putting

$$P = (I + GX)^{-1}H,$$

in which case we have

$$Y = F - EXP.$$

We can think of these two relations as expressing two of the unknown operators (F and H) in terms of the other two (E and G) and the known operators X, Y and P. The condition $\rho\rho^* = \iota$ then gives us an equation for E and G which we can solve.

Proof of Lemma 2. Let E, G be operators and let F, H be given by

$$F = Y + EXP,$$

$$H = (I + GX)P.$$

Then

(13) $$\rho = \begin{bmatrix} E & Y + EXP \\ G & P + GXP \end{bmatrix} = \begin{bmatrix} 0 & Y \\ 0 & P \end{bmatrix} + \begin{bmatrix} E \\ G \end{bmatrix} [I \quad XP].$$

Let us write

$$w = [E^* \quad G^*] : \mathscr{K} \oplus \mathscr{H} \to \mathscr{K},$$

so that (13) becomes

(14) $$\rho = [0 \quad b^*] + w^*[I \quad XP].$$

With this choice of ρ, $\rho\rho^* = \iota$ is equivalent to

$$w^*(I + XPP^*X^*)w + w^*XPb + b^*P^*X^*w + b^*b = \iota.$$

To complete the square in this quadratic equation for w let us define $z \in \mathscr{L}(\mathscr{K} \oplus \mathscr{H}, \mathscr{K})$ by

$$z = (I + XPP^*X^*)^{1/2}w + (I + XPP^*X^*)^{-1/2} XPb.$$

The equation becomes

$$z^*z = \imath - b^*b + b^*P^*X^*(I + XPP^*X^*)^{-1}XPb =$$

$$= \imath - b^*[I - P^*X^*(I + XPP^*X^*)^{-1}XP]b =$$

$$= \imath - b^*[I - (I + P^*X^*XP)^{-1}P^*X^*XP]b =$$

$$= \imath - b^*(I + P^*X^*XP)^{-1}b =$$

$$= \imath - b^*(bb^*)^{-1}b.$$

Write $\pi = b^*(bb^*)^{-1}b$. Clearly $\pi^2 = \pi$ and $\pi^* = \pi$, so that π is a Hermitian projection. Furthermore

$$\operatorname{Ker}\pi = \operatorname{Ker}[(bb^*)^{-1/2}b]^*(bb^*)^{-1/2}b =$$

$$= \operatorname{Ker}(bb^*)^{-1/2}b = \operatorname{Ker}b.$$

Hence $\imath - \pi$ is the orthogonal projection on $\operatorname{Ker}b$; that is, $\imath - \pi = \sigma$.

We have shown that, if ρ is given by (13), the equation $\rho\rho^* = \imath$ is equivalent to

$$z^*z = \sigma.$$

The general solution of this is $z = v$, where $v \in \mathscr{L}(\mathscr{K} \oplus \mathscr{H}, \mathscr{K})$ is a partial isometry with initial space $\operatorname{Ker}b$. In terms of w this becomes

$$w = -(I + XPP^*X^*)^{-1}XPb + (I + XPP^*X^*)^{-1/2}v =$$

$$= -XP(I + P^*X^*XP)^{-1}b + (I + XPP^*X^*)^{-1/2}v.$$

Hence

$$w^* = -b^*M^2P^*X^* + v^*(I + XPP^*X^*)^{-1/2}.$$

From (14),

$$\rho = [w^* \quad b^* + w^*XP].$$

Now

$$b^* + w^*XP = b^*(I - M^2P^*X^*XP) + v^*(I + XPP^*X^*)^{-1/2}XP =$$

$$= b^*M^2 + v^*XPM.$$

Hence

$$(15) \qquad \rho = [-b^*M^2P^*X^* + v^*(I + XPP^*X^*)^{-1/2} \quad b^*M^2 + v^*XPM] =: \rho_1\rho_2.$$

We have shown that, subject to the relations (12), $\rho\rho^* = \imath$ is actually equivalent to $\rho = \rho_1\rho_2$ for some choice of partial isometry v with initial space $\operatorname{Ker}b$.

There is a converse to Lemma 2.

LEMMA 2c. *Let X, $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be contractions and let Ψ_ρ be a symplectic transformation such that $\Psi_\rho(X) = Y$. Then $\rho = \rho_1\rho_2$, where ρ_1 and ρ_2 are given by* (10) *and* (11), *for some invertible operator $P \in \mathcal{L}(\mathcal{H})$ satisfying*

$$(16) \qquad I - Y^*Y = P^*(I - X^*X)P$$

and some partial isometry $v \in \mathcal{L}(\mathcal{K} \oplus \mathcal{H}, \mathcal{K})$ with initial space $\operatorname{Ker}[Y^* \ P^*]$.

Proof. Let E, F, G and H be as in (9), and let

$$P = (I + GX)^{-1}H.$$

Then (12) is satisfied, and so, by Lemma 1, is (16). It follows from the above proof (see the final sentence) that $\rho = \rho_1\rho_2$ for some choice of v.

LEMMA 3. *The operator ρ described in Lemma 2 is unitary if and only if the partial isometry $v: \mathcal{K} \oplus \mathcal{H} \to \mathcal{K}$ is surjective.*

Proof. As the reader may verify, ρ_2 is unitary. Since we already know that $\rho\rho^* = \iota$, it follows that ρ is unitary if and only if $\rho_1^*\rho_1 = \iota$. Now

$$\rho_1^*\rho_1 = \begin{bmatrix} vv^* & vb^*M \\ Mbv^* & Mbb^*M \end{bmatrix} = \begin{bmatrix} vv^* & 0 \\ 0 & I \end{bmatrix},$$

since $bv^* = 0$. Thus ρ is unitary if and only if $vv^* = I_{\mathcal{K}}$; since v is a partial isometry, vv^* is the projection with range equal to $\operatorname{Range} v$, and so $vv^* = I$ if and only if v is surjective.

Note that v is surjective if and only if $v^*: \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ is an isometry with range $\operatorname{Ker} b$.

Our search for a symplectic transformation Φ such that $\Phi(X) = Y$ will be completed if we can choose the isometry v^* in such a way that H, the (2,2) entry of ρ above, is invertible, for then Ψ_ρ is symplectic, $I + GX$ is invertible and $\Psi_\rho(X) = Y$. This invertibility condition is the most delicate part of the construction, and as the statement of Theorem 1 indicates, it can only be satisfied subject to a further hypothesis.

Incidentally, one might ask whether a unitary ρ could be found such that $\Psi_\rho(X) = Y$, irrespective of the invertibility of H. The ρ constructed above will do provided $I + GX$ is invertible, but in view of (12) the invertibility of $I + GX$ is in this case equivalent to that of H, so we cannot obtain any more than is stated in Theorem 1.

To enable us to write down H let us put

$$v = [V_1 \ \ V_2] : \mathscr{H} \oplus \mathscr{H} \to \mathscr{H}.$$

Then H in Lemma 2 is given by (cf. (15))

$$H = PM^2 + V_2^* XPM =$$

$$= PM^2 + V_2^* XP(I + P^* X^* XP)^{1/2} M^2 =$$

$$= [I + V_2^*(I + XPP^* X^*)^{1/2} X] PM^2.$$

Since PM^2 is invertible we deduce:

LEMMA 4. *If ρ is as in Lemma 2 then Ψ_ρ is a symplectic transformation if and only if the partial isometry $v = [V_1 \ \ V_2]$ is surjective onto $\mathscr{H}$ and is such that*

$$I_{\mathscr{H}} + V_2^*(I + XPP^* X^*)^{1/2} X \tag{17}$$

is invertible.

LEMMA 5. *The partial isometries $v : \mathscr{H} \oplus \mathscr{H} \to \mathscr{H}$, with initial space* $\operatorname{Ker} b$ *which satisfy the conditions of Lemma 4 are precisely the operators of the form*

$$v = U^*(I + YP^{-1}P^{*-1}Y^*)^{-1/2} [I \ \ -YP^{-1}] \tag{18}$$

where U is a unitary operator on $\mathscr{H}$ such that $I - TU$ is invertible, where

$$T = (I + XPP^* X^*)^{1/2} XP^{*-1} Y^* (I + YP^{-1}P^{*-1}Y^*)^{-1/2}. \tag{19}$$

Proof. Suppose $v^* : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ is an isometry with range $\operatorname{Ker} b$. Then $bv^* = 0$, which is to say $Y^* V_1^* + P^* V_2^* = 0$. Thus

$$v = V_1 [I \ \ -YP^{-1}], \tag{20}$$

and the relation $vv^* = I$ is equivalent to

$$V_1 (I + YP^{-1}P^{*-1}Y^*) V_1^* = I,$$

that is, $(I + YP^{-1}P^{*-1}Y^*)^{1/2} V_1^*$ is an isometry: let us call it U. Then, from (20),

$$v = U^*(I + YP^{-1}P^{*-1}Y^*)^{-1/2} [I \ \ -YP^{-1}]. \tag{21}$$

Now $\operatorname{Ker} b = \{(k, -P^{*-1}Y^* k) : k \in \mathscr{H}\}$, so that the projection of $\operatorname{Ker} b$ onto $\mathscr{H}$ is the whole of $\mathscr{H}$. On the other hand (21) shows that the projection onto $\mathscr{H}$ of $\operatorname{Range} v^*$ is

$$(I + YP^{-1}P^{*-1}Y^*)^{-1/2} \operatorname{Range} U.$$

This shows that Range v^* is the whole of Ker b if and only if U is surjective, or in other words, U is unitary.

From (21),

$$V_2 = -U^*(I + YP^{-1}P^{*-1}Y^*)^{-1/2}YP^{-1}.$$

Condition (17) in Lemma 4 is thus equivalent to the invertibility of

$$I - P^{*-1}Y^*(I + YP^{-1}P^{*-1}Y^*)^{-1/2}U(I + XPP^*X^*)^{1/2}X.$$

Since $I + AB$ is invertible if and only if $I + BA$ is invertible, whenever A, B are operators for which AB and BA are both defined, condition (17) is equivalent to the invertibility of $I - TU$.

LEMMA 6. *The operator T of Lemma 5 is a contraction on $\mathscr{K}$, and*

$$I - TT^* = (I + XPP^*X^*)^{1/2}(I - XX^*)(I + XPP^*X^*)^{1/2},$$

$$I - T^*T = (I + YP^{-1}P^{*-1}Y^*)^{1/2}(I - YY^*)(I + YP^{-1}P^{*-1}Y^*)^{1/2}.$$

Proof. Let $N = (I + XPP^*X^*)^{1/2}$. Then

$$N^{-1}TT^*N^{-1} = XP^{*-1}Y^*(I + YP^{-1}P^{*-1}Y^*)^{-1}YP^{-1}X^* =$$

$$= X(I + P^{*-1}Y^*YP^{-1})^{-1}P^{*-1}Y^*YP^{-1}X^* =$$

$$= X[I - (I + P^{*-1}Y^*YP^{-1})^{-1}]X^* =$$

$$= XX^* - XP(P^*P + Y^*Y)^{-1}P^*X^* =$$

$$= XX^* - XP(I + P^*X^*XP)^{-1}P^*X^* =$$

$$= XX^* - (I + XPP^*X^*)^{-1}XPP^*X^* =$$

$$= XX^* - [I - (I + XPP^*X^*)^{-1}] =$$

$$= XX^* - I + N^{-2}.$$

Hence

$$N^{-1}(I - TT^*)N^{-1} = I - XX^*,$$

and so

$$I - TT^* = N(I - XX^*)N,$$

as required. It follows at once that $I - TT^* \geqslant 0$, that is, T is a contraction.

To prove the second identity let $L = (I + YP^{-1}P^{*-1}Y^*)^{1/2}$. Then

$$L^{-2} = (I + YP^{-1}P^{*-1}Y^*)^{-1} =$$
$$= I - YP^{-1}P^{*-1}Y^*(I + YP^{-1}P^{*-1}Y^*)^{-1} =$$
$$= I - YP^{-1}(I + P^{*-1}Y^*YP^{-1})^{-1}P^{*-1}Y^* =$$
$$= I - Y(bb^*)^{-1}Y^*.$$

$$L^{-1}T^*TL^{-1} =$$
$$= (I + YP^{-1}P^{*-1}Y^*)^{-1}YP^{-1}X^*(I + XPP^*X^*)XP^{*-1}Y^*(I + YP^{-1}P^{*-1}Y^*)^{-1} =$$
$$= Y(P^*P + Y^*Y)^{-1}P^*X^*(I + XPP^*X^*)XP(P^*P + Y^*Y)^{-1}Y^* =$$
$$= Y(bb^*)^{-1}(I + P^*X^*XP)P^*X^*XP(bb^*)^{-1}Y^* =$$
$$= Y(bb^*)^{-1}\, bb^*(bb^* - I)(bb^*)^{-1}Y^* =$$
$$= Y[I - (bb^*)^{-1}]Y^*.$$

Thus

$$L^{-1}(I - T^*T)L^{-1} = I - YY^*.$$

We defer to the next section a characterization of those contractions T such that $I - TU$ is invertible for some unitary U. Assuming this result (Theorem 3) we can conclude the proof of Theorem 1.

Suppose, then, that $X, Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ are contractions and that $P \in \mathscr{L}(\mathscr{H})$ is invertible and satisfies

$$I - Y^*Y = P^*(I - X^*X)P.$$

Suppose also that $I - YY^*$ is congruent to $I - XX^*$ and that condition (*ii*) of the the theorem holds: that is, either

(a) X is not essentially unitary, or

(b) X is essentially unitary and

$$\operatorname{ind} X = \operatorname{ind} Y.$$

In case (a) either $I - X^*X$ or $I - XX^*$ is not compact; by virtue of symmetry we may suppose $I - XX^*$ not compact.

Let T be as in Lemma 5; we wish to show that there is a unitary U on $\mathscr{K}$ such that $I - TU$ is invertible. Lemma 4 tells us that T is a contraction, and Theorem 3 ensures that there does exist U with the desired properties unless T is a compact perturbation of a non-unitary isometry or co-isometry V, say.

Suppose that T is a compact perturbation of some such V. Then either $I - TT^*$ or $I - T^*T$ is compact. By Lemma 6 $I - TT^*$, $I - T^*T$ are congruent to $I - XX^*$, $I - YY^*$ respectively, and since the latter two operators are supposed congruent, $I - TT^*$ and $I - T^*T$ are congruent to each other, and hence are both compact, as also are $I - XX^*$ and $I - YY^*$. This excludes alternative (a) above, and so we can assume that X and Y are essentially unitary. T is Fredholm, and so therefore is its compact perturbation V. As V is an isometry or co-isometry but not a unitary, $\operatorname{ind} V \neq 0$ and hence $\operatorname{ind} T \neq 0$. Clearly

$$\operatorname{ind} T = \operatorname{ind} XP^{*-1}Y^* = \operatorname{ind} X - \operatorname{ind} Y,$$

and so $\operatorname{ind} X \neq \operatorname{ind} Y$, contradicting alternative (b). We infer that T is not a compact perturbation of any such V, and hence there does exist a unitary U on $\mathscr{K}$ such that $I - TU$ is invertible.

Choose such a U. Lemma 5 now shows how to construct a partial isometry v with the properties described in Lemma 4, and the latter in conjunction with Lemma 2 tells us that if $\rho = \rho_1\rho_2$ is as stated then Ψ_ρ is symplectic (so that H is invertible) and

$$Y = F - EXP,$$

$$H = (I + GX)P.$$

Since both H and P are invertible, $I + GX$ is too and we may eliminate P to obtain

$$Y = F - EX(I + GX)^{-1}H,$$

i.e. $\Psi_\rho(X) = Y$.

To prove the converse we simply reverse the steps. Suppose that $\Psi_\rho(X) = Y$ for some symplectic transformation Ψ_ρ. Let $P = (I + GX)^{-1}H$, in our usual notation. Then P is invertible and so, by Lemma 1, the congruence conditions (i) of Theorem 1 are satisfied. Now suppose X is essentially unitary: to prove (ii) we must show that $\operatorname{ind} X = \operatorname{ind} Y$.

Lemma 2c tells us that ρ is of the form $\rho = \rho_1\rho_2$ for some choice of the partial isometry v and Lemmas 3, 4 and 5 then imply that there exists a unitary U on $\mathscr{K}$ such that $I - TU$ is invertible for T as in Lemma 5. Lemma 6 shows that T is essentially unitary. The well-known result of Brown, Douglas and Fillmore [3] tells us that T is a compact perturbation of an isometry or co-isometry S, and that S is unitary if and only if $\operatorname{ind} T = 0$. Theorem 3 shows that in the present instance S must in fact be unitary, and therefore $\operatorname{ind} T = 0$. Consequently

$$0 = \operatorname{ind} T = \operatorname{ind} XP^{*-1}Y^* = \operatorname{ind} X - \operatorname{ind} Y,$$

and condition (ii) is established.

We can extract from the above proof a description of *all* symplectic transformations Φ such that $\Phi(X) = Y$.

THEOREM 2. *Let $X, Y \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be contractions satisfying conditions* (i) *and* (ii) *of Theorem* 1. *The symplectic transformations Φ such that $\Phi(X) = Y$ are precisely those of the form $\Phi = \Psi_\rho$ where*

$$\rho = \begin{bmatrix} I & Y \\ -P^{*-1}Y^* & P \end{bmatrix} \begin{bmatrix} (I + YP^{-1}P^{*-1}Y^*)^{-1/2}U & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} (I + XPP^*X^*)^{-1/2} & XPM \\ -MP^*X^* & M \end{bmatrix}, \tag{18}$$

where $P \in \mathcal{L}(\mathcal{H})$ is an invertible operator such that

$$I - Y^*Y = P^*(I - X^*X)P,$$

$M \in \mathcal{L}(\mathcal{H})$ is given by

$$M = (I + P^*X^*XP)^{-1/2}$$

and U is a unitary operator on $\mathcal{H}$ such that $I - TU$ is invertible, where

$$T = (I + XPP^*X^*)^{1/2}XP^{*-1}Y^*(I + YP^{-1}P^{*-1}Y^*)^{-1/2}.$$

4. THE INVERTIBILITY OF $I - TU$

We shall complete the proof of Theorem 1 by establishing the promised characterization. We shall denote the closure of the range of an operator A by $\mathcal{R}(A)$ and the dimension of the kernel of A by $\nu(A)$.

LEMMA 7. *Let $A \geqslant 0$ be a contraction such that $A - A^2$ is compact. Then $A^{1/2}$ is a compact perturbation of a Hermitian projection P such that $\mathcal{R}(P) \subseteq \mathcal{R}(A)$.*

Proof. Let the spectral decomposition of A be

$$A = \int_{[0,1]} \lambda E(\mathrm{d}\lambda).$$

For $0 < c < 1/2$ we have

$$A - A^2 = \int_{[0,1]} (\lambda - \lambda^2)E(\mathrm{d}\lambda) \geqslant (c - c^2)E([c, 1 - c]).$$

Since $A - A^2$ is compact, $E([c, 1-c])$ is a projection of finite rank. Let $P = E([1/2, 1])$. Then $\mathscr{R}(P) \subseteq \mathscr{R}(A)$ and, for small $c > 0$,

$$A^{1/2} - P = \int_{[0,c)} + \int_{[c,1/2)} \lambda^{1/2} E(\mathrm{d}\lambda) +$$

$$+ \int_{[1/2,1-c)} + \int_{[1-c,1]} (\lambda^{1/2} - 1) E(\mathrm{d}\lambda).$$

The first and last integrals are operators tending to zero in norm as $c \downarrow 0$, while the two middle ones are finite-rank operators. Thus $A^{1/2} - P$ is compact.

For the following result see [4].

LEMMA 8. *If M is a Hermitian operator and K is compact on $\mathscr{K}$ then $\sigma(M+K)$ contains only countably many non-real points.*

THEOREM 3. *The following are equivalent for any contraction T on $\mathscr{K}$:*

(i) *there is no unitary U such that $I - TU$ is invertible;*

(ii) *$T - TT^*T$ is compact and $\nu(T_1) \neq \nu(T_1^*)$ for every compact perturbation T_1 of T;*

(iii) *T is a compact perturbation of an isometry or co-isometry which is not unitary.*

Proof. (i) $\Rightarrow$ (ii). Suppose there is a compact perturbation $T_1 = T + K$ of T such that $\nu(T_1) = \nu(T_1^*)$. We have

$$T_1 = V(T_1^* T_1)^{1/2}$$

where $V\colon \mathscr{R}(T_1^*) \to \mathscr{R}(T_1)$ is unitary. Since

$$\dim \mathscr{R}(T_1^*)^{\perp} = \nu(T_1) = \nu(T_1^*) = \dim \mathscr{R}(T_1)^{\perp},$$

V extends to a unitary operator on $\mathscr{K}$. Let $U = -V^*$: then $-UT_1 = (T_1^* T_1)^{1/2}$, and hence

$$I - \lambda UT = I + \lambda(M + K_1),$$

where $M \geqslant 0$ and K_1 is compact, for any $\lambda \in \mathbf{C}$. Now $\sigma(M + K_1)$ contains only countably many points off the real axis. Hence there exists λ such that $|\lambda| = 1$ and $-\bar{\lambda} \notin \sigma(M + K_1)$. Then λU is unitary and $I - \lambda UT$ is invertible: thus also $I - T(\lambda U)$ is invertible.

Alternatively, suppose that $T - TT^*T$ is not compact. We shall construct a unitary U such that $I - TU$ is invertible.

Step 1. There is an infinite-dimensional subspace E of $\mathscr{R}(T^*)$, invariant under T^*T, such that $\|T|E\| < 1$.

Let P be the Hermitian projection on $\mathscr{R}(T^*)$ and let

$$I - T^*T = \int_{[0,1]} \lambda E(\mathrm{d}\lambda)$$

be the spectral decomposition. $E(\{1\})$ is the projection onto the eigenspace of $I - T^*T$ corresponding to the eigenvalue 1, i.e. onto $\operatorname{Ker} T^*T = \operatorname{Ker} T$, and hence $E(\{1\}) = I - P$.

As $c \downarrow 0$,

$$(19) \qquad \int_{[c,1)} \lambda E(\mathrm{d}\lambda) \to \int_{[0,1)} \lambda E(\mathrm{d}\lambda) = I - T^*T - (I - P) = P - T^*T$$

in norm. Now $P - T^*T$ is not compact. To see this observe that $T^* - T^*TT^* = (T - TT^*T)^*$ is not compact, and hence $(I - T^*T)|\mathscr{R}(T^*)$ is not compact. $P - T^*T$ is the orthogonal direct sum of $(I - T^*T)|\mathscr{R}(T^*)$ and the zero operator on $\operatorname{Ker} T$.

It follows that $P - T^*T$ is not a norm limit of finite rank operators, and so, from (19), there exists $c > 0$ such that $E([c, 1))$ is the projection onto a space E of infinite dimension. E is invariant under T^*T, and for $x \in E$,

$$((I - T^*T)x, x) \geqslant c(x, x),$$

so that $\|T|E\| \leqslant \sqrt{(1 - c)} < 1$.

Step 2. Polar decomposition of T.

$$T = V(T^*T)^{1/2}$$

where $V: \mathscr{R}(T^*) \to \mathscr{R}(T)$ is unitary. Let $F = E^\perp$, and let the restrictions of T^*T to the reducing subspaces F, E be M^2, N^2 respectively, where $M, N \geqslant 0$. By Step 1, $\|N\| < 1$. And $T = V(M \oplus N)$.

Step 3. Construction of U.

For $x \in VF$ define Ux to be $-V^*x$. This defines $U: VF \to F$ as a unitary operator. It can be extended to a unitary operator on $\mathscr{H}$ provided that

$$\dim(VF)^\perp = \dim F^\perp = \dim E = \infty.$$

Now

$$(VF)^\perp = V^{*-1}(F^\perp) = V^{*-1}(E),$$

and since $V^*|\mathscr{R}(T)$ is unitary into $\mathscr{R}(T^*) \supseteq E$, it follows that $\dim(VF)^\perp \geqslant \dim E = \infty$. Hence U does extend to a unitary on $\mathscr{K}$.

Step 4. Conclusion.

Consider $x = f + e$, $f \in F$, $e \in E$.

$$UTx = UV(M \oplus N)(f + e) =$$

$$= UV(Mf + Ne).$$

Here $Mf \in F$, $Ne \in E$. Since $UVy = -y$ for $y \in F$ we have

$$UTx = -Mf + UVNe,$$

where $\|UVN\| < 1$. Thus, with respect to the orthogonal decomposition $\mathscr{K} = F \oplus E$, $I - UT$ has a matrix of the form

$$\begin{bmatrix} I + M & * \\ 0 & I + N_1 \end{bmatrix}$$

where $M \geqslant 0$ and $\|N_1\| < 1$. It follows that $I - UT$ is invertible.

(ii) $\Rightarrow$ (iii). Assume (ii). $TT^* - (TT^*)^2$ is compact, and so, by Lemma 7, here is a Hermitian projection P onto a subspace of $\mathscr{R}(T)$ such that

$$(TT^*)^{1/2} = P + K,$$

with K compact. Now

$$T = (TT^*)^{1/2} V$$

where V is a partial isometry, isometric from $\mathscr{R}(T^*)$ onto $\mathscr{R}(T)$, zero on $\operatorname{Ker} T$. Then

$$T = PV + KV.$$

Since $\mathscr{R}(P) \supseteq \mathscr{R}(T) = \mathscr{R}(V)$, $S = PV$ is a partial isometry. Thus T is a compact perturbation of the partial isometry S, and since also S is a compact perturbation of T, we have by hypothesis $\nu(S) \neq \nu(S^*)$.

If $\nu(S) < \nu(S^*)$ then $\nu(S)$ is finite. Pick an isometry $J\colon \operatorname{Ker} S \to \mathscr{R}(S)^\perp$: we can do this since the latter space has dimension $\nu(S^*) > \nu(S)$. Note that $\mathscr{R}(J)$ is not the whole of $\mathscr{R}(S)^\perp$. J is compact, and so the operator

$$V = S|(\operatorname{Ker} S)^\perp \oplus J$$

on $\mathscr{K}$ is a compact perturbation of S, and hence of T. V is an isometry and $\mathscr{R}(V)$ is a proper subset of $\mathscr{K}$, so V is not unitary.

If, on the other hand, $\nu(S) > \nu(S^*)$ we can apply the above reasoning to S^* to deduce that T is a compact perturbation of a non-unitary co-isometry.

(iii) $\Rightarrow$ (i). Suppose that (iii) is true but (i) is false: that is, T is a compact perturbation of a non-unitary isometry or co-isomety S, but $I - TU$ is invertible for some unitary U. By passing to T^* if necessary we can suppose that S is a co-isometry. Then we have

$$TT^* = I + K_1$$

(K_i denotes a compact operator throughout). Then

$$I - TU = TT^* - TU + K_2 =$$

$$= TU(U^*T^* - I) + K_2$$

and hence

$$T = (I - TU)(I - TU)^{*-1}U^* + K_3.$$

This shows that T is a compact perturbation of an invertible operator, and the same is therefore true of S. This makes S a co-isometry which is Fredholm of index 0, that is, a unitary, contrary to hypothesis. Hence (iii) $\Rightarrow$ (i).

We note that, for operators T having closed range, the property $\nu(T) \neq \nu(T^*)$ tis preserved under compact perturbations. If $\nu(T)$ and $\nu(T^*)$ are both finite this follows from the invariance of the Fredholm index, while $\nu(T) = \infty$ and $\nu(T^*) < \infty$ if and only if T has a right inverse but no left inverse modulo the compact operators. Hence, for operators with closed range, we can replace condition (ii) in Theorem 3 by (ii') $T - TT^*T$ is compact and $\nu(T) \neq \nu(T^*)$.

In general, however, (ii) and (ii') are different: it is easy to write down a compact weighted shift T for which $\nu(T) \neq \nu(T^*)$, while for the compact perturbation $T_1 = 0$ of T, $\nu(T_1^*) = \nu(T_1)$.

5. AN EXAMPLE

The orbits of the unit sphere are characterized in Theorem 1 by the congruence condition (i) and, in the exceptional case of essentially unitary operators, the index condition (ii). It is just conceivable that for these very special operators the index condition might be a consequence of the congruence condition, in which case we could omit condition (ii) altogether from the statement of the theorem. In fact this is not the case, as the following example shows.

Let $\mathscr{H} = \mathscr{K}$ and let $(e_n)_1^\infty$ be an orthonormal basis of $\mathscr{H}$. Let $\alpha_n = \sqrt{(1 - 2^{-n})}$, $n = 1, 2, \ldots$, and let X be the backward shift with weights α_n — that is,

$$Xe_1 = 0,$$

$$Xe_n = \alpha_{n-1}e_{n-1}, \quad n = 2, 3, \ldots .$$

Let

$$Y = \mathrm{diag}\{1/\sqrt{2},\ (2-\alpha_1^2)^{-1/2},\ (2-\alpha_2^2)^{-1/2},\ldots\},$$

in a self-explanatory notation. X and Y are compact perturbations on the backward shift and the identity operator respectively, and we have

$$\mathrm{ind}\, X = 1, \quad \mathrm{ind}\, Y = 0.$$

However, the congruence conditions (i) do hold.

$$Y^{*-1}Y^{-1} = Y^{-2} = \mathrm{diag}\{2, 2-\alpha_1^2,\ 2-\alpha_2^2, \ldots\} = 2I - X^*X,$$

so that

$$I - X^*X = Y^{*-1}Y^{-1} - I = Y^{*-1}(I - Y^*Y)Y^{-1}.$$

Moreover

$$I - XX^* = \mathrm{diag}\{1-\alpha_1^2,\ 1-\alpha_2^2, \ldots\},$$

$$I - YY^* = \mathrm{diag}\left\{\frac{3}{4}, \frac{1-\alpha_1^2}{2-\alpha_1^2}, \frac{1-\alpha_2^2}{2-\alpha_2^2}, \ldots\right\}.$$

The ratio of the nth diagonal entries in these two diagonal operators is always non-zero, and tends to $1/2$ as $n \to \infty$. Thus $I - XX^*$ is congruent to $I - YY^*$.

It is rather strange that, for Fredholm operators X which are not unitary modulo the compacts, $\mathrm{ind}\, X$ is not relevant to the orbit of X.

6. THE ORBIT OF A PARTIAL ISOMETRY

It is natural to ask about the orbits of shifts and projections: these are both ypes of partial isometries, which are the appropriate entities to consider in the context of $\mathscr{L}(\mathscr{H}, \mathscr{K})$.

Let us call a contraction $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ a *shriek operator* if it is the orthogonal direct sum of an isometry and a strict contraction (i.e. an operator of norm less than 1). It is elementary to show that the following are equivalent for a contraction X:

(i) X is a shriek operator;

(ii) 1 is not in the closure of $\sigma(X^*X)\setminus\{1\}$;

(iii) $I - X^*X$ has closed range.

Condition (ii) explains the sensational terminology: the spectrum of X^*X looks like an exclamation mark on its side (typically).

THEOREM 4. *Let $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ be a contraction. The orbit of X under the symplectic transformations contains a partial isometry if and only if X is a shriek operator.*

This characterization was obtained by Yu. M. Šmulian [13]: we show how it follows from our result.

Proof. ($\Rightarrow$). Let the orbit of X contain a partial isometry Y. $I - Y^*Y$ is a Hermitian projection and so has closed range. $I - X^*X$, being congruent to $I - Y^*Y$, also has closed range.

($\Leftarrow$). Let X be a shriek operator. X acts as an isometry on $\mathrm{Ker}(I - X^*X)$ and as a strict contraction on its orthogonal complement, $\mathscr{R}(I - X^*X)$. X clearly maps $\mathrm{Ker}(I - X^*X)$ into $\mathrm{Ker}(I - XX^*)$, and in fact the mapping is onto, for if $y \in \mathrm{Ker}(I - XX^*)$ then $x = X^*y \in \mathrm{Ker}(I - X^*X)$ and $Xx = y$. Thus $X = X_1 \oplus X_2$ where X_1 is a unitary mapping from $\mathrm{Ker}(I - X^*X)$ onto $\mathrm{Ker}(I - XX^*)$, and X_2 is a strict contraction from $\mathscr{R}(I - X^*X)$ into $\mathscr{R}(I - XX^*)$.

Let $Y = X_1 \oplus 0$. Then Y is a partial isometry, and we have

$$I - X^*X = 0 \oplus (I - X_2^*X_2),$$

(20)

$$I - Y^*Y = 0 \oplus I$$

and hence

$$I - Y^*Y = P^*(I - X^*X)P$$

where

$$P = I \oplus (I - X_2^*X_2)^{-1/2}$$

is invertible in $\mathscr{L}(\mathscr{H})$. Likewise

$$I - YY^* = Q(I - XX^*)Q^*$$

where $Q = I \oplus (I - X_2X_2^*)^{-1/2}$. Thus the congruence conditions (i) of Theorem 1 are satisfied.

Now suppose $I - X^*X$ is compact. Since $I - X_2^*X_2$ is invertible, (20) shows that $\mathscr{R}(I - X^*X)$ and hence also $\mathscr{R}(I - XX^*)$ are finite-dimensional. Thus Y is a perturbation of the Fredholm operator X by a finite rank operator, and so $\mathrm{ind}\, Y = \mathrm{ind}\, X$. Condition (ii) of Theorem 1 is satisfied, and so Y lies in the orbit of X.

In conclusion let us note the position in the finite-dimensional case. If either of $\mathscr{H}$ or $\mathscr{K}$ has finite dimension then every contraction X lies in the orbit of a partial isometry since $\sigma(X^*X)$ is finite. In this case, moreover, condition (ii) of Theorem 1 is automatically satisfied: if both of $\mathscr{H}$ and $\mathscr{K}$ have finite dimension then all operators in $\mathscr{L}(\mathscr{H}, \mathscr{K})$ have the same index, while if only one space is finite-dimensional then no member of $\mathscr{L}(\mathscr{H}, \mathscr{K})$ is essentially unitary. It follows that the orbit of X is determined by the congruence classes of $I - X^*X$ and $I - XX^*$, and it is not hard to see that these depend only on the multiplicity with which 1 is a singular value of X. Thus, if

$$n = \min\{\dim \mathscr{H}, \dim \mathscr{K}\} < \infty$$

then the closed unit ball of $\mathscr{L}(\mathscr{H}, \mathscr{K})$ splits up into $n+1$ orbits under the action of the symplectic transformations.

An interpretation of these results in terms of the geometry of Kreĭn spaces is given in [14].

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