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REDHEFFER PRODUCTS AND THE LIFTING OF CONTRACTIONS ON HILBERT SPACE

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It is the purpose of this note to show how Redheffer products [3] can be used to obtain a simple proof of the main result in [1]. In this paper all spaces are Hilbert spaces and all operators are bounded. We follow the standard notation in [2], [5].

We begin by recalling some properties of Redheffer products. Throughout

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad L_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \tag{1}$$

are bounded operators mapping $\mathscr{X} \oplus \mathscr{U}$ $[\mathscr{X}_1 \oplus \mathscr{U}_1]$ into $\mathscr{Z} \oplus \mathscr{Y}$ $[\mathscr{Z}_1 \oplus \mathscr{Y}_1]$, respectively, where $\mathscr{Z} = \mathscr{U}_1$ and $\mathscr{X} = \mathscr{Y}_1$. It is also assumed that the range of C_1 and D_1B $[C^*$ and $A^*B_1^*]$ are contained in $\overline{(I - D_1A)\mathscr{X}}$ $[\overline{(I - A^*D_1^*)\mathscr{Z}}]$, respectively. The Redheffer product is defined by

$$M = L_1 \circ L = \begin{bmatrix} A_1 + B_1A(I - D_1A)^{-1}C_1 & B_1A(I - D_1A)^{-1} D_1B + B_1B \\ C(I - D_1A)^{-1}C_1 & C(I - D_1A)^{-1}D_1B + D \end{bmatrix} \tag{2}$$

where the inverse is the pseudo-inverse: If $T^{-1}y = x$ then x is the unique element orthogonal to $\ker(T)$ such that $Tx = y$. The Redheffer product exists whenever the entries in (2) exist as unique bounded operators coinciding with the linear transformations indicated by the entries. (Redheffer did not use pseudo-inverses in his original paper [3]. However, his results extend to our setting.) Consider the system

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \tag{3}$$

$$z = u_1 \quad \text{and} \quad x = y_1$$

where $u, x, y, z, u_1, x_1, y_1$ and z_1 are elements in the appropriate spaces. The Redheffer product (2) is obtained by solving (3) in the following way:

$$\begin{pmatrix} z_1 \\ y \end{pmatrix} = [L_1 \circ L] \begin{pmatrix} x_1 \\ u \end{pmatrix} = M \begin{pmatrix} x_1 \\ u \end{pmatrix}. \tag{4}$$

The operator J is defined by

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{5}$$

where I is the identity operator on the appropriate space. J acts like the identity for the Redheffer product. To be precise $L = L \circ J = J \circ L$. If T is a contraction mapping $\mathscr{H}$ into $\mathscr{H}'$ then D_T is the positive square root of $(I - T^*T)$ and $\mathscr{D}_T = \overline{D_T \mathscr{H}}$. We are ready for

LEMMA 1. *Assume that $M = L_1 \circ L$ where L_1 and L in (1) are contractions. Then M is a contraction and*

$$\|D_M(x_1 \oplus u)\|^2 = \|D_{L_1}(x_1 \oplus u_1)\|^2 + \|D_L(x \oplus u)\|^2 \tag{6}$$

where x, u_1 are obtained by (3).

Proof. The proof is similar to some of the results in [3]. For completeness it is given. Using (3) and (4) we have

$$\begin{aligned} \|D_M(x_1 \oplus u)\|^2 &= \|x_1 \oplus u\|^2 - \|M(x_1 \oplus u)\|^2 = \\ &= \|x_1\|^2 + \|u\|^2 - (\|z_1\|^2 + \|y\|^2) = \|x_1\|^2 + \|u\|^2 + \\ &+ \|u_1\|^2 + \|x\|^2 - (\|z_1\|^2 + \|y\|^2 + \|z\|^2 + \|y_1\|^2) = \\ &= \|x_1 \oplus u_1\|^2 - \|z_1 \oplus y_1\|^2 + \|x \oplus u\|^2 - \|z \oplus y\|^2 = \\ &= \|D_{L_1}(x_1 \oplus u_1)\|^2 + \|D_L(x \oplus u)\|^2 \geqslant 0. \end{aligned} \tag{7}$$

Therefore M is a contraction and (6) holds. This completes the proof.

Let A be a contraction mapping $\mathscr{X}$ into $\mathscr{Z}$. It is well known that the operators U $[U_1]$ mapping $\mathscr{X} \oplus \mathscr{D}_{A^*}$ $[\mathscr{D}_A \oplus \mathscr{Z}]$ into $\mathscr{Z} \oplus \mathscr{D}_A$ $[\mathscr{D}_{A^*} \oplus \mathscr{X}]$, defined by

$$U = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}, \quad U_1 = \begin{bmatrix} -A & D_{A^*} \\ D_A & A^* \end{bmatrix} \tag{8}$$

respectively, are unitary [2]. Using $D_{A^*}A = AD_A$ with (2) gives

$$J = U_1 \circ U = U \circ U_1 . \tag{9}$$

Finally we need

LEMMA 2. ([4]). *The operator*

$$M = \begin{bmatrix} 0 & X \\ Y & Z \end{bmatrix} \tag{10}$$

mapping $\mathscr{X}_1 \oplus \mathscr{U}$ *into* $\mathscr{X}_1 \oplus \mathscr{Y}$ *is a contraction if and only if* X, Y *are contractions and* $Z = D_{Y^*}\Gamma D_X$, *where* Γ *is a uniquely determined contraction mapping* $\mathscr{D}_X$ *into* $\mathscr{D}_{Y^*}$. *Furthermore,* $\mathscr{D}_M$ $[\mathscr{D}_{M^*}]$ *can be identified with* $\mathscr{D}_Y \oplus \mathscr{D}_\Gamma$ $[\mathscr{D}_{X^*} \oplus \mathscr{D}_{\Gamma^*}]$, *respectively.*

The above allows us to give a simple proof of the main result in [1].

THEOREM 1. ([1]). *The operator* L *in* (1) *is a contraction if and only if*

$$L = \begin{bmatrix} A & D_{A^*}X \\ YD_A & D_{Y^*}\Gamma D_X - YA^*X \end{bmatrix} \tag{11}$$

where $X\colon \mathscr{U} \to \mathscr{D}_{A^*}$, $Y\colon \mathscr{D}_A \to \mathscr{Y}$ *and* $\Gamma\colon \mathscr{D}_X \to \mathscr{D}_{Y^*}$ *are all (uniquely) determined contractions. Furthermore, the spaces* $\mathscr{D}_L$ $[\mathscr{D}_{L^*}]$ *can be identified with* $\mathscr{D}_Y \oplus \mathscr{D}_\Gamma$ $[\mathscr{D}_{X^*} \oplus \oplus \mathscr{D}_{\Gamma^*}]$, *respectively.*

Proof. Assume L in (1 is a contraction. Clearly the $[A, B]$ and $[A^*, C^*]^*$ are contractions. By Lemma 2 (with $\mathscr{X}_1 = \{0\}$, respectively $\mathscr{X}_1 = \{0\}$) there exist contractions $X\colon \mathscr{U} \to \mathscr{D}_{A^*}$ and $Y\colon \mathscr{D}_A \to \mathscr{Y}$ such that

$$L = \begin{bmatrix} A & D_{A^*}X \\ YD_A & D \end{bmatrix}. \tag{12}$$

Equations (2), (12) and $D_{A^*}A = AD_A$ give:

$$M = U_1 \circ L = \begin{bmatrix} 0 & X \\ Y & YA^*X + D \end{bmatrix}. \tag{13}$$

Lemma 1 shows that M is a contraction. Thus Lemma 2 provides $YA^*X + D = D_{Y^*}\Gamma D_X$ with the adequate contraction Γ mapping $\mathscr{D}_X$ into $\mathscr{D}_{Y^*}$. This yields (11).

The other part follows by a similar argument. Assume L is given by (11). Let M be the contraction in Lemma 2. Using (2) or (9) it is easy to verify that $L = U \circ M$. Since M is a contraction, L is a contraction.

Using $L_1 = U_1$ in (6) we have

$$\|D_M(x_1 \oplus u)\|^2 = \|D_L(x \oplus u)\|^2. \tag{14}$$

A simple calculation given below shows that x in $\mathscr{X}$ and x_1 in $\mathscr{D}_A$ are related by

$$x_1 = D_A x - A^*Xu. \tag{15}$$

This, $A^*\mathscr{D}_{A^*} \subseteq \mathscr{D}_A$ and (14) implies $\mathscr{D}_L$ can be identified with $\mathscr{D}_M$. Lemma 2 shows that $\mathscr{D}_L$ can be identified with $\mathscr{D}_Y \oplus \mathscr{D}_\Gamma$. The other identification follows by duality, i.e., use (11) and apply the above result to L^*.

To complete the proof we verify (15). Here u, x, y, z, u_1, x_1, y_1 and z_1 are defined by (1), (3) and (4) where L is given by (11) and $L_1 = U_1$. By (3) and (4) we also have

$$\begin{pmatrix} z_2 \\ x \end{pmatrix} = (U \circ U_1) \begin{pmatrix} x_2 \\ z \end{pmatrix} \tag{16}$$

where

$$\begin{pmatrix} z_1 \\ x \end{pmatrix} = U_1 \begin{pmatrix} x_1 \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_2 \\ x_1 \end{pmatrix} = U \begin{pmatrix} x_2 \\ z_1 \end{pmatrix}. \tag{17}$$

Since $U \circ U_1 = J$, the relation (16) yields $z_2 = z$, $x = x_2$. Now, equations (8) and (17) imply $x_1 = D_A x - A^* z_1$. The form of M $(= U_1 \circ L)$ in (4) and (13) shows that $z_1 = Xu$. This yields (15) and completes the proof.

Finally, it is noted that M and L in (13) do not necessarily have the same norm, even though U_1 is unitary. To see this choose $A = 1/2$, $X = Y = \Gamma = 0$. Then $\|A\| \neq \|M\|$.

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