J. OPERATOR THEORY
11(1984), 215–241

CROSSED PRODUCTS BY LOCALLY UNITARY AUTOMORPHISM GROUPS AND PRINCIPAL BUNDLES

JOHN PHILLIPS and IAIN RAEBURN

Let A be a continuous trace C^*-algebra with spectrum T, and let $\alpha: \mathbf{Z} \to \operatorname{Aut} A$ be a group consisting of $C(T)$-automorphisms with generator $\beta = \alpha_1$. We proved in [13, Section 2] that there is a cohomology class $\eta(\beta)$ in the Čech group $H^2(T, \mathbf{Z})$, which vanishes precisely when β is implemented by a unitary element of the multiplier algebra $\mathcal{M}(A)$, or, equivalently, when α is implemented by a homomorphism $u: \mathbf{Z} \to \mathcal{M}(A)$. Elements of $H^2(T, \mathbf{Z})$ are associated with isomorphism classes of principal S^1-bundles over T, and in fact the class $\eta(\beta)$ is constructed in [13] as the transition functions of such a bundle. The starting point of our present work was the observation that this principal bundle appears naturally as the spectrum of the crossed product C^*-algebra $A \times_\alpha \mathbf{Z}$: the action of S^1 on $(A \times_\alpha \mathbf{Z})^\wedge$ comes from the dual action of $S^1 = \hat{\mathbf{Z}}$ on $A \times_\alpha \mathbf{Z}$, and the bundle projection $p: (A \times_\alpha \mathbf{Z})^\wedge \to \hat{A}$ is given by sending $\pi \times U \in (A \times_\alpha \mathbf{Z})^\wedge$ to the (irreducible) representation π of A. We shall show here that Theorem 2.1 of [13] is a special case of general theorems which relate a class of abelian automorphism groups of a type I C^*-algebra A to locally trivial principal bundles over $\hat{A}$.

Let A be a type I C^*-algebra, G a locally compact abelian group and $\alpha: G \to \operatorname{Aut} A$ a strongly continuous automorphism group. We shall say that α is implemented by $u: G \to \mathcal{M}(A)$ in the representation π of A if

$$\pi(\alpha_g(a)) = \pi(u_g a u_g^*) \quad \text{for } g \in G, \ a \in A;$$

we call α locally unitary if there are maps of G into $\mathcal{M}(A)$ which implement α locally in $\hat{A}$. A theorem of Russell [17] shows that singly generated groups of $C(\hat{A})$-automorphisms of continuous trace C^*-algebras are always locally unitary, and we show in Section 1 that there are other circumstances in which automorphism groups are automatically locally unitary.

Suppose $\alpha: G \to \operatorname{Aut} A$ is locally unitary. The dual action $\hat{\alpha}$ of $\hat{G}$ on $A \times_\alpha G$ induces an action of $\hat{G}$ on $(A \times_\alpha G)^\wedge$, and it is not hard to see that this is free. It turns out that the irreducible representations of $A \times_\alpha G$ all have the form $\pi \times U$ for some irreducible representation π of A, and we show that $\pi \times U \to \pi$ defines a continuous surjection p of $(A \times_\alpha G)^\wedge$ onto $\hat{A}$. Our first main result (Theorem 2.2) says that the dual action of $\hat{G}$ and the projection p make $(A \times_\alpha G)^\wedge$ into a locally trivial principal $\hat{G}$-bundle over $\hat{A}$. When $G = \mathbf{Z}$ and A has continuous trace, the class of this bundle in $H^2(T, \mathbf{Z})$ is the class $\eta(\alpha_1)$ defined in [13] or [17, Section 5]. In this case $\eta(\alpha_1)$ determines α_1 up to multiplication by inner automorphisms, and in general, provided $\hat{A}$ is Hausdorff, the isomorphism class of the bundle $(A \times_\alpha G)^\wedge$ determines α up to exterior equivalence.

After proving these basic results in Section 2, we consider the problem of constructing locally unitary groups $\alpha: G \to \operatorname{Aut} A$ such that the spectrum of $(A \times_\alpha G)^\wedge$ is a given principal $\hat{G}$-bundle. We prove first that any principal $\hat{G}$-bundle over a locally compact paracompact space T can be realised as the spectrum of $C_\infty(T, K(H)) \times_\alpha G$ for some locally unitary automorphism group α, and with a bit more work we can replace $C_\infty(T, K)$ by an arbitrary stable C^*-algebra with spectrum T. The case $G = \mathbf{Z}$ represents a slight generalisation of [13, Theorem 2.1]; however, our results are also interesting from a different viewpoint. If X and Y are compact spaces, then $C(X) \otimes C(Y)$ is naturally isomorphic to $C(X \times Y)$, and so we can regard the C^*-algebraic tensor product as an algebraic version of the cartesian product. In the same spirit, our results show that crossed products by locally unitary actions can be viewed as an algebraic interpretation of non-trivial principal bundles. We observe that the algebra in question is always non-commutative — in fact, to capture all principal bundles we need to assume it is stable and hence homogeneous of infinite degree. These results are all in Section 3, and constitute the main part of the paper.

In our final section we discuss the possibility of decomposing a continuous trace C^*-algebra whose spectrum is a principal bundle as a crossed product. This cannot always be done, but the obstruction can be neatly described using the Dixmier-Douady class of the algebra.

Many of our results hold with suitable modifications for more general automorphism groups. In particular, for the first part of Section 2 it is only necessary that the group consist of universally weakly inner (or π-inner) automorphisms. The results of Section 3 seem to hold for pointwise unitary groups (see Section 1); just as locally unitary actions by G correspond to locally trivial principal $\hat{G}$-bundles, it seems likely that, at least for discrete G, pointwise unitary actions will correspond to arbitrary free actions of $\hat{G}$. Since our results in this direction are fragmentary, and the techniques required are substantially different, we plan to discuss them elsewhere.

We shall assume throughout that our C^*-algebras are type I. This is not strictly necessary as far as Section 2 goes, since it can be shown that locally unitary groups always consist of π-inner automorphisms. The proofs would require modification, though, and we have therefore preferred to stress the analogy between locally unitary groups and principal bundles in the technically easier type I case.

NOTATION. Let A be a C^*-algebra. We shall denote the group of *-automorphisms of A by $\operatorname{Aut} A$, the multiplier algebra of A by $\mathcal{M}(A)$, and the spectrum of A by $\hat{A}$. We shall frequently confuse irreducible representations of A with their class in $\hat{A}$. If π is a nondegenerate representation of A on a Hilbert space H_π, we shall write $\overline{\pi}$ for its canonical extension to $\mathcal{M}(A)$, and if $\varphi\colon A \to B$ is an isomorphism, we write $\mathcal{M}(\varphi)$ for its extension to $\mathcal{M}(A)$. We denote the algebra of compact operators on a Hilbert space H by $K(H)$, and if $\mathscr{H}$ is a continuous field of Hilbert spaces we denote by $K(\mathscr{H})$ the continuous field of elementary C^*-algebras associated with $\mathscr{H}$ and by $\Gamma_\infty(K(\mathscr{H}))$ the C^*-algebra defined by $\mathscr{H}$ [3, Section 10]. Our notation on tensor products is straightforward: we denote the algebraic tensor product by $A \odot B$ and only write $A \otimes B$ if the C^*-norm on $A \odot$ B is unique. We say A is stable if $A \otimes K(H) \cong A$, where H is a separable infinite-dimensional Hilbert space.

Let G be a locally compact group, and $\alpha\colon G \to \operatorname{Aut} A$ a strongly continuous automorphism group. We shall denote the crossed product C^*-algebra (see [12]) by $A \times_\alpha G$, and if (π, U) is a covariant representation of (A, G, α) we denote the corresponding representation of $A \times_\alpha G$ by $\pi \times U$. There is a natural injection $i_\alpha\colon A \to \mathcal{M}(A \times_\alpha G)$ defined on the dense subalgebra $C_c(G, A)$ of $A \times_\alpha G$ by

$$(i_\alpha(a)\varphi)(g) = a\varphi(g), \quad (\varphi i_\alpha(a))(g) = \varphi(g)\alpha_g(a).$$

If G is abelian, there is an action $\hat{\alpha}$ of the dual group $\hat{G}$ on $A \times_\alpha G$ given by

$$\hat{\alpha}_\gamma(\varphi)(g) = \overline{\gamma(g)}\varphi(g) \quad (\gamma \in \hat{G},\ \varphi \in C_c(G, A)).$$

Finally, we denote the sheaf of germs of continuous G-valued (respectively, $\hat{G}$-valued) functions on a space T by $\mathscr{G}$ (respectively, $\hat{\mathscr{G}}$).

1. LOCALLY UNITARY AUTOMORPHISM GROUPS

Let A be a C^*-algebra and $\alpha\colon G \to \operatorname{Aut} A$ a strongly continuous automorphism group. We shall say α is *pointwise unitary* if for each $\pi \in \hat{A}$ there is a (strongly continuous unitary) representation U of G on H_π such that

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \quad \text{for all } g \in G \text{ and } a \in A;$$

we say U implements α in the representation π. We call α *locally unitary* if for each $\pi \in \hat{A}$ there is a neighbourhood N of π and a strictly continuous map $u\colon G \to \mathcal{M}(A)$ such that, for each $\rho \in N$, $\overline{\rho} \circ u$ is a representation of G on H_ρ which implements α.

Every locally unitary automorphism group is pointwise unitary (take $U = \overline{\pi} \circ u$). A pointwise unitary group $\alpha\colon G \to \operatorname{Aut} A$ consists of automorphisms α_g which act trivially on the primitive ideal space Prim A and the spectrum $\hat{A}$, and if A is of type I, this is equivalent by a theorem of Lance and Elliott [5] to saying that α consists of universally weakly inner (or π-inner) automorphisms. In the rest of this section we shall discuss to what extent the converses of these statements are valid. The algebra A will always be of type I.

First of all, we observe that singly generated automorphism groups which fix Prim$A$ are pointwise unitary. Secondly, when A has continuous trace, π-inner automorphisms are locally implemented by multipliers ([17, Theorem 3.4]; see also Lemma 1.4 below). Thus in this case singly generated groups which act trivially on Prim$A$ are locally unitary, and the same result can be used to show that for other groups G, and A with continuous trace, pointwise unitary implies locally unitary (Proposition 1.1). However, as the succeeding example shows, even discrete abelian automorphism groups of continuous trace C^*-algebras can be pointwise unitary without being locally unitary.

PROPOSITION 1.1. *Let A be a continuous trace C^*-algebra, let G be a finitely generated abelian group, and suppose that α: $G \to$ Aut$A$ is pointwise unitary. Then α is locally unitary.*

Proof. We write G as a sum $\oplus G_i$ of N cyclic groups, n_i for the order of G_i (we allow $n_i = \infty$), and g_i for a generator of G_i. Let $\pi \in \hat{A}$, and choose a compact neighbourhood N of π such that $A|N$ is isomorphic to the C^*-algebra $\Gamma(K(\mathscr{H}))$ defined by a continuous field of Hilbert spaces over N, and such that there is a section ξ of $\mathscr{H}$ with $\|\xi(\rho)\| = 1$ for all $\rho \in N$. By Theorem 3.4 of [17] we can shrink N so that there are $u_i \in M(A)$ satisfying

$$\rho(\alpha_{g_i}(a)) = \bar{\rho}(u_i)\rho(a)\bar{\rho}(u_i^*) \quad \text{for } 1 \leqslant i \leqslant N, \ a \in A.$$

Since α is pointwise unitary, there is a representation V of G on $\mathscr{H}_\rho$ which implements α; since ρ is irreducible we conclude that each $\bar{\rho}(u_i)$ has the form $\lambda_i V(g_i)$ for some scalars $\lambda_i \in S^1$. It follows immediately that the $\bar{\rho}(u_i)$ commute, and that, for those i with $n_i < \infty$, there are functions $\mu_i : N \to S^1$ such that

$$\bar{\rho}(u_i)^{n_i} = \mu_i(\rho)1 \qquad \text{for } \rho \in N.$$

Lemma 3.5 of [17] shows that for each multiplier m of A, $\rho \to \bar{\rho}(m)\xi(\rho)$ is a continuous section of $\mathscr{H}$, and so

$$\rho \to (\bar{\rho}(u_i)^{n_i}\xi(\rho) \mid \xi(\rho)) = \mu_i(\rho)$$

is continuous. We can thus by shrinking N again assume that each μ_i has a continuous n_ith root v_i, and that $\rho \to \overline{v_i(\rho)}\bar{\rho}(u_i)$ extends to an element u_i of $\mathfrak{M}(A)$. We now define $u: G \to \mathcal{M}(A)$ by

$$u(\sum_i m_i g_i) = u_1^{m_1} u_2^{m_2} \ldots u_N^{m_N}.$$

Then $\bar{\rho} \circ u$ is a representation of G which implements α for each $\rho \in N$, and α is locally unitary. ▨

EXAMPLE 1.2. Let $\{X_i\}$ be a sequence of compact Hausdorff spaces each homeomorphic to the 2-sphere, and let X be the one point compactification of the disjoint union of the X_i. For each i we choose an automorphism α_i of $C(X_i, K(H))$ such that α_i is π-inner but not inner — that is, not implemented by a multiplier; this is possible by Theorem 2.1 of [13] since $H^2(X_i, \mathbf{Z}) = \mathbf{Z}$ for all i. We then define, $A = C(X, K(H))$ and $\alpha: \bigoplus_{i=1}^{\infty} \mathbf{Z} \to \operatorname{Aut} A$ by

$$\alpha\left(\sum_{i=1}^{k} n_i\right)(f)(x) = \begin{cases} \alpha_i^{n_i}(f \mid X_i)(x) & \text{if } x \in X_i \text{ for } 1 \leqslant i \leqslant k \\ f(x) & \text{if } x \in X \setminus \bigcup_{i=1}^{k} X_i. \end{cases}$$

It is easy to see that α is pointwise unitary, but it is not locally unitary near the point $\infty \in X$. For if it were, there would be a neighbourhood N of ∞ and a map $u: \bigoplus \mathbf{Z} \to \mathcal{M}(A)$ such that u implements α in every representation $f \to f(x)$ for $x \in N$. Since N must contain a set of the form $\{X_i : i \geqslant I\}$ this would imply in particular that α_I is inner, which is not the case.

There is nothing special about the choice of the X_i in this example apart from the fact that $C(X_i, K(H))$ had outer π-inner automorphisms — we could, for example, have obtained an example with A 2-homogeneous by taking X_i homeomorphic to the projective unitary group PU(2), and α_i the canonical automorphism of $C(\mathrm{PU}(2), \mathcal{M}_2(\mathbf{C}))$ given by the identity map of PU(2) into $\operatorname{Aut} \mathcal{M}_2(\mathbf{C})$ (see [10, Example d]). ▨

Now suppose that A is a separable type I C^*-algebra, G is separable and $\alpha: G \to \operatorname{Aut} A$ induces the trivial action on $\operatorname{Prim} A = \hat{A}$. For each $\pi \in \hat{A}$ there is a multiplier ω_π of G and an ω_π-representation $U: G \to U(H_\pi)$ such that

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \quad \text{for } a \in A,\ g \in G$$

[19, Theorem 2.6]; we can choose a genuine representation $U: G \to U(H_\pi)$ which implements α in the representation π if and only if the multiplier ω_π is trivial. Thus in particular the automorphism group α will be pointwise unitary if the (Moore) cohomology group $H^2(G, \mathbf{T})$ vanishes. This is the case, for example, if $G = \mathbf{R}$ or $\mathbf{T}$. We can also use some cohomology theory to see that there are groups G, other than those in Proposition 1.1, for which pointwise unitary implies locally unitary. We thank the referee for pointing this out to us, and for drawing our attention to [21].

PROPOSITION 1.3. *Let A be a separable continuous trace C^*-algebra with spectrum T, and let G be a separable compact group. Then every pointwise unitary group $\alpha: G \to \mathrm{Aut}\, A$ is locally unitary. If in addition $H^2(G, \mathbf{T}) = 0$, then every group $\alpha: G \to \mathrm{Aut}\, A$ of $C(T)$-module automorphisms is locally unitary.*

We shall need the following version of [13, Proposition 2.6].

LEMMA 1.4. *Let A be a C^*-algebra with compact spectrum T, and let $p \in A$ be a projection such that $\pi(p)$ has rank one for all $\pi \in T$. Then there is a continuous map*

$$\gamma: M = \{\varphi \in \mathrm{Aut}_{C(T)} A: \|\varphi(p) - p\| < 1\} \to U\mathcal{M}(A)$$

such that $\mathrm{Ad}\, \gamma(\varphi) = \varphi$ *for all* $\varphi \in M$.

Proof. We can view A as the sections of the field $\{K(A(t)p(t))\}$ of elementary C^*-algebras defined by the field of Hilbert spaces $\{A(t)p(t); t \in T\}$. Given $\varphi \in M$, define a field $u(t) \in B(A(t)p(t))$ by

$$u(t)(a(t)p(t)) = \varphi(a)(t)\varphi(p)(t)p(t)/\|\varphi(p)(t)p(t)\|.$$

As in the proof of [13, Proposition 2.6], $u(t)$ is a unitary operator for each $t \in T$, and it is easy to check that $t \to u(t)$ defines an element of $\mathcal{M}(A)$ (using, for example, [17, Lemma 3.5]). We then define $\gamma(\varphi) = u$. ▨

Proof of Proposition 1.3. A standard compactness argument using Lemma 1.4 shows that for each $\pi \in \hat{A}$ there are a compact neighbourhood M of π and a Borel map $u: G \to \mathcal{M}(A)$ such that u_g implements α_g in each representation $\rho \in M$. We can then define a Borel cocycle $\omega: G \times G \to C(M, \mathbf{T})$ by

$$\bar{\rho}(u_g u_h) = \omega(g, h)(\rho)\bar{\rho}(u_{gh});$$

notice that as α is pointwise unitary, the cocycle $w(\cdot, \cdot)(\rho)$ is trivial for each fixed ρ. Let $\mathcal{C}^1(G, \mathbf{T})$ denote the Polish group of Borel maps: $G \to \mathbf{T}$ (with maps agreeing a.e. identified) in the topology of convergence in measure, and let $\mathrm{Hom}(G, \mathbf{T})$ denote the subgroup of continuous homomorphisms. As in the proof of [21, Theorem 2.6], ω defines a continuous map of M into the quotient $\mathcal{C}^1(G, \mathbf{T})/\mathrm{Hom}(G, \mathbf{T})$, and ω will continuously trivialise locally if this map has local continuous liftings to $\mathcal{C}^1(G, \mathbf{T})$. However, as G is compact $\mathrm{Hom}(G, \mathbf{T})$ is a discrete subgroup of $\mathcal{C}^1$, so $\mathcal{C}^1 \to \mathcal{C}^1/\mathrm{Hom}$ is a locally trivial fibre bundle and these local liftings always exist. We can use this trivialisation to adjust $u: G \to \mathcal{M}(A)$ and cut down the neighbourhood M to ensure that u is a.e. a Borel homomorphism of G into $U\mathcal{M}(A|M)$. Then a standard application of the closed graph theorem for Borel homomorphisms implies that u is equal a.e. to a strictly continuous map of G into $U\mathcal{M}(A|M)$, which also implements α over M. Thus α is locally unitary. As $C(T)$-module automorphisms fix $\hat{A}$, the remarks preceding the proposition show that if $H^2(G, \mathbf{T}) = 0$ every group of such automorphisms is pointwise unitary, so the second statement follows from the first. ▨

Finally, we observe that even if A has compact spectrum T and $\alpha: G \to \operatorname{Aut} A$ is a locally unitary group consisting of inner automorphisms, it need not follow that α is unitary. For, as in the proof of Proposition 1.3, there is another obstruction in $H^2(G, C(T, \mathbf{T}))$ which need not vanish even if $H^2(G, \mathbf{T}) = 0$ (see [21, proof of Theorem 2.6 and Example 2.8]). In the next two sections we shall study another obstruction to solving this problem.

2. THE SPECTRUM OF A CROSSED PRODUCT BY A LOCALLY UNITARY GROUP

Let A be a type I C^*-algebra and let $\alpha: G \to \operatorname{Aut} A$ be abelian and locally unitary. We shall prove in this section that $(A \times_\alpha G)^\wedge$ is a locally trivial principal $\hat{G}$-bundle with base $\hat{A}$. Our first step, the construction of the bundle map $p: (A \times_\alpha G)^\wedge \to \hat{A}$, works equally well for pointwise unitary automorphism groups.

PROPOSITION 2.1. *Let A be a type* I *C^*-algebra, G a locally compact abelian group and $\alpha: G \to \operatorname{Aut} A$ a pointwise unitary automorphism group. Then for each $\pi \times U \in (A \times_\alpha G)^\wedge$ the representation π of A is irreducible, and the map $p: \pi \times U \to \pi$ is a continuous surjection of $(A \times_\alpha G)^\wedge$ onto $\hat{A}$, such that*

$$p^{-1}(p(\pi \times U)) = \{\pi \times \gamma U : \gamma \in \hat{G}\}.$$

Proof. Let $\pi \times U \in (A \times_\alpha G)^\wedge$. Since A is type I, each α_g is π-inner, and so we can choose unitary operators $M_g \in \pi(A)''$ which implement α_g in the representation π. It follows from the covariance of (π, U) that $M_g^* U_g \in \pi(A)'$. Suppose that E belongs to the centre $\pi(A)'' \cap \pi(A)'$ of $\pi(A)''$. Then for each $g \in G$

$$EU_g = EM_g M_g^* U_g = M_g E(M_g^* U_g) = M_g(M_g^* U_g)E = U_g E,$$

so that E commutes with the range of U as well as the range of π, and hence belongs to $(\pi \times U)(A \times_\alpha G)' = \mathbf{C}1$. We deduce that π is a factor representation, type I since A is, and therefore can be realised as $\rho \otimes 1$ acting on $H_\rho \otimes H$ for some $\rho \in \hat{A}$. If $W: G \to U(H_\rho)$ implements α in the representation ρ, then each $(W_g \otimes 1)^* U_g$ belongs to $\pi(A)' = 1 \otimes B(H)$ and so has the form $1 \otimes Y_g$. Using the fact that $W_g \otimes 1 \in B(H_\rho) \otimes 1$ we see that Y is a representation of G, and because $(\pi, U) = (\rho \otimes 1, W \otimes Y)$ is irreducible Y must be too. But G is abelian, so this implies that H is one-dimensional and $\pi = \rho$ is irreducible. The surjectivity of p is automatic since α is pointwise unitary, and the irreducibility of π shows that if $\pi \times U$ and $\pi \times V$ both belong to $(A \times_\alpha G)^\wedge$ then $U = \gamma V$ for some $\gamma \in \hat{G}$.

Thus it only remains to check the continuity of p. We note first that if (π, U) is a covariant representation of (A, G) then π can be recovered from $\pi \times U$ as the restriction of the representation to the subalgebra $A \cong i_\alpha(A)$ of $\mathfrak{M}(A \times_\alpha G)$. If J is an ideal in A, so that $N = \{\pi \in \hat{A} : \ker \pi \not\supset J\}$ is a typical open set in $\hat{A}$, then

$$p^{-1}(N) = \{(\pi \times U) : \ker (\pi \times U)^- \not\supset i_\alpha(J)\}.$$

Since this latter set is open in the open subset $(A \times_\alpha G)^\wedge$ of $\mathfrak{M}(A \times_\alpha G)^\wedge$, we conclude that p is continuous. ∎

REMARK. If A, G, α are as in Proposition 2.1, we shall call the map p of $(A \times_\alpha G)^\wedge$ onto $\hat{A}$ defined by $p(\pi \times U) = \pi$ *the restriction map.* We justify this by observing that $p(\pi \times U)$ can be viewed as the restriction of the representation $(\pi \times U)^-$ of $\mathfrak{M}(A \times_\alpha G)$ to the subalgebra $i_\alpha(A)$ of $\mathfrak{M}(A \times_\alpha G)$.

THEOREM 2.2. *Let A be a type I C^*-algebra, let G be a locally compact abelian group, and let $\alpha : G \to \operatorname{Aut} A$ be a locally unitary automorphism group. Then the restriction map $p : (A \times_\alpha G)^\wedge \to \hat{A}$ is a locally trivial principal $\hat{G}$-bundle relative to the dual action of $\hat{G}$. Moreover, if α is implemented by $u : G \to \mathfrak{M}(A)$ over the open set N, then*

$$(\pi, \gamma) \to \pi \times \gamma\bar{\pi}(u)$$

is a $\hat{G}$-isomorphism of $N \times \hat{G}$ onto $p^{-1}(N)$.

The proof of this theorem consists of localising to an ideal I of A where α is implemented by a unitary and then observing that in this case $(I \times_\alpha G)^\wedge$ is homeomorphic to $\hat{I} \times \hat{G}$. We state these well-known facts as separate results since we have been unable to find a satisfactory reference.

LEMMA 2.3. *Let A be a type I C^*-algebra and $\alpha : G \to \operatorname{Aut} A$ a pointwise unitary abelian automorphism group. Let I be an ideal in A, and observe that I is invariant under α.*

a) *The inclusion $C_c(G, I) \subset C_c(G, A)$ induces an isometric embedding i of $I \times_\alpha G$ as an ideal in $A \times_\alpha G$.*

b) *If we denote by i^* and j^* the embeddings of $(I \times_\alpha G)^\wedge$ and $\hat{I}$ as open subsets of $(A \times_\alpha G)^\wedge$ and $\hat{A}$ respectively (as in* [3, 3.2.1]*), then i^* preserves the action of $\hat{G}$, and the following diagram commutes:*

$$\begin{array}{ccc} (I \times_\alpha G)^\wedge & \xrightarrow{i^*} & (A \times_\alpha G)^\wedge \\ \downarrow p & & \downarrow p \\ \hat{I} & \xrightarrow{j^*} & \hat{A} \end{array} .$$

Proof. Part (a) is contained in [7, Proposition 12] (and is true for all C^*-algebras). The embeddings i^* and j^* consist of taking the unique extension of [3, 2.10.4]; it is routine to check that the extension $E(\pi \times U)$ of $\pi \times U$ is just $E(\pi) \times U$, and the rest is easy. ▨

LEMMA 2.4. *Let A be a C^*-algebra, G a locally compact group and suppose that $u: G \to \mathfrak{M}(A)$ is a strictly continuous unitary representation of G. Then the map $\varphi: C_c(G) \odot A \to C_c(G, A)$ defined by $\varphi(\sum_i z_i \otimes a_i)(g) = \sum_i z_i(g) a_i u_g^*$ extends to an isomorphism of $C^*(G) \otimes_{\max} A$ onto $A \times_{\mathrm{Ad}\, u} G$. If G is abelian and we identify $(C^*(G) \otimes A)^\wedge$ with $\hat{A} \times \hat{G}$, then the induced homeomorphism $\hat{\varphi}$ on spectra is given by $\hat{\varphi}(\pi \times \gamma\bar{\pi}(u)) = (\pi, \gamma)$.*

Proof. That φ is an algebraic $*$-isomorphism is easy to check. It is also easy to see that commuting pairs of representations of $C_c(G)$ and A correspond bijectively under φ with covariant pairs of representations of $C_c(G, A)$. Thus φ is isometric and hence extends to an isomorphism of $C^*(G) \otimes_{\max} A$ onto $A \times_{\mathrm{Ad}\, u} G$. The last assertion follows from a straightforward computation. ▨

Proof of Theorem 2.2. Let $\pi \in \hat{A}$, and let N be an open neighbourhood of π such that on N the group α is implemented by a map $u: G \to \mathfrak{M}(A)$. We write I for the ideal corresponding to the closed subset $\hat{A} \setminus N$ of $\hat{A}$, so that there is a natural homeomorphism j^* of $\hat{I}$ onto N. Lemma 2.3 implies that $p^{-1}(N)$ is homeomorphic to $(I \times_\alpha G)^\wedge$. A simple approximate identity argument shows that each multiplier $m \in \mathfrak{M}(A)$ defines a multiplier $\tilde{m} \in \mathfrak{M}(I)$, and it is easy to see that $g \to \tilde{u}_g$ is strictly continuous. If $\rho \in \hat{I}$, then $\rho = \sigma \mid I$ for some $\sigma \in N$, and $\bar{\rho}(\tilde{u}_g) = \bar{\sigma}(u_g)$, so $g \to \bar{\rho}(\tilde{u}_g)$ is a representation. Further, for $a \in I$ we have

$$\rho(\alpha_g(a)) = \sigma(\alpha_g(a)) = \sigma(u_g a u_g^*) = \rho(\tilde{u}_g a \tilde{u}_g^*),$$

so that $\alpha \mid I = \mathrm{Ad}\, \tilde{u}$. By the preceding lemma, therefore, $(I \times_\alpha G)^\wedge$ is homeomorphic to $\hat{I} \times \hat{G} = N \times \hat{G}$, and we have proved that $p^{-1}(N)$ is homeomorphic to $N \times \hat{G}$. A simple check of the results we have used shows that the homeomorphism is given by:

$$N \times \hat{G} \to \hat{I} \times \hat{G} \to (I \times_{\mathrm{Ad}\, \tilde{u}} G)^\wedge = (I \times_\alpha G)^\wedge \to p^{-1}(N)$$

$$(\pi, \gamma) \to (\pi \mid I, \gamma) \to (\pi \mid I) \times \gamma\bar{\pi}(u) \to \pi \times \gamma\bar{\pi}(u).$$

A calculation on $C_c(G, A)$ shows that the dual action of $\hat{G}$ on $(A \times_\alpha G)^\wedge$ is given by $\gamma(\rho \times U) = \rho \times \gamma U$; hence our homeomorphism preserves the $\hat{G}$-actions and the theorem follows. ▨

Let (A, G, α) be as in Theorem 2.2. If $\{N_i\}$ is an open cover of $\hat{A}$ and $u: N_i \to \mathfrak{M}(A)$ implements α over N_i, then on the intersection $N_{ij} = N_i \cap N_j$ we have a $\hat{G}$-homeomorphism

$$N_{ij} \times \hat{G} \to p^{-1}(N_{ij}) \to N_{ij} \times \hat{G};$$

if we write $\gamma_{ij}(\pi)$ for the unique character of G with $\bar{\pi}(u^j) = \gamma_{ij}(\pi)\bar{\pi}(u^i)$, then this homeomorphism is given by

$$\varphi_{ij}\colon (\pi, \gamma) \to (\pi, \gamma\gamma_{ij}(\pi)),$$

so that the γ_{ij} are the transition functions of $(A \times_\alpha G)^\wedge \to \hat{A}$. (We note that since φ_{ij} is a homeomorphism the γ_{ij} are automatically continuous. It could take a bit of work to prove this directly, especially since the u^i are not necessarily continuous.) The cocycle $\{N_i, \gamma_{ij}\}$ with values in the sheaf $\hat{\mathscr{G}}$ of continuous $\hat{G}$-valued functions defines a cohomology class $\zeta_A(\alpha) \in H^1(X, \hat{\mathscr{G}})$, and it is routine to verify that $\zeta_A(\alpha)$ depends only on α and not on any of the choices we have made. If in particular we take $G = \mathbf{Z}$, then the γ_{ij} take values in $S^1 = \hat{\mathbf{Z}}$, $\zeta(\alpha)$ belongs to $H^1(X, \mathscr{S})$, and hence we have a class $\eta(\alpha) = \delta(\zeta(\alpha))$ in $H^2(X, \mathbf{Z})$. In this case, the γ_{ij} depend only on the multipliers $u_i \in \mathfrak{M}(A)$ which locally implement the generator $\varphi = \alpha_1$; when A has continuous trace, $\eta(\alpha)$ is therefore the class $\eta(\varphi)$ defined in [13, Theorem 2.1] and [17, Section 5].

It is natural to ask what the isomorphism class of the principal $\hat{G}$-bundle $p\colon (A \times_\alpha G)^\wedge \to \hat{A}$ (or, equivalently, the cohomology class $\zeta_A(\alpha)$) tells us about the group α. The answer appears to be the exterior equivalence class [12, p. 357] of α, but we have only been able to prove this when the spectrum of A is Hausdorff.

PROPOSITION 2.5. *Let G be a locally compact abelian group, and suppose that $\alpha, \beta\colon G \to \operatorname{Aut} A$ are two locally unitary automorphism groups of a type I C^*-algebra A. If α and β are exterior equivalent then the two principal $\hat{G}$-bundles $p_\alpha\colon (A \times_\alpha G)^\wedge \to \hat{A}$ and $p_\beta\colon (A \times_\beta G)^\wedge \to \hat{A}$ are isomorphic. If $\hat{A}$ is Hausdorff, the converse holds.*

Proof. Suppose first that α and β are exterior equivalent, so that there is a strictly continuous map $u\colon G \to U\mathfrak{M}(A)$ satisfying

$$(1) \qquad u_{gh} = u_g \alpha_g(u_h) \quad \text{and} \quad \beta_g = (\operatorname{Ad} u_g) \circ \alpha_g \quad \text{for } g, h \in G.$$

We define $\varphi\colon C_c(G, A) \to C_c(G, A)$ by $\varphi(x)(g) = x(g)u_g^*$. It is unpleasant but routine to verify that φ is a $*$-homomorphism from the subalgebra of $A \times_\alpha G$ to the subalgebra of $A \times_\beta G$, and in fact φ is an isomorphism since we can write down its inverse. If (π, U) is a covariant representation of (A, G, α), then π is a non-degenerate representation of A and therefore extends to a representation $\bar{\pi}$ of $\mathfrak{M}(A)$; it is routine to check that

$$(2) \qquad \pi \times U \to \pi \times \bar{\pi}(u)U = (\pi \times U) \circ \varphi^{-1}$$

induces a bijection of $(A \times_\alpha G)^\wedge$ onto $(A \times_\beta G)^\wedge$ (the fact that u is an α-cocycle (1) and the covariance of (π, U) show that $\bar{\pi}(u)U$ is a representation of G). A straightforward computation now shows that φ is isometric, and hence extends to an isomorphism, also denoted φ, of $A \times_\alpha G$ onto $A \times_\beta G$ with $\hat{\varphi}$ given by (2). Further calculations show that

$$\begin{array}{ccc} \mathcal{M}(A \times_\alpha G) & \xrightarrow{\mathcal{M}(\varphi)} & \mathcal{M}(A \times_\beta G) \\ & {\scriptstyle i_\alpha} \nwarrow \quad \nearrow {\scriptstyle i_\beta} & \\ & A & \end{array}$$

commutes. Thus if $\pi \times U \in (A \times_\alpha G)^\wedge$, then for $a \in A$ we have

$$p_\beta(\hat{\varphi}(\pi \times U))(a) = \hat{\varphi}(\pi \times U)^-(i_\beta(a)) = (\pi \times U)^-(\mathcal{M}(\varphi)^{-1} \circ i_\beta(a)) =$$
$$= (\pi \times U)^-(i_\alpha(a)) = p_\alpha(\pi \times U)(a),$$

so that $p_\beta \circ \hat{\varphi} = p_\alpha$. We conclude that $\hat{\varphi}$ is an isomorphism of $\hat{G}$-bundles, and we have proved the first part of the proposition.

We now suppose that $\hat{A}$ is Hausdorff, and that $(A \times_\alpha G)^\wedge$ and $(A \times_\beta G)^\wedge$ are isomorphic $\hat{G}$-bundles. Let $\{N_i\}$ be an open cover of $\hat{A}$ and suppose $u^i, v^i : G \to \mathcal{M}(A)$ implement α, β respectively over N_i; then the transition functions of the corresponding $\hat{G}$-bundles are the maps γ_{ij}, χ_{ij} satisfying

$$\bar{\pi}(u^j) = \gamma_{ij}(\pi)\bar{\pi}(u^i), \quad \bar{\pi}(v^j) = \chi_{ij}(\pi)\bar{\pi}(v^i) \quad \text{for } \pi \in N_{ij}.$$

We can therefore assume by passing to a subcover that there are maps $\lambda_i : N_i \to \hat{G}$ such that $\chi_{ij} = \lambda_i^{-1}\gamma_{ij}\lambda_j$ on N_{ij}. The map $(\pi, g) \to \lambda_i(\pi)(g)$ is continuous on $N_i \times G$, and we may assume by shrinking the N_i that each of these extends to a continuous function $\tilde{\lambda}_i : \hat{A} \times G \to \mathbf{C}$ such that $\{\pi \in \hat{A} : \tilde{\lambda}_i(\pi, g) \neq 0 \text{ for some } g\}$ has compact closure in $\hat{A}$ (this uses the fact that $\hat{A}$ is Hausdorff). If we set $\lambda_i^g(\pi) = \tilde{\lambda}_i(\pi, g)$, then $g \to \lambda_i^g$ is a continuous map of G into $C_c(\hat{A})$, and $g \to \tilde{u}_g^i = \lambda_i^g u_g^i$ is a strictly continuous map of g into $\mathcal{M}(A)$. It is clear that $\tilde{u}_g^i$ also implements α_g over N_i, and on N_{ij} we have

$$\bar{\pi}(\tilde{u}_g^j) = \chi_{ij}(\pi)\bar{\pi}(\tilde{u}_g^i) \quad \text{and} \quad \bar{\pi}(v_g^i(\tilde{u}_g^i)^*) = \bar{\pi}(v_g^j(\tilde{u}_g^j)^*).$$

Since $\hat{A}$ is Hausdorff, A is defined by a continuous field of elementary C^*-algebras over $\hat{A}$, and we can define elements of A locally. In particular, for $g \in G$ we can define aw_g, $w_g a \in A$ by the formulas

$$\pi(aw_g) = \pi(av_g^i(\tilde{u}_g^i)^*), \quad \pi(w_g a) = \pi(v_g^i(\tilde{u}_g^i)^* a) \quad (\pi \in N_i),$$

and it is easy to check that this in turn defines a multiplier $w_g \in \mathcal{M}(A)$. The map $g \to w_g$ is strictly continuous locally, and since the elements of A vanish at infinity on $\hat{A}$ it follows that $g \to w_g$ is strictly continuous. Routine calculations show that w is an α-cocycle, and that $\beta_g = \operatorname{Ad} w_g \circ \alpha_g$ for $g \in G$, and we are done. ▨

Let A be a C^*-algebra with Hausdorff spectrum and G a locally compact abelian group. We have now constructed a map ζ_A from the collection $\mathrm{LU}(G, A)$ of locally unitary groups $\alpha: G \to \mathrm{Aut}\, A$ into the Čech cohomology group $H^1(A, \hat{\mathscr{G}})$, and we have shown that ζ_A is an injection modulo exterior equivalence. The main result of our next section asserts that ζ_A is also surjective when A is stable.

3. THE CONSTRUCTION OF CROSSED PRODUCTS WITH GIVEN SPECTRUM

We have seen that a locally unitary automorphism group $\alpha: G \to \mathrm{Aut}\, A$ gives rise to a principal $\hat{G}$-bundle $p: (A \times_\alpha G)^\wedge \to \hat{A}$. In this section we shall prove that every principal $\hat{G}$-bundle E over a locally compact and paracompact space T can be realised this way for a variety of C^*-algebras A with spectrum T. All our results depend on the following theorem, which shows that locally unitary actions are dual to locally trivial bundles in a natural way.

THEOREM 3.1. *Let G be a locally compact abelian group, and suppose that $q: E \to T$ is a locally trivial principal G-bundle over a locally compact paracompact space T. Then the dual action of $\hat{G}$ on $A = C^*(G, E)$ is locally unitary, and the bundle $p: (A \times \hat{G})^\wedge \to \hat{A}$ is isomorphic to $q: E \to T$.*

Given such a bundle $q: E \to T$, our strategy will be to construct a map $\xi \to \pi_\xi \times U$ of E into $(C^*(G, E) \times \hat{G})^\wedge$ and prove that it is a homeomorphism. We begin by defining the representations π_ξ of $C^*(G, E)$ and looking at their properties. For each $\xi \in E$, the representation π_ξ acts on $L^2(G)$ and is given by

$$(\pi_\xi(z)x)(g) = \int_G z(h)\,(g\cdot\xi)\,x(h^{-1}g)\,\mathrm{d}h \qquad (z \in C_c(G, C_c(E)),\, x \in L^2(G)).$$

This is the representation of $C^*(G, E)$ induced from the one-dimensional representation $f \to f(\xi)$ of $C_\infty(E)$ in the sense of Takesaki [19].

LEMMA 3.2. *Let $z \in C^*(G, E), x \in L^2(G)$. Then the map $\xi \to \pi_\xi(z)x$ is continuous.*

Proof. It is enough to work with $x \in C_c(G)$ and z of the form $g \to f(g)y$ for $f \in C_c(G)$, $y \in C_c(E)$: we then have to show that

$$\int_G |y(g\cdot\xi_\alpha)(f * x)(g) - y(g\cdot\xi)(f * x)(g)|^2\,\mathrm{d}g \to 0$$

whenever $\xi_\alpha \to \xi$ in E. This follows by a routine compactness argument using the fact that $f * x$ has compact support. ▨

LEMMA 3.3. *For each $\xi \in E$, π_ξ is an irreducible representation of $C^*(G, E)$. Two such representations π_ξ and π_η are equivalent if and only if $G\cdot\xi = G\cdot\eta$, and every irreducible representation of $C^*(G, E)$ is equivalent to one of the form π_ξ. The map $\xi \to \pi_\xi$ induces a homeomorphism of $T = E/G$ onto $C^*(G, E)^\wedge$.*

Proof. We observe first that $\pi_\xi = M_\xi \times V$, where for $y \in C_c(E)$, $x \in L^2(G)$ and h, $g \in G$ we have

$$[M_\xi(y)x](h) = y(h\cdot\xi)x(h), \quad (V_g x)(h) = x(g^{-1}h).$$

The kernel of M_ξ is the subalgebra $C_\infty(E\setminus G\cdot\xi)$ of $C_\infty(E)$ consisting of those functions which vanish on $G\cdot\xi$, and so it follows from [6, Lemma 1 (ii)] that $\ker\pi_\xi$ is the ideal $C^*(G, E\setminus G\cdot\xi)$. The same lemma shows that the map φ_ξ of $C_c(G, C_c(E))$ into $C_c(G, C_c(G))$ defined by

$$[\varphi_\xi(z)(g)](h) = z(g)(h\cdot\xi) \quad (g, h \in G)$$

extends to an isomorphism of $C^*(G, E)$ onto $C^*(G, G)$ with kernel $C^*(G, E\setminus G\cdot\xi)$. It is well-known that if M is the representation of $C_\infty(G)$ on $L^2(G)$ by multiplication operators and V is as above, then $M \times V$ is an isomorphism of $C^*(G, G)$ onto $K(L^2(G))$ (see, for example, Lemma 3 of [6]). Simple calculations show that $M_\xi \times V = (M \times V)\circ\varphi_\xi$, and we deduce that π_ξ is the essentially unique irreducible representation of $C^*(G, E)$ with kernel $C^*(G, E\setminus G\cdot\xi)$. Since the G-orbits in E are closed, [6, Lemma 1 (ii)] shows that every irreducible representation of $C^*(G, E)$ has a kernel of this form, and so the map $\xi \to \pi_\xi$ gives a bijection of E/G onto $C^*(G, E)$. If M is open in E/G, then, again using Lemma 1 of [6], we see that

$$\begin{aligned} C^*(G, E)^\wedge\setminus h(M) &= \{[\pi_\xi] : q^{-1}(M)\cap G\cdot\xi = \emptyset\} = \\ &= \{[\pi_\xi] : C_\infty(E\setminus G\cdot\xi) \supset C_\infty(q^{-1}(M))\} = \\ &= \{[M_\xi\times V] : \ker M_\xi \supset C_\infty(q^{-1}(M))\} = \\ &= \{[\pi_\xi] : \ker\pi_\xi \supset C^*(G, q^{-1}(M))\}, \end{aligned}$$

which is closed in $C^*(G, E)^\wedge$. We conclude that $h(M)$, and hence the map h, is open. Lemma 3.2 shows that $\xi \to [\pi_\xi]$ is continuous, and it follows from this that h is continuous and so a homeomorphism. ▨

REMARK. This construction and these results are essentially contained, modulo some notational differences and separability hypotheses, in Lemma 16 and in the proof of Theorem 14 of [6].

Although π_ξ and $\pi_{g\cdot\xi}$ are equivalent representations, we can use them to construct inequivalent covariant representations of the system $(C^*(G, E), \hat{G}, \alpha)$, where α denotes the dual action of $\hat{G}$. We define $U: \hat{G} \to (U(L^2(G))$ by

$$(U_\gamma x)(g) = \overline{\gamma(g)}x(g) \quad (x \in L^2(G),\ g \in G,\ \gamma \in \hat{G});$$

a routine check shows that each (π_ξ, U) is a covariant representation. To see that $\pi_\xi \times$ U and $\pi_\eta \times U$ are only equivalent if $\xi = \eta$, note that η must have the form $g\cdot\xi$ by Lemma 3.3, and hence the only unitary operators which can intertwine $\pi_\xi \times U$ and $\pi_\eta \times U$ are scalar multiples of $W \in U(L^2(G))$ defined by $Wx(h) = x(hg)$. But then

$$(WU_\gamma W^* x)(h) = (U_\gamma W^* x)(hg) = \overline{\gamma(hg)}(W^* x)(hg) = \gamma(g)(U_\gamma x)(h),$$

so if W intertwines $\pi_\xi \times U$ and $\pi_{g\cdot\xi} \times U$ then $\gamma(g) = 1$ for all $\gamma \in \hat{G}$. This implies that $g = 1$ and $\xi = \eta$.

We aim to prove that $\varphi(\xi) = \pi_\xi \times U$ defines a homeomorphism of E onto $(C^*(G, E) \times_\alpha \hat{G})^\wedge$. The following simple (and presumably very standard) observation about fibre bundles will be useful.

LEMMA 3.4. *Let $q: E \to X$ and $p: F \to Y$ be two locally trivial principal G-bundles and suppose that $\varphi: E \to F$ is a continuous G-equivariant surjection which induces a homeomorphism $\psi: X \to Y$. Then φ is a homeomorphism.*

Proof. It follows easily from $\psi \circ q = p \circ \varphi$ that φ is one-to-one, and we therefore only have to prove that φ^{-1} is continuous. This is a local problem, and since ψ is a homeomorphism we can assume $E = X \times G$ and $F = Y \times G$. Then $\varphi(x, g) = (\psi(x), h_x(g))$. Since φ is equivariant we see that $h_x(g_1 g_2) = h_x(g_1)g_2$ for $g_1, g_2 \in G$. Letting $g_1 = e$ shows that $x \to h_x(e)$ is continuous and $\varphi(x, g) = (\psi(x), h_x(e)g)$. Thus $\varphi^{-1}(y, g) = (\psi^{-1}(y), [h_{\psi^{-1}(y)}(e)]^{-1}g)$, and this is continuous since inversion is continuous in the group G. ∎

Proof of Theorem 3.1. We begin by proving that α is locally unitary. Let $t_0 \in T$, and suppose that N is a neighbourhood of t_0 such that there is a continuous section $\zeta: N \to E$ of $q: E \to T$. We choose a continuous function $f: T \to [0,1]$ such that $f = 0$ off N and $f = 1$ on some neighbourhood M of t_0. For $z \in C_c(G, C_c(E))$ and $\gamma \in \hat{G}$ we define

$$(m_\gamma z)(g)(\eta) = \overline{\gamma(h)}z(g)(\eta)f(q(\eta)) \quad \text{where } \eta = h\cdot\zeta(q(\eta)) \in E$$

$$(zm_\gamma)(g)(\eta) = \overline{\gamma(k)}z(g)(\eta)f(q(\eta)) \quad \text{where } g^{-1}\cdot\eta = k\cdot\zeta(q(\eta));$$

note that $\eta \to h$ and $(g, \eta) \to k$ are continuous since the bundle $E \to X$ is trivial over N. (We are thinking of m_γ as the function from G into $C(E)$ defined by

$$m_\gamma(g)(\eta) = \delta_e(g)\gamma(h)f(q(\eta)) \quad \text{where } \eta = h\cdot\zeta(q(\eta)).)$$

If $\xi \in E$ and $y \in L^2(G)$ then

$$(\pi_\xi(m_\gamma z)y)(g) = \overline{\gamma(h)}f(q(\xi))(\pi_\xi(z)y)(g) \quad \text{where } g\cdot\xi = h\cdot\zeta(q(\xi)).$$

Since $|\gamma(h)| = 1$ and $f(q(\xi)) \in [0,1]$, it follows that

$$\|\pi_\xi(m_\gamma z)y\| = f(q(\xi))\|\pi_\xi(z)y\| \leqslant \|\pi_\xi(z)\|\,\|y\|,$$

so that

$$\|m_\gamma z\|_{C^*(G,E)} = \sup_{\rho\in C^*(G,E)^\wedge}\|\rho(m_\gamma z)\| = \sup_{\xi\in E}\|\pi_\xi(m_\gamma z)\| \leqslant \|z\|_{C^*(G,E)}.$$

Thus the map $z \to m_\gamma z$ extends to a bounded linear operator on $C^*(G, E)$ of norm 1. Similarly, if $y \in L^2(G)$ and $\xi \in E$, then

$$(\pi_\xi(zm_\gamma)y)(g) = f(q(\xi))(\pi_\xi(z)y_1)(g)$$

where y_1 is given by $y_1(g) = \gamma(k)y(g)$ where $g\cdot\xi = k\cdot\zeta(q(\xi))$. Since $\|y_1\| = \|y\|$, we can deduce as above that $z \to zm_\gamma$ extends to $C^*(G, E)$ in fact, it is not hard to see that we have defined a multiplier m_γ of $C^*(G, E)$. We claim that $\gamma \to m_\gamma$ is the map of $\hat{G}$ into $M(C^*(G, E))$ required to prove that α is locally unitary.

If $z \in C_c(G, C_c(E))$, then a standard compactness argument together with the continuity of $\eta = h\cdot\zeta(q(\eta)) \to h$ shows that $\gamma \to m_\gamma z$ and $\gamma \to zm_\gamma$ are continuous maps of $\hat{G}$ into $L^1(G, C_\infty(E))$; this implies that $\gamma \to m_\gamma$ is strictly continuous. A simple calculation shows that for $\xi \in q^{-1}(M)$, $x \in L^2(G)$ and $g \in G$ we have

$$(\bar{\pi}_\xi(m_\gamma)x)(g) = \overline{\gamma(h)}x(g) \quad \text{where } g\cdot\xi = h\cdot\zeta(q(\xi)),$$

and it follows easily that $\gamma \to \bar{\pi}_\xi(m_\gamma)$ is a representation of $\hat{G}$. The adjoint m_γ^* of m_γ is defined by

$$(m_\gamma^* z)(g)(\eta) = \gamma(h)f(q(\eta))z(g)(\eta), \quad \eta = h\cdot\zeta(q(\eta))$$
$$(zm_\gamma^*)(g)(\eta) = \gamma(k)f(q(\eta))z(g)(\eta), \quad g^{-1}\cdot\eta = k\cdot\zeta(q(\eta)),$$

and so for $\xi \in q^{-1}(M)$, $\gamma \in L^2(G)$ we have

$$(\pi_\xi(m_\gamma zm_\gamma^*)x)(g) = \int \overline{\gamma(h)}z(g_1)(g\cdot\xi)\gamma(k)x(g_1^{-1}g)\mathrm{d}g_1,$$

where $g\cdot\xi = h\cdot\zeta(q(\xi))$ and $g_1^{-1}g\cdot\xi = k\cdot\zeta(q(\xi))$. Since the action of G is free, this implies $k = g_1^{-1}h$, and

$$(\pi_\xi(m_\gamma zm_\gamma^*)x)(g) = \int z(g_1)(g\cdot\xi)\overline{\gamma(g_1)}x(g_1^{-1}g)\,\mathrm{d}g_1 = (\pi_\xi(\alpha_\gamma z)x)(g),$$

so that m_γ implements α_γ in the representation π_ξ for $\xi \in q^{-1}(M)$. Since $\{\pi_\xi : q(\xi) \in M\}$ is an open set in $C^*(G, E)^\wedge$ by Lemma 3.3, we deduce that α is locally unitary.

As in the earlier discussion, we define $\varphi: E \to (A \times_\alpha \hat{G})^\wedge$ by $\varphi(\xi) = \pi_\xi \times U$. The dual action of $G = (\hat{G})^\wedge$ on $(A \times_\alpha \hat{G})^\wedge$ is given by $g\cdot(\pi_\xi \times U) = \pi_\xi \times gU$, and it is easy to verify that the unitary operator V on $L^2(G)$ defined by $(Vx)(h) = x(gh)$ intertwines $\pi_\xi \times gU$ and $\pi_{g\cdot\xi} \times U$, so φ is G-equivariant. Theorem 2.2 shows that $(A \times_\alpha \hat{G})^\wedge$ is a principal G-bundle over $\hat{A}$, and Lemma 3.3 shows that φ induces a homeomorphism ψ of $T = E/G$ onto $\hat{A}$, so φ is surjective, and the result will follow from Lemma 3.4 if we can show that φ is continuous. To do this it is enough to prove that if $\xi_\alpha \to \xi$ then $[(\pi_{\xi_\alpha} \times U)f]x \to [(\pi_\xi \times U)f]x$ for each $f \in C_c(\hat{G}, A)$, $x \in L^2(G)$. For $w \in C_c(G)$ and $z \in C_c(G, C_c(E))$ we denote by $w \otimes z$ the function $\gamma \to w(\gamma)z$ belonging to $C_c(G, A)$. A straightforward calculation shows that for $\eta \in E$ and $x \in L^2(G)$ we have

$$[(\pi_\eta \times U)(w \otimes z)]x = \pi_\eta(z)[\hat{w}x],$$

where $\hat{w}$ denotes the usual Fourier transform of w. We have already shown in Lemma 3.2 that $\pi_{\xi_\alpha}(z)y \to \pi_\xi(z)y$ for each $y \in L^2(G)$, and since elements of the form $w \otimes z$ span a dense subspace of $C_c(\hat{G}, A)$, it follows that φ is continuous. This completes the proof of Theorem 3.1. ▨

COROLLARY 3.5. *Let G be an infinite locally compact abelian group, let $q: E \to T$ be a locally trivial principal G-bundle over a locally compact paracompact space T, and let A denote the C^*-algebra $C_\infty(T, K(L^2(G)))$. Then there is a locally unitary automorphism group $\alpha: G \to \operatorname{Aut} A$ such that the principal G-bundle $p: (A \times_\alpha \hat{G})^\wedge \to T$ is isomorphic to $q: E \to T$.*

Proof. Corollary 15 and Theorem 14 of [6] show that there is an isomorphism Φ of $C_\infty(T, K(L^2(G)))$ onto $C^*(G, E)$ such that the induced homeomorphism $\Phi^\wedge$ of T onto $C^*(G, E)^\wedge$ is the homeomorphism $G\cdot\xi \to [\pi_\xi]$ of Lemma 3.3. (Note that the unitary transformation $U: L^2(G) \to L^2(G\cdot\xi)$ defined by $(Uy)(g\cdot\xi) = y(g)$ intertwines π_ξ and the representation $L_{G\cdot\xi}$ constructed in the proof of Theorem 14 in [6].) This isomorphism carries the dual action of $\hat{G}$ on $C^*(G, E)$, which is locally unitary by the theorem, into a locally unitary automorphism group $\alpha: \hat{G} \to \operatorname{Aut} A$. There is a natural isomorphism $\Phi \times \hat{G}$ of $A \times_\alpha \hat{G}$ onto $C^*(G, E) \times \hat{G}$, and it is routine to verify that the following diagram commutes:

$$\begin{array}{ccc}
(A \times_\alpha \hat{G})^\wedge & \xrightarrow{(\Phi \times \hat{G})^\wedge} & (C^*(G, E) \times \hat{G})^\wedge \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\hat{A} = T & \xrightarrow{\Phi^\wedge} & C^*(G, E)^\wedge.
\end{array}$$

Since the homeomorphism $\varphi: E \to (C^*(G, E) \times \hat{G})^\wedge$ constructed in Theorem 2.1 also induces the map $G\cdot\xi \to [\pi_\xi]$ on base spaces, we deduce that $\varphi^{-1} \circ (\Phi \times \hat{G})^\wedge$

is a G-preserving homeomorphism of $(A \times_\alpha \hat{G})^\wedge$ onto E which induces the identity map on T — in other words, $\varphi^{-1} \circ (\Phi \times \hat{G})^\wedge$ is an isomorphism of principal G-bundles over T. ▨

COROLLARY 3.6. *Let G be a separable locally compact abelian group, let H be a separable infinite-dimensional Hilbert space, and let $q: E \to T$ be a locally trivial principal G-bundle over a locally compact and paracompact space T. Then there is a locally unitary automorphism group $\alpha: \hat{G} \to \mathrm{Aut}\, A$ of the C^*-algebra $A = C_\infty(T, K(H))$ such that the restriction map $p: (A \times_\alpha \hat{G})^\wedge \to T$ is G-isomorphic to $q: E \to T$.*

Proof. When G is infinite $L^2(G)$ is isomorphic to H, so the result follows from the preceding corollary. So suppose $|G| = n$ is finite. By Theorem 14 of [6] there is a locally trivial field $\mathscr{H}$ of Hilbert spaces of dimension n over T (in other words, $\mathscr{H}$ is a Hermitian vector bundle over T) such that $C^*(G, E)$ is isomorphic to the C^*-algebra $\Gamma_\infty(K(\mathscr{H}))$ defined by $\mathscr{H}$. It is easy to see that $\Gamma_\infty(K(\mathscr{H})) \otimes K(H)$ is isomorphic to $\Gamma_\infty(K(\mathscr{H} \otimes (X \times H)))$ [13, Lemma 1.11], and $\mathscr{H} \otimes (X \times H)$ is locally trivial since $\mathscr{H}$ is. Hence by [4, Théorème 1] $\mathscr{H} \otimes (X \times H)$ is isomorphic to the trivial field $X \times H$ and $C^*(G, E) \otimes K(H)$ is isomorphic to A in such a way that the induced homeomorphism of $T = C^*(G, E)^\wedge$ onto $T = \hat{A}$ is the identity. The result now follows by the reasoning of the previous corollary applied to the tensor product action $\beta \otimes \mathrm{id}$ on $C^*(G, E) \otimes K(H)$, where β is the dual action of $\hat{G}$ on $C^*(G, E)$ and id the trivial action on $K(H)$. ▨

We had to handle the case of a finite group separately because, although [6, Theorem 14] shows that $C^*(G, E) = \Gamma_\infty(K(\mathscr{H}))$ for a locally trivial field $\mathscr{H}$ of rank $|G|$ over E/G, we can only invoke [4, Théorème 1] to conclude that $\mathscr{H}$ is trivial if the fibres are infinite dimensional. As the following example shows, $C^*(G, E)$ can fail to be isomorphic to $C(E/G, M_n)$ for $n = |G|$. In fact, we prove that there is a principal $\mathbf{Z}_n$-bundle $q: E \to T$ which cannot be realised as the spectrum of a crossed product of $C(E/G, M_n)$. This shows that Corollary 3.5 is not true for finite groups, and since Theorem 3.1 shows that E is G-isomorphic to $(C^*(G, E) \times G)^\wedge$, it also shows that $C^*(G, E)$ is not isomorphic to $C(E/G, M_n)$.

Suppose that $\alpha: \mathbf{Z}_n \to \mathrm{Aut}\, C(T, M_n(\mathbf{C}))$ is locally unitary. Then according to the remarks following Theorem 2.2, the bundle $p: (C(T, M_n) \times_\alpha \mathbf{Z}_n)^\wedge \to T$ defines a class $\zeta(\alpha)$ in $H^1(T, \mathbf{Z}_n) \subset H^1(T, \mathscr{S})$, which can be described as follows: if $\{N_i\}$ is an open cover of T such that α_1 is implemented over N_i by $u_i: N_i \to SU_n$, then $\zeta(\alpha)$ is represented by the cocycle $\{N_i, \lambda_{ij}\}$, where $\lambda_{ij}: N_{ij} \to \mathbf{Z}_n \subset S^1$ satisfy $\lambda_{ij} u_i = u_j$. If we denote by L the line bundle with transition functions λ_{ij}, then this last statement says that $\lambda_{ij} \cdot 1 = u_i^* u_j$, so that the bundle $nL = L \oplus \ldots \oplus L$ (n times) is trivial, and has total Chern class $\mathrm{c}(nL) = 1$. The theory of Chern classes [9] shows that

$$\mathrm{c}(nL) = \mathrm{c}(L)^n \quad \text{and} \quad \mathrm{c}(L) = 1 + \mathrm{c}_1(L),$$

and further that $c_1(L)$ is the image of $\zeta(\alpha)$ in $H^2(T, \mathbf{Z}) \cong H^1(T, \mathscr{S})$. Hence the line bundle L must satisfy

$$(1 + c_1(L))^n = 1 + nc_1(L) + \dots + c_1(L)^n = 1 \quad \text{in } H^*(T, \mathbf{Z}).$$

If we take for T the real projective space $\mathbf{RP}^4$, then $H^2(T, \mathbf{Z}) = \mathbf{Z}_2$, and if h is non-zero in $H^2(T, \mathbf{Z})$ then h^2 generates $H^4(T, \mathbf{Z}) = \mathbf{Z}_2$ [8, 3.9.6, 4.3.1], so $(1+h)^2 \neq 1$. Thus the complex line bundle L over T with $c_1(L) = h$ cannot arise from a locally unitary automorphism group of $C(T, M_2)$; however, consideration of the exact sequence in sheaf cohomology induced by the covering map $z \to z^2 \colon S^1 \to S^1$ shows that L does come from a principal $\mathbf{Z}_2$-bundle. This $\mathbf{Z}_2$-bundle over $\mathbf{RP}^4$ cannot therefore be realised as the spectrum of a crossed product of $C(\mathbf{RP}^4, M_2)$. (We thank Shaun Disney for showing us this example; the relevance of the higher Chern classes was earlier noticed by Paulsen [11].)

Thus not every G-bundle E over T arises as the spectrum of a crossed product of $C(T, M_n)$ for $n = |G|$. Provided the base T is compact, however, it is always possible to realise E as the spectrum of $C(T, M_m) \times_\alpha \hat{G}$ for some value of m. The idea for the proof comes from [15, Lemma 3].

PROPOSITION 3.7. *Let G be a finite abelian group, and let $q \colon E \to T$ be a principal G-bundle over a compact space T. Then there is a positive integer m and a locally unitary automorphism group $\alpha \colon \hat{G} \to \operatorname{Aut} C(T, M_m(\mathbf{C}))$ such that $(C(T, M_m) \times_\alpha \hat{G})^\wedge$ is G-isomorphic to E.*

Proof. As before, we write $C^*(G, E) = \Gamma(K(\mathscr{H}))$ where $\mathscr{H}$ is a Hermitian vector bundle over T. Let β denote the dual action of $\hat{G}$ on $C^*(G, E)$, so that by Theorem 3.1 the class $\zeta(\beta)$ in $H^1(T, \mathscr{G})$ is that of E. There are an open cover $\{N_i\}$ of T, $*$-isomorphisms $\varphi_i \colon \mathscr{H}|N_i \to N_i \times \mathbf{C}^n$ and maps $u^i_\gamma \colon N_i \to U_n$ such that $\gamma \to u^i_\gamma$ is a homomorphism for each i and

$$\varphi_i(\beta_\gamma(\varphi_i^{-1}(f)))(t) = u^i_\gamma(t) f(t) u^i_\gamma(t)^* \quad \text{for } t \in N_i, f \colon N_i \to M_n.$$

On the intersections we have bundle isomorphisms $\varphi_i \varphi_j^{-1}$ given by continuous maps $v_{ij} \colon N_{ij} \to U_n$. Then the class of $\zeta(\beta)$ in $H^1(T, \mathscr{G})$ is defined by $\{N_i, g_{ij}\}$, where $g_{ij} \colon N_{ij} \to G$ and

$$v_{ij}(t) u^j_\gamma(t) v_{ij}(t)^* = g_{ij}(t) u^i_\gamma(t) \quad \text{for } t \in N_{ij}.$$

Let η be a vector bundle satisfying $\mathscr{H} \otimes \eta \cong T \times \mathbf{C}^m$ for some m (such a bundle exists by [1, Proposition IX. 4.6]), and suppose η has transition functions $w_{ij} \colon N_{ij} \to U_{m/n}$ so that $\mathscr{H} \otimes \eta$ has transition functions $v_{ij} \otimes w_{ij}$. We define $t^i_\gamma \colon N_{ij} \to U_m$ by $t^i_\gamma = u^i_\gamma \otimes 1$; each $\gamma \to t^i_\gamma$ is a homomorphism and on the intersections we have

$$[v_{ij}(t) \otimes w_{ij}(t)] t^j_\gamma(t) [v_{ij}(t)^* \otimes w_{ij}(t)^*] = g_{ij}(t) t^i_\gamma(t).$$

Thus the t_γ^i define a locally unitary automorphism group δ of $\Gamma(K(\mathscr{H} \otimes \eta))$ whose class $\zeta(\delta)$ in $H^1(T, \mathscr{G})$ is just $\zeta(\beta)$, and hence a locally unitary group $\alpha: \hat{G} \to \to \operatorname{Aut} C(T, M_m)$ whose class is $\zeta(\beta)$. This proves the proposition. ▨

Let G be a separable locally compact abelian group. Corollary 3.6 asserts that every principal G-bundle over a locally compact paracompact space T can be realised as the spectrum of a crossed product of $C_\infty(T, K(H))$. We now prove that $C_\infty(T, K(H))$ can be replaced by any other stable C^*-algebra with spectrum T.

THEOREM 3.8. *Let A be a stable C^*-algebra with paracompact spectrum T. Let G be a separable locally compact abelian group and let $q: E \to T$ be a locally trivial principal G-bundle. Then there is a locally unitary automorphism group $\alpha: \hat{G} \to \to \operatorname{Aut} A$ such that the principal G-bundle $p: (A \times_\alpha \hat{G})^\wedge \to T$ is isomorphic to E.*

In the remarks following the proof of Theorem 2.2 we constructed a map ζ_A which associated to each $\alpha \in \mathrm{LU}(\hat{G}, A)$ the cohomology class $\zeta_A(\alpha)$ in $H^1(T, \mathscr{G})$ of the bundle $p: (A \times_\alpha \hat{G})^\wedge \to T$. We shall prove that ζ_A is surjective; since the isomorphism class of E is determined by its class in $H^1(T, \mathscr{G})$, this will establish the theorem. Our proof will depend on the special case in Corollary 3.6 and on general properties of the map ζ_A. In particular, we shall need to know what ζ_A looks like when A is the central tensor product of two C^*-algebras with spectrum T.

Let A, B, Z be C^*-algebras with Z abelian. Suppose that A and B are Banach-Z-modules where $\|az\| \leqslant \|a\| \|z\|$, $\|bz\| \leqslant \|b\| \|z\|$ and the action of Z commutes with everything. Let I be the closed ideal in $A \otimes_\gamma B$ generated by $\{af \otimes b - a \otimes \otimes fb : a \in A, b \in B, f \in Z\}$, where γ is some C^*-cross-norm on $A \odot B$. We denote the C^*-algebra $A \otimes_\gamma B/I$ by $A \otimes_{\gamma, Z} B$. When A and B have the same Hausdorff spectrum T, and Z equals either $C_b(T)$ or $C_\infty(T)$, it is easy to see that the ideal I of $A \otimes B$ corresponds to the closed subset of $\operatorname{Prim}(A \otimes B) = T \times T$ given by the diagonal $\Delta = \{(t, t) : t \in T\}$, so that $A \otimes_{C_b(T)} B = A \otimes_{C_\infty(T)} B$.

We shall need the following lemma, for which we do not have a suitable reference.

LEMMA 3.9. *Let A, Z, D be C^*-algebras with Z commutative and either A or D nuclear. Let A be a Z-module and suppose that the span of $A \cdot Z$ is dense in A. Then the map φ of the algebraic tensor product $A \odot Z \odot D$ into $A \odot D$ defined by $\varphi(a \otimes f \otimes d) = af \otimes d$ induccs an isomorphism of $A \otimes_Z (Z \otimes D)$ onto $A \otimes D$.*

Proof. Since $|||x||| = \max\{\|x\|, \|\varphi(x)\|\}$ is a C^*-cross-norm on $A \odot Z \odot D$, we deduce that $|||x||| = \|x\|$ and φ is norm-decreasing; hence φ extends to a homomorphism of $A \otimes (Z \otimes D)$ into $A \otimes D$. Since $A \cdot Z$ spans A, the extension is onto $A \otimes D$. Let J be the closed ideal in $A \otimes (Z \otimes D)$ generated by elements of the form $af \otimes (g \otimes d) - a \otimes (fg \otimes d)$; clearly $J \subset \ker \varphi$. Now, any $x \in A \otimes (Z \otimes D)$ can be written as a limit

$$x = \lim_{n \to \infty} \sum_{i=1}^{m_n} a_i^n \otimes (f_i^n \otimes d_i^n),$$

where for each n the supports of the $(f_i^n)^\wedge$ are contained in a compact set $K_n \subset \hat{Z}$. Let g_n in Z be such that $(g_n)^\wedge = 1$ on K_n and $\|g_n\| = 1$. If $x \in \ker\varphi$, we have

$$0 = \lim_{n\to\infty} \varphi\left(\sum_{i=1}^{m_n} a_i^n \otimes f_i^n \otimes d_i^n\right) = \lim_{n\to\infty} \sum_{i=1}^{m_n} a_i^n f_i^n \otimes d_i^n,$$

and therefore we also have

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} a_i^n f_i^n \otimes g_n \otimes d_i^n = 0.$$

Thus

$$x = \lim_{n\to\infty} \left(\sum_i a_i^n \otimes f_i^n \otimes d_i^n - \sum_i a_i^n f_i^n \otimes g_n \otimes d_i^n\right) =$$
$$= \lim_{n\to\infty} \left(\sum_i a_i^n \otimes f_i^n g_n \otimes d_i^n - \sum_i a_i^n f_i^n \otimes g_n \otimes d_i^n\right)$$

belongs to J. Thus $\ker\varphi = J$, and we have proved that φ is an iomorphism of $A \otimes (Z \otimes D)/J$ onto $A \otimes D$. ∎

Now suppose that $\alpha: G \to \operatorname{Aut} A$ and $\beta: G \to \operatorname{Aut} B$ are locally unitary automorphism groups, where the algebras A, B have Hausdorff spectrum T as above. It is easy to see that π-inner automorphisms are automatically $C(T)$-module automorphisms, so that α and β consist of $C(T)$-module automorphisms. It follows easily that for each $g \in G$ the tensor product automorphism $\alpha_g \otimes \beta_g$ of $A \otimes B$ (which exists by [20, V.4.22]) preserves the ideal I, and so defines an automorphism $\alpha_g \otimes_{C(T)} \beta_g$ of $A \otimes_{C(T)} B$. We can therefore define an automorphism group $\alpha \otimes_{C(T)} \beta$ by

$$(\alpha \otimes_{C(T)} \beta)_g = \alpha_g \otimes_{C(T)} \beta_g \quad (g \in G).$$

PROPOSITION 3.10. *Let A, B, T, α and β be as above with T paracompact. Then $\alpha \otimes_{C(T)} \beta$ is a locally unitary automorphism group of $A \otimes_{C(T)} B$, and*

$$\zeta_{A \otimes_{C(T)} B}(\alpha \otimes_{C(T)} \beta) = \zeta_A(\alpha)\zeta_B(\beta) \quad \text{in } H^1(T, \hat{\mathcal{G}}).$$

Proof. We first observe that if J is an ideal in a C^*-algebra C, then any multiplier m of C preserves J and so defines a multiplier m_J of the quotient C/J. Suppose that α and β are implemented over N_i by $u^i: G \to \mathcal{M}(A)$ and $v^i: G \to \mathcal{M}(B)$; we claim that $\alpha \otimes_{C(T)} \beta$ is then implemented over N_i by $w_g^i = [j(u_g^i \otimes v_g^i)]_I$, where j denotes the canonical embedding of $\mathcal{M}(A) \otimes \mathcal{M}(B)$ in $\mathcal{M}(A \otimes B)$. Standard approximation arguments show that $g \to w_g^i$ is strictly continuous, and if $t \in N_i$ then it is easy to check that

$$\overline{\pi_t \otimes \rho_t}(j(u_g^i \otimes v_g^i)_I) = \bar{\pi}_t(u_g^i) \otimes \bar{\rho}_t(v_g^i) \in B(H(\pi_t) \otimes H(\rho_t)).$$

From this it follows easily that $g \to \overline{\pi_t \otimes \rho_t}(w_g^i)$ is a representation of G, and straightforward calculations show that w_g^i implements $\alpha_g \otimes_{C(T)} \beta_g$ in the representation $\pi_t \otimes \rho_t$. This establishes the claim, and we have proved in particular that $\alpha \otimes_{C(T)} \beta$ is locally unitary. The cohomology classes $\zeta_A(\alpha)$, $\zeta_B(\beta)$ are represented by the cocycles

$\{N_i, \gamma_{ij}\}$ and $\{N_i, \chi_{ij}\}$ where

$$\bar{\pi}_t(u_g^j) = \gamma_{ij}(t)(g)\bar{\pi}_t(u_g^i), \quad \bar{\rho}_t(v_g^j) = \chi_{ij}(t)(g)\bar{\rho}_t(v_g^i)$$

for $t \in N_{ij}$ and $g \in G$. Since

$$\overline{\pi_t \otimes \rho_t}(w_g^j) = \bar{\pi}_t(u_g^j) \otimes \bar{\rho}_t(v_g^j) = \gamma_{ij}(t)(g)\chi_{ij}(t)(g)[\bar{\pi}_t(u_g^i) \otimes \bar{\rho}_t(v_g^i)]$$

we deduce that the product cocycle represents $\zeta(\alpha \otimes_{C(T)} \beta)$, and the proposition is proved. ▨

LEMMA 3.11. *Let A be as above and suppose $\psi: A \to D$ is an isomorphism. Define* $\operatorname{Aut}\psi: \mathrm{LU}(G, A) \to \mathrm{LU}(G, D)$ *by* $\operatorname{Aut}\psi(\alpha)_g = \psi \circ \alpha_g \circ \psi^{-1}$. *Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{LU}(G, A) & \xrightarrow{\zeta_A} & H^1(\hat{A}, \hat{\mathscr{G}}) \\ {\scriptstyle \operatorname{Aut}\psi}\downarrow & & \downarrow{\scriptstyle (\hat{\psi})^*} \\ \mathrm{LU}(G, D) & \xrightarrow{\zeta_D} & H^1(\hat{D}, \hat{\mathscr{G}}). \end{array}$$

Proof. If $\alpha \in \mathrm{LU}(G, A)$ is implemented by $u^i: G \to \mathcal{M}(A)$ over N_i, then $\operatorname{Aut}\psi(\alpha)$ is implemented over $\hat{\psi}^{-1}(N_i)$ by $\mathcal{M}(\psi) \circ u^i: G \to \mathcal{M}(D)$. Note that if $\pi \in \hat{A}$ then $\overline{\hat{\psi}^{-1}(\pi)} \circ \mathcal{M}(\psi) = \bar{\pi}$. Thus if

$$\bar{\pi}_t(u_g^j) = \gamma_{ij}(t)(g)\bar{\pi}_t(u_g^i) \quad \text{for } t \in N_{ij},$$

then for $\pi_{\hat{\psi}^{-1}(t)} = \pi_t \circ \psi^{-1} \in \hat{\psi}^{-1}(N_{ij})$ we have

$$\overline{\pi_t \circ \psi^{-1}}(\mathcal{M}(\psi)u_g^j) = \bar{\pi}_t(u_g^j) = \gamma_{ij}(t)(g)\,\bar{\pi}_t(u_g^i) =$$

$$= \gamma_{ij}(\hat{\psi}(\hat{\psi}^{-1}(t)))(g)\overline{\pi_t \circ \psi^{-1}}(\mathcal{M}(\psi)u_g^i),$$

and $\zeta_D(\operatorname{Aut}\psi(\alpha))$ is represented by $\{\hat{\psi}^{-1}(N_i), \gamma_{ij} \circ \hat{\psi}\}$. But this is exactly what the induced map in Čech cohomology does to cocycles. ▨

Proof of Theorem 3.8. Because A is stable, Lemma 3.9 implies that there is an isomorphism ψ of $A \otimes_{C(T)} C_\infty(T, K(H))$ onto A. It follows from Lemma 3.11 that to prove ζ_A surjective it is enough to prove $\zeta_{A \otimes_{C(T)} C_\infty(T, K(H))}$ surjective. But if we denote by $\mathrm{id}: G \to \operatorname{Aut} A$ the trivial automorphism group, then Lemma 3.10 shows that for any $\alpha \in \mathrm{LU}(G, C_\infty(T, K(H)))$ we have

$$\zeta_{A \otimes_{C(T)} C_\infty(T, K(H))}(\mathrm{id} \otimes_{C(T)} \alpha) = \zeta_A(\mathrm{id})\,\zeta_{C_\infty(T, K(H))}(\alpha) = \zeta_{C_\infty(T, K(H))}(\alpha).$$

Since Corollary 3.6 implies that every class in $H^1(T, \hat{\mathscr{G}})$ has this latter form, we conclude that ζ_A is surjective. This proves the theorem. ▨

Let A be a C^*-algebra with paracompact spectrum. An automorphism group $\alpha: \mathbf{Z} \to \operatorname{Aut} A$ is determined by its generator α_1, and α is locally unitary precisely when α_1 is locally implemented by multipliers; in this case we say α_1 is locally inner and write $\alpha_1 \in \operatorname{Loc}\operatorname{Inn} A$. The map $\alpha \to \zeta_A(\alpha)$ of Section 2 now gives a homomorphism η from $\operatorname{Loc}\operatorname{Inn} A$ into $H^2(T, \mathbf{Z}) \cong H^1(T, \mathscr{S})$, and η has as kernel the group $\operatorname{Inn} A$ of automorphisms implemented by multipliers. Theorem 3.8 therefore completes the following generalisation of Theorem 2.1 of [13]:

COROLLARY 3.12. *Let A be a C^*-algebra with paracompact spectrum T. Then there is an exact sequence*

$$0 \to \operatorname{Inn} A \to \operatorname{Loc}\operatorname{Inn} A \xrightarrow{\eta} H^2(T, \mathbf{Z}).$$

If A is stable, then η is surjective.

To see that this includes Theorem 2.1 of [13], we only have to observe that if A has continuous trace then the locally inner automorphisms are the π-inner automorphisms by Theorem 3.4 of [17], and hence the $C(T)$-module automorphisms by Corollary 1.9 of [13].

4. LOCALLY UNITARY ACTIONS ON CONTINUOUS TRACE C^*-ALGEBRAS

We consider here the structure of the crossed product $A \times_\alpha G$ when A has continuous trace and α is locally unitary. We prove that $A \times_\alpha G$ has continuous trace, and we identify the Dixmier-Douady class $\delta(A \times_\alpha G)$ as the pull-back of $\delta(A)$ via the restriction map $p: (A \times_\alpha G)^\wedge \to \hat{A}$ (for the definition of $\delta(A)$, see [3, Section 10]). This enables us to decide when a given stable continuous trace C^*-algebra whose spectrum is a principal bundle can be decomposed as a crossed product.

PROPOSITION 4.1. *Let A be a continuous trace C^*-algebra, G a locally compact abelian group, and $\alpha: G \to \operatorname{Aut} A$ a locally unitary automorphism group. Let $p: (A \times_\alpha G)^\wedge \to \hat{A}$ be the restriction map. Then $A \times_\alpha G$ has continuous trace and the induced homomorphism p^* on third Čech cohomology satisfies $p^*(\delta(A)) = \delta(A \times_\alpha G)$.*

Proof. We realize A as the algebra of sections of a field of elementary C^*-algebras over $T = \hat{A}$ and use the notation of [3, Section 10]. Let $\{T_i\}$ be an open cover of T such that there are maps $u^i: G \to \mathcal{M}(A)$ implementing α over T_i and elements $\{p_i\}$, $\{v_{ij}\}$ of A satisfying

a) $p_i(t)$ is a rank one projection for each $t \in T_i$;

b) $v_{ij}(t)$ is a partial isometry for each $t \in T_{ij}$ and

$$v_{ij}(t)v_{ij}(t)^* = p_i(t), \quad v_{ij}(t)^* v_{ij}(t) = p_j(t).$$

Then on the intersection T_{ijk} the partial isometries $v_{ij}v_{jk}$ and v_{ik} have the same initial and range projections, so there are continuous maps $\mu_{ijk}\colon T_{ijk} \to S^1$ such that $v_{ij}v_{jk} = \mu_{ijk}v_{ik}$. The cocycle $\{T_i, \mu_{ijk}\}$ represents an element $\gamma(A)$ of $H^2(T, \mathscr{S})$ whose image $\delta(A)$ in $H^3(T, \mathbf{Z})$ is the Dixmier-Douady class of A. It is independent of any of the choices we have made. (See [14, Section 2] for this description of $\delta(A)$.)

By Theorem 2.2 the map $h_i\colon (t, \gamma) \to t \times \gamma u^i(t)$ is a homeomorphism of $T_i \times \hat{G}$ onto $p^{-1}(T_i)$. Let $\{Y_s\}$ be an open cover of $\hat{G}$ by relatively compact sets, and fix functions $f_s \in L^1(G)$ such that $\hat{f}_s \geqslant 0$ and $\hat{f}_s = 1$ on Y_s (this is possible by [16, Theorem 2.6.1]). For each pair of indices (i, s) we define a function $f_{si} \in L^1(G, A)$ by

$$f_{si}(g) = p_i f_s(g)(u_g^i)^* \quad \text{for } g \in G.$$

Simple calculations show that if $(t, \gamma) \in T_i \times Y_s$ then

$$h_i(t, \gamma)(f_{si}) = (t \times \gamma u^i(t))(f_{si}) = p_i(t)\hat{f}_s(\gamma) = p_i(t),$$

and in particular $h_i(t, \gamma)(f_{si})$ is a rank one projection. Thus $A \times_\alpha G$ satisfies Fell's condition, and since $(A \times_\alpha G)^\wedge$, being a principal bundle over a Hausdorff space, is Hausdorff, this is enough to show that $A \times_\alpha G$ has continuous trace [3, 4.5.4]. We now let $N_{si} = h_i(T_i \times Y_s)$, and define $v_{(si)(rj)} \in L^1(G, A)$ by

$$v_{(si)(rj)}(g) = v_{ij} f_s(g)(u_g^i)^* \quad (g \in G).$$

As above, if $(t, \gamma) \in T_i \times Y_s$

$$h_i(t, \gamma)v_{(si)(rj)} = v_{ij}(t)\hat{f}_s(\gamma) = v_{ij}(t),$$

so that on $N_{si} \cap N_{rj}$ the elements $v_{(si)(rj)}$ are partial isometries. (Note that if we chose to write $h_i(t, \gamma)$ in the form $h_j(t, \chi)$ for $\chi \in Y_r$, then the irreducibility of t implies that $\chi u^j = \gamma u^i$, so the same answer would result.) It is now clear that on $N_{si} \cap N_{rj}$ we have

$$h_j(t, \gamma)(v_{(si)(rj)}v^*_{(si)(rj)}) = p_i(t) = h_i(t, \gamma)f_{si}$$

$$h_i(t, \gamma)(v^*_{(si)(rj)}v_{(si)(rj)}) = v_{ij}(t)^* v_{ij}(t) = p_j(t).$$

Since $h_i(t, \gamma)$ is also in N_{rj} it can be written as $h_j(t, \chi)$, and then it follows that

$$h_i(t, \gamma)(f_{rj}) = h_j(t, \chi)(f_{rj}) = p_j(t).$$

Thus if we define $\lambda_{(si)(rj)(qk)}\colon N_{si} \cap N_{rj} \cap N_{qk} \to S^1$ by

$$v_{(si)(rj)}v_{(rj)(qk)} = \lambda_{(si)(rj)(qk)}v_{(si)(qk)},$$

then the resulting cocycle represents the class $\delta(A\times_\alpha G)$ in $H^3((A\times_\alpha G)^\wedge, \mathbf{Z})$. However, if $\pi\in(A\times_\alpha G)^\wedge$ belongs to the triple intersection, then putting π in the forms $h_i(t,\gamma)$ and $h_j(t,\chi)$ and applying it to the defining equation for λ shows that

$$v_{ij}(t)v_{jk}(t) = \lambda_{(si)(rj)(qk)}(\pi)v_{ik}(t).$$

Thus $\lambda_{(si)(rj)(qk)} = \mu_{ijk}\circ p$. This proves that $p^*(\delta(A)) = \delta(A\times_\alpha G)$, and the result follows at once. ▨

LEMMA 4.2. *Let A and B be C^*-algebras, G a locally compact group, and $\alpha: G\to \operatorname{Aut} A$ a strongly continuous automorphism group. Then the map Φ of $C_c(G,A)\odot B$ into $C_c(G, A\odot B)$ defined by*

$$\Phi\Big(\sum_i f_i\otimes b_i\Big)(g) = \sum_i f_i(g)\otimes b_i$$

extends to an isomorphism of $(A\times_\alpha G)\otimes_{\max} B$ onto $(A\otimes_{\max} B)\times_{\alpha\otimes \mathrm{id}} G$.

Proof. Straightforward: it is a special case of [18, Proposition 2.4]. ▨

LEMMA 4.3. *Let A be a stable C^*-algebra with spectrum T. Then there is an isomorphism $\varphi: A\to A\otimes K(H)$ such that the induced homeomorphism $\hat\varphi: T = (A\otimes K(H))^\wedge \to T$ is the identity.*

Proof. Let $\rho: A\to A\otimes K(H)$ be any isomorphism and let $\psi: K(H)\to K(H)\otimes K(H)$ be any isomorphism. We then define $\varphi: A\to A\otimes K(H)$ to be the composition:

$$A\xrightarrow{\rho} A\otimes K(H)\xrightarrow{\mathrm{id}\otimes\psi} A\otimes(K(H)\otimes K(H))\cong(A\otimes K(H))\otimes K(H)\xrightarrow{\rho^{-1}\otimes\mathrm{id}} A\otimes K(H).$$

It is not hard to see that $\hat\varphi: T = (A\otimes K(H))^\wedge\to T$ is the identity. ▨

THEOREM 4.4. *Let A be a separable stable continuous trace C^*-algebra and G a separable locally compact abelian group. Suppose that G acts on $\hat A$ in such a way that $q:\hat A\to \hat A/G$ is a locally trivial principal G-bundle. Then the following are equivalent:*

a) *There is a stable continuous trace C^*-algebra B and a locally unitary automorphism group $\alpha:\hat G\to \operatorname{Aut} B$ such that $A\cong B\times_\alpha \hat G$ and $p:(B\times_\alpha\hat G)^\wedge\to\hat B$ is G-isomorphic to $q:\hat A\to\hat A/G$.*

b) *The Dixmier-Douady class $\delta(A)$ belongs to the range of the induced homomorphism*

$$q^*: H^3(\hat A/G, \mathbf{Z})\to H^3(\hat A,\mathbf{Z}).$$

Proof. (a) $\Rightarrow$ (b) follows from Proposition 4.1. Now suppose that $\delta(A)$ belongs to the range of q^*. The Dixmier-Douady theorem [3, 10.8.4] asserts that δ gives a bijection between $H^3(T,\mathbf{Z})$ and (isomorphism classes of) locally trivial fields of elementary C^*-algebras over T. Since the section algebras of these are precisely the separable stable continuous trace C^*-algebras with spectrum T [13, Proposition 1.12], we deduce that δ classifies these. In particular, there is such an algebra B with spectrum $\hat{A}/G$ and $q^*(\delta(B)) = \delta(A)$. According to Theorem 3.8, there is a locally unitary action $\alpha\colon \hat{G} \to \operatorname{Aut} B$ and a G-equivariant homeomorphism φ such that

$$\begin{array}{ccc} (B \times_\alpha \hat{G})^\wedge & \xrightarrow{\ \varphi\ } & \hat{A} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \hat{B} = \hat{A}/G & \xrightarrow{\text{identity}} & \hat{A}/G \end{array}$$

commutes. Then by Proposition 4.1 we have

$$\delta(B \times_\alpha \hat{G}) = p^*(\delta(B)) = (q \circ \varphi)^*(\delta(B)) = \varphi^*(\delta(A)).$$

If we can prove that $B \times_\alpha G$ is stable, then, by regarding $B \times_\alpha G$ as the section algebra of a field over A and applying the Dixmier-Douady theorem, we can deduce that $B \times_\alpha G \cong A$.

So it remains for us to show that $B \times_\alpha G$ is stable. Lemma 3.3 shows that there is an isomorphism $\varphi\colon B \otimes K(H) \to B$ such that $\hat{\varphi}$ is the identity, and Lemma 4.2 shows that

$$(B \times_\alpha \hat{G}) \otimes K(H) \cong (B \otimes K(H)) \times_{\alpha \otimes \mathrm{id}} \hat{G}.$$

We denote by β the automorphism group $\operatorname{Aut}\varphi(\alpha \otimes \mathrm{id})$ of B; since $\hat{\varphi}$, and hence also $(\hat{\varphi})^*$, is the identity, Lemma 3.11 implies that $\zeta_B(\beta) = \zeta_{B \otimes K(H)}(\alpha \otimes \mathrm{id})$. Now it is quite easy to see that the isomorphism of Lemma 3.9 carries the automorphism group $\alpha \otimes \mathrm{id}$ of $B \otimes K(H)$ into the group $\alpha \otimes_{C(T)} \mathrm{id}$ acting on $B \otimes_{C(T)} C_\infty(T, K(H))$, and induces the identity map on spectra. Thus another application of Lemma 3.11 and one of Theorem 3.10 give

$$\zeta_B(\beta) = \zeta_{B \otimes K(H)}(\alpha \otimes \mathrm{id}) = \zeta_{B \otimes_{C(T)} C_\infty(T, K(H))}(\alpha \otimes_{C(T)} \mathrm{id}) = \zeta_B(\alpha).$$

By Proposition 2.5 the groups α and β are exterior equivalent, and so $B \times_\beta \hat{G}$ is isomorphic to $B \times_\alpha \hat{G}$ (see the first part of the proof of Proposition 2.5). Thus we have

$$B \times_\alpha \hat{G} \cong B \times_\beta \hat{G} \cong (B \otimes K(H)) \times_{\alpha \otimes \mathrm{id}} \hat{G} \cong (B \times_\alpha \hat{G}) \otimes K(H),$$

and the theorem follows. ▨

REMARK. We observe that if the map q^* is not injective then there may be several possible choices for the C^*-algebra B; the proof shows that for each of these there is a locally unitary automorphism group $\alpha : \hat{G} \to \mathrm{Aut}\, B$ such that $A \cong B \times_\alpha \hat{G}$. However, once we have fixed an algebra B with $q^*(\delta(B)) = \delta(A)$ the group α is unique up to exterior equivalence by Proposition 2.5.

EXAMPLE. Let $q: S^3 \to S^2$ be the Hopf fibration, so that the fibres of q are the orbits of the action of S^1 on S^3 defined by

$$\lambda \cdot (z, w) = (\lambda z, \lambda w) \quad \text{for } |\lambda| = 1 \text{ and } |z|^2 + |w|^2 = 1.$$

Recall that $H^3(S^2, \mathbf{Z}) = 0$ and $H^3(S^3, \mathbf{Z}) = \mathbf{Z}$, so the range of the map q^* is 0. The only separable stable continuous trace C^*-algebra A with spectrum S^3 and vanishing Dixmier-Douady class is $C(S^3, K(H))$, and so Theorem 4.4 gives us a decomposition

$$C(S^3, K(H)) \cong C(S^2, K(H)) \times_\alpha \mathbf{Z}.$$

The automorphism which generates the group α is unique up to perturbation by inner automorphisms; it is certainly not inner, since then the crossed product would have spectrum $S^2 \times S^1$. No other stable continuous trace C^*-algebra with spectrum S^3 can be so decomposed relative to the Hopf fibration.

REFERENCES

1. BASS, H., *Algebraic K-theory*, Benjamin, New York, 1968.
2. DIXMIER, J., Champs continus d'espaces hilbertiens et de C^*-algèbres. II, *J. Math. Pures Appl.*, **42**(1963), 1–20.
3. DIXMIER, J., *C^*-algebras*, North-Holland, Amsterdam, 1977.
4. DIXMIER, J.; DOUADY, A., Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. France*, **91**(1963), 227–284.
5. ELLIOTT, G. A., Ideal-preserving automorphisms of postliminary C^*-algebras, *Proc. Amer. Math. Soc.*, **27**(1971), 107–109.
6. GREEN, P., C^*-algebras of transformation groups with smooth orbit space, *Pacific J. Math.*, **72**(1977), 71–97.
7. GREEN, P., The local structure of twisted covariance algebras, *Acta Math.*, **140**(1978), 191–250.
8. HILTON, P. J.; WYLIE, S., *Homology theory*, Cambridge University Press, 1960.
9. HUSEMOLLER, D., *Fibre bundles* (second edition), Springer-Verlag, New York, 1975.
10. KADISON, R. V.; RINGROSE, J. R., Derivations and automorphisms of operator algebras, *Comm. Math. Phys.*, **4**(1967), 32–63.
11. PAULSEN, V., Continuous canonical forms for matrices under unitary equivalence, *Pacific J. Math.*, **76**(1978), 129–142.
12. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.
13. PHILLIPS, J.; RAEBURN, I., Automorphisms of C^*-algebras and second Čech cohomology, *Indiana Univ. Math. J.*, **29**(1980), 799–822.
14. PHILLIPS, J.; RAEBURN, I., Perturbations of C^*-algebras. II, *Proc. London Math. Soc.* (3), **43**(1981), 46–72.

15. PHILLIPS, J.; RAEBURN, I.; TAYLOR, J. L., Automorphisms of certain C^*-algebras and torsion in second Čech cohomology, *Bull. London Math. Soc.*, **14**(1982), 33–38.
16. RUDIN, W., *Fourier analysis on groups*, Interscience. New York, 1962.
17. RUSSELL, M. J., Automorphisms and derivations of continuous trace C^*-algebras, *J. London Math. Soc.* (2), **22**(1980), 139–145.
18. TAKAI, H., On a duality for crossed products of C^*-algebras, *J. Functional Analysis*, **19**(1975), 25–39.
19. TAKESAKI, M., Covariant representations of C^*-algebras and their locally compact automorphism groups, *Acta Math.*, **119**(1967), 273–303.
20. TAKESAKI, M., *Theory of operator algebras*. I, Springer-Verlag, New York, 1979.
21. HERMAN, R. H.; ROSENBERG, J., Norm-close group actions on C^*-algebras, *J. Operator Theory*, **6**(1981), 25–37.

JOHN PHILLIPS and IAIN RAEBURN
School of Mathematics,
University of New South Wales, P.O. Box 1,
Kensington, 2033, N.S.W.
Australia.

Usual address of J. PHILLIPS:
Department of Mathematics,
Dalhousie University, Halifax,
Nova Scotia B3H 4H8,
Canada.

Received December 24, 1981; revised July 18, 1983.