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A COMPLETE TREATMENT OF LOW-ENERGY SCATTERING IN ONE DIMENSION

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1. INTRODUCTION

The purpose of this paper is to provide a systematic analysis of low-energy scattering on the entire real line, taking into account explicitly the possibility of zero-energy resonances of the Hamiltonian. Such an analysis has been carried out very recently for three dimensions [1], [2], [13]. In one (and two) dimensions, this problem is more involved due to the well-known additional difficulty that the free Green's function has a square root (logarithmic) singularity in the limit as the energy tends to zero.

Different aspects of the one-dimensional scattering problem have received much attention in the past, especially in connection with inverse scattering techniques, which are used extensively in quantum mechanical problems (cf. [14], [16], [34] and the references therein), and quantum field theory (cf. [17], [46] for a review). More recently, new rigorous results have appeared ([3], [7], [15], [19], [20], [26—30], [33], [36], [38], [39], [41], [47]). In particular, one has studied the ground-state properties of one-dimensional Schrödinger operators with various potentials, including long-range ones, especially in the limit of weak coupling [7], [26], [27], [38], [39], [41]. Also bounds for the number of bound states [26], [27], [36] as well as for the imaginary parts of resonances [20] have been obtained. Other results are concerned with the limit situation where some negative eigenvalues approach zero as the coupling constant approaches a "critical value" [29], [30]. Furthermore, scaling techniques have been applied to analyse in detail the limit of one-dimensional short-range interactions converging to point interactions [3]. We remark that the latter paper contains an extensive list of earlier one-dimensional results which are not explicitly mentioned here.

Let us now give a short description of the results obtained in this paper. In Section 2 we study the occurrence and properties of zero-energy resonances of the one-dimensional Schrödinger Hamiltonian H. This leads to a classification of essentially two cases i.e. H has no zero-energy resonances (= generic case) and H has a zero-energy resonance with multiplicity one. This study is based upon and extends some of the results of Klaus [29] and Klaus and Simon [30] in the sense that the class of potentials V can be extended from $C_0^\infty(\mathbf{R})$ to those satisfying $(1+|x|^2)V \in L^1(\mathbf{R})$.

Section 3 describes in detail the low-energy behavior of the transition operator $T(k)$, assuming roughly exponential fall off for V at infinity. It establishes recursion relations for the coefficients in the Taylor expansion for $T(k)$ (generic case) or the Laurent expansion for $T(k)$ (other cases). Analogous results for the resolvent and the evolution group of general elliptic differential operators have been obtained by Murata [33] (thereby extending the work of Jensen [22], [23] and Jensen and Kato [24]).

In Section 4 we present Taylor expansions for the reflection and transmission coefficients. In the generic case, we thus obtain results that are more detailed than the ones available in the literature (cf. e.g. [14], [16], [34]). For the other cases, the results are new.

Section 5 derives two sets of trace relations involving the continuous spectrum, i.e. negative energy-moments of the trace of the difference between the full and free resolvent, on one side, and the point spectrum, i.e. negative-energy bound states and zero-energy resonances, on the other side. Such trace relations for positive-energy moments were initially introduced by Gelfand and Levitan [18]. (For a list of further references we refer to [11], [13].) As a special case of these relations we obtain Levinson's theorem for scattering on the line. We find that its structure completely changes in comparison with three dimensions.

If one is interested in asymptotic expansions of the scattering parameters instead of analytic ones, Section 6 briefly indicates how the exponential fall off conditions can be relaxed.

A brief outline of this analysis has appeared before [10]. A similar analysis for two dimensions which is technically more complicated because of the logarithmic nature of the free Green's function singularity and the possible existence of zero-energy bound states besides zero-energy resonances, is in preparation [9].

2. ZERO-ENERGY PROPERTIES OF H

In this section we study the one-dimensional Schrödinger operator, allowing the possible occurrence of zero-energy resonances.

We define the Schrödinger Hamiltonian H in $L^2(\mathbf{R})$ as the form sum

$$
\begin{aligned}
&H := H_0 \dotplus \lambda_0 V, \quad \lambda_0 \in \mathbf{R}\setminus\{0\}, \\
(2.1)\qquad &H_0 = -\frac{d^2}{dx^2} \quad \text{on} \quad \mathcal{D}(H_0) := H^{2,2}(\mathbf{R}).
\end{aligned}
$$

Throughout this paper we assume the potential $V(x)$ to be real and satisfy

(2.2) $$\int_{\mathbf{R}} dx(1 + |x|^2)|V(x)| < \infty, \quad \int_{\mathbf{R}} dx V(x) \neq 0.$$

Introducing

(2.3) $$v(x) = |V(x)|^{1/2}, \quad u(x) = |V(x)|^{1/2} \operatorname{sign} V(x), \quad u \cdot v = V,$$

the transition operator $T(k)$ in $L^2(\mathbf{R})$ is defined as

(2.4) $$T(k) = (1 + \lambda_0 u R_0(k) v)^{-1}, \quad \operatorname{Im} k \geqslant 0,\ k \neq 0,\ k^2 \notin \Sigma_p(H),$$

where $R_0(k)$ denotes the free resolvent

(2.5) $$R_0(k) = (H_0 - k^2)^{-1}, \quad \operatorname{Im} k > 0$$

with kernel

(2.6) $$R_0(k, x, y) = (i/2k)e^{ik|x-y|}.$$

In order to isolate the first order pole in $R_0(k)$ as $k \to 0$ we follow ref. [41] and decompose

(2.7) $$u R_0(k) v = (i/2k)(v, \cdot)u + M(k), \quad \operatorname{Im} k \geqslant 0,\ k \neq 0,$$

where $M(k) \in \mathscr{B}_2(L^2(\mathbf{R}))$ for all $\operatorname{Im} k \geqslant 0$. If

(2.8) $$\int_{\mathbf{R}} dx\, e^{a|x|}|V(x)| < \infty \quad \text{for some } a > 0,$$

then $M(k)$ is analytic with respect to k in the region $\operatorname{Im} k > -a/2$, and

(2.9) $$M(k) = \sum_{n=0}^{\infty} (ik)^n M_n,$$

where the M_n are Hilbert-Schmidt operators with kernels

(2.10) $$M_n(x, y) = -2^{-1} u(x) \frac{|x - y|^{n+1}}{(n + 1)!} v(y), \quad n = 0, 1, 2 \ldots .$$

The expansion (2.9) converges in $\mathscr{B}_2$-norm. Defining

(2.11) $$P = (v, u)^{-1} (v, \cdot)u, \quad Q = 1 - P,$$

we thus obtain for $T(k)$

(2.12) $$T(k) = [1 + (i\lambda_0(v, u)/2k)P + \lambda_0 M(k)]^{-1}, \quad \operatorname{Im} k \geqslant 0,\ k \neq 0,\ k^2 \in \Sigma_p(H).$$

Obviously the low-energy behavior of $T(k)$ strongly depends on the zero-energy behavior of H. As a first step in our analysis of possible zero-energy resonances of H, we state a slightly extended version of Lemma 7.3 of [30].

LEMMA 2.1. *Let V satisfy condition* (2.2). *Assume that* -1 *is an eigenvalue of* $\lambda_0 Q M_0 Q$ *and let*

$$\mathscr{V} = \{\varphi \in L^2(\mathbf{R}) \mid \lambda_0 Q M_0 Q \varphi = -\varphi\},$$

$$\mathscr{W} = \{\chi \in \mathscr{V} \mid (v, (z - \lambda_0 M_0)^{-1}\chi) = 0 \textit{ for some } |z| > \|\lambda_0 M_0\|\}.$$

Then

(i) $\mathscr{W}$ *is independent of* z.

(ii) $\dim \mathscr{W} = \dim \mathscr{V}$ *or* $\dim \mathscr{W} = \dim \mathscr{V} - 1$.

(iii) $(z - \sigma P - \lambda_0 M_0)^{-1}\chi = (z - \lambda_0 M_0)^{-1}\chi$ *for all* $\chi \in \mathscr{W}$, $\sigma \in \mathbf{C}$.

(iv) $\lambda_0 M_0 \chi = -\chi$ *for all* $\chi \in \mathscr{W}$.

(v) *If* $\varphi_0 \in \mathscr{V} \setminus \mathscr{W}$ *then*

$$\lambda_0 M_0 \varphi_0 = -\varphi_0 + (v, u)^{-1}\lambda_0 (v, M_0 \varphi_0) u.$$

(vi) *If* $\chi \in \mathscr{V}$ *then* $(v, (z - \lambda_0 M_0)^{-1}\chi) = 0$ *is equivalent to* $(v, M_0 \chi) = 0$. *Consequently*

$$\mathscr{W} = \{\chi \in \mathscr{V} \mid (v, M_0 \chi) = 0\}.$$

Proof. Let $\sigma \in \mathbf{C}$, then for $|z|$ large enough (compared to $|\sigma|$ and $\|\lambda_0 M_0\|$) one derives

$$(z - \sigma P - \lambda_0 M_0)^{-1} = (z - \lambda_0 M_0)^{-1} + \tag{2.13}$$
$$+ \frac{\sigma}{1 - \sigma(v, (z - \lambda_0 M_0)^{-1}u)/(v, u)} (z - \lambda_0 M_0)^{-1} P (z - \lambda_0 M_0)^{-1}.$$

Since the spectrum of $\lambda_0 M_0$ is a compact subset of the real line (which follows from $(\operatorname{sign} V)\ M_0 = M_0^*(\operatorname{sign} V)$ cf. e.g. [2], [30]), equality (2.13) extends to all $z \notin \Sigma(\lambda_0 M_0) \cup \{z \mid (v, u) = \sigma(v, (z - \lambda_0 M_0)^{-1}u)\}$. Expanding (2.13) with respect to σ finally yields [30]

$$(z - \sigma P - \lambda_0 M_0)^{-1} = Q(z - \lambda_0 Q M_0 Q)^{-1} Q - \tag{2.14}$$
$$- \sigma^{-1}(v, (z - \lambda_0 M_0)^{-1}u)^{-2}(v, u)^2 (z - \lambda_0 M_0)^{-1} P (z - \lambda_0 M_0)^{-1} + O(\sigma^{-2})$$

for $|\sigma|$ large enough.

Since $\chi \in \mathscr{W}$ is characterized by the fact that $\chi \in \mathscr{V}$ and $((\bar{z} - \lambda_0 M_0^*)^{-1}v, \chi) = 0$, $\dim \mathscr{W} = \dim \mathscr{V}$ or $\dim \mathscr{W} = \dim \mathscr{V} - 1$ which proves (ii). Since $P(z - \lambda_0 M_0)^{-1}\chi =$

$= 0$ for all $\chi \in \mathscr{W}$ (iii) directly follows from (2.13). Taking the limit $\sigma \to \infty$ in (2.14) we obtain from (iii)

$$\operatorname*{s-lim}_{\sigma\to\infty} (z - \sigma P - \lambda_0 M_0)^{-1}\chi = Q(z - \lambda_0 Q M_0 Q)^{-1} Q\chi =$$
$$= (z+1)^{-1} Q\chi = (z - \lambda_0 M_0)^{-1}\chi, \quad \chi \in \mathscr{W}.$$

Noting that $P\chi = 0$, $Q\chi = \chi$ we get (iv). If $\varphi_0 \in \mathscr{V} \setminus \mathscr{W}$ and $\eta_0 = \lambda_0 M_0 \varphi_0 + \varphi_0$ we have that $Q\eta_0 = 0$ and $\eta_0 = (v, u)^{-1}(v, \eta_0)u$. Thus

$$(v, \eta_0) = \lambda_0(v, M_0\varphi_0) + (v, \varphi_0) = \lambda_0(v, M_0\varphi_0)$$

which proves (v), (vi) and (i). ▨

REMARK 2.1. Lemma 2.1 (i) — (iv) coincides with Lemma 7.3 of [30], where it was used in the context of coupling constant thresholds (cf. [26], [27], [29], [30], [32], [38], [39], [44]) for Schrödinger Hamiltonians in two dimensions. (We have reproduced here a full proof for the convenience of the reader.)

Next we give

LEMMA 2.2. *Let V satisfy condition* (2.2). *Assume that*

$$\lambda_0 Q M_0 Q\varphi = -\varphi, \quad \varphi \in L^2(\mathbf{R}),$$

and define the zero-energy resonance function ψ by

$$\psi(x) = -(v, u)^{-1}\lambda_0(v, M_0\varphi) - 2^{-1}\lambda_0 \int_{\mathbf{R}} dy\,|x - y|\,v(y)\varphi(y). \tag{2.15}$$

Then

(i) $\psi \in L^\infty(\mathbf{R})$ *and* $H\psi = 0$ *in the sense of distributions.*

(ii) $\psi \notin L^2(\mathbf{R})$.

(iii) $u(x)\psi(x) = -\varphi(x)$ a.e..

(iv) $\psi + (v, u)^{-1}\lambda_0(v, M_0\varphi) - 2^{-1}\lambda_0 \operatorname{sign}(\cdot)((\cdot)v, \varphi) \in L^2(\mathbf{R})$,

$$\psi(\pm\infty) = -(v, u)^{-1}\lambda_0(v, M_0\varphi) \pm 2^{-1}\lambda_0((\cdot)v, \varphi).$$

(v) ψ *is unique and thus the nonzero eigenvalues of $\lambda_0 Q M_0 Q$ are simple.*

Proof. Obviously $\psi \in L^\infty_{\mathrm{loc}}(\mathbf{R})$. From $(v, \varphi) = 0$ we infer

$$\psi(x) = -(v, u)^{-1}\lambda_0(v, M_0\varphi) + 2^{-1}\lambda_0 \frac{x}{|x|}((\cdot)v, \varphi) - \begin{cases} \lambda_0 \int_{-\infty}^{x} dy(x - y)v(y)\varphi(y), & x < 0, \\ \lambda_0 \int_{x}^{\infty} dy(y - x)v(y)\varphi(y), & x > 0. \end{cases} \tag{2.16}$$

Taking e.g. $x > 0$ the estimate

$$\left|\lambda_0 \int_x^\infty dy(y-x)v(y)\varphi(y)\right| \leqslant 2|\lambda_0| \int_x^\infty dy\, y v(y)\, |\varphi(y)| \leqslant 2|\lambda_0|\, \|\varphi\|_2 \left(\int_x^\infty dy\, y^2 |V(y)|\right)^{1/2}$$

proves that

$$\psi(+\infty) = -(v, u)^{-1}\lambda_0(v, M_0\varphi) + 2^{-1}\lambda_0((\cdot)v, \varphi).$$

A similar argument holds for $\psi(-\infty)$. Thus $\psi \in L^\infty(\mathbf{R})$. Moreover, multiplying (2.15) with $u(x)$ yields (iii) by Lemma 2.1 (v), equality (2.10), and the compactness of M_0. Since by (2.15) ψ is locally absolutely continuous we get

$$\psi'(x) = \lambda_0 \int_x^\infty dy\, v(y)\varphi(y). \tag{2.17}$$

This shows that ψ' is also locally absolutely continuous. Differentiating (2.17) once again we finally obtain

$$\psi''(x) = -\lambda_0 v(x)\varphi(x) = \lambda_0 V(x)\psi(x) \quad \text{a.e.} \tag{2.18}$$

This proves statement (i) of the lemma.

Next, we define

$$\tilde\psi(x) = \psi(x) + (v, u)^{-1}\lambda_0(v, M_0\varphi) - 2^{-1}\lambda_0 \operatorname{sign}(x)((\cdot)v, \varphi). \tag{2.19}$$

Using equality (2.15) in (2.19) and the fact that $|x|(v, \varphi) = 0$ we arrive at (e. g. $x > 0$)

$$\tilde\psi(x) = -\lambda_0 \int_x^\infty dy\,(y-x)\, v(y)\, \varphi(y).$$

Employing (iii) and $\psi \in L^\infty(\mathbf{R})$ we obtain

$$|\tilde\psi(x)| \leqslant |\lambda_0|\, \|\psi\|_\infty\, x^{-1} \int_x^\infty dy\, y^2 |V(y)|,$$

and similarly for $x < 0$. This completes the proof of (iv).

Finally let us assume that $\psi \in L^2(\mathbf{R})$. Then, because of (iv), we have $(v, M_0\varphi) = ((\cdot)v, \varphi) = 0$, and by (iii) and (2.16) we obtain the equation

$$\psi(x) = \begin{cases} \lambda_0 \displaystyle\int_{-\infty}^x dy(x-y)V(y)\psi(y), & x < 0, \\ \lambda_0 \displaystyle\int_x^\infty dy(y-x)V(y)\psi(y), & x > 0. \end{cases} \tag{2.20}$$

Iterating (2.20) (this type of Volterra operator is quasinilpotent) yields e.g. for $x > 0$

$$|\psi(x)| \leqslant |\lambda_0| \int_x^\infty dy_1 |y_1 - x| V(y_1) \int_{y_1}^\infty dy_2 |y_2 - y_1| \, |V(y_2)| \ldots$$

$$\ldots \int_{y_{n-1}}^\infty dy_n |y_n - y_{n-1}| \, |V(y_n)| \, |\psi(y_n)| \leqslant$$

$$\leqslant |\lambda_0| \, \|\psi\|_\infty (n!)^{-1} \left[2 \int_x^\infty dy (y) \, |V(y)| \right]^n, \quad n = 1, 2, \ldots .$$

This means that $\psi = 0$ such that (ii) is proved. Uniqueness of ψ and hence (v) follows in exactly the same way from equality (2.16). ▨

The converse of Lemma 2.2 is contained in

LEMMA 2.3. *Let V satisfy condition* (2.2). *Assume that $\psi \in L^\infty(\mathbf{R})$ and $H\psi = 0$ in the sense of distributions. Define*

$$\varphi(x) = u(x) \left\{ \left(2 \int_{\mathbf{R}} dx' V(x') \right)^{-1} \lambda_0 \int_{\mathbf{R}^2} dx' dy' V(x') |x' - y'| V(y') \psi(y') \right\} -$$

(2.21)

$$- 2^{-1} \lambda_0 \int_{\mathbf{R}} dy \, u(x) |x - y| V(y) \psi(y)$$

then $\varphi \in L^2(\mathbf{R})$, $\lambda_0 Q M_0 Q \varphi = - \varphi$ and again (ii) — (v) *of Lemma* 2.2 *hold.*

Proof. Since the norm of the second term on the right-hand-side of (2.21) is bounded by $\|\psi\|_\infty^2 [\|(\cdot)^2 V\|_1 \|V\|_1^2 + 3\|V\|_1 \|(\cdot) V\|_1^2]$ we get $\varphi \in L^2(\mathbf{R})$. Next we introduce the function $\Psi(x)$ satisfying

$$\Psi(x) = - 2^{-1} (v, u)^{-1} \lambda_0 \int_{\mathbf{R}^2} dx' dy' V(x') |x' - y'| V(y') \psi(y') +$$

(2.22)

$$+ 2^{-1} \lambda_0 \int_{\mathbf{R}} dy |x - y| V(y) \psi(y)$$

such that

$$\varphi(x) = - u(x) \Psi(x) \text{ a.e. } .$$

Since Ψ is locally absolutely continuous we obtain

$$\Psi'(x) = 2^{-1}\lambda_0 \int_{-\infty}^{x} dy V(y)\psi(y) - 2^{-1}\lambda_0 \int_{x}^{\infty} dy V(y)\psi(y)$$

and differentiating once again

$$\Psi''(x) = \lambda_0 V(x)\psi(x) = \psi''(x) \text{ a.e.}.$$

Consequently

$$\Psi(x) = \psi(x) + cx + d \tag{2.23}$$

for some constants c and d. From equalities (2.17), (2.18), (2.22) and (2.23), and $\psi \in L^\infty(\mathbf{R})$ we get

$$\lim_{x\to\pm\infty} x^{-1}\Psi(x) = c = \pm\, 2^{-1}\lambda_0 \int_{\mathbf{R}} dy\, V(y)\psi(y) = 0.$$

Moreover a direct calculation shows that $P\varphi = 0$ and thus

$$(v, \varphi) = -\int_{\mathbf{R}} dx\, V(x)\Psi(x) = 0,$$

which proves $d = 0$. Thus $\Psi(x) = \psi(x)$ and after multiplying equality (2.22) with $u(x)$ we get

$$-\varphi(x) = u(x)\psi(x) = -(v, u)^{-1}\lambda_0(v, M_0\varphi)u(x) + \lambda_0(M_0\varphi)(x). \tag{2.24}$$

Applying Q on both sides of (2.24) (observing $Qu = 0$, $Q\varphi = \varphi$) finally yields

$$-\varphi = \lambda_0 Q M_0 Q \varphi.$$

From here one can follow the proof of Lemma 2.2. ▨

REMARK 2.2. Different proofs of most of the results of Lemmas 2.2 and 2.3 under the assumption $V \in C_0^\infty(\mathbf{R})$ have appeared in [29] (cf. also [16]).

With the help of Lemmas 2.1—2.3 we are able to distinguish the following cases in the zero-energy behavior of H. If the potential V obeys condition (2.2), then we have

Case I. -1 is not an eigenvalue of $\lambda_0 Q M_0 Q$ (i.e. H has no zero-energy resonance).

Case II. -1 is a simple eigenvalue of $\lambda_0 Q M_0 Q$, $\lambda_0 Q M_0 Q \varphi_0 = -\varphi_0$ for some $\varphi_0 \in L^2(\mathbf{R})$ (i.e. H has a zero-energy resonance) and

a) $c_1 = 0, \quad c_2 \neq 0$

or

b) $c_1 \neq 0, \quad c_2 = 0$

or

c) $c_1 \neq 0, \quad c_2 \neq 0$

where

(2.25) $$c_1 = (v, u)^{-1}(v, M_0\varphi_0), \quad c_2 = 2^{-1}((\cdot)v, \varphi_0).$$

Note that in the Cases II a) — c) we have $(v, \varphi_0) = 0$. Furthermore, in these cases there exists precisely one resonance function $\psi_0 \notin L^2(\mathbf{R})$ (up to multiplicative constants) given by equality (2.15). Since H has no zero-energy bound states (or equivalently $(v, M_0\varphi_0)$ and $((\cdot)v, \varphi_0)$ do not vanish simultaneously) and nonzero eigenvalues of $\lambda_0 QM_0Q$ are simple, the above list of cases is complete. It is trivial to realize all Cases I, II a) — c) in the example of an asymmetric square well.

3. LOW-ENERGY BEHAVIOR OF $T(k)$-RECURSION RELATIONS

We discuss in detail the low-energy behavior of the transition operator $T(k)$ for the different cases presented in Section 2. In particular we establish recursion relations for the coefficients in its Laurent series around $k = 0$.

We start with

LEMMA 3.1. *Let $\varepsilon \in C\setminus\{0\}$ small enough. Then the norm convergent expansion*

(3.1) $$(1 + \lambda_0 QM_0Q + \varepsilon)^{-1} = \frac{P_0}{\varepsilon} + \sum_{m=0}^{\infty} (-\varepsilon)^m T_0^{m+1}$$

holds. Here P_0 denotes the projection onto the (at most one-dimensional) eigenspace of $\lambda_0 QM_0Q$ to the eigenvalue -1.

$P_0 = 0$ in Case I,

$P_0 = (\tilde{\varphi}_0, \varphi_0)^{-1}(\tilde{\varphi}_0, \cdot)\varphi_0$ in Cases II a) — c),

where

$$\lambda_0 QM_0Q\varphi_0 = -\varphi_0, \quad \varphi_0 \in L^2(\mathbf{R}),\ \tilde{\varphi}_0(x) = \operatorname{sign} V(x)\varphi_0(x).$$

T_0 denotes the corresponding reduced resolvent viz.

(3.2) $$T_0 = \text{n-}\lim_{\varepsilon \to 0} (1 + \lambda_0 QM_0Q + \varepsilon)^{-1}Q_0, \quad Q_0 = 1 - P_0.$$

Proof. In principle one could follow the proof of Lemma 3.1 in [2] step by step but we prefer another argument based on [25, p. 180] (cf. also [22], Theorem 4.3). It turns out that (3.1) holds if we can show that $(\lambda_0 QM_0Q + 1)^2 g = 0$, $g \in L^2(\mathbf{R})$, implies that $(\lambda_0 QM_0Q + 1)g = 0$.

Assume $(\lambda_0 QM_0Q + 1)^2 g = 0$ and define $f = (\lambda_0 QM_0Q + 1)g$. Then $(\lambda_0 QM_0Q + 1)f = 0$ and consequently

$$(\tilde{f}, f) = ((\lambda_0 Q^* M_0^* Q^* + 1)\tilde{g}, (\lambda_0 QM_0Q + 1)g) =$$

$$= (\tilde{g}, (\lambda_0 QM_0Q + 1)^2 g) = 0,$$

where

$$\tilde{f} = (\lambda_0 Q^* M_0^* Q^* + 1)\tilde{g}, \quad \tilde{g} = (\operatorname{sign} V)g.$$

Furthermore

$$0 = -(\tilde{f}, \lambda_0 QM_0Qf) = -(\tilde{f}, \lambda_0 QuH_0^{-1}vQf) =$$

$$= -\lambda_0(H_0^{-1/2}uQ^*\tilde{f}, H_0^{-1/2}vQf) = -\lambda_0\|H_0^{-1/2}vQf\|_2^2$$

implies $vQf = 0$ and hence $f = 0$ (since $f = -\lambda_0 QuH_0^{-1}vQf$). Thus the eigenvalue -1 of $\lambda_0 QM_0Q$ has algebraic and geometric multiplicity equal to one. ☐

Next we collect some relations which turn out to be useful in the sequel:

$$PP_0 = P_0P = 0, \quad QP_0 = P_0Q = P_0,$$

$$PQ_0 = Q_0P = P, \quad QQ_0 = Q_0Q = Q_0 - P = Q - P_0,$$

$$P_0 + P + QQ_0 = 1,$$

(3.3)

$$QT_0 = T_0Q = T_0 - P, \quad PT_0 = T_0P = P, \quad PT_0Q = 0,$$

$$\lambda_0 P_0M_0P_0 = -P_0, \quad \lambda_0 P_0M_0Q_0 = (\tilde{\varphi}_0, \varphi_0)^{-1}\lambda_0 c_1^*(v, \cdot)\varphi_0,$$

$$(\tilde{\varphi}_0, M_1\varphi_0) = 2|c_2|^2.$$

We then prove

THEOREM 3.1. *Assume* $(v, u) \neq 0$ *and* $e^{a|\cdot|}V \in L^1(\mathbf{R})$ *for some* $a > 0$. *Then* $T(k)$ *has the following norm convergent Laurent (resp. Taylor) expansion around* $k = 0$

$$T(k) = \sum_{n=q}^{\infty} (ik)^n t_n \tag{3.4}$$

where $q = 0$ *in Case* I *and* $q = -1$ *in Cases* II a)—c).

Proof. Case I. From equalities (2.4) and (2.7) we get

$$T(k) = [1 + (i\lambda_0(v, u)/2k)P + \lambda_0 M(k)]^{-1} =$$

$$= \left\{1 + \lambda_0\left[1 - \frac{i\lambda_0(v, u)\, P}{2k + i\lambda_0(v, u)}\right] M(k)\right\}^{-1} \left[1 - \frac{i\lambda_0(v, u)P}{2k + i\lambda_0(v, u)}\right] =$$

$$= [1 + \lambda_0 Q M_0 + O(k)]^{-1} [Q + O(k)].$$

This proves analyticity of $T(k)$ around $k = 0$ since

$$(1 + \lambda_0 Q M_0)^{-1} = 1 - \lambda_0(1 + \lambda_0 Q M_0 Q)^{-1} Q M_0$$

exists.

Cases II a) — c). Take $\varepsilon \in C \setminus \{0\}$ small enough. Then we have

$$(1 + \lambda_0 Q M_0 + \varepsilon)^{-1} = (1 + \lambda_0 Q M_0 Q + \varepsilon)^{-1}[1 + \lambda_0 Q M_0 P(1 + \lambda_0 Q M_0 Q + \varepsilon)^{-1}]^{-1} =$$

$$= \left[\frac{P_0}{\varepsilon} + \sum_{m=0} (-\varepsilon)^m T_0^{m+1}\right]\left[1 - \frac{\lambda_0 Q M_0 P}{1 + \varepsilon}\right],$$

where we have used Lemma 3.1 and the relation

$$\left(1 + \frac{\lambda_0 Q M_0 P}{1 + \varepsilon}\right)^{-1} = \left(1 - \frac{\lambda_0 Q M_0 P}{1 + \varepsilon}\right).$$

After some straightforward calculations using some of the equalities in (3.3) this leads to

$$(3.5) \qquad (1 + \lambda_0 Q M_0 + \varepsilon)^{-1} = \frac{-\lambda_0 P_0 M_0}{\varepsilon} + O(1).$$

Next, we consider the operator $T(k)$ (see equality (2.4)) which can be written as

$$T(k) = \{[1 + (i\lambda_0(v, u)/2k)P][1 + (1 + (i\lambda_0(v, u)/2k)P)^{-1}\lambda_0 M(k)]\}^{-1} =$$

$$= \left[1 + \lambda_0 Q M(k) + \frac{2k\lambda_0 P M(k)}{2k + i\lambda_0(v, u)}\right]^{-1} \left(Q + \frac{2kP}{2k + i\lambda_0(v, u)}\right).$$

After some manipulations we get by expanding $M(k)$ (see equality (2.9)) to order k^2

$$T(k) = \left[1 + \left(1 + \lambda_0 Q M_0 - \frac{2ik}{\lambda_0(v, u)}\right)^{-1}\left(ik\lambda_0 Q M_1 - \frac{2ikM_0}{(v, u)} + O(k^2)\right)\right]^{-1} \times$$

$$\times \left(1 + \lambda_0\, Q M_0 - \frac{2ik}{\lambda_0(v, u)}\right)^{-1} \left(Q - \frac{2ik}{\lambda_0(v, u)}\right),$$

or using equality (3.5) with $\varepsilon = -2ik(\lambda_0(v, u))^{-1}$ we arrive at

$$T(k) = \left\{1 + \left[1 + \frac{(B^*\tilde{\varphi}_0, \cdot)\varphi_0}{(\tilde{\varphi}_0, \varphi_0)}\right]^{-1} \mathrm{O}(k)\right\}^{-1} \left[1 + \frac{(B^*\tilde{\varphi}_0, \cdot)\varphi_0}{(\tilde{\varphi}_0, \varphi_0)}\right]^{-1} \times \tag{3.6}$$
$$\times \left[\frac{\lambda_0^2(v, u)}{2ik} P_0 M_0 + \mathrm{O}(1)\right]\left(Q - \frac{2ik}{\lambda_0(v, u)}\right),$$

where

$$B = \lambda_0^2(v, u)\left[\frac{\lambda_0 M_0 Q M_1}{2} - \frac{M_0 P M_0}{(v, u)} - \frac{M_0 Q M_0}{(v, u)}\right].$$

Finally, we calculate the inverse appearing in equality (3.6). Employing

$$\frac{(B^*\tilde{\varphi}_0, \varphi_0)}{(\tilde{\varphi}_0, \varphi_0)} = -1 - \frac{\lambda_0^2(v, u)}{(\tilde{\varphi}_0, \varphi_0)}(|c_1|^2 + |c_2|^2), \tag{3.7}$$

we get

$$\left[1 + \frac{(B^*\tilde{\varphi}_0, \cdot)\varphi_0}{(\tilde{\varphi}_0, \varphi_0)}\right]^{-1} = 1 + \frac{1}{(|c_1|^2 + |c_2|^2)\lambda_0^2(v, u)}(B^*\tilde{\varphi}_0, \cdot)\varphi_0.$$

Inserting this into equality (3.6) and calculating the terms up to O(1) we obtain

$$T(k) = [2ik\lambda_0(|c_1|^2 + |c_2|^2)]^{-1}(\tilde{\varphi}_0, \cdot)\varphi_0 + \mathrm{O}(1),$$

where we have used again equality (3.7) and the relation $\lambda_0 P_0 M_0 Q = -P_0$ (see equality (3.3)).

This completes the proof of Theorem 3.1. ▨

Assuming the potential condition (2.8) and of course $(v, u) \neq 0$ throughout the rest of this section we now derive a systematic way to calculate the coefficients t_n in the Laurent series (3.4) for $T(k)$.

We start from the integral equation satisfied by $T(k)$, viz.

$$T(k) = 1 - \lambda_0[(\mathrm{i}(v, u)/2k)P + M(k)]T(k). \tag{3.8}$$

Following [13] by defining

$$P(k) = P_0 T(k), \quad Q(k) = Q_0 T(k), \quad \mathrm{Im}\, k > -a/2, k \neq 0 \tag{3.9}$$

equality (3.8) leads to the following set of coupled equations

$$2ik\lambda_0(\tilde{\varphi}_0, \varphi_0)^{-1}|c_2|^2 P(k) = P_0 - \lambda_0(\tilde{\varphi}_0, \varphi_0)^{-1} c_1^*(Q(k)^* v, \cdot)\varphi_0 - \tag{3.10}$$
$$-\lambda_0 P_0 M^{(2)}(k) P(k) - \lambda_0 P_0 M^{(1)}(k) Q(k)$$

and

$$Q(k) = Q_0 - (i\lambda_0/2k)(Q(k)^* v, \cdot)u - \lambda_0 Q_0 M(k) P(k) - \lambda_0 Q_0 M(k) Q(k) \tag{3.11}$$

where

$$M^{(j)}(k) = \sum_{n=j}^{\infty} (ik)^n M_n, \quad j = 1,2, \ldots . \tag{3.12}$$

Rewriting equality (3.11) using

$$[1 + (i\lambda_0/2k)(v, \cdot)u + \lambda_0 Q M_0 Q + \varepsilon]^{-1} =$$

$$= \varepsilon^{-1} P_0 + T_0\{1 + [\lambda_0(v, u)/(2ik - \lambda_0(v, u))]P\} + O(\varepsilon),$$

one obtains

$$\begin{aligned} Q(k) = T_0\{1 + [\lambda_0(v, u)/(2ik - \lambda_0(v, u))]P\} [Q_0 - \lambda_0 Q_0 M^{(1)}(k) P(k) - \\ - \lambda_0 Q_0 M_0 P Q(k) - \lambda_0 Q_0 M^{(1)}(k) Q(k)] - \\ - [2ik/(2ik - \lambda_0(v, u))][(\tilde{\varphi}_0, \varphi_0)^{-1} c_1 \lambda_0 (P(k)^* \tilde{\varphi}_0, \cdot)u + \lambda_0 P M_0 Q Q(k)]. \end{aligned} \tag{3.13}$$

Inserting (3.13) into the second term on the right-hand-side of (3.10) leads to, after some calculations,

$$\begin{aligned} P(k) = (\lambda_0 ik)^{-1}(\tilde{\varphi}_0, \varphi_0)c[P_0 - \lambda_0 P_0 M^{(2)}(k) P(k) - \lambda_0 P_0 M^{(1)}(k) Q(k)] + \\ + [2cc_1^*/(\lambda_0(v, u) - 2ik)](\{1 - \lambda_0 M^{(1)}(k) P(k) - \lambda_0 M(k) Q(k)\}^* v, \cdot)\varphi_0 - \\ - [4\, ik\, c|c_1|^2/(\lambda_0(v, u) - 2ik)]P(k), \end{aligned} \tag{3.14}$$

where

$$c = [2(|c_1|^2 + |c_2|^2)]^{-1}. \tag{3.15}$$

Equalities (3.13) and (3.14) may then be rewritten as

$$P(k) = P_0(k) - P_1(k)P(k) - P_2(k)Q(k), \tag{3.16}$$

and

$$Q(k) = Q_0(k) - Q_1(k)P(k) - Q_2(k)Q(k) \tag{3.17}$$

where the explicit expressions for $P_0(k), \ldots, Q_2(k)$ can be easily read off from equalities (3.13) and (3.14).

Next, from Theorem 3.1 we infer the existence of the norm convergent expansions

$$P(k) = \sum_{n=-1}^{\infty} (ik)^n p_n, \quad Q(k) = \sum_{n=0}^{\infty} (ik)^n q_n . \tag{3.18}$$

In order to calculate the coefficients p_n, q_n in all Cases I, II a) — c) we expand

$$P_0(k) = \sum_{n=-1}^{\infty} (ik)^n A_n, \tag{3.19}$$

$$P_1(k) = \sum_{n=1}^{\infty} (ik)^n B_n, \tag{3.20}$$

$$P_2(k) = \sum_{n=0}^{\infty} (ik)^n D_n, \tag{3.21}$$

and insert these expansions (3.18)—(3.21) into equality (3.16). We get

$$\begin{aligned} p_{-1} &= A_{-1}, \\ p_n &= A_n - \sum_{l=0}^{n} B_{n-l+1} p_{l-1} - \sum_{l=0}^{n} D_{n-l} q_l, \quad n \geqslant 0, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} A_{-1} &= \lambda_0^{-1} c(\tilde{\varphi}_0, \cdot)\varphi_0, \\ A_n &= c c_1^* [2/\lambda_0(v, u)]^{n+1}(v, \cdot)\varphi_0, \quad n \geqslant 0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} B_n = {} & 2c|c_1|^2[2/\lambda_0(v, u)]^n P_0 + c(\tilde{\varphi}_0, M_{n+1}\varphi_0)P_0 + \\ & + \lambda_0 c c_1^* \sum_{l=0}^{n-1} [2/\lambda_0(v, u)]^{n-l}(v, M_{l+1}\varphi_0)P_0, \quad n \geqslant 1, \end{aligned} \tag{3.24}$$

$$D_n = (\tilde{\varphi}_0, \varphi_0) c P_0 M_{n+1} Q_0 + \lambda_0 c c_1^* \sum_{l=0}^{n} [2/\lambda_0(v, u)]^{n-l+1}(\{M_l Q_0\}^* v, \cdot)\varphi_0, \quad n \geqslant 0. \tag{3.25}$$

Similarly, writing

$$Q_0(k) = \sum_{n=0}^{\infty} (ik)^n \tilde{F}_n, \tag{3.26}$$

$$Q_1(k) = \sum_{n=1}^{\infty} (ik)^n \tilde{K}_n, \tag{3.27}$$

$$Q_2(k) = \sum_{n=0}^{\infty} (ik)^n \tilde{L}_n, \tag{3.28}$$

we obtain from equality (3.17)

$$q_n = \tilde{F}_n - \sum_{l=0}^{n} \tilde{K}_{n-l+1} p_{l-1} - \sum_{l=0}^{n} \tilde{L}_{n-l} q_l, \tag{3.29}$$

where we do not need to write down at this moment the known explicit expressions for the $\tilde{F}_n$, $\tilde{K}_n$ and $\tilde{L}_n$. The reason is that since q_n occurs also on the right-hand-side of (3.29) we have to bring the term $\tilde{L}_0 q_n = \lambda_0 T_0 Q M_0 P q_n$ to the left and invert $(1 + \lambda_0 T_0 Q M_0 P)$. Doing this we finally obtain from (3.29)

$$\begin{aligned} q_0 &= F_0 - K_1 p_{-1}, \\ q_n &= F_n - \sum_{l=0}^{n} K_{n-l+1} p_{l-1} - \sum_{l=0}^{n-1} L_{n-l} q_l, \quad n \geqslant 1, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} F_0 &= T_0 Q, \\ F_n &= -[2/\lambda_0(v, u)]^n (1 - \lambda_0 T_0 Q M_0) P, \quad n \geqslant 1, \end{aligned} \tag{3.31}$$

$$K_1 = \lambda_0 T_0 Q M_1 P_0 - 2(\tilde{\varphi}_0, \varphi_0)^{-1}(v, u)^{-1} c_1(\tilde{\varphi}_0, \cdot)[1 - \lambda_0 T_0 Q M_0] u,$$

$$\begin{aligned} K_n = \lambda_0 T_0 Q M_n P_0 &- \lambda_0 (\tilde{\varphi}_0, \varphi_0)^{-1} [2/\lambda_0(v, u)]^n c_1(\tilde{\varphi}_0, \cdot)[1 - \lambda_0 T_0 Q M_0] u - \\ &- \lambda_0 \sum_{l=1}^{n-1} [2/\lambda_0(v, u)]^{n-l} (1 - \lambda_0 T_0 Q M_0) P M_l P_0, \quad n \geqslant 2, \end{aligned} \tag{3.32}$$

$$L_0 = \lambda_0 T_0 Q M_0 Q_0,$$

$$L_n = \lambda_0 T_0 Q M_n Q_0 - \lambda_0 \sum_{l=0}^{n-1} [2/\lambda_0(v, u)]^{n-l} (1 - \lambda_0 T_0 Q M_0) P M_l Q_0), \quad n \geqslant 1. \tag{3.33}$$

Equalities (3.12) and (3.30) represent the final result. They allow us to compute all p_n and q_n recursively. Below we list a few coefficients explicitly and state certain matrix elements needed for later purposes:

Case I.

$$p_n = 0, \quad n \geqslant -1, \tag{3.34}$$

$$t_0 = q_0 = T_0 Q, \tag{3.35}$$

$$\begin{aligned} t_1 = q_1 = -\lambda_0 T_0 Q M_1 T_0 Q &- [2/\lambda_0(v, u)][P - \lambda_0 T_0 Q M_0 P - \lambda_0 P M_0 T_0 Q + \\ &+ \lambda_0^2 T_0 Q M_0 P M_0 T_0 Q], \end{aligned} \tag{3.36}$$

(3.37) $$(v,\ t_0 u) = (v,\ q_0 u) = 0,$$

(3.38) $$(v,\ t_1 u) = (v,\ q_1 u) = -2/\lambda_0\,,$$

(3.39) $$(v,\ t_2 u) = (v,\ q_2 u) = -\,[2/\lambda_0(v,\ u)]^2[(v,\ (1 + \lambda_0 M_0)u) - \\ - \lambda_0^2(v,\ M_0 T_0 Q M_0 u)].$$

Cases II a) — c).

(3.40) $$p_{-1} = c\lambda_0^{-1}(\tilde{\varphi}_0, \cdot)\varphi_0\,,$$

(3.41) $$q_0 = T_0 Q - cc_2(\tilde{\varphi}_0\,, \cdot)T_0(\cdot)u + (v,\ u)^{-1}cc_2((\cdot)v,\ u)\ (\tilde{\varphi}_0, \cdot)u + \\ + [2/\lambda_0(v,\ u)]cc_1(\tilde{\varphi}_0\,, \cdot)\ (1 - \lambda_0 T_0 Q M_0)u,$$

(3.42) $$p_0 P = 2\lambda_0^{-1}(v,\ u)^{-1}cc_1^*(v, \cdot)\varphi_0$$

(3.43) $$p_n^* v = 0, \quad n \geqslant -1,$$

(3.44) $$(v,\ t_{-1}u) = (v,\ p_{-1}u) = 0,$$

(3.45) $$(v,\ t_0 u) = (v,\ q_0 u) = 0,$$

(3.46) $$(v,\ t_1 u) = (v,\ q_1 u) = -\ 4\lambda_0^{-1}c|c_2|^2.$$

4. S-MATRIX, REFLECTION AND TRANSMISSION COEFFICIENTS

In this section we apply the preceding results to get low-energy expansions for the on-shell scattering matrix on the line and thus for the reflection and transmission coefficients. Throughout we assume condition (2.8) and $(v,\ u) \neq 0$ on V.

Combining Jost functions techniques (cf. e.g. [34]) with Fredholm methods (cf. e.g. equality (2.24)), one arrives at the following expression for the one-dimensional on-shell scattering amplitude

(4.1) $$f_{\varepsilon_1 \varepsilon_2}(k) = (2ik)^{-1}\lambda_0(\Phi_0^+(\varepsilon_1\,,\ k),\ T(k)\Phi_0^-(\varepsilon_2\,,\ k)), \quad \operatorname{Im} k > -\ a/2,\ \ k \neq 0,$$

where

$$\varepsilon_j = \pm 1, \quad j = 1,2\,,$$

and

(4.2) $$\Phi_0^+(\varepsilon_j\,,\ k,\ x) = v(x)\mathrm{e}^{i\varepsilon_j kx}, \quad \Phi_0^-(\varepsilon_j\,,\ k,\ x) = u(x)\,\mathrm{e}^{i\varepsilon_j kx}, \quad j = 1,2\,.$$

The on-shell scattering matrix is then given by (cf. e.g. [16], [34])

$$S_{\varepsilon_1\varepsilon_2}(k) = \delta_{\varepsilon_1\varepsilon_2} + f_{\varepsilon_1\varepsilon_2}(k), \quad \operatorname{Im} k > -a/2, \ k \neq 0, \tag{4.3}$$

or equivalently,

$$S(k) = \begin{pmatrix} T^{\mathrm{l}}(k) & R^{\mathrm{r}}(k) \\ R^{\mathrm{l}}(k) & T^{\mathrm{r}}(k) \end{pmatrix}, \quad \operatorname{Im} k > -a/2, \ k \neq 0. \tag{4.4}$$

Here the matrix elements

$$S_{++}(k) = T^{\mathrm{l}}(k) = T^{\mathrm{r}}(k) = S_{--}(k) \tag{4.5}$$

denote the transmission coefficients and

$$S_{-+}(k) = R^{\mathrm{l}}(k), \quad S_{+-}(k) = R^{\mathrm{r}}(k) \tag{4.6}$$

represent the reflection coefficients for left and right incidence respectively. Clearly $S(k)$ is analytic in $\operatorname{Im} k > -a/2$, $k \neq 0$.

Employing the low-energy expansion (3.4) for $T(k)$ in equality (4.1) we arrive at

THEOREM 4.1. *Assume* $(v, u) \neq 0$ *and* $e^{a|\cdot|}V \in L^1(\mathbf{R})$ *for some* $a > 0$. *Then the on-shell scattering matrix* $S(k)$ *is analytic with respect to* k *in* $\operatorname{Im} k > -a/2$. *In particular we have the following Taylor expansion in* k *around* $k = 0$

$$S_{\varepsilon_1\varepsilon_2}(k) = \sum_{n=0}^{\infty} (\mathrm{i}k)^n s^{(n)}_{\varepsilon_1\varepsilon_2}, \tag{4.7}$$

where

$$s^{(n)}_{\varepsilon_1\varepsilon_2} = \delta_{n0}\delta_{\varepsilon_1\varepsilon_2} + 2^{-1}\lambda_0 \sum_{l=0}^{n+1-q} \sum_{m=0}^{n+1-q-l} (-\varepsilon_1)^l(\varepsilon_2)^m(l!\,m!)^{-1}((\cdot)^l v, t_{n+1-l-m}(\cdot)^m u) \tag{4.8}$$

with $q = 0$ *in Case* I *and* $q = -1$ *in Cases* II a) – c). *Moreover, the leading coefficients in Case* I *read*

$$s^{(0)}_{\varepsilon_1\varepsilon_2} = \delta_{\varepsilon_1\varepsilon_2} - 1, \tag{4.9}$$

$$\begin{aligned} s^{(1)}_{\varepsilon_1\varepsilon_2} = {} & -2\lambda_0^{-1}(v,u)^{-2}[(v,u)+\lambda_0(v,M_0u)+\lambda_0^2(v,u)^{-1}(v,M_0u)^2-\lambda_0^2(v,M_0T_0M_0u)]+ \\ & + (v,u)^{-1}(\varepsilon_1-\varepsilon_2)[((\cdot)v,u)-\lambda_0((\cdot)v,T_0M_0u)+\lambda_0(v,u)^{-1}((\cdot)v,u)(v,M_0u)]- \\ & - 2^{-1}\lambda_0\varepsilon_1\varepsilon_2((\cdot)v,T_0(\cdot)u) + 2^{-1}\lambda_0\varepsilon_1\varepsilon_2(v,u)^{-1}((\cdot)v,u)^2, \end{aligned} \tag{4.10}$$

and in Cases II a) — c)

$$(4.11)\quad s^{(0)}_{\varepsilon_1\varepsilon_2} = \delta_{\varepsilon_1\varepsilon_2} - 1 + [|c_1|^2 - \varepsilon_1 c_1^* c_2 + \varepsilon_2 c_1 c_2^* - \varepsilon_1\varepsilon_2 |c_2|^2]/[|c_1|^2 + |c_2|^2]$$

where T_0 and M_0 are given by equalities (3.2) *and* (2.10) *respectively, and the c_1, c_2 are given by equality* (2.25).

Proof. Because of equalities (3.4) and (4.1) we know that $f_{\varepsilon_1\varepsilon_2}(k)$ has a Laurent expansion of the type

$$(4.12)\qquad f_{\varepsilon_1\varepsilon_2}(k) = \sum_{n=-1+q}^{\infty} (ik)^n f^{(n)}_{\varepsilon_1\varepsilon_2}.$$

The coefficients $f^{(n)}_{\varepsilon_1\varepsilon_2}$ are found by expanding $T(k)$ and $\Phi_0^{\pm}(\varepsilon_j, k)$ in k. The result is

$$(4.13)\quad f^{(n)}_{\varepsilon_1\varepsilon_2} = 2^{-1}\lambda_0 \sum_{l=0}^{n+1-q} \sum_{m=0}^{n+1-q-l} (-\varepsilon_1)^l(\varepsilon_2)^m (l!\, m!)^{-1}((\cdot)^l v,\, t_{n+1-l-m}(\cdot)^m u).$$

In Case I (i.e. $q = 0$), insertion of expressions (3.34)—(3.39) in equality (4.13) immediately gives (4.9)—(4.10). Similarly, in Case II (i.e. $q = -1$), inserting expressions (3.40) — (3.46) in equality (4.13) leads directly to equality (4.11). ▨

So, up to $O(k^2)$, the transmission and reflection coefficients in Case I are given by

$$(4.14)\qquad \begin{aligned} T^l(k) = T^r(k) &\underset{k\to 0_+}{=} iks^{(1)}_{++} + O(k^2),\\ R^l(k) &\underset{k\to 0_+}{=} -1 + iks^{(1)}_{-+} + O(k^2),\\ R^r(k) &\underset{k\to 0_+}{=} -1 + iks^{(1)}_{+-} + O(k^2), \end{aligned}$$

where the $s^{(1)}_{\varepsilon_1\varepsilon_2}$ can be read off from equality (4. 10). These results are more detailed than the ones available in the literature (cf. e.g. [14], [16], [34]). In Cases II a) — c) the transmission and reflection coefficients are

$$(4.15)\qquad \begin{aligned} T^l(k) = T^r(k) &\underset{k\to 0_+}{=} \frac{|c_1|^2 - |c_2|^2}{|c_1|^2 + |c_2|^2} + O(k),\\ R(k) &\underset{k\to 0_+}{=} \frac{2c_1c_2^*}{|c_1|^2 + |c_2|^2} + O(k),\\ R^r(k) &\underset{k\to 0_+}{=} -\frac{2c_1c_2^*}{|c_1|^2 + |c_2|^2} + O(k). \end{aligned}$$

(Here we have used $c_1^* c_2 = c_1 c_2^*$ since φ is unique up to multiplicative constants.) These results are new.

5. TRACE RELATIONS

This section investigates the low-energy behavior of the trace of the difference between the full and the free resolvent. This is then used to derive so called trace relations (or sum rules) involving moments of the phase shift derivative (or time delay) on one side, and bound state and zero-energy resonance contributions on the other side. Such (positive moment) trace relations were initially introduced by Gelfand and Levitan [18]. For a list of further references we refer to [11]—[13]. Here we are interested in proving the zero-th and negative moment relations allowing the possible occurrence of a zero-energy resonance. As a special case we obtain Levinson's theorem for scattering on the line.

In order to be able to apply the results of Section 3 we assume condition (2.8) and $(v, u) \neq 0$ on the potential. First we discuss

LEMMA 5.1. *In all Cases* I, II a) – c), $\mathrm{Tr}[R(k) - R_0(k)]$ *has the following Laurent expansion in* k *around* $k = 0$

$$\mathrm{Tr}[R(k) - R_0(k)] = \sum_{n=q-2}^{\infty} (\mathrm{i}k)^n \Delta_n, \tag{5.1}$$

where

$$\Delta_{q-2} = 4^{-1}\lambda_0(v, t_{q+1}u),$$

$$\Delta_n = 4^{-1}\lambda_0(v, t_{n+3}u) + 2^{-1}\lambda_0 \sum_{l=0}^{n+1-q} (n + 2 - l - q)\mathrm{Tr}[M_{n+2-l-q}t_{l+q}], \quad n \geqslant q - 1 \tag{5.2}$$

with $q = 0$ *in Case* I *and* $q = -1$ *in Cases* II a) – c).

Proof. Define

$$G(k) = kuR_0(k)v. \tag{5.3}$$

Then, by mimicking the proof of Proposition 5.6 in [43] one gets

$$\|G(k)\|_1 \leqslant c\int_{\mathbf{R}} \mathrm{d}x\, \mathrm{e}^{2\theta(-\mathrm{Im}\,k)\,|\mathrm{Im}\,k|\,|x|}(1 + |x|^{1+\delta})|V(x)| < \infty, \tag{5.4}$$

for $\mathrm{Im}\,k > -a/2$ and some $c, \delta > 0$ ($\theta(s)$ denotes the step function: $\theta(s) = 1$, $s > 0$; $\theta(s) = 0$, $s \leqslant 0$). Similarly, defining $G'(k)$ to be the operator with kernel

$$G'(k, x, y) = -\,2^{-1}u(x)|x - y|\, \mathrm{e}^{\mathrm{i}k|x-y|}v(y) \tag{5.5}$$

one obtains

$$\|G'(k)\|_1 \leqslant c\int_{\mathbf{R}} \mathrm{d}x\, \mathrm{e}^{2\theta(-\mathrm{Im}\,k)|\mathrm{Im}\,k||x|}(1 + |x|^{3+\delta})|V(x)| < \infty,$$

$$c, \delta > 0, \quad \mathrm{Im}\,k > -\,a/2. \tag{5.6}$$

Applying this method of proof once again one finally arrives at

$$(5.7)\quad \left\| \frac{G(k)-G(k_0)}{k-k_0} - G'(k_0) \right\|_1 \leqslant c|k-k_0| \int_{\mathbf{R}} dx\, e^{2\theta(-\operatorname{Im}\tilde{k})|\operatorname{Im}\tilde{k}||x|} (1+|x|^{5+\delta})|V(x)| < \infty,$$

$$c,\ \delta > 0,\ \operatorname{Im} k > -a/2,\ \operatorname{Im} k_0 > -a/2$$

for some $\tilde{k}$, $\operatorname{Im}\tilde{k} > -a/2$ depending on k and k_0. (The estimate (5.7) also proves that $uR_0(k)v$ is analytic in trace norm around any $k \neq 0$, $\operatorname{Im} k > -a/2$ and has a Laurent expansion around $k = 0$ convergent in trace norm.) Consequently

$$\operatorname{Tr}[R(k) - R_0(k)] = -\lambda_0 \operatorname{Tr}[R_0(k)vT(k)uR_0(k)] = -\lambda_0 \operatorname{Tr}[uR_0^2(k)vT(k)] =$$

$$(5.8)\quad = -\lambda_0(2k)^{-1} \operatorname{Tr}\left\{\left[\frac{d}{dk}(uR_0(k)v)\right] T(k)\right\} =$$

$$= 4^{-1}\lambda_0(ik)^{-3}(v, T(k)u) - 2^{-1}\lambda_0 k^{-1} \operatorname{Tr}\left\{\left[\frac{d}{dk} M(k)\right] T(k)\right\}, \quad \operatorname{Im} k > -a/2,\ k \neq 0,$$

where we have used the cyclic properties of the trace and equality (2.7). Insertion of equalities (2.9) and (3.4) into (5.8) yields, after interchanging $\sum$ and Tr (which is allowed by the above arguments since $k\left[\frac{d}{dk} M(k)\right] T(k)$ is analytic around $k = 0$ in trace norm)

$$\operatorname{Tr}[R(k) - R_0(k)] = 4^{-1}\lambda_0 \sum_{n=q-3}^{\infty} (ik)^n (v, t_{n+3}u) +$$

$$+ 2^{-1}\lambda_0 \sum_{n=q-1}^{\infty} (ik)^n \sum_{l=0}^{n+1-q} (n+2-l-q) \operatorname{Tr}\left\{M_{n+2-l-q} t_{l+q}\right\}$$

for $k \neq 0$, $|k|$ small enough. Since $(v, t_q u) = 0$ by equalities (3.37) respectively (3.44), (5.1) follows. ▨

Next we establish the high-energy behavior of $\operatorname{Tr}[R(k) - R_0(k)]$.

LEMMA 5.2. *In all Cases* I, II a) – c) *there exists a* $k_0 > 0$ *and a* $c > 0$ *(depending on* k_0*) such that*

$$(5.9)\quad \|R(k) - R_0(k)\|_1 \leqslant c|k|^{-2}, \quad |k| \geqslant k_0 > 0,\ \operatorname{Im} k > -a/2.$$

Proof. From the estimate

$$(5.10)\quad \|\lambda_0 uR_0(k)v\|_2^2 \leqslant |\lambda_0|^2(2|k|)^{-2}\| e^{-2\operatorname{Im} k|\cdot|} V\|_1^2, \quad k \neq 0,\ \operatorname{Im} k > -a/2,$$

we obtain

(5.11) $\|T(k)\| = \|(1 + \lambda_0 u R_0(k) v)^{-1}\| \leqslant \text{const}, \; |k| \geqslant k_0 > 0, \; \operatorname{Im} k > -a/2.$

On the other hand equalities (5.4) and (5.6) imply that

(5.12) $\|uR_0^2(k)v\|_1 = (2|k|^2)^{-1}\|G'(k) - k^{-1}G(k)\|_1 \leqslant c|k|^{-2}, \quad |k| \geqslant k_0 > 0, \; \operatorname{Im} k > -a/2.$

Thus equality (5.9) follows from

$$\|R(k) - R_0(k)\|_1 \leqslant |\lambda_0| \|T(k)\| \, \|uR_0^2(k)v\|_1, \quad k \neq 0, \; \operatorname{Im} k > -a/2. \quad \blacksquare$$

LEMMA 5.3. *In all Cases* I, II a) — c) *there exists a* $k_0 > 0$ *and a* $c > 0$ *(depending on* k_0*) such that*

(5.13) $|\operatorname{Tr}[R(k) - R_0(k)]| \leqslant c|k|^{-3}, \quad |k| \geqslant k_0 > 0, \; \operatorname{Im} k \geqslant 0.$

Proof. Equality (3.8) implies

(5.14) $\operatorname{Tr}[R(k) - R_0(k)] = -\lambda_0 \operatorname{Tr}[uR_0^2(k)v] + \lambda_0^2 \operatorname{Tr}[uR_0^2(k)vuR_0(k)vT(k)].$

By equalities (5.10) and (5.12) the second term on the right-hand-side of (5.14) is $O(|k|^{-3})$ as $|k| \to \infty$, $\operatorname{Im} k \geqslant 0$. The first term can be treated as follows. Writing $uR_0^2(k)v = (2k^3)^{-1}G(k) - (2k^2)^{-1}G'(k)$ (cf. equalities (5.3) and (5.5)) we get

(5.15)
$$\operatorname{Tr}[uR_0^2(k)v] = (i/4k^3)\int_{\mathbf{R}} dx V(x) - (1/2k^2)\operatorname{Tr}[G'(k)] =$$
$$= (i/4k^3)\int_{\mathbf{R}} dx V(x), \quad k \neq 0, \; \operatorname{Im} k > 0$$

since for $\operatorname{Im} k > 0$

$$\|uR_0(k)\|_2^2 = \|R_0(k)v\|_2^2 = (1/4|k|^2 \operatorname{Im} k)\|V\|_1 < \infty.$$

Trace norm continuity of $uR_0^2(k)v$ with respect to k in $\operatorname{Im} k \geqslant 0$ (cf. the proof of Lemma 5.1) then proves equality (5.15) for all $\operatorname{Im} k \geqslant 0$. $\blacksquare$

For asymptotic expansions of $R(k)$ as $|k| \to \infty$ we refer to [50] and references therein.

In the next lemma we relate the trace of the resolvent difference $R(k) - R_0(k)$ with the phase shift $\delta(k)$.

LEMMA 5.4. *Define the phase shift* $\delta(k)$ *by*

$$\det S(k) = e^{2i\delta(k)}, \quad k > 0 \tag{5.16}$$

with $\lim_{k\to\infty} \delta(k) = 0$. *Then* $\delta(k)$ *is continuously differentiable in* $k > 0$ *and*

$$\operatorname{Im}\operatorname{Tr}[R(k) - R_0(k)] = -(i/4k)\operatorname{Tr}[S^*(k)S'(k)] = (1/2k)\delta'(k), \quad k > 0. \tag{5.17}$$

Proof. Since $R(k) - R_0(k)$ is trace class, Kreĭn's theorem [31] (cf. also [6], [21], [45]) implies

$$\operatorname{Tr}[R(k+i\varepsilon) - R_0(k+i\varepsilon)] = -\int_{\mathbf{R}} 2p\mathrm{d}p \frac{\xi(p)}{[p^2 - (k+i\varepsilon)^2]^2}, \quad \varepsilon,\ k > 0 \tag{5.18}$$

where ξ is real and $(1 + |\cdot|^2)\xi \in L^1(\mathbf{R})$. Secondly, writing $S(k) = e^{2i\Delta(k)}$, it also implies that

$$\operatorname{Tr}\Delta(k) = -\pi\xi(k) = \delta(k), \quad k > 0. \tag{5.19}$$

Then, from equalities (4.1), (4.3) and (5.11) we get the high-energy estimates

$$|\det S(k) - 1| \leqslant ck^{-1}, \quad |\delta(k)| \leqslant c'k^{-1}, \quad k \geqslant k_0 > 0 \tag{5.20}$$

for some $c, c' > 0$ depending on k_0. Next, by hypothesis (2.8) the $\Phi_0^{\pm}(\varepsilon_j, k)$ are infinitely many times strongly differentiable with respect to k. Furthermore, since $uR_0(k)v$, $\operatorname{Im} k > -a/2$, $k \neq 0$ is analytic in trace norm (cf. the proof of Lemma 5.1), $T(k)$ and, using (4.1), also $S(k)$ are norm analytic in $k > 0$. Thus $\delta(k)$ is C^∞ in $k > 0$. Moreover, an estimate similar to (5.20) shows that $\left(\frac{\mathrm{d}}{\mathrm{d}k} f_{++}\right)(k) \underset{k\to\infty}{=} O(k^{-2})$ and hence

$$|\delta'(k)| = 2^{-1}\left|\left(\frac{\mathrm{d}}{\mathrm{d}k}\det S\right)(k)\right| \leqslant c''k^{-2}, \quad k \geqslant k_0 > 0. \tag{5.21}$$

Next, integration by parts in the integral from 0 to ∞ in (5.18) using (5.20) and (5.21) yields

$$\begin{aligned}\lim_{\varepsilon\to 0_+} \operatorname{Im}\operatorname{Tr}[R(k+i\varepsilon) - R_0(k+i\varepsilon)] &= -\lim_{\varepsilon\to 0_+} 4\varepsilon \int_{-\infty}^{0} p\mathrm{d}p \frac{\xi(p)(p^2-k^2)}{[(p^2-k^2)^2+\varepsilon^2]^2} + \\ &+ \lim_{\varepsilon\to 0_+} \int_0^\infty \mathrm{d}p^2\,(2p)^{-1}\operatorname{Tr}\Delta'(p) \frac{\varepsilon/\pi}{(p^2-k^2)^2+\varepsilon^2}.\end{aligned} \tag{5.22}$$

Dominated convergence in the first term on the right-hand-side of (5.22) (observing $k > 0$) and a standard δ-function computation ([40], p. 128, using the fact that $(2p)^{-1}\mathrm{Tr}\,\Delta'(p)$ is continuous and bounded in a small k-dependent neighborhood of $p = k > 0$) then proves equality (5.17). ▨

Given the above results we can now state the main theorems of this section.

THEOREM 5.1. *Assume* $(v, u) \neq 0$ *and* $e^{a|\cdot|}V \in L^1(\mathbf{R})$ *for some* $a > 0$. *Then the following trace relations (sum rules) hold*

$$2\int_0^\infty dk\, k^{-2N+1}\left\{\mathrm{Im\,Tr}[R(k) - R_0(k)] - \sum_{n=q-2}^{N-2}(-1)^n k^{2n+1}\Delta_{2n+1}\right\} =$$

(5.23)

$$= -\pi\sum_{j=1}^{N_b}(-\varkappa_j^2)^{-N} + \pi(-1)^{N-1}\Delta_{2N-2}, \quad N = 0,1,2,\ldots$$

where $-\varkappa_j^2$, $j = 1, \ldots, N_b$ *denote the (negative and simple) eigenvalues of* H, *and the* Δ_m *are defined by equality* (5.2).

Proof. Following [8], [37], we introduce the function

$$(5.24)\quad F_N(k) = 2k^{-2N+1}\left\{\mathrm{Tr}[R(k) - R_0(k)] - \sum_{n=q-2}^{2N-2}(ik)^n\Delta_n\right\}, \quad N = 0,1,\ldots$$

where $q = 0$ in Case I and $q = -1$ in Cases II a) — c). Clearly $F_N(k)$ is analytic in the open upper k-half plane with possible poles on the positive imaginary axis. Since $(1 + |\cdot|)V \in L^1(\mathbf{R})$ these poles are finite in number [26], [34], [42] and non-degenerate (e.g. by well-known Volterra integral equation arguments similar to that in the proof of Lemma 2.2). Moreover H has no nonnegative eigenvalues and purely absolutely continuous spectrum on $[0, \infty)$ [48], [49].

In order to derive (5.23) we apply contour integration techniques and integrate $F_N(k)$ along the following paths: From $-R + i\varepsilon$ to $R + i\varepsilon$ avoiding the origin by a semi-circle $C_{\eta,\varepsilon} = \{\eta e^{i\theta} \mid \theta \in [\pi - \arcsin(\varepsilon/\eta), \arcsin(\varepsilon/\eta)]\}$, along the semi-circle $C_{R,\varepsilon} = \{Re^{i\theta} \mid \theta \in [\arcsin(\varepsilon/R), \pi - \arcsin(\varepsilon/R)]\}$, and finally encircling all bound state energy positions $i\varkappa_j$ clockwise $C_{r_j} = \{r_j e^{i\theta_j} \mid \theta_j \in [0, 2\pi],\ r_j$ sufficiently small$\}$, $1 \leqslant j \leqslant N_b$.

We then study the different contributions. From $C_{R,\varepsilon}$ we get by Lemma 5.3

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0_+}\int_{C_{R,\varepsilon}} dk\, F_N(k) = -\lim_{R\to\infty} 2i\int_0^\pi d\theta (Re^{i\theta})^{-2N+2}\sum_{n=q-2}^{2N-2}(iRe^{i\theta})^n\Delta_n =$$

(5.25)

$$= 2\pi i(-1)^N\Delta_{2N-2}.$$

Since $R(k)$ has a first-order pole at $k = i\varkappa_j$ we get from C_{r_j}

$$\sum_{j=1}^{N_b} \oint_{C_{r_j}} dk F_N(k) = 2\pi i \sum_{j=1}^{N_b} (-\varkappa_j^2)^{-N}. \tag{5.26}$$

To get the contribution from $C_{\eta,\varepsilon}$ we insert (5.1) into (5.24) to obtain

$$\lim_{\eta\to 0_+}\lim_{\varepsilon\to 0_+} \int_{C_{\eta,\varepsilon}} dk F_N(k) = \lim_{\eta\to 0_+} 2i \int_{\pi}^{0} d\theta (\eta e^{i\theta})^{-2N+2} \sum_{n=2N-1}^{\infty} (i\eta e^{i\theta})^n \Delta_n = 0. \tag{5.27}$$

Finally we calculate the contribution form $C_{\varepsilon,\eta,R} = \{k + i\varepsilon \mid -R \leqslant k \leqslant -\eta$ or $\eta \leqslant k \leqslant R\}$:

$$\lim_{R\to\infty}\lim_{\eta\to 0_+}\lim_{\varepsilon\to 0_+} \int_{C_{\varepsilon,\eta,R}} dk\, F_N(k) =$$

$$= \lim_{R\to\infty}\lim_{\eta\to 0_+}\lim_{\varepsilon\to 0_+} 4i \int_{\eta}^{R} dk\, k^{-2N+1} \operatorname{Im}\Big\{\operatorname{Tr}[R(k+i\varepsilon) - R_0(k+i\varepsilon)] -$$

(5.28)

$$- \sum_{n=q-2}^{2N-2} [i(k+i\varepsilon)]^n \Delta_n\Big\} = \lim_{R\to\infty}\lim_{\eta\to 0_+} 4i \int_{\eta}^{R} dk\, k^{-2N+1} \Big\{\operatorname{Im}\operatorname{Tr}[R(k) - R_0(k)] -$$

$$- \sum_{n=-2}^{N-2} (-1)^n k^{2n+1} \Delta_{2n+1}\Big\} = 4i \int_{0}^{\infty} dk\, k^{-2N+1} \Big\{\operatorname{Im}\operatorname{Tr}[R(k) - R_0(k)] -$$

$$- \sum_{n=-2}^{N-2} (-1)^n k^{2n+1} \Delta_{2n+1}\Big\},$$

where we have used dominated convergence, the low-energy behavior of $\operatorname{Tr}[R(k) - R_0(k)]$ in (5.1) and the high-energy bound (5.13). Noting then that

$$\int_C dk\, F_N(k) = 0, \quad C = C_{R,\varepsilon} \cup \Big\{\bigcup_{j=1}^{N_b} C_{r_j}\Big\} \cup C_{\eta,\varepsilon} \cup C_{\varepsilon,\eta,R}$$

equality (5.23) results after summing up (5.25)—(5.28). ▨

The sum rule for $N = 0$ in (5.23) corresponds to Levinson's theorem for scattering on the line:

COROLLARY 5.1. *Let* $(v, u) \neq 0$ *and assume* $e^{a|\cdot|}V \in L^1(\mathbf{R})$ *for some* $a > 0$. *Then*

$$\delta(0_+) = \pi(N_b + \Delta_{-2}), \quad \delta(\infty) = 0 \tag{5.29}$$

where

$$\Delta_{-2} = \begin{cases} -1/2 & \text{in Case I}, \\ 0 & \text{in Cases II a) -- c).} \end{cases}$$

Proof. Insertion of (5.17) into the left-hand-side of (5.23) and taking $N = 0$ immediately yields (5.29). By equalities (5.2) and (3.38) we obtain in Case I

$$\Delta_{-2} = 4^{-1}\lambda_0(v, t_1 u) = -1/2.$$

In Cases II we employ equalities (5.2), (3.46) and the relation $(\tilde{\varphi}_0, M_1\varphi_0) = 2|c_2|^2$ (see equalities (3.3)) to get

$$\Delta_{-2} = 4^{-1}\lambda_0(v, t_1 u) + 2^{-1}\lambda_0 \operatorname{Tr}[M_1 t_{-1}] =$$

$$= -c|c_2|^2 + c|c_2|^2 = 0.$$

▨

The structure of Levinson's theorem for scattering on the line is completely different from its analogue in three dimensions. Indeed, when there is no zero-energy resonance present (Case I), we not only get on the right hand side of (5.29) the term proportional to the number of negative-energy bound states, πN_b, but an additional factor $-\pi/2$ appears. In the case of a zero-energy resonance (Cases II), we simply obtain the term πN_b. This difference is of course due to the additional Dirichlet boundary conditions at the origin when considering Schrödinger operators on the half line. A one dimensional Levinson's theorem has been studied recently for scattering by a local impurity in a periodic potential [35] and in the context of (1+1)-dimensional field theoretical models ([5] and references cited therein).

Our second main result of this section reads

THEOREM 5.2. *Assume* $(v, u) \neq 0$ *and* $e^{a|\cdot|}V \in L^1(\mathbf{R})$ *for some* $a > 0$. *Then*

$$2\int_0^\infty dk\, k^{-2M} \left\{ \operatorname{Re}\operatorname{Tr}[R(k) - R_0(k)] - \sum_{n=-1}^{M-1} (-1)^n k^{2n} \Delta_{2n} \right\} =$$

$$= -i\pi \sum_{j=1}^{N_b} (-\varkappa_j^2)^{-M-1/2} + \pi(-1)^M \Delta_{2M-1}, \quad M = 0,1,2,\ldots, \tag{5.30}$$

where $-\varkappa_j^2, j = 1, \ldots, N_b$, *denote the eigenvalues of* H, *and the* Δ_n *are defined by equality* (5.2).

Proof. Analogous to that of Theorem 5.1 after replacing $F_N(k)$ by

$$(5.31) \quad G_M(k) = 2k^{-2M}\left\{\mathrm{Tr}[R(k) - R_0(k)] - \sum_{n=q-2}^{2M-1} (\mathrm{i}k)^n \Delta_n\right\}, \quad M = 0,1,2,\ldots .$$

Combining Corollary 5.1 and Theorem 4.1 we obtain the following low-energy behavior of the phase shift in Case I

$$(5.32) \quad \begin{aligned} \delta(k) \underset{k\to 0_+}{=} & \pi(N_b - 1/2) + \{2\lambda_0^{-1}(v, u)^{-2}[(v, u) + \lambda_0(v, M_0u) + \lambda_0^2(v,u)^{-1}(v, M_0u)^2 - \\ & - \lambda_0^2(v, M_0T_0M_0u)] - 2^{-1}\lambda_0((\cdot)v, T_0(\cdot)u) + 2^{-1}\lambda_0(v, u)^{-1}((\cdot)v, u)^2\}k + \mathrm{O}(k^2). \end{aligned}$$

For the Cases II a) — c) we get

$$(5.33) \quad \delta(k) \underset{k\to 0_+}{=} \pi N_b + \mathrm{O}(k),$$

where N_b denotes the number of (negative and simple) eigenvalues of H.

6. POSSIBLE GENERALIZATIONS

If one is interested in asymptotic expansions (instead of Laurent expansions) of the various quantities discussed before, Condition (2.8) on the potential, which roughly implies exponential fall off at infinity, can be relaxed considerably. In particular we shall now briefly indicate how the condition $(1 + |\cdot|^p)V \in L^1(\mathbf{R})$ for suitable $p \geqslant 2$ (depending on the order of the asymptotic expansions involved) can be shown to suffice to derive most of the results of this paper.

Assume $(v, u) \neq 0$ and

$$(6.1) \quad \int_{\mathbf{R}} \mathrm{d}x(1 + |x|^{2m+2\varepsilon})|V(x)| < \infty \quad \text{for some } \varepsilon \geqslant 0,\ m = 1,2,3,\ldots$$

then, by dominated convergence, $M(k)$ has the asymptotic expansion

$$(6.2) \quad M(k) \underset{k\to 0_+}{=} \sum_{n=0}^{m-1} (\mathrm{i}k)^n M_n + \mathrm{o}(k^{m+\varepsilon-1}), \quad m \geqslant 1,$$

which is valid in Hilbert-Schmidt norm (i.e. $\lim_{k\to 0} k^{1-\varepsilon-m}\|o(k^{m+\varepsilon-1})\|_2 = 0$). Similarly $T(k)$ has the asymptotic expansion

(6.3)
$$T(k) \underset{k\to 0_+}{=} \sum_{n=0}^{m-1} (ik)^n t_n + o(k^{m+\varepsilon-1}), \quad m \geqslant 1 \text{ in Case I, and}$$
$$T(k) \underset{k\to 0_+}{=} \sum_{n=-1}^{m-3} (ik)^n t_n + o(k^{m+\varepsilon-3}), \quad m \geqslant 2 \text{ in Cases II a) -- c)}$$

which are valid in norm.

Given the expansion (6.3) one infers e.g.

(6.4)
$$S_{\varepsilon_1\varepsilon_2}(k) \underset{k\to 0_+}{=} \sum_{n=0}^{m-2} (ik)^n s^{(n)}_{\varepsilon_1\varepsilon_2} + o(k^{m+\varepsilon-2}), \quad m \geqslant 2 \text{ in Case I, and}$$
$$S_{\varepsilon_1\varepsilon_2}(k) \underset{k\to 0_+}{=} \sum_{n=0}^{m-4} (ik)^n s^{(n)}_{\varepsilon_1\varepsilon_2} + o(k^{m+\varepsilon-4}), \quad m \geqslant 4 \text{ in Cases II a) -- c).}$$

Analogously equalities (4.14) (with $O(k^2)$ replaced by $o(k)$) are true if condition (6.1) with $m = 3$ holds. In the same way equalities (4.15) (with $O(k)$ replaced by $o(1)$) are valid with $m = 4$ in (6.1). In addition for $k \neq 0$, $|k|$ small enough

(6.5)
$$\mathrm{Tr}[R(k) - R_0(k)] \underset{k\to 0_+}{=} \sum_{n=-2}^{m-4} (ik)^n \Delta_n + o(k^{m+\varepsilon-4}), \quad m \geqslant 2 \text{ in Case I, and}$$
$$\mathrm{Tr}[R(k) - R_0(k)] \underset{k\to 0_+}{=} \sum_{n=-3}^{m-6} (ik)^n \Delta_n + o(k^{m+\varepsilon-6}), \quad m \geqslant 3 \text{ in Cases II a) -- c).}$$

Finally, in order to indicate that the trace relations (5.23) and (5.30) hold under considerably weaker assumptions on V it suffices to discuss Levinson's theorem i.e. $N = 0$ in (5.23). Following the proof of Theorem 5.1 step by step shows that in Case I

$$\delta(0_+) = \pi(N_b - 1/2), \quad \delta(\infty) = 0$$

if we take $m = 2$ in (6.1). Similarly, in Cases II a) — c), one gets

$$\delta(0_+) = \pi N_b\,, \quad \delta(\infty) = 0$$

by taking $m = 4$ in (6.1). (It is reasonable to expect that these conditions on V in the case of Levinson's theorem may be improved.)

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