OPERATORS OF TYPE A AND LOCAL ABSOLUTE CONTINUITY

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Let E be a Banach lattice, F a Banach space and $T \in L(E, F)$. T is said to be of *type A* provided that

$$0 \leqslant x_n \downarrow \quad \text{in } E \text{ implies } \{Tx_n\}_n \text{ is norm convergent.}$$

This class of operators was first considered by Dodds [6] who noted the connection with the class of all weakly compact operators defined on $C(S)$ spaces. The terminology is motivated by the fact that the identity of a σ-complete Banach lattice E is of type A if, and only if, E is of type A in the Kantorovich's terminology, i.e. E has order continuous norm.

The aim of the present paper is to develop a parallel to the measure theory and to prove the analogue of the following result due to Bartle, Dunford and Schwartz : Given a (finite additive) measure m on a σ-algebra $\mathscr{T}$, m is σ-additive if, and only if, m is absolutely continuous with respect to a suitable σ-additive measure $\mu: \mathscr{T} \to \mathbf{R}_+$.

The concept of absolute continuity was extended in [12] to operators given on Banach lattices.

Let E and F be as above and let $x' \in E'$, $x' \geqslant 0$.

We shall say that an operator $T \in L(E, F)$ is *locally absolutely continuous* with respect to x' (i.e., $T \ll_{\text{loc}} x'$) provided that for every $\varepsilon > 0$ and every $x \in E$, $x \geqslant 0$, there exists a $\delta > 0$ such that

$$y \in E, |y| \leqslant x \quad \text{and} \quad x'(|y|) < \delta \quad \text{implies} \quad \|Ty\| < \varepsilon.$$

Our main result asserts that if E has a quasi-interior point then an operator $T \in L(E, F)$ is of type A if, and only if, T is locally absolutely continuous with respect to a suitable $x' \in E'$, $x' > 0$; see Theorem 2.1 below. This result includes the Bartle-Dunford-Schwartz theorem mentioned above and is strongly related to a result of Amemya which states that "all Hausdorff order continuous locally solid topologies on a vector lattice induce the same topology on an order bounded set".

As a consequence we are able to prove that the operators of type A admit nice facto-
rizations through order continuous Banach lattices; see Theorem 3.1 below. The
latter fact is used to derive operator theoretical generalizations of Lozanovskii's cha-
racterizations for reflexivity and weak sequential completeness as well as of the renorm-
ing results obtained by Davis, Ghoussoub and Lindenstrauss [5].

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1. PRELIMINARIES

The main ingredients which we need to characterize the operators of type A
in terms of local absolute continuity are a very general scheme to associate AM-spaces
to a given Banach lattice and some consequences of Grothendieck's criterion of
weak compactness in a space $C(S)'$.

Let E be a Banach lattice and let $x \in E$, $x > 0$. Consider the ideal E_x gene-
rated by x in E

$$E_x = \{y \in E ; (\exists) \lambda > 0 \text{ such that } |y| \leq \lambda x\}$$

endowed with the norm

$$\|y\|_x = \inf \{\lambda ; |y| \leq \lambda x\}.$$

Then E_x is an AM-space with a strong order unit (which is x) and thus lattice iso-
metric to a space $C(S_x)$ for a suitable compact Hausdorff space S_x. The canonical
inclusion $i_x : E_x \to E$ is an interval preserving continuous mapping.

Dually, to each positive functional $x' \in E'$ we can associate an AL-space as
follows. Consider on E the following relation of equivalence

$$x \sim y \quad \text{if, and only if,} \quad x'(|x - y|) = 0.$$

The completion of $E/\sim$ with respect to the norm

$$\|x\|_{L^1(x')} = x'(|x|)$$

is an AL-space, which will be denoted by $L^1(x')$. The adjoint of the canonical surjec-
tion $j_{x'} : E \to L^1(x')$ is $i_{x'}$.

The connection between the operators of type A and Grothendieck's theory
on weakly compact operators defined on $C(S)$-spaces is outlined below.

1.1. THEOREM (P. G. Dodds [6]). *Let E be a Banach lattice, F a Banach
space and $T \in L(E, F)$. Then T is of type A if, and only if, T verifies one of the follow-
ing equivalent conditions:*

i) *T maps every order interval into a relatively weakly compact subset, i.e.
$T \circ i_x$ is weakly compact for every $x \in E$, $x > 0$;*

ii) *T maps every order bounded sequence of pairwise disjoint positive elements of E into a norm convergent sequence;*

iii) T'' *maps* $I_E = \{x \in E''; (\exists) y \in E, |x| \leqslant |y|\}$, *the ideal generated by E in E'', into F;*

iv) *If $x_a \downarrow 0$ in I_E then $\|T''x_a\| \to 0$.*

As noted in [10], every weak compactness criterion for operators defined on a Banach space E is equivalent to a weak compactness criterion for bounded subsets of the dual space E'. The same argument allows a translation of Theorem 1.1 for bounded subsets of E':

1.2. THEOREM (Burkinshaw [4] and Fremlin [8]). *Let E be a Banach lattice and $A \subset E'$ a bounded subset. Then the following assertions are equivalent:*

i) $\sup\{|x'(x_m - x_n)|; x' \in A\} \to 0$ *as $m, n \to \infty$ for each monotone order bounded sequence $\{x_n\}_n \subset E$;*

ii) $\sup\{|x'(x_n)|; x' \in A\} \to 0$ *as $n \to \infty$ for each order bounded sequence $\{x_n\}_n$ of pairwise disjoint positive elements of E;*

iii) $x_a \downarrow 0$ *in I_E implies* $\inf_a \sup\{|x'|(x_a); x' \in A\} = 0$;

iv) *The solid hull of A is relatively $\sigma(E', I_E)$-compact.*

It should be noted that Theorem 1.2 was established previous to 1.1.

1.3. LEMMA. *Let E be a Banach lattice and F an order complete Banach lattice with a strong order unit. Then the subspace of all operators of type A belonging to $L(E, F)$ constitutes a closed ideal.*

Proof. Let $T \in L(E, F)$ an operator of type A. Then by [17], Proposition IV.1.3, T^+ exists in $L(E, F)$ and

$$T^+x = \sup_{0 \leqslant y \leqslant x} Ty$$

for every $x \in 0$; by Theorem 1.1 ii) above, T^+ is an operator of type A. The same argument shows that if $0 \leqslant S \leqslant T$ and T is an operator of type A then S is also an operator of type A.   ▨

The notion of a quasi-interior point is due to Schaefer. A positive element u of a Banach lattice E is called a *quasi-interior point* provided $\|x \wedge nu - x\| \to 0$ for every $x \in E, x > 0$. It was noted in [17] that every separable Banach lattice has a quasi-interior point. On the other hand it is clear that every $C(S)$-space and also every Banach lattice with order continuous norm and weak order unit has a quasi-interior point.

A useful remark concerning the connection between the weak order units and the quasi-interior points is the following:

1.4. LEMMA. *Let E be a Banach lattice with a weak order unit $u > 0$. Then u is a weak order unit for I_E if, and only if, u is a quasi-interior point for E.*

Proof. If u is a quasi-interior point of E then $E \subset u^{\perp\perp}$ ($\perp$ is considered in E''); since $u^{\perp\perp}$ is an ideal, it follows that $u^{\perp\perp} \supset I_E$ and thus u is a weak order unit for I_E.

If u is a weak order unit for I_E then $x = \sup(x \wedge nu)$ in I_E for each $x \in E$, $x > 0$. Consequently, $x'(x) = \lim_{n \to \infty} x'(x \wedge nu)$ for every $x' \in E'$ and thus Dini's lemma implies $\|x - x \wedge nu\| \to 0$.   ◪

In the next section the assumption of a quasi-interior point is shown to allow a good reduction of the study of operators of type A to that of weakly compact operators defined on $C(S)$-spaces.

The natural generalization of the concept of a quasi-interior point is that of a topological orthogonal system. According to [17], a family $\{u_a\}_{a \in A}$ of pairwise disjoint non-zero elements of E_+ is a *topological orthogonal system* (t.o.s.) for E provided each closed ideal E_a generated by u_a is a projection band and for each $x \in E$

$$x = \sum_{a \in A} P_a(x),$$

the series being unconditionally convergent in E. Here P_a denotes the band projection associated to u_a, $a \in A$.

2. THE ANALOGUE OF THE BARTLE-DUNFORD-SCHWARTZ THEOREM

The aim of this section is to prove that the behaviour of each operator of type A can be controlled by a suitable positive functional. The particular case where $T = I_E$ was first considered in [12]:

2.1. THEOREM. *Let E be a Banach lattice with a quasi-interior point $u > 0$, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent:*

i) *T is of type A;*

ii) *There exists a positive functional $x' \in E'$ such that $T \leqslant_{\mathrm{loc}} x'$;*

iii) *There exists a positive functional $x' \in E'$ such that $T'' \mid I_E \leqslant_{\mathrm{loc}} x'$;*

iv) *There exists a positive functional $x' \in E'$ such that if $\{x_n\}_n \subset E$ is an order bounded sequence with $x'(|x_n|) \to 0$ then $\|Tx_n\| \to 0$.*

Proof. The implications ii) $\Rightarrow$ iv) and iii) $\Rightarrow$ ii) are immediate, while iv) $\Rightarrow$ i) follows from Theorem 1.1 ii) above.

The implication i) $\Rightarrow$ iii) constitutes the objective of the next two lemmata. The first one extends the well-known fact that every Banach lattice with a quasi-interior point and order continuous topology admits strictly positive functionals (see [17], Theorem II.6.6).

2.2. LEMMA. *Let E be a Banach lattice with a quasi-interior point $u > 0$, F a Banach space and $T \in L(E, F)$ an operator of type A.*

Then there exists a positive functional $x' \in E'$ such that

$$x \in I_E, \quad x'(|x|) = 0 \quad implies \quad T''x = 0.$$

Proof. Let $\mathscr{A} = \{|T'y'| ; y' \in F', \|y'\| \leqslant 1\}$. We shall prove first the following assertion

(*)   For each $\varepsilon > 0$ there exist a $\delta > 0$ and a finite subset $\mathscr{A}_\varepsilon \subset \mathscr{A}$ such that

$$x \in I_E, \ 0 \leqslant x \leqslant u \ \text{ and } \ \sup_{x' \in \mathscr{A}_\varepsilon} x'(x) < \delta \ \text{ implies } \sup_{x' \in \mathscr{A}} x'(x) < \varepsilon.$$

In fact, if the contrary is true, then there are an $\varepsilon_0 > 0$, a sequence $\{x_n\}_n \subset I_E$ and a sequence $\{x'_n\}_n \subset \mathscr{A}$ such that
   a) $0 \leqslant x_n \leqslant u$
   b) $\sup\limits_{1 \leqslant k \leqslant n} x'_k(x_n) \leqslant 2^{-n-1}$
   c) $x'_{n+1}(x_n) \geqslant \varepsilon_0$
for each $n \in \mathbf{N}$. Put $\bar{x}_n = \sup\{x_k ; k \geqslant n\}$ and $\bar{x} = \inf \bar{x}_n$ in I_E. Then $\sup\{x'_k(\bar{x}_n) ; 1 \leqslant k \leqslant n\} \leqslant 2^{-n}$ and thus $x'_k(\bar{x}) = \lim\limits_{n \to \infty} x'_k(\bar{x}_n) = 0$. By Theorem 1.2 above it follows that

$$x'(\bar{x}_n) \to x'(\bar{x})$$

uniformly for $x' \in \mathscr{A}$, in contradiction with c).

Then, according to (*), the positive functional

$$x' = \left(\sum_{n=1}^{\infty} (2^n \, \mathrm{Card} \, \mathscr{A}_{1/n})^{-1} \cdot \sum_{y' \in \mathscr{A}_{1/n}} y' \right) \in E'$$

satisfies the following condition:

(**) $x \in I_E, \ \ 0 \leqslant x \leqslant u \ \text{ and } \ x'(x) = 0 \ \text{ implies } \ T''x = 0.$

The proof ends with the remark that for each $x \in I_E$, $x > 0$, we have $x = \sup(x \wedge nu)$ in I_E and thus from Theorem 1.1 iv) it follows that $T''x = \lim\limits_{n \to \infty} T''(x \wedge nu)$; if in addition $x'(x) = 0$ then $x'(x \wedge nu) = 0$ for all $n \in \mathbf{N}$ and it remains to apply (**).   ▨

2.3. LEMMA. *Let E and F be two Banach lattices, $T \in L(E, F)$ a positive operator of type A and $x' \in E'$ a positive functional such that*

$$x \in I_E, \quad x'(|x|) = 0 \quad implies \quad T''x = 0.$$

Then $T'' \mid I_E \leqslant_{\mathrm{loc}} x'$.

Proof. Suppose that the contrary is true. Then there are an $\varepsilon_0 > 0$, an $x \in I_E$, $x > 0$, and a sequence $\{x_n\}_n \subset [0, x]$ such that

$$x'(x_n) \leqslant 2^{-n} \quad \text{and} \quad \|T''x_n\| \geqslant \varepsilon_0$$

for every $n \in \mathbf{N}$. Put $\bar{x}_n = \sup\{x_k ; k \geqslant n\}$ and $\bar{x} = \inf \bar{x}_n$ in I_E. Then

$$0 \leqslant x'(\bar{x}) = \lim_{n \to \infty} x'(\bar{x}_n) = \lim_{n \to \infty} \sup_{m \geqslant n} x' (\sup_{n \leqslant k \leqslant m} x_k) \leqslant \limsup_{n \to \infty} \sum_{m \geqslant n}^{m} x'(x_k) = 0.$$

On the other hand $\|T''\bar{x}\| \geqslant \varepsilon_0$ (see Theorem 1.1 iv) above), contradiction.   ▨

Now we can prove the implication i) $\Rightarrow$ iii) in Theorem 2.1 above as follows: Let $T \in L(E, F)$ an operator of type A and $\varphi : F \to \ell^\infty(\Gamma)$ an isometry. By Lemma 1.3, $S = |\varphi \circ T|$ is also an operator of type A and thus from Lemmata 2.2 and 2.3 it follows that $S'' | I_E \ll_{\mathrm{loc}} x'$ for a suitable $x' \in E'$, $x' > 0$; or $S'' = |\varphi \circ T|'' \geqslant \geqslant |(\varphi \circ T)''|$, which implies that $\varphi'' \circ T'' | I_E$ (and thus $T'' | I_E$) is locally absolutely continuous with respect to x'.   ▨

The next corollary extends a result due to Ghoussoub and Steele [9] but our argument is different.

2.4. COROLLARY. *Let E be a Banach lattice, F a Banach space and $T \in L(E, F)$ an operator of type* A.

   i) *If $0 \leqslant x_n \leqslant x$ and $x_n \overset{w}{\to} x$ in E then $\|Tx_n - Tx\| \to 0$;*

   ii) *If $0 \leqslant x_n \leqslant y_n$ ($n \in \mathbf{N}$), $x_n \overset{w}{\to} x$ and $\|y_n - x\| \to 0$ then $\|Tx_n - Tx\| \to 0$.*

Proof. i) Clearly, we may restrict ourselves to the case when E is also separable. Then by Theorem 2.1 above there exists an $x' \in E'$, $x' > 0$, such that $T \ll_{\mathrm{loc}} x'$; since $x'(|x - x_n|) = x'(x - x_n) \to 0$ and $0 \leqslant x - x_n \leqslant x$, it follows that $\|Tx - Tx_n\| \to 0$.

   ii) Since $0 \leqslant x_n \leqslant x_n \vee x \leqslant y_n \vee x$, we have $f(x_n) \leqslant f(x_n \vee x) \leqslant f(y_n \vee x)$ and thus $\{x_n \vee x\}_n$ converges weakly to x. Since $x_n \vee x + x_n \wedge x = x_n + x$, we also see that $\{x_n \wedge x\}_n$ converges weakly to x. By i) and Lemma 1.3 it follows that $\|T^\pm(x_n \wedge x) - T^\pm(x)\| \to 0$. Since $0 \leqslant x_n \wedge x \leqslant x_n \leqslant y_n$, we have also that $\|T^\pm(x_n) - T^\pm(x)\| \to 0$, which implies that $\|Tx_n - Tx\| \to 0$.   ▨

As noted in [2], if E is a Banach lattice and $x', y' \in E'$, $x' > 0$, then $y' \ll_{\mathrm{loc}} x'$ if, and only if, y' belongs to the band generated by x' in E'. By combining this fact with Theorem 2.1 above we obtain the following:

2.5. COROLLARY. *Let E be a Banach lattice with a quasi-interior point, F a Banach space and $T \in L(E, F)$ an operator of type* A. *Then there exists a positive functional $x' \in E'$ such that $\mathrm{Im}\, T'$ is contained in the band generated by x'.*

Other applications of Theorem 2.1 will be indicated in the next sections.

2.6. REMARKS. i) The assumption of the existence of a quasi-interior point cannot be dropped in Theorem 2.1 above e.g., consider the case where $T = 1_{\ell^2(\Gamma)}$ with Card $\Gamma > \aleph_0$.

ii) The proof of Lemma 2.3 shows that I_E may be replaced by any σ-complete sublattice $\mathscr{I} \subset E''$ such that $\mathscr{I} \supset E$ and $E' \subset \mathscr{I}'_n$ (= the subset of all order continuous functionals on $\mathscr{I}$); e.g., if $E = C(S)$ then we can consider as $\mathscr{I}$ the space $M(S)$ of all Borel measurable bounded real functions defined on S. If E is order continuous then $E = I_E$. Combining this fact with a remark before Lemma 2.2 we obtain a direct proof of the fact that the identity of an order continuous Banach lattice with a quasi-interior point is locally absolutely continuous.

iii) Let S be a compact Hausdorff space and $\mathscr{B}(S)$ the σ-algebra of all Borel subsets of S. As noted in [1], the weakly compact operators $T \in L(C(S), F)$ correspond to the σ-additive regular mesures $m : \mathscr{B}(S) \to F$ by the formula

$$Tf = \int f \, dm.$$

Given such a pair (m, T), by Theorem 2.1 above there exists an $x' \in C(S)'$, $x' > 0$, such that $T'' \mid I_{C(S)} \ll_{\text{loc}} x'$; since $I_{C(S)} \supset M(S)$ and $T''(\chi_A) = m(A)$ for every $A \in \mathscr{B}(S)$, it follows that $m \ll_{\text{loc}} x'$. The latter result was previously obtained by Bartle, Dunford and Schwartz [1]. It is also true that $m \ll_{\text{loc}} x'$ implies $T \ll_{\text{loc}} x'$.

iv) Theorem 2.1 as well as its main consequences depends heavily on the possibility to extend every operator $T \in L(E, F)$ of type A to I_E. At this point we generalize a result due to Meyer-Nieberg [11] by indicating another extension property of the operators of type A.

Let $T \in L(E, F)$ and $x' \in E'$ a strictly positive functional such that $T \ll_{\text{loc}} x'$. Since x' is strictly positive, the canonical mapping $j_{x'}$ is injective and thus E can be identified with a sublattice of $L^1(x')$. We shall show that T can be extended (as a linear mapping) to the order ideal generated by E in $L^1(x')$.

For, let $y \in L^1(x')$ and $x \in E$ with $|y| \leqslant x$. Then there exists a sequence $\{y_n\}_n \subset E$ such that $\|y_n - y\|_{L^1(x')} \to 0$. By considering $(y_n \wedge x) \vee (-x)$ instead of y_n if necessary, we may assume also that $|y_n| \leqslant x$. Since $T \ll_{\text{loc}} x'$, the limit $\lim_{n \to \infty} Ty_n$ exists in F and we shall put

$$Ty = \lim_{n \to \infty} Ty_n.$$

3. THE STRUCTURE OF THE OPERATORS OF TYPE A

The next theorem allows a good reduction of the study of operators of type A to that of order continuous Banach lattices.

3.1. THEOREM. *Let E be a Banach lattice with a quasi-interior point, F a Banach space and $T \in L(E, F)$. Then T is an operator of type A (if and) only if there exist an order continuous Banach lattice X, a lattice homomorphism R from E into X and an operator $S \in L(X, F)$ such that*

$$T = S \circ R.$$

Moreover, for each $\eta > 0$, R and S can be chosen such that in addition $\|R\| \leqslant \leqslant \eta + \|T\|$ and $\|S\| \leqslant 1$.

Proof. Suppose T is an operator of type A. Then by Theorem 2.1 iii), there exists a positive $x' \in E'$ such that $T''|_{I_E} \leqslant_{\mathrm{loc}} x'$ and $\|x'\| = \eta$. Consider on I_E the following solid semi-norm

$$|||x||| = x'(|x|) + \sup_{|y| \leqslant |x|} \|T''y\|.$$

Then $|||\cdot||| \leqslant (\eta + \|T\|)\|\cdot\|$ and

$$x \in I_E, \quad x'(|x|) = 0 \quad \text{implies} \quad |||x||| = 0.$$

Let $\mathfrak{N} = \{x \in I_E; x'(|x|) = 0\} = \{x \in I_E; |||x||| = 0\}$; $\mathfrak{N}$ is a closed ideal of I_E and thus $I_E/\mathfrak{N}$ is a vector lattice. We shall denote by X the completion of $I_E/\mathfrak{N}$ with respect to the quotient topology associated to $|||\cdot|||$.

Claim: X is an order continuous Banach lattice.

To prove this Claim it suffices (see [17], Theorem II.5.10) to show that

(∗) $\quad 0 \leqslant \hat{x}_n \downarrow$ in X implies $\{\hat{x}_n\}_n$ is a Cauchy sequence in X.

For, notice first that (∗) holds for decreasing sequences of positive elements x_n of I_E (and thus of $I_E/\mathfrak{N}$). In fact, in this case there is an $x \in I_E$ such that $x_n \downarrow x$ By Theorem 1.1 iv) and Lemma 1.3 above, $|(\varphi \circ T)''|$ is also an operator of type A for any isometry $\varphi: F \to l^\infty(\Gamma)$ and

$$|||x_n - x||| \leqslant x'(x_n - x) + |||(\varphi \circ T)''|(x_n - x)|| \to 0.$$

In the general case, let $\varepsilon > 0$ and choose for each $n \in \mathbf{N}$ a positive $\hat{y}_n \in I_E/\mathfrak{N}$ such that $|||\hat{x}_n - \hat{y}_n||| < \varepsilon \cdot 2^{-n-3}$. Put

$$\hat{z}_n = \inf\{\hat{y}_k; 1 \leqslant k \leqslant n\}, \quad n \in \mathbf{N}.$$

Then $0 \leqslant \hat{z}_n \downarrow$ in $I_E/\mathfrak{N}$ and

$$|||\hat{x}_n - \hat{z}_n||| < \varepsilon/4, \quad n \in \mathbf{N}.$$

By a remark above there exists an $N \in \mathbf{N}$ such that

$$|||\hat{z}_m - \hat{z}_n||| < \varepsilon/2 \quad \text{for all} \quad m, n \geqslant N.$$

Then

$$|||\hat{x}_m - \hat{x}_n||| \leqslant |||\hat{x}_m - \hat{z}_m||| + |||\hat{z}_m - \hat{z}_n||| + |||\hat{z}_n - \hat{x}_n||| <$$
$$< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon$$

for all $m, n \geqslant N$ and thus (∗) holds.

Then the desired factorization is obtained for R the composition of the quotient mapping $E \to I_E/\mathfrak{N}$ with the canonical inclusion $I_E/\mathfrak{N} \to X$ and S the extension of T'' to X. ▨

3.2. COROLLARY. *The result of Theorem* 3.1 *above extends to the case of Banach lattices E possessing topological orthogonal systems* $\{u_a\}_{a \in A}$.

Proof. Let $T \in L(E, F)$ an operator of type A and let E_a and P_a as in the definition of a topological orthogonal system. By Theorem 3.1, for each $a \in A$ there are an order continuous Banach lattice X_a, a lattice homomorphism $R_a \in L(E_a, X_a)$ and an operator $S_a \in L(X_a, F)$ such that $T|E_a = S_a \circ R_a$, $\|R_a\| \leqslant 1 + \|T\|$ and $\|S_a\| \leqslant 1$. Consider the mapping $R: E \to \prod_{a \in A} X_a$ given by $R(x) = \{R_a P_a x\}_a$, and let $\varphi \in L(F, \ell^\infty(\Gamma))$ an isometry. Then the completion X of $R(E)$ with respect to the norm

$$\|R(x)\| = \sup \{\| |\varphi \circ T|(|x|)\| \vee \|R_a P_a x\| \, ; a \in A\}$$

is an order continuous Banach lattice. In fact, as in the proof of Theorem 3.1 we can prove that

(∗)   If $0 \leqslant y_n \downarrow$ in X then $\{y_n\}_n$ is a Cauchy sequence in X.

Then $T = S \circ R$ where $S \in L(X, F)$ denotes the operator given by $S(Rx) = Tx$ for $x \in E$. ▨

3.3. COROLLARY. *Let E be a Banach lattice having a topological orthogonal system, F a Banach space and $T \in L(E, F)$ an operator of type A. If Z is a subspace of E isomorphic to c_0 such that $T|Z$ is an isomorphism, then Z is complemented in E.*

Proof. By Corollary 3.2 there are an order continuous Banach lattice X and operators $R \in L(E, X)$ and $S \in L(X, F)$ such that $T = S \circ R$. Since $R(Z)$ is isomorphic to c_0 and X has an order continuous norm, there is also a projection Q of X onto $R(Z)$ (see [3], § 3, Remark 3 d)). Then $P = R^{-1} \circ Q \circ R$ provides a projection of E onto Z. ▨

3.4. COROLLARY. ([13]). *Every operator $T \in L(E, F)$ of type A has the Pełczyński's property* (u), *i.e. for every weak Cauchy sequence $\{x_n\}_n \subset E$ there exists a weakly summable sequence $\{y_n\}_n \subset \overline{T(E)}$ such that*

$$Tx_n - \sum_{k=1}^{n} y_k \overset{w}{\to} 0.$$

Proof. Since every separable subspace of E is contained in a separable sublattice it suffices to consider only the case when E is also separable and thus has a quasi-interior point. As noted by Lozanovskii [3], the identity of any order continuous Banach lattice has the property (u) and thus it remains to make use of Theorem 3.1 above. ▨

The next result is an operator-theoretical generalization of the fact that any weakly sequentially complete Banach lattice is complemented in its second dual.

3.5. PROPOSITION. *Let E be a Banach lattice with a quasi-interior point $u > 0$ (or merely with a t.o.s.), F and G two Banach spaces, $T \in L(E, F)$ an operator of type A and $S \in L(F, G)$ an operator which fixes no copy of c_0.*

Then $(S \circ T)''$ maps the band generated by E in E'' into F and thus $S \circ T$ extends to E''.

Recall that an operator $U \in L(X, Y)$ fixes no copy of the Banach space Z provided no restriction of U to a subspace isomorphic to Z is an isomorphism.

Proof. By Theorem 1.1 above, $T''|I_E$ is an operator of type A and $T''(I_E) \subset F$. On the other hand every operator which fixes no copy of c_0 maps weakly summable sequences into summable sequences (see [14], Lemma 1). By combining this fact with Corollary 3.4 above we obtain that $(S \circ T)''| I_E = S \circ (T''|I_E)$ maps weak Cauchy sequences into weakly convergent sequences.

For each positive x belonging to the band $u^{\perp\perp}$ generated by E in E'' we have

$$x = \sup(x \wedge nu).$$

Then $\{x \wedge nu\}_n$ is a weak Cauchy sequence in I_E and $x \wedge nu \xrightarrow{w'} x$, which implies that $(S \circ T)''x \in F$.

Since the band $u^{\perp\perp}$ is the image of a positive projection on E'', $S \circ T$ extends to E''. ▨

Under the assumptions of Proposition 3.5, the operator $U = S \circ T$ maps weak Cauchy sequences into weakly convergent sequences. We do not know whether every operator U which maps weak Cauchy sequences into weakly convergent sequences admits a factorization as in Proposition 3.5.

The next result answers a question raised in [13] and extends the well-known criterion of reflexivity due to Lozanovskii [3]:

3.6. PROPOSITION. *Let E be a Banach lattice and $T \in L(E, E)$ an operator which fixes no copy of c_0 and ℓ^1. Then T^3 is weakly compact.*

Proof. Since T fixes no copy of ℓ^1, a result due to H. P. Rosenthal [16] shows that T maps bounded sequences into sequences with weak Cauchy subsequences. Since T fixes no copy of c_0, by Theorem 1.1 ii) above it follows that T is an operator of type A. Then a remark above yields that T^3 is weakly compact. ▨

We end this section by an example which shows that T^2 might be not weakly compact and thus the power 3 is the best possible. In fact, it was exhibited in [7] an example of a lattice homomorphism S from a separable Banach lattice X (which contains no isomorph of ℓ^1) onto c_0. Consider the operator $T: X \oplus c_0 \oplus \ell^1 \to \to X \oplus c_0 \oplus \ell^1$ given by

$$T(x, y, z) = (\pi(z), S(x), 0)$$

where π denotes any operator from ℓ^1 onto X. Then T fixes no copy of c_0 and ℓ^1 and T^2 is not weakly compact.

4. TYPE A AND LOCAL UNIFORM CONVEXITY

In this section we shall apply Corollary 3.2 to the problem of renorming Banach spaces to improve the isometric properties of operators defined on them.

Let E and F be two Banach spaces and $T \in L(E, F)$.

T is said to be *locally uniformly convex* provided that T verifies the following condition

(LUC) $\|x_n\| \to \|x\|$ and $\|x_n + x\| \to 2\|x\|$ in E implies $\|Tx_n - Tx\| \to 0$.

T is said to have the *Kadec-Klee property* provided that

(H) $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$ implies $\|Tx_n - Tx\| \to 0$.

Clearly, (LUC) implies (H). Kadec has shown that every separable Banach space E can be renormed such that 1_E have (LUC). This result was extended by Troyanski to the class of all weakly compactly generated Banach spaces and by Davis, Ghoussoub and Lindenstrauss [5] to the class of all Banach lattices with an order continuous norm. Due to Corollary 3.2, the later result can be used to improve the isometric properties of operators of type A.

4.1. THEOREM. *Let E be a Banach lattice with a topological orthogonal system, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent:*

   i) *T is of type A;*

   ii) *E can be renormed (as a Banach lattice) such that T satisfies the condition* (LUC);

   iii) *E can be renormed (as a Banach lattice) such that T satisfies the condition* (H).

Proof. i) $\Rightarrow$ ii). Since T is of type A then by Corollary 3.2 there are an order continuous Banach lattice X, a lattice homomorphism $R \in L(E, X)$ and an operator $S \in L(X, F)$ such that $T = S \circ R$. By [5], we may assume that X is locally uniformly convex.

We shall denote by $\tilde{c}_0$ the vector space c_0 endowed with the norm

$$\|(a_n)_n\|_{\tilde{c}_0} = \sup\left(\sum_{n\in F}\left|\frac{a_n}{2^n}\right|^2\right)^{1/2}$$

where the sup is taken over all finite subsets $F \subset \mathbf{N}$. This norm is equivalent to the sup norm and makes c_0 a locally uniformly convex Banach lattice (see [15]).

Consider on E the following equivalent solid norm

$$|\|x\|| = \|(\|x\|, \|Rx\|, 0, 0, \ldots)\|_{\tilde{c}_0}.$$

Due to the nature of the norm on $\tilde{c}_0$, if $|\|x_n\|| \to |\|x\||$ and $|\|x_n + x\|| \to 2|\|x\||$ then $\|Rx_n\| \to \|Rx\|$ and $\|Rx_n + Rx\| \to 2\|Rx\|$. Since X is supposed to be locally uniformly convex it follows that $\|Rx_n - Rx\| \to 0$ and thus $\|Tx_n - Tx\| \to 0$. Consequently T becomes locally uniformly convex.

The implication ii) $\Rightarrow$ iii) is clear.

The implication iii) $\Rightarrow$ i) is a consequence of the following.

4.2. LEMMA. *Let E be a Banach lattice, F a Banach space and $T \in L(E, F)$ an operator which is not of type* A. *Then there are an $\alpha > 0$, an $x \in E_+$ and a sequence $\{d_n\}_n$ of pairwise disjoint elements of $[0, x]$ such that $\|Td_n\| \geqslant \alpha$ for every $n \in \mathbf{N}$.*

Consequently T satisfies the condition (H) *for no lattice norm on E which is equivalent to the given norm.*

Proof. The first part follows from Theorem 1.1 above. The sequence $\{d_n\}_n$ is equivalent to the natural basis of c_0 and thus $x - d_n \xrightarrow{w} x$. Let $\|\cdot\|_1$ be a lattice norm on E which is equivalent to the given norm $\|\cdot\|$. Then $\|x\|_1 \leqslant \leqslant \varliminf_{n\to\infty}\|x - d_n\|_1 \leqslant \varlimsup_{n\to\infty}\|x - d_n\|_1 \leqslant \|x\|_1$, which implies $\|x - d_n\|_1 \to \|x\|_1$. It remains to remark that $\|T(x - d_n) - Tx\|_1 \geqslant \alpha' > 0$ for every $n \in \mathbf{N}$.   ▨

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