J. OPERATOR THEORY
13(1985), 73–86

TOWARD A CHARACTERIZATION OF REFLEXIVE CONTRACTIONS

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In memory of James P. Williams (1938–1983)

Throughout this paper, we consider bounded linear operators on complex, separable Hilbert spaces. For an operator T, let $\{T\}'$, $\{T\}''$ and $\operatorname{Alg} T$ denote, respectively, its commutant, double commutant and the weakly closed algebra generated by T and I. Let $\operatorname{Lat} T$ denote the invariant subspace lattice of T and $\operatorname{Alg}\operatorname{Lat} T = \{S : \operatorname{Lat} T \subseteq \operatorname{Lat} S\}$. Recall that an operator T is *reflexive* if $\operatorname{Alg}\operatorname{Lat} T = \operatorname{Alg} T$.

The study of reflexive operators was initiated by Sarason [2]. He showed the reflexivity of normal operators and analytic Toeplitz operators. Since then, various classes of operators are known to be reflexive. Among contractions, it is now known that $C_{\cdot 0}$ contractions with unequal defect indices and $C_{1\cdot}$ contractions with at least one finite defect index are reflexive (cf. [6] and [12], resp.). Moreover, the characterization of reflexive operators among c_0 contractions and completely non-unitary weak contractions with finite defect indices has been reduced to that of $S(\varphi)$, the compression of the shift on $H^2 \ominus \varphi H^2$, φ inner (cf. [1] and [12]). It is generally agreed that reflexivity is difficult to characterize unless we have a rather deep understanding of the structure of the operators under study. In this paper we propose a characterization of reflexive contractions with at least one finite defect index which summarizes all the above-mentioned results (at least when restricted to the class of operators we consider). We have not been successful with the most general case. What we can handle is under the additional condition that the outer factor of the characteristic function of the contraction admits a right outer scalar multiple (see Section 1 for the definition). However the results obtained, together with our previous experiences, seem to indicate that this is the appropriate criterion. Our proofs for the more restricted case depend heavily on the scalar multiple condition. More refined methods will be needed to deal with the general case.

In Section 1 below we fix the notation and review some basic facts needed in the subsequent discussions. Then in Section 2 we prove an approximate decomposition theorem. This theorem generalizes the major theorem in [11] and its applicability extends from weak contractions and $C_{1\cdot}$ contractions, as discussed in [11], to $C_{\cdot 0}$ contractions. Using this theorem as a tool, we prove, in Section 3, the double commutant property of certain contractions. Section 4 contains the characterization of reflexivity. The main results proved in these sections (Theorems 3.1 and 4.1) generalize the corresponding ones for $C_{\cdot 0}$ and $C_{1\cdot}$ contractions. We conclude in Section 5 with conjectures concerning the double commutant property and reflexivity of contractions with at least one finite defect index.

1. PRELIMINARIES

In this paper we will use extensively the contraction theory of Sz.-Nagy and Foiaş. The main reference is their book [5].

Let T be a contraction on the Hilbert space H. Let $\mathscr{D}_T = \overline{(I - T^*T)^{1/2}H}$ and $\mathscr{D}_{T^*} = \overline{(I - TT^*)^{1/2}H}$ be the *defect spaces* and $d_T = \operatorname{rank}(I - T^*T)^{1/2}$ and $d_{T^*} = \operatorname{rank}(I - TT^*)^{1/2}$ the *defect indices* of T. T is *completely non-unitary* (*c.n.u.*) if there exists no non-trivial reducing subspace on which T is unitary. Any contraction can be decomposed as the direct sum of a unitary operator and a c.n.u. contraction. A unitary operator can be further decomposed as the direct sum of a singular unitary operator and an absolutely continuous unitary operator. T is *of class* $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x \not\to 0$ (resp. $T^{*n}x \not\to 0$) for any $x \neq 0$; T is *of class* $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x \to 0$ (resp. $T^{*n}x \to 0$) for any x. $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$ for $\alpha, \beta = 0, 1$. T is a *weak contraction* if its spectrum $\sigma(T)$ does not fill the open unit disk $\mathbf{D}$ and $I - T^*T$ is of finite trace. Any contraction has *canonical triangulations* of types

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}.$$

For a contraction T, let $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T(\lambda)\}$ denote its *characteristic function*. If T is c.n.u., then we can consider its *functional model*, that is, consider T being defined on the space

$$H = [H^2(\mathscr{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathscr{D}_T)}] \ominus \{\Theta_T w \oplus \Delta_T w : w \in H^2(\mathscr{D}_T)\}$$

by

$$T(f \oplus g) = P(e^{it} f \oplus e^{it} g),$$

where $\Delta_T = (I - \Theta_T^* \Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. A contractive analytic function $\{\mathscr{D}, \mathscr{D}_*, \Theta(\lambda)\}$ is said to admit a *right* (resp. *left*) *scalar multiple* $\delta(\lambda)$ if $\delta(\lambda) \not\equiv 0$ is a sclar-valued analytic function and there exists a contractive analytic function $\{\mathscr{D}_*, \mathscr{D}, \Omega(\lambda)\}$ such that $\Theta(\lambda)\Omega(\lambda) = \delta(\lambda) I_{\mathscr{D}_*}$ (resp. $\Omega(\lambda)\Theta(\lambda) = \delta(\lambda) I_{\mathscr{D}}$) for all λ in $\mathbf{D}$. If $\Theta(\lambda)\Omega(\lambda) = \delta(\lambda) I_{\mathscr{D}_*}$ and $\Omega(\lambda)\Theta(\lambda) = \delta(\lambda) I_{\mathscr{D}}$ both hold, then we say $\Theta(\lambda)$ admits the *scalar multiple* $\delta(\lambda)$ (*from both sides*).

For operators T_1 on H_1 and T_2 on H_2, $T_1 \prec T_2$ denotes that there exists an injection $X: H_1 \to H_2$ with dense range, called *quasi-affinity*, such that $XT_1 = T_2X$. $T_1 \overset{\text{cd}}{\prec} T_2$ denotes that there exists a family $\{X_\alpha\}$ of operators $X_\alpha: H_1 \to H_2$ such that $X_\alpha T_1 = T_2 X_\alpha$ for each α and $H_2 = \bigvee_\alpha X_\alpha H_1$. T_1 and T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$.

2. AN APPROXIMATE DECOMPOSITION THEOREM

In this section, we prove an approximate decomposition theorem. It generalizes [11], Theorem 2.1 in two respects: (1) it applies to triangulations of T other than the canonical triangulations; (2) the characteristic function of T_1 is only required to admit a right scalar multiple instead of a scalar multiple from both sides.

THEOREM 2.1. *Let T be a c.n.u. contraction on H and let*

$$T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$$

be a triangulation on $H = H_1 \oplus H_2$. If the characteristic function of T_1 admits a right outer scalar multiple $\delta(\lambda)$, then $T \sim T_1 \oplus T_2$. Moreover, there are quasi-affinities $Y: H \to H_1 \oplus H_2$ and $Z: H_1 \oplus H_2 \to H$ intertwining T and $T_1 \oplus T_2$ and such that $YZ = \delta(T_1 \oplus T_2)$ and $ZY = \delta(T)$.

Proof. We will consider T in its functional model. To the triangulation

$$T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix},$$

there corresponds a regular factorization $\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda)$ of $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T(\lambda)\}$ into the product of two contractive analytic functions $\{\mathscr{D}_T, \mathscr{F}, \Theta_1(\lambda)\}$ and $\{\mathscr{F}, \mathscr{D}_{T^*}, \Theta_2(\lambda)\}$ such that H_1 and H_2 can be represented as

$$H_1 = \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathscr{F}),\ v \in \overline{\Delta_1 L^2(\mathscr{D}_T)}\} \ominus \{\Theta_T w \oplus \Delta_T w : w \in H^2(\mathscr{D}_T)\}$$

and

$$H_2 = [H^2(\mathscr{D}_{T^*}) \oplus Z^{-1}(\overline{\Delta_2 L^2(\mathscr{F})} \oplus \{0\})] \ominus \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus 0) : u \in H^2(\mathscr{F})\},$$

where $\Delta_j = (I - \Theta_j^* \Theta_j)^{1/2}$, $j = 1, 2$, and Z is the unitary operator from $\overline{\Delta_T L^2(\mathscr{D}_T)}$ onto $\overline{\Delta_2 L^2(\mathscr{F})} \oplus \overline{\Delta_1 L^2(\mathscr{D}_T)}$ defined by $Z(\Delta_T v) = \Delta_2 \Theta_1 v \oplus \Delta_1 v$ (cf. [5], p. 288). Note that the characteristic function of T_1 coincides with the purely contractive part of Θ_1 (cf. [5], p. 289). Let $\{\mathscr{F}, \mathscr{D}_T, \Omega(\lambda)\}$ be a contractive analytic function such that $\Theta_1 \Omega = \delta I_{\mathscr{F}}$.

Define the operator $S\colon H_2 \to H_1$ by

$$S(x \oplus Z^{-1}(y \oplus 0)) = P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega\, \Theta_2^* x - \Delta_1 \Omega \Delta_2 y)))$$

for $x \oplus Z^{-1}(y \oplus 0)$ in H_2. We first check that S satisfies $T_1 S - S T_2 = \delta(T_1) X$. Assume that, for $x \oplus Z^{-1}(y \oplus 0)$ in H_2,

$$X(x \oplus Z^{-1}(y \oplus 0)) = \Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v),$$

where $u \in H^2(\mathscr{F})$ and $v \in \overline{\Delta_1 L^2(\mathscr{D}_T)}$. Then

$$T_2(x \oplus Z^{-1}(y \oplus 0)) = P_2 T(x \oplus Z^{-1}(y \oplus 0)) =$$

$$= P_2((e^{it} x \oplus e^{it} Z^{-1}(y \oplus 0)) - (\Theta_T w \oplus \Delta_T w)) =$$

$$= (e^{it} x \oplus Z^{-1}(e^{it} y \oplus 0)) - (\Theta_T w \oplus Z^{-1}(\Delta_2 \Theta_1 w \oplus \Delta_1 w)) - (\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v)) =$$

$$= (e^{it} x - \Theta_T w - \Theta_2 u) \oplus Z^{-1}((e^{it} y - \Delta_2 \Theta_1 w - \Delta_2 u) \oplus (-\Delta_1 w - v)),$$

where P_2 denotes the (orthogonal) projection from H onto H_2 and $w \in H^2(\mathscr{D}_T)$. In particular, since $T_2(x \oplus Z^{-1}(y \oplus 0))$ is in H_2, we have $-\Delta_1 w - v = 0$. Hence

$$(T_1 S - S T_2)(x \oplus Z^{-1}(y \oplus 0)) =$$

$$= T_1 P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_2^* x - \Delta_1 \Omega \Delta_2 y))) -$$

$$- S((e^{it} x - \Theta_T w - \Theta_2 u) \oplus Z^{-1}((e^{it} y - \Delta_2 \Theta_1 w - \Delta_2 u) \oplus 0)) =$$

$$= P(0 \oplus e^{it} Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_2^* x - \Delta_1 \Omega \Delta_2 y))) -$$

$$- P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_2^* (e^{it} x - \Theta_T w - \Theta_2 u) - \Delta_1 \Omega \Delta_2 (e^{it} y - \Delta_2 \Theta_1 w - \Delta_2 u)))) =$$

$$= P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_2^* \Theta_T w - \Delta_1 \Omega \Theta_2^* \Theta_2 u - \Delta_1 \Omega \Delta_2^2 \Theta_1 w - \Delta_1 \Omega \Delta_2^2 u))) =$$

$$= P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_1 w - \Delta_1 \Omega u))),$$

where in the last equality we make use of the identity $\Delta_2^2 = I - \Theta_2^* \Theta_2$.

On the other hand,

$$\delta(T_1) X(x \oplus Z^{-1}(y \oplus 0)) = \delta(T_1)(\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v)) = P(\delta \Theta_2 u \oplus \delta Z^{-1}(\Delta_2 u \oplus v)) =$$

$$= P(\Theta_T \Omega u \oplus Z^{-1}(\Delta_2 \Theta_1 \Omega u \oplus \Delta_1 \Omega u)) + P(0 \oplus Z^{-1}(0 \oplus (\delta v - \Delta_1 \Omega u))) =$$

$$= P(\Theta_T \Omega u \oplus \Delta_T \Omega u) + P(0 \oplus Z^{-1}(0 \oplus (-\delta \Delta_1 w - \Delta_1 \Omega u))) =$$

$$= P(0 \oplus Z^{-1}(0 \oplus (-\delta \Delta_1 w - \Delta_1 \Omega u))).$$

We need to check that

$$P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_1 w - \Delta_1 \Omega u))) = P(0 \oplus Z^{-1}(0 \oplus (-\delta \Delta_1 w - \Delta_1 \Omega u)))$$

or, equivalently,

$$P(0 \oplus Z^{-1}(0 \oplus (-\Delta_1 \Omega \Theta_1 w + \delta \Delta_1 w))) = 0.$$

Indeed, the last expression equals

$$P(0 \oplus Z^{-1}(\Delta_2 \Theta_1(\delta - \Omega \Theta_1)w \oplus \Delta_1(\delta - \Omega \Theta_1)w)) =$$

$$= P(0 \oplus \Delta_T(\delta - \Omega \Theta_1)w) =$$

$$= P(\Theta_T(\delta - \Omega \Theta_1)w \oplus \Delta_T(\delta - \Omega \Theta_1)w) = 0.$$

We conclude that $T_1 S - S T_2 = \delta(T_1) X$ as asserted.

Let

$$Y = \begin{bmatrix} \delta(T_1) & S \\ 0 & I \end{bmatrix} : H \to H_1 \oplus H_2$$

and

$$Z = \begin{bmatrix} I & V - S \\ 0 & \delta(T_2) \end{bmatrix} : H_1 \oplus H_2 \to H,$$

where V is the operator appearing in the triangulation of $\delta(T)$ on $H_1 \oplus H_2$:

$$\delta(T) = \begin{bmatrix} \delta(T_1) & V \\ 0 & \delta(T_2) \end{bmatrix}.$$

The proof that Y and Z implement the quasi-similarity of T and $T_1 \oplus T_2$ and satisfy $YZ = \delta(T_1 \oplus T_2)$ and $ZY = \delta(T)$ is the same as in the proof of [11], Theorem 2.1. We omit the details.

As shown in [11], the preceding result is applicable to weak contractions and $C_{1\cdot}$ contractions with at least one finite defect index. Corollary 2.2 below shows that it also applies to $C_{\cdot 0}$ contractions. In this case, it covers part of [14], Theorem 2.

COROLLARY 2.2. *Let T be a $C_{\cdot 0}$ contraction on H and let* $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *on $H = H_1 \oplus H_2$ be the canonical triangulation of type* $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. *If the characteristic function of T_2 admits a left outer scalar multiple $\delta(\lambda)$, then $T \sim T_1 \oplus T_2$. Moreover, there are quasi-affinities Y: $H \to H_1 \oplus H_2$ and Z: $H_1 \oplus H_2 \to H$ intertwining T and $T_1 \oplus T_2$ and such that $YZ = \delta(T_1 \oplus T_2)$ and $ZY = \delta(T)$.*

Proof. Since $C_{\cdot 0}$ contractions are c.n.u., the assertions follow by applying Theorem 2.1 to T^*.

3. DOUBLE COMMUTANT PROPERTY

An operator T is said to satisfy the *double commutant property* (*DCP*) if $\{T\}'' = \mathrm{Alg}\, T$. Necessary and sufficient conditions for a c.n.u. weak contraction with finite defect indices to satisfy DCP have been given in [7], Theorem 4.4. The remaining case, under the additional assumption on the scalar multiple, is covered in Theorem 3.1 below, the main theorem of this section. We think that this extra assumption is superfluous (cf. Section 5). But, in any case, this theorem already generalizes [6], Theorem 1 and [11], Theorem 3.13.

THEOREM 3.1. *Let T be a contraction with at least one finite defect index. Assume that the outer factor of the characteristic function of T admits a right outer scalar multiple.*

(1) *If T is not a weak contraction, then T satisfies* DCP.

(2) *If T is not a weak contraction and has no singular unitary summand, then* $\{T\}'' = \mathrm{Alg}\, T = \{\varphi(T) : \varphi \in H^\infty\}$.

For the proof, we start by observing that for a contraction T repeated applications of canonical triangulations yield a triangulation of type

$$\begin{bmatrix} C_{01} & & & & * \\ & C_{00} & & & \\ & & C_{11} & & \\ & 0 & & C_{00} & \\ & & & & C_{10} \end{bmatrix}.$$

This triangulation is, in general, not unique as can be easily seen by considering a direct sum of contractions of various classes. In case T has at least one finite defect index, then more can be said.

LEMMA 3.2. *If T is a contraction with at least one finite defect index, then T has a unique triangulation of type*

$$(*) \qquad \begin{bmatrix} C_{01} & & & * \\ & C_{11} & & \\ & 0 & C_{00} & \\ & & & C_{10} \end{bmatrix}.$$

Proof. Let

$$T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$$

on $H = H_1 \oplus H_2$ be the canonical triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ with the corresponding canonical factorization $\Theta_T = \Theta_i \Theta_e$, where $\{\mathcal{D}_T, \mathcal{F}, \Theta_e(\lambda)\}$ and $\{\mathcal{F}, \mathcal{D}_{T^*}, \Theta_i(\lambda)\}$ are the outer and inner factors of $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$, respectively. We have $d_T = \dim \mathcal{D}_T \geqslant \dim \mathcal{F}$ and $d_{T^*} = \dim \mathcal{D}_{T^*} \geqslant \dim \mathcal{F}$ (cf. [5], p. 192).

Our assumption on the defect indices of T implies that $\dim \mathscr{F} < \infty$. Since Θ_{T_1} coincides with the purely contractive part of Θ_e, we infer that $d_{T_1^*} < \infty$. Hence T_1 can be triangulated of type $\begin{bmatrix} C_{01} & * \\ 0 & C_{11} \end{bmatrix}$ (cf. [11], Lemma 3.2). Together with

$$C_{\cdot 0} = \begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix},$$

this yields a triangulation of type (*).

Now for the uniqueness. If

$$T = \begin{bmatrix} T_3 & & & * \\ & T_4 & & \\ & 0 & T_5 & \\ & & & T_6 \end{bmatrix}$$

on $H = H_3 \oplus H_4 \oplus H_5 \oplus H_6$ is a triangulation of type (*), then, by considering their corresponding regular factorizations, it is easily seen that $\begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ and $\begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix}$ are of classes $C_{\cdot 1}$ and $C_{\cdot 0}$, respectively. Hence

$$T = \begin{bmatrix} \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix} & * \\ 0 & \begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix} \end{bmatrix}$$

coincides with the canonical triangulation

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$$

of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Thus

$$T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix}$$

coincide with the canonical triangulations of T_1 and T_2 of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. This proves the uniqueness.

Note that in the preceding lemma, T is a weak contraction if and only if its triangulation of type (*) is of the form $\begin{bmatrix} C_{11} & * \\ 0 & C_{00} \end{bmatrix}$.

In view of the approximate decomposition theorem in Section 2, it seems reasonable to expect a contraction (with at least one finite defect index) to behave as the direct sum of the diagonals appearing in its triangulation of type (*), at least when the DCP and reflexivity property are concerned. Then known results about these summand operators can be exploited to achieve our desired result for the contraction.

To make our proof of Theorem 3.1 more readable, we single out some repeatedly used arguments and state them as the following two lemmata.

LEMMA 3.3. *Let*

$$T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$$

on $H=H_1\oplus H_2$ *be a contraction without singular unitary summand. If* $\{T_2\}''=\{\varphi(T_2):\varphi\in H^\infty\}$, $T_2 \overset{cd}{\prec} T_1$ *and* $T_1S-ST_2=\delta(T_1)X$ *for some operator* $S: H_2\to H_1$ *and outer function* δ, *then* $\{T\}''=\{\varphi(T):\varphi\in H^\infty\}$.

Note that the assumption on the absence of the singular unitary summand is needed to guarantee that $\varphi(T)$ is well-defined for all $\varphi\in H^\infty$. We also remark that this lemma has already been used in the proof of [11], Theorem 3.13.

Proof. Let

$$R=\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\in\{T\}''.$$

From the relation $T_1S-ST_2=\delta(T_1)X$, it is easily seen that

$$U\equiv\begin{bmatrix} \delta(T_1) & S \\ 0 & 0 \end{bmatrix}$$

is in $\{T\}'$. Hence $RU=UR$. A simple calculation yields $R_{21}\delta(T_1)=0$. Since $\delta(T_1)$ has dense range in H_1, $R_{21}=0$ and therefore $R_{22}\in\{T_2\}'$. We check that actually $R_{22}\in\{T_2\}''$. Let $J\in\{T_2\}'$, and let

$$Y=\begin{bmatrix} \delta(T_1) & S \\ 0 & I \end{bmatrix}: H\to H_1\oplus H_2$$

and

$$Z=\begin{bmatrix} I & V-S \\ 0 & \delta(T_2) \end{bmatrix}: H_1\oplus H_2\to H$$

be as in the proof of Theorem 2.1. As before, we have $YZ=\delta(T_1\oplus T_2)$ and $ZY=\delta(T)$. Moreover, it is easily checked that $Z(I\oplus J)Y\in\{T\}'$. Hence $Z(I\oplus J)YR=RZ(I\oplus J)Y$. A simple calculation yields $\delta(T_2)JR_{22}=R_{22}\delta(T_2)J=\delta(T_2)R_{22}J$. Since $\delta(T_2)$ is an injection, we have $JR_{22}=R_{22}J$ whence $R_{22}\in\{T_2\}''$ as asserted. (A similar argument shows that $R_{11}\in\{T_1\}''$.) Thus there exists $\varphi\in H^\infty$

such that $R_{22} = \varphi(T_2)$. We have $\varphi(T_1)W = W\varphi(T_2) = WR_{22}$ for any $W: H_2 \to H_1$ satisfying $T_1 W = WT_2$. On the other hand, since

$$K \equiv \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix} \in \{T\}',$$

we have $RK = KR$. It follows that $R_{11}W = WR_{22}$ whence $R_{11}W = \varphi(T_1)W$. From our assumption $T_2 \overset{\text{cd}}{\prec} T_1$, we conclude that $R_{11} = \varphi(T_1)$. Thus R is triangulated as $\begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}$. But we also have

$$\varphi(T) = \begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}.$$

Hence $R - \varphi(T) = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix} \in \{T\}''$. To complete the proof, it suffices to show that $Q = 0$. Since $U \in \{T\}'$, we have $U(R - \varphi(T)) = (R - \varphi(T))U$. A simple calculation yields $\delta(T_1)Q = 0$. Since $\delta(T_1)$ is an injection, we conclude that $Q = 0$, completing the proof.

LEMMA 3.4. *Let T_1 be a $C_{\cdot 1}$ contraction on H_1 ($\neq \{0\}$) with $d_{T_1^*} < \infty$ and without singular unitary summand, and let T_2 be a $C_{\cdot 0}$ contraction on H_2 ($\neq \{0\}$) with $d_{T_2} < \infty$.*

(1) *If $d_{T_2} \neq d_{T_2^*}$, then $T_2 \overset{\text{cd}}{\prec} T_1$.*

(2) *If $d_{T_1} \neq d_{T_1^*}$, then $T_1^* \overset{\text{cd}}{\prec} T_2^*$.*

Proof. Let $T_1 = U \oplus \tilde{T}_1$, where U is an absolutely continuous unitary operator and $\tilde{T}_1$ is c.n.u. . By [11], Theorem 3.5, $\tilde{T}_1^* \prec T_3 \oplus T_4$, where T_3 is another absolutely continuous unitary operator and T_4 is a unilateral shift. On the other hand, there are a C_0-Jordan operator $T_5 = S(\varphi_1) \oplus \ldots \oplus S(\varphi_n)$ and a unilateral shift T_6 such that $T_2 \prec T_5 \oplus T_6$, where φ_j's are inner functions and, for each inner φ, $S(\varphi)$ denotes the compression of the unilateral shift on the space $H^2 \ominus \varphi H^2$ (cf. [3], Theorem 3).

(1) If $d_{T_2} \neq d_{T_2^*}$, then T_6 is not missing. Let S denote the simple unilateral shift and M denote the operator of multiplication by e^{it} on $L^2(E)$, E being a Borel subset of the unit circle. It is known that $S \prec M$ and $S \prec S^*$ (cf. [9], Lemma 2.5 and [4], Proposition 5, resp.). Thus we infer that $T_6 \overset{\text{cd}}{\prec} U \oplus T_3^*$ and $T_6 \overset{\text{cd}}{\prec} T_4^*$. Combining these relations with the facts that $T_2 \prec T_5 \oplus T_6$ and $U \oplus T_3^* \oplus T_4^* \prec T_1$, we obtain that $T_2 \overset{\text{cd}}{\prec} T_1$ as asserted.

(2) In this case, T_4 is not missing. Let S on H^2 be as in (1). The (orthogonal) projection from H^2 onto $H^2 \ominus \varphi H^2$ defines an operator intertwining S and $S(\varphi)$.

Hence we infer that $T_4 \overset{\text{cd}}{\prec} T_5^*$. Combining this with $T_4 \overset{\text{cd}}{\prec} T_6^*$, $T_1^* \prec U^* \oplus T_3 \oplus T_4$ and $T_5^* \oplus T_6^* \prec T_2^*$, we conclude that $T_1^* \overset{\text{cd}}{\prec} T_2^*$, completing the proof.

Now we are ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. By virtue of [12], Lemma 1.3, we only need to prove (2). Let $T = U \oplus \tilde{T}$ on $H = H_0 \oplus \tilde{H}$, where U is an absolutely continuous unitary operator and $\tilde{T}$ is c.n.u.. Let

$$\tilde{T} = \begin{bmatrix} T_1 & \tilde{X} \\ 0 & T_2 \end{bmatrix}$$

on $\tilde{H} = H_1 \oplus H_2$ be the canonical triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. From our assumption on the defect indices of T, we infer that $d_{T_1^*} < \infty$ and $d_{T_2} < \infty$ (cf. proof of Lemma 3.2). There are four cases to consider:

(i) $H_0 \oplus H_1 = \{0\}$. Then $T = T_2$ is of class $C_{\cdot 0}$. Since T is not a weak contraction, its defect indices are unequal. Thus $\{T\}'' = \{\varphi(T) : \varphi \in H^\infty\}$ by [6], Theorem 1.

(ii) $H_2 = \{0\}$. Then $T = U \oplus T_1$ and T_1 are $C_{\cdot 1}$ contractions with unequal defect indices. Let

$$T_1 = \begin{bmatrix} T_3 & X \\ 0 & T_4 \end{bmatrix}$$

on $H_1 = H_3 \oplus H_4$ be the canonical triangulation of type $\begin{bmatrix} C_{01} & * \\ 0 & C_{11} \end{bmatrix}$ (cf. [11], Lemma 3.2). Note that $H_3 \neq \{0\}$ for otherwise $T_1 = T_4$, being of class C_{11}, will have equal defect indices. If $H_0 \oplus H_4 = \{0\}$, then $T = T_3$ and $\{T\}'' = \{\varphi(T) : \varphi \in H^\infty\}$ by [11], Theorem 3.13. Hence we may assume that $H_0 \oplus H_4 \neq \{0\}$. We want to apply Lemma 3.3 to

$$T^* = \begin{bmatrix} V & Y \\ 0 & T_3^* \end{bmatrix},$$

where

$$V = \begin{bmatrix} U^* & 0 \\ 0 & T_4^* \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 \\ X^* \end{bmatrix}.$$

To verify the conditions there, note that since T_3^* is a $C_{\cdot 0}$ contraction with unequal defect indices, we have $\{T_3^*\}'' = \{\varphi(T_3^*) : \varphi \in H^\infty\}$ by [6], Theorem 1, and $T_3^* \overset{\text{cd}}{\prec} V$ by Lemma 3.4 (1). On the other hand, apply Theorem 2.1 to T_1^* and obtain $S: H_3 \to H_4$ such that $T_4^* S - S T_3^* = \delta(T_4^*) X^*$ for some outer δ. Let $S' = \begin{bmatrix} 0 \\ S \end{bmatrix} : H_3 \to H_0 \oplus H_4$. Then $VS' - S'T_3^* = \delta(V)Y$. Thus Lemma 3.3 is applicable and we conclude that $\{T^*\}'' = \{\varphi(T^*) : \varphi \in H^\infty\}$. It follows that $\{T\}'' = \{\varphi(T) : \varphi \in H^\infty\}$ as asserted.

(iii) $H_0 \oplus H_1 \neq \{0\}$, $H_2 \neq \{0\}$ and $d_{T_2} \neq d_{T_2^*}$. As in (ii), apply Lemma 3.3 to

$$T = \begin{bmatrix} V & Y \\ 0 & T_2 \end{bmatrix}, \quad \text{where } V = \begin{bmatrix} U & 0 \\ 0 & T_1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ \tilde{X} \end{bmatrix}.$$

We leave the details to the readers.

(iv) $H_0 \oplus H_1 \neq \{0\}$, $H_2 \neq \{0\}$ and $d_{T_1} \neq d_{T_1^*}$. In this case, we need a more elaborate argument. Consider

$$T = \begin{bmatrix} V & Y \\ 0 & T_2 \end{bmatrix}$$

as in (iii). Since Theorem 2.1 is applicable to $\tilde{T}$, $T_1 S - ST_2 = \delta(T_1)\tilde{X}$ for some operator $S: H_2 \to H_1$ and outer function δ. Let $S' = \begin{bmatrix} 0 \\ S \end{bmatrix} : H_2 \to H_0 \oplus H_1$. Then $VS' - S'T_2 = \delta(V)Y$. Let $R \in \{T\}''$. Using the above relation and arguing as in the first part of the proof of Lemma 3.3, we obtain that

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where $R_{11} \in \{V''\}$. Since V is a $C_{\cdot 1}$ contraction with unequal defect indices, $R_{11} = \varphi(V)$ for some $\varphi \in H^\infty$ as proved in (ii). For any $W: H_2 \to H_0 \oplus H_1$ satisfying $WT_2 = VW$, we have

$$Q \equiv \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix} \in \{T\}'.$$

Hence $QR = RQ$ yields $WR_{22} = R_{11}W$. On the other hand, we also have $W\varphi(T_2) = \varphi(V)W = R_{11}W$. It follows that $WR_{22} = W\varphi(T_2)$ or $R_{22}^* W^* = \varphi(T_2)^* W^*$. Since, by Lemma 3.4 (2), we have $V^* \overset{cd}{\prec} T_2^*$ or $\bigvee_{WT_2 = VW} W^*(H_0 \oplus H_1) = H_2$, we conclude that $R_{22}^* = \varphi(T_2)^*$ or $R_{22} = \varphi(T_2)$. Thus

$$R = \begin{bmatrix} \varphi(V) & R_{12} \\ 0 & \varphi(T_2) \end{bmatrix}.$$

Since $R - \varphi(T) \in \{T\}''$, we infer that

$$R - \varphi(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & P \\ 0 & 0 & 0 \end{bmatrix}$$

for some operator $P: H_2 \to H_1$. Note that

$$M \equiv \begin{bmatrix} \delta(T_1) & S \\ 0 & 0 \end{bmatrix} \in \{\tilde{T}\}' \quad \text{and} \quad N \equiv \begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix} \in \{\tilde{T}\}''.$$

From $MN = NM$, we obtain $\delta(T_1)P = 0$. Since $\delta(T_1)$ is an injection, $P = 0$ follows. Thus $R = \varphi(T)$, completing the proof.

4. REFLEXIVITY

In [12], Theorem 2.3, we gave necessary and sufficient conditions for a c.n.u. weak contraction with finite defect indices to be reflexive. Our next theorem covers the remaining case, under the additional assumption on the scalar multiple. It generalizes [6], Theorem 2 and [12], Theorem 1.4.

THEOREM 4.1. *Let T be a contraction with at least one finite defect index. Assume that the outer factor of the characteristic function of T admits a right outer scalar multiple. If T is not a weak contraction, then T is reflexive.*

The idea of the proof is to reduce the consideration of the reflexivity of T to that of the direct sum of the diagonals appearing in its triangulation of type (*). It is similar in spirit to the proof for the reflexivity of $C_{1\cdot}$ contractions (cf. [12], Section 1).

Proof. By virtue of [12], Lemma 1.3, we may assume that T has no singular unitary summand. Let $T = U \oplus \tilde{T}$ on $H = H_0 \oplus \tilde{H}$ and

$$\tilde{T} = \begin{bmatrix} T_1 & \tilde{X} \\ 0 & T_2 \end{bmatrix}$$

on $\tilde{H} = H_1 \oplus H_2$ be as in the proof of Theorem 3.1. Since Theorem 2.1 is applicable to $\tilde{T}$, $\tilde{T} \sim T_1 \oplus T_2$ and there are quasi-affinities $\tilde{Y}$ and $\tilde{Z}$ intertwining $\tilde{T}$ and $T_1 \oplus T_2$ and such that $\tilde{Y}\tilde{Z} = \delta(T_1 \oplus T_2)$ and $\tilde{Z}\tilde{Y} = \delta(\tilde{T})$ for some outer function δ. Let $Y = \delta(U) \oplus \tilde{Y}$ and $Z = I_{H_0} \oplus \tilde{Z}$. Then Y and Z are quasi-affinities intertwining T and $M \equiv U \oplus T_1 \oplus T_2$ and satisfying $YZ = \delta(M)$ and $ZY = \delta(T)$. For $K \in \operatorname{Lat} T$ and $L \in \operatorname{Lat} M$, the mappings $K \to \overline{YK}$ and $L \to \overline{ZL}$ preserve the lattice operations in $\operatorname{Lat} T$ and $\operatorname{Lat} M$ and are inverses to each other. Hence invariant subspaces of T and M are of the forms $\overline{ZL}$ and $\overline{YK}$, where $L \in \operatorname{Lat} M$ and $K \in \operatorname{Lat} T$. Arguing as in [12], Lemma 1.1 by using these facts, we may show that T is reflexive if and only if M is. Next we make a further reduction. Let

$$T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$$

on $H_1 = H_3 \oplus H_4$ be the canonical triangulation of type $\begin{bmatrix} C_{01} & * \\ 0 & C_{11} \end{bmatrix}$ (cf. [11], Lemma 3.2). Since Theorem 2.1 is applicable to T_1^*, we may argue as above to show that M is reflexive if and only if $N \equiv U \oplus T_3 \oplus T_4 \oplus T_2$ is. Note that T_4 is a c.n.u. C_{11} contraction with finite defect indices. Hence T_4 is quasi-similar to an absolutely continuous unitary operator, say V, and the quasi-similarity is implemented by quasi-affinities P and Q satisfying $PQ = \eta(V)$ and $QP = \eta(T_4)$ for some outer function η (cf. [10], Lemma 2.1). As above, we infer that N is reflexive if and only if $K \equiv U \oplus V \oplus T_3 \oplus T_2$ is.

In the following we show the reflexivity of K. For simplicity, let $W = U \oplus V$. Since $C_{\cdot 0}$ contractions with unequal defect indices and $C_{1\cdot}$ contractions with at least one finite defect index are known to be reflexive (cf. [6], Theorem 2 and [12], Theorem 1.4, resp.), we need only to show the reflexivity of the following direct sums whose summands are non-trivial:

(i) $W \oplus T_2$. Since this is a direct sum of an absolutely continuous unitary operator and a $C_{\cdot 0}$ contraction with unequal defect indices, its reflexivity has been proved in [13], Lemma 1.

(ii) $T_3 \oplus T_2$. If $d_{T_2} = d_{T_2^*}$, then T_2 is a C_{00} contraction. Hence $T_3 \oplus T_2$, being a $C_{\cdot 0}$ contraction with unequal defect indices, is reflexive. Thus we may assume that $d_{T_2} \neq d_{T_2^*}$. Let $R \in \operatorname{Alg Lat}(T_3 \oplus T_2)$. Then $R = R_3 \oplus R_2$, where $R_j \in \operatorname{Alg Lat} T_j$, $j=2, 3$. There is φ_j in H^∞ such that $R_j = \varphi_j(T_j)$, $j=2, 3$ (cf. [6], Theorem 2). For any operator $J: H_2 \to H_3$ satisfying $JT_2 = T_3J$, consider the (closed) subspace $G = \{Jx \oplus x : x \in H_2\}$ in $\operatorname{Lat}(T_3 \oplus T_2)$. We infer from $RG \subseteq G$ that, for any $x \in H_2$, $\varphi_3(T_3)Jx \oplus \varphi_2(T_2)x = Jy \oplus y$ for some $y \in H_2$. It follows that $\varphi_3(T_3)J = J\varphi_2(T_2) = \varphi_2(T_3)J$. However $T_2 \overset{\text{cd}}{\prec} T_3$ by Lemma 3.4 (1). We conclude that $\varphi_3(T_3) = \varphi_2(T_3)$ whence $\varphi_3 = \varphi_2$ a.e.. This shows that $R = \varphi_2(T_3 \oplus T_2) \in \operatorname{Alg}(T_3 \oplus T_2)$ and the reflexivity of $T_3 \oplus T_2$ follows.

(iii) $K = W \oplus T_3 \oplus T_2$. If $d_{T_2} = d_{T_2^*}$, then, as in (ii), $T_3 \oplus T_2$ is a $C_{0\cdot}$ contraction with unequal defect indices. The reflexivity of K follows as in (i). Next we consider the case $d_{T_2} \neq d_{T_2^*}$. By [13], Lemma 1, $W \oplus T_j$ is reflexive and $\operatorname{Alg}(W \oplus T_j) = \{\varphi(W \oplus T_j) : \varphi \in H^\infty\}$, $j=2, 3$. Let $R \in \operatorname{Alg Lat} K$. Then $R = R_0 \oplus R_3 \oplus R_2$ with $R_0 \oplus R_j \in \operatorname{Alg Lat}(W \oplus T_j)$, $j = 2, 3$. Hence $R_0 \oplus R_j = \varphi_j(W \oplus T_j)$ for some $\varphi_j \in H^\infty$. We infer from $R_0 = \varphi_2(W) = \varphi_3(W)$ that $\varphi_2 = \varphi_3$ a.e.. Thus $R = \varphi_2(K) \in \operatorname{Alg} K$. This shows the reflexivity of K and completes the proof.

5. CONJECTURES

As we remarked before, it is quite plausible that the extra assumption on the scalar multiple in Theorems 3.1 and 4.1 can be dropped, that is, a contraction with at least one finite defect index behaves, as far as the DCP and reflexivity property are concerned, like the direct sum of the diagonals in its triangulation of type (*). We state them as the following conjectures:

CONJECTURE 5.1. *Let T be a contraction with at least one finite defect index.*

(1) *If T is not a weak contraction, then T satisfies* DCP.

(2) *If T is not a weak contraction and has no singular unitary summand, then* $\{T\}'' = \operatorname{Alg} T = \{\varphi(T) : \varphi \in H^\infty\}$.

CONJECTURE 5.2. *Let T be a contraction with at least one finite defect index. If T is not a weak contraction, then T is reflexive.*

If Conjecture 5.2 is true, it will reduce the characterization of the reflexivity of contractions with at least one finite defect index to that of $S(\varphi)$, the compression of the shift on $H^2 \ominus \varphi H^2$, φ inner (cf. [12], Theorem 2.3 and [8] or [1]).

The preceding two conjectures can be summarized as the following:

CONJECTURE 5.3. *A contraction with unequal defect indices satisfies* DCP *and is reflexive.*

This research was partially supported by National Science Council of Taiwan.

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Received August 24, 1983; revised October 25, 1983.

Added in proof. Conjectures 5.1 and 5.2 have since been known to be true for any contraction T for which $(1 - T^*T)^{1/2}$ is of Hilbert-Schmidt class. The former is proved by K. Takahashi [15] and the latter by him and H. Bercovici.