J. OPERATOR THEORY
13(1985), 119–129

A GENERALIZED WEYL THEOREM AND L^p-SPECTRA OF SCHRÖDINGER OPERATORS

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1. INTRODUCTION

Weyl-type theorems occupy the unique place among tools of finding spectra of (pseudo) differential operators. The classical theorem of Weyl states (see [8, vol. I]) that if A and B are self-adjoint and $A-B$ is compact, then $\mathrm{ess\,spec}(A) = \mathrm{ess\,spec}(B)$ (the definitions are given below). Known generalizations of this theorem (see [8, vol. I, IV; 5]) replace the compactness requirement on $A-B$ by the condition that $(A-z)^{-1} - (B-z)^{-1}$ is compact for $z \in \rho(A) \cap \rho(B)$ and relax to various degrees the self-adjointness restriction on A. A generalization to closed operators which uses a different definition of the essential spectrum (W(A), defined below, it coincides with our definition for self-adjoint operators) was given in [5, p. 244, Theorem 5.35]. The results below (Theorems 1 and 5) go beyond these. We use also one of our results to solve a special case of one of the problems posed by B. Simon [12, 13, 14]: we prove that the spectra of Schrödinger operators on the L^p-spaces are independent of p.

All operators below are densely defined. For a closed operator A on a Banach space we adopt the following definitions (see [8, vol. I, IV])

$\mathrm{spec}_X(A)$ = the spectrum of A on X;

$\mathrm{disc\,spec}_X(A)$ = the discrete spectrum of A on X = the set of all isolated eigenvalues of finite (algebraic) multiplicities;

$\mathrm{ess\,spec}_X(A)$ = the essential spectrum of A on $X = \mathrm{spec}_X(A) \setminus \mathrm{disc\,spec}_X(A)$;

$\rho_X(A)$, $\mathrm{D}_X(A)$ = the resolvent set and domain of A, respectively, on X.

When the underlying space X is obvious from the context we omit the subindex X.

$\mathrm{W}(A)$=the Weyl spectrum of $A = \{\lambda \in \mathbf{C} \mid \|(\lambda - A)u_n\| \to 0$ for some sequence $u_n \in \mathrm{D}(A)$, $\|u_n\| \not\to 0$, $u_n \xrightarrow{\mathrm{w}} 0\}$;

A', X' = the dual operator and space, respectively.

It is a pleasure to thank W. Hunziker and Y. Kannai for discussions and J. Voigt for a very useful correspondence.

2. GENERALIZED WEYL THEOREMS

THEOREM 1. *Let L be a closed operator on a reflexive Banach space X and let G be an open complex set. Let there exist a family $F: G \to \mathscr{B}(X)$ such that*

(a) *$F(\lambda)(L - \lambda) - \mathbf{1}$ is compact for each $\lambda \in G$.*

(i) *If* (R) *each connected component of G contains at least one point of $\rho(L)$, then* $\operatorname{ess\,spec}(L) \cap G = \emptyset$.

(ii) *Furthermore, condition* (R) *holds, provided*

(b) $\operatorname{Ker} F(\lambda) = \{0\}$,

(c) *$F(\lambda)(L - \lambda)$ is analytic, and*

(d) *each connected component of G contains at least one point λ_0 such that*

$$\operatorname{Ker}[F(\lambda_0)(L - \lambda_0)] = \{0\}.$$

Proof. (i) We check the Weyl criterion for $\lambda \in G$. Let $u_n \in D(L)$ obey $\|u_n\| \leqslant M < \infty$, $u_n \xrightarrow{w} 0$ and $(L - \lambda)u_n \to 0$ as $n \to \infty$. Due to the compactness of $F(\lambda)(L - \lambda) - \mathbf{1}$ we have that $[F(\lambda)(L - \lambda) - \mathbf{1}]u_n \to 0$ which implies that $u_n \to 0$. Thus G does not intersect the "Weyl spectrum" of L and therefore $\operatorname{ess\,spec}(L) \cap G = \emptyset$ by Corollary 2.3 of [15], quoted at the end of this section.

(ii) We call λ a singular point for the family $A(\lambda) \equiv F(\lambda)(L - \lambda)$ iff $0 \in \sigma(A(\lambda))$. By Theorem 1.9, Section VII, p. 370 of [5], $A(\lambda)$ has at most a finite number of singular points in each compact subset of G. If λ is not a singular point of $A(\lambda)$ then $L - \lambda$ has the bounded left inverse $A(\lambda)^{-1}F(\lambda)$. This and condition (b) imply that $L - \lambda$ is onto (so $A(\lambda)^{-1}F(\lambda)$ is the inverse of $L - \lambda$). Hence $\lambda \in \rho(L)$ and therefore $\rho(L) \cap G \supset G \setminus \{\text{singular points of } A(\lambda)\}$. ▨

REMARK 2. This theorem can be regarded as a generalization of the classical Weyl theorem (see [8, vol. I, IV]) as can be seen from Theorem 5 below. Condition (d) cannot be relaxed (see Example 1, p. 110 of [8]).

Before proceeding to the analysis of consequences of Theorem 1 we give, for the sake of completeness, some known definitions and auxiliary statements. Here we adopt a restricted definition of relative compactness/boundedness. Let A be a closed operator on a Banach space X with $\rho(A) \neq \emptyset$. B is said to be A-compact/A-bounded iff $D(B) \supset D(A)$ and $B(A - \lambda)^{-1}$ is compact/bounded for some, and therefore all, $\lambda \in \rho(A)$. One can show (cf. [10] and Lemma 3 below) that the latter statement is equivalent to requiring that $\rho(A) \neq \emptyset$, $D(B) \supset D(A)$ and B, as an operator from $D(A)$, equipped with the graph norm, to X, is compact/bounded.

LEMMA 3. *Let X be a reflexive Banach space with the approximation property (see [2, 7]). Let A be a closed operator on X and let B be closable. The following statements hold:*

(α) *If $D(B) \supset D(A)$, then for some $a, b > 0$*

$$\|Bu\| \leqslant a\|Au\| + b\|u\| \quad \forall u \in D(A). \tag{1}$$

If, moreover, $\rho(A) \neq \emptyset$, then B is A-bounded.

(β) *If B is A-compact, then for any $a > 0$, there is $b > 0$ such that* (1) *holds.*
(γ) *If B obeys* (1) *with $a < 1$, then $A + B$ is defined and closed on* $\mathrm{D}(A)$.
(δ) *If B is A-compact and $\rho(A + B) \cap \rho(A) \neq \emptyset$, then B is $(A + B)$-compact.*

REMARK 4. The approximation property of X is needed only in the proof of (β) (which is used then in the proof of (δ)).

Proof. (α) By the closed graph theorem, B, considered as an operator from $\mathrm{D}(A)$, equipped with the graph norm, to X, is bounded. This implies (1). Now let $\lambda \in \rho(A) \neq \emptyset$. We have

$$\|B(A - \lambda)^{-1}u\| \leqslant C(\|(A - \lambda)^{-1}u\| + \|A(A - \lambda)^{-1}u\|) \leqslant C_1\|u\|,$$

where $C_1 = C[\|(A - \lambda)^{-1}\|(1 + |\lambda|) + 1]$.

(β) Let $\lambda \in \rho(A)$. Since $\mathrm{D}(A)$ is dense and $B(A - \lambda)^{-1}$ is compact, for any $a > 0$ there is a decomposition $B(A - \lambda)^{-1} = F + G$, where F is a finite-rank operator such that FA is bounded and $\|G\| \leqslant a$ (one uses a cannonical form of the finite rank operators). This gives

$$\|Bu\| \leqslant \|F(A - \lambda)\| \, \|u\| + a\|(A - \lambda)u\| \leqslant a\|Au\| + b\|u\|$$

with $b = \|F(A - \lambda)\| + a|\lambda|$.

(γ) Let $u_n \in \mathrm{D}(A)$ be such that $u_n \to u$ and $(A + B)u_n \to v$. Then

$$\|A(u_n - u_m)\| \leqslant \|(A + B)(u_n - u_m)\| + \|B(u_n - u_m)\|.$$

Using now relation (1) with $a < 1$ we obtain

$$\|Au_n - Au_m\| \leqslant (1 - a)^{-1}(\|(A + B)(u_n - u_m)\| + b\|u_n - u_m\|) \to 0 \quad \text{as } n, m \to \infty.$$

Hence Au_n converges. Since A is closed, this implies that $u \in \mathrm{D}(A)$ and $Au_n \to Au$. Hence $\|B(u_n - u)\| \leqslant a\|A(u_n - u)\| + b\|u_n - u\| \to 0$. Thus Bu_n converges to Bu.

(δ) In virtue of the equation (here $\lambda \in \rho(A) \cap \rho(A + B)$)

$$B(A + B - \lambda)^{-1} = B(A - \lambda)^{-1}[1 - B(A + B - \lambda)^{-1}]$$

it suffices to show that B is $(A + B)$-bounded. By (β) and (γ) $\mathrm{D}(A + B) = \mathrm{D}(A) \subset \mathrm{D}(B)$. Hence, by ($\alpha$), B is $(A + B)$-bounded. One can also use a direct computation:

$$\|B(A + B - \lambda)^{-1}u\| \leqslant a\|A(A + B - \lambda)^{-1}u\| + b\|(A + B - \lambda)^{-1}u\| \leqslant$$

$$\leqslant a\|B(A + B - \lambda)^{-1}u\| + b_1\|u\|,$$

where $b_1 = 1 + (|\lambda| + b)\|(A + B - \lambda)^{-1}\|$. Since a can be taken less than 1, this implies the boundedness of $B(A + B - \lambda)^{-1}$. ▨

THEOREM 5. *Let X be a reflexive Banach space with the approximation property. Let L and T be closed operators on X such that* $D(L') \cap D(T')$ *is a core for L' and T' and $L' - T'$ extends to a T'-compact operator. Assume moreover that each connected component of $\rho(T)$ (resp. of $\rho(L)$) contains a point λ_0 such that* $\operatorname{Ker}[(T-\lambda_0)^{-1}(L-\lambda_0)] = \{0\}$ *(resp.* $\operatorname{Ker}[(L-\lambda_0)^{-1}(T-\lambda_0)] = \{0\}$*). Then* $\operatorname{ess\,spec}(L) = \operatorname{ess\,spec}(T)$.

Proof. Taking $F(\lambda) = (T-\lambda)^{-1}$ we see readily that the conditions of Theorem 1 are obeyed for $G = \rho(T)$ (one uses here that $\rho(T') = \rho(T)$) and therefore $\rho(T) \cap \operatorname{ess\,spec}(L) = \emptyset$. Reversing the roles of L and T and using Lemma 3 (here we need the approximation property) we get: $\rho(L) \cap \operatorname{ess\,spec}(T) = \emptyset$. Thus $\operatorname{ess\,spec}(L) \subset \operatorname{spec}(T)$ and $\operatorname{spec}(L) \supset \operatorname{ess\,spec}(T)$.

It remains now to check the discrete points. Let $\lambda_0 \in \operatorname{disc\,spec}(T)$ and let $F(\lambda)$ be the reduced resolvent of T, i.e. $F(\lambda) = (T-\lambda)^{-1}(\mathbf{1} - P)$ where $P = (2\pi i)^{-1} \oint_\Gamma (T-\lambda)^{-1} d\lambda$ with Γ a contour in $\rho(T)$ encircling λ_0. Since $\lambda_0 \in \operatorname{disc\,spec}(T)$, P is finite-rank. By the construction, $F(\lambda)$ is analytic in $\rho(T) \cup \{\lambda_0\}$ (see [5, p. 178]) and $\operatorname{Ker} F(\lambda) = PX$. Moreover, $F(\lambda)(L-\lambda) - \mathbf{1} = -P + F(\lambda)(L-T)$ is compact for each $\lambda \in \rho(T) \cup \{\lambda_0\}$. Hence, by (i), $\lambda_0 \notin \operatorname{ess\,spec}(L)$. Thus $\operatorname{disc\,spec}(T) \cap \operatorname{ess\,spec}(L) = \emptyset$. Reversing, again, the roles of L and T, we get: $\operatorname{disc\,spec}(L) \cap \operatorname{ess\,spec}(T) = \emptyset$. ▨

In conclusion we quote a result from [15] which played an important role in our proofs.

COROLLARY 2.3. *Let A be a closed operator on a Hilbert space. Let Ω be an open, connected complex set with $\Omega \cap \rho(A) \neq \emptyset$. Let $\Omega \cap W(A) = \emptyset$. Then* $\Omega \cap \operatorname{ess\,spec}(A) = \emptyset$, i.e. *$A$ has purely discrete spectrum in Ω. Moreover,* [15] *remarks that the proof of the theorem from which Corollary* 2.3 *follows holds in reflexive Banach spaces.*

3. APPLICATION: HVZ THEOREM ON THE L^p-SPACES

We apply Theorem 1 to show that the spectrum of a many-body Schrödinger operator on L^p is independent of p (and is given by the famous HVZ-expression [4; 8, vol. IV]). This solves a special case of a problem posed by B. Simon [12] (see also [13, 14]).

We begin with definitions of the many-body Schrödinger operators.

a. N-PARTICLE SCHRÖDINGER OPERATORS. Consider a system of N particles in $\mathbf{R}^\nu$, $\nu \geqslant 3$, with masses m_i and interacting via pair potentials $V_l(x^l)$. Here l labels pairs of indices and $x^l = x_i - x_j$ for $l = (ij)$. The *configuration space* of the system in the center-of-mass frame is defined as $X = \{x \in \mathbf{R}^{\nu N} \mid \sum m_i x_i = 0\}$ with the inner product $(x, \tilde{x}) = 2\sum m_i x_i \cdot \tilde{x}_i$. Denote by V^l and V_l the multiplication operators on $L^2(\mathbf{R}^\nu)$ and $L^2(X)$ by the functions $V_l(y)$ and $V_l(x^l)$, respectively. We say that V^l is Δ-compact on $L^p(\mathbf{R}^\nu)$ if it is compact as operator from the Sobolev space $H_2^p(R^\nu)$ to $L^p(\mathbf{R}^\nu)$.

The Schrödinger operator for the system in question (in the center-of-mass frame) is

$$H = T + V, \text{ where } T = -\text{Laplacian in } X \text{ and } V = \sum V_l .$$

To define H on $L^p(X)$ we require

(A) V^l are Δ-compact on $L^1(\mathbf{R}^\nu)$ and are real.

Under (A) H generates a bounded and continuous Schrödinger semigroup e^{-Ht} on $L^1(X)$ (see [6, 14]) which by duality and interpolation produces the bounded and continuous Schrödinger semigroup on $L^p(X)$, $1 \leqslant p \leqslant \infty$. The generator, H, of the latter semigroup defines the closed Schrödinger operator on $L^p(X)$. For $p = 2$ this operator is self-adjoint. Using the Hölder and Young inequalities (see [8, vol. II] we find that if

$$V^l \in L^r(\mathbf{R}^\nu) + (L^\infty(\mathbf{R}^\nu))_\varepsilon , \quad r \geqslant \frac{\nu}{2} , \tag{2}$$

where ε indicates that the L^∞-component can be taken arbitrary small, then V^l is Δ-compact on $L^p(\mathbf{R}^\nu)$ for $1 \leqslant p \leqslant \nu/2$, provided that $\nu \geqslant 3$. On the other hand condition (A) is stronger than the condition imposed in [14] that the potential V belongs to a certain explicitly defined class K_m (see Theorem A.2.3 of [14]). So we will use freely results from [14].

In conjuncture with condition (A) we use also

(B) $D_{L^p}(H) = \{f \in L^p \mid Hf \in L^p\}$ for each p, $1 \leqslant p \leqslant \infty$.

It is shown in [1] that if $\sum V_l \in L^{p'}_{\mathrm{loc}}(X)$ and is bounded from below then (B) holds. Of course, if V^l are Δ-compact on $L^p(\mathbf{R}^\nu)$, $1 \leqslant p \leqslant \infty$, then (B) is true.

PARTITIONS. Let $a = \{C_i\}$ be a partition of the set $\{1, \ldots, N\}$ into nonempty, disjoint subsets C_i, called clusters. Denote by $\mathscr{A}$ the set of all such partitions. $\mathscr{A}$ is a lattice if $b \subset a$ is set for b a *refinement* of a: the clusters of b are subsets of the clusters of a. $\#(a)$ stands for the number of clusters in partition a.

We denote by $a_{\max}$ the maximal element in $\mathscr{A}$. A pair l will be identified with the decomposition on $N - 1$ clusters, one of which is the pair itself and the others are free particles. The unions and intersections appearing below of sets labeled by partitions are understood to be taken over all partitions excluding $a_{\max}$.

With each partition a we associate the *truncated Schrödinger operator* H_a which is obtained from H by neglecting the potentials linking the different clusters in the partition a:

$$H_a = H - I_a = T + V_a \quad \text{with } I_a = \sum_{l \not\subseteq a} V_l , \quad V_a = \sum_{l \subseteq a} V_l . \tag{3}$$

Finally, the configuration space of an N-particle system in question when the centers-of-mass of subsystems $C_j \in a$ are fixed is

(4) $$X^a = \{x \in X \mid \sum_{i \in C_j} m_i x_i = 0 \quad \forall C_j \in a\}.$$

Let T^a and T_a be the $-$ Laplacians on X^a and on X_a, respectively. Here $X_a = X \ominus X^a$. Define as above $H^a = T^a + \sum_{l \subseteq a} V_l$ on $L^p(X^a)$. Then we have

(5) $$H_a = H^a \otimes \mathbf{1}_a + \mathbf{1}^a \otimes T_a$$

(the removal of the center-of-mass motion in H_a).

Now we can formulate our result (the HVZ theorem on the L^p-spaces).

THEOREM 6. *Assume the potentials V_l satisfy conditions* (A) *and* (B). *Then* $\operatorname{spec}_{L^p}(H)$ *is independent of p and the essential spectrum is given by the* HVZ *formula:*

(6) $$\operatorname{ess\,spec}_{L^p}(H) = \bigcup \operatorname{spec}_{L^2}(H_a) = \bigcup \operatorname{disc\,spec}_{L^2}(H^a) \cup \{0\} + \mathbf{R}^+ .$$

REMARK 7. Strengthening the smoothness conditions on the potentials we can extend this theorem to general complex potentials (in this case one uses in the proof results of [11] instead of [14]).

To prove this theorem we use the method of geometric parametrix introduced in [9] and developed further in [11] (see these papers and [16] for other applications).

b. GEOMETRIC PARAMETRICES. We introduce our principal tool used in the analysis of N-body systems. The definitions and basic statements are borrowed from [9, 11]. In this section we assume that V^l are Δ-compact on $L^p(X)$, $1 \leqslant p \leqslant \infty$.

Let $J = \{j_a\}$ be a partition of unity on $L^p(X)$: j_a are L^∞-function on X and $\sum j_a = 1$. We specify further the partitions we use as follows: assume j_a are homogeneous functions of degree 0 for $|x| \geqslant 1$ localized as

(7) $$\operatorname{supp} j_a \cap \{x \in X \mid |x| \geqslant 1\} \subset \{x \in X \mid |x|_a > d|x|\}$$

for some $d > 0$ and form a regular partition for $|x| < 1$. Here

$$|x|_a = \min_{l \not\subseteq a} |x^l| \quad \text{(the distance between the clusters in } a).$$

It is not difficult to see that the domains on the r.h.s. of (7) with different a's cover entire X.

The partitions described above will be called the *Simon partitions* (after B. Simon, who was first to introduce and apply such partitions [12]). They have the following important property (we assume now $j_a \in C^\infty$)

(8) $$j_a I_a \text{ and } [H_0, j_a] \text{ are } H_0\text{-compact on } L^p(X).$$

The relative compactness of $[H_0, j_a] = -(\Delta j_a) - 2(\nabla j_a)\cdot\nabla$ follows from the degree 0 homogeneity of j_a for $|x| > 1$. To show the relative compactness of $j_a I_a$ one approximates the potentials V_l in I_a by $C_0^\infty(\mathbf{R}^\nu)$ functions U_l and then notices that $U_l j_a$ with $l \not\subseteq a$ has a compact support in the entire space X: $U_l = 0$ for $|x^l| > A$ for some $A > 0$ and therefore $U_l j_a = 0$ for $|x| > d^{-1}A$.

For a Simon partition J we define the geometric parametrix

$$F_J(\lambda) = \sum j_a R_a(\lambda). \tag{9}$$

Here and henceforth, unless mentioned to the contrary, the sum over a extends over all $a \neq a_{\max}$. Using that $H = H_a + I_a$ we derive the equation

$$F_J(\lambda)(H - \lambda) = \mathbf{1} + K_J(\lambda), \tag{10}$$

where

$$K_J(\lambda) = \sum j_a R_a(\lambda) I_a. \tag{11}$$

LEMMA 8. *$K_J(\lambda)$ defines an analytic in $\lambda \in \bigcap \rho(H_a)$ family of compact operators on $L^p(X)$.*

Proof. The boundedness and analyticity are obvious (remember that I_a are H_a-bounded). To prove the compactness we let $\{j_a'\}$ be a Simon C^∞-partition such that $j_a' = 1$ on $\operatorname{supp} j_a$. Commuting j_a' one step to the right and using the relation

$$[j_a', R_a(\lambda)] = R_a(\lambda)[H_0, j_a']R_a(\lambda),$$

we obtain

$$K_J(\lambda) = \sum j_a R_a(\lambda)(I_a j_a' + R_a(\lambda)[j_a', H_0]).$$

Since the expressions in the parentheses on the r.h.s. are H_a-compact, by the choice of j_a', the result follows. ▨

Equation (10) implies readily

LEMMA 9. *Let $H\varphi = \lambda\varphi$ and $\varphi \in L^p(X)$ for $\lambda \in \bigcap \rho(H_a)$. Then $\varphi + K_J(\lambda)\varphi = 0$.*

Denote by $J^s = \{j_a^s\}$ a *sharp* Simon partition of unity, i.e. with each j_a^s indicator function of some subset of $\{x \in X \mid |x|_a > d|x|\}$.

LEMMA 10. *For any sharp Simon partition of unity J^s,* $\operatorname{Ker} F_{J^s}(\lambda) = \{0\}$.

Proof. The equation

$$\sum j_a^s R_a(\lambda)\varphi = 0$$

implies $R_a(\lambda)\varphi = 0$ on $\operatorname{supp} j_a^s \setminus \partial(\operatorname{supp} j_a^s)$ for any a. Hence

$$\varphi(x) = (H_a - \lambda)(R_a(\lambda)\varphi)(x) = 0 \quad \text{on } \operatorname{supp} j_a^s \setminus \partial(\operatorname{supp} j_a^s) \text{ for each } a$$

(remember that H_a is a local operator). Since $\varphi \in L^p$ and the Lebesque measure of $\bigcup(\partial \operatorname{supp} j_a^s)$ is 0 we conclude that $\varphi \equiv 0$, i.e. $\operatorname{Ker} F_{J^s}(\lambda) = \{0\}$. ▨

LEMMA 11. *For any sharp Simon partition* J^s,

$$\mathrm{Ker}(\mathbf{1} + K_{J^s}(\lambda)) = \mathrm{Ker}(H - \lambda). \tag{12}$$

Proof. The statement follows from Lemma 10, equation (10) and the implication: $H\varphi = \lambda\varphi$, $\lambda \neq 0$, weakly in $L^p(X) \Rightarrow \varphi \in \mathrm{D}_{L^p}(H)$. ▨

c. PROOF OF THEOREM 6. We prove this theorem by the induction on the partitions. For H^a with $a = \{(1) \ldots (N)\}$ the statement is trivial. Assume the theorem holds for every b with $b \subsetneq a$ and prove it for a. Henceforth the superindex a is omitted and we act as if H^a were H and X^a were X. This saves us introducing extra notations.

(i) We assume first that V^l are Δ-compact on $L^p(\mathbf{R}^\nu)$, $1 \leqslant p \leqslant \infty$. By Lemma 11 and Theorem 13 below

$$\mathrm{disc\,spec}_{L^p}(H) \text{ is independent of } p \tag{13}$$

(and the eigenfunctions of H belong to $\bigcap_{1 \leqslant p \leqslant \infty} L^p(X)$).

Since $F_{J^s}(\lambda)$ and $G = \bigcap_{b \subsetneq a} \rho_{L^p}(H_b)$ obey conditions (a)—(c) of Theorem 1 we obtain

$$\mathrm{ess\,spec}_{L^p}(H) \subset \bigcup_{b \subsetneq a} \mathrm{spec}_{L^p}(H_b). \tag{14}$$

LEMMA 12. *We have*

$$\mathrm{spec}_{L^p}(H_b) = \mathrm{spec}_{L^p}(H^b) + \mathrm{spec}(T_b) = \mathrm{spec}_{L^p}(H^b) + \bar{\mathbf{R}}^+.$$

Proof. The direction

$$\mathrm{spec}_{L^p}(H_b) \supset \mathrm{spec}_{L^p}(H^b) + \mathrm{spec}(T_b)$$

follows by the straightforward application of the Weyl criterion (Corollary 2.3 of [15] and the remark preceding it, both quoted at the end of Section 2). The opposite direction follows from the estimate

$$\|\mathrm{e}^{-H_b t}\|_{p \to p} \leqslant C(1 + t)^{m/2}\,\mathrm{e}^{-t\mu_b}, \tag{15}$$

where $m = \nu(N - \#(b)) = \dim X^b$ and $\mu_b = \inf[\mathrm{spec}_{L^2}(H^b)]$ and the formula

$$(A - \lambda)^{-1} = \int_0^\infty \mathrm{e}^{-At}\,\mathrm{e}^{\lambda t}\,\mathrm{d}t \tag{16}$$

which, being applied to $A = H_b$, defines the resolvent of H_b on $L^p(X)$ for all $\lambda < \mu_b$. Estimate (15) follows from equation (5), Theorem B.5.1 of [14] and the estimate

$$\|e^{-T_b t}\|_{p\to p} \leqslant \text{const.},$$

obtained by representing $\exp(-T_b t)$ as the convolution operator by const$\cdot$ $\cdot t^{-\nu/2(\#(b)-1)}\exp[-\|x_b\|^2/2t]$, where $x_b \in X_b$. Note that $\nu(\#(b)-1) = \dim X_b$. ▨

Applying this lemma and using the induction assumption we get

$$\operatorname{spec}_{L^p}(H_b) = \bigcup_{c\subseteq b} \operatorname{disc\,spec}_{L^p}(H^c) \cup \{0\} + \overline{\mathbf{R}^+}.$$

Hence, due to (14) and the induction assumption

$$\operatorname{ess\,spec}_{L^p}(H) \subset \bigcup_b \operatorname{disc\,spec}_{L^2}(H^b) \cup \{0\} + \overline{\mathbf{R}^+}.$$

This proves, what is traditionally called, the difficult direction of the theorem (under the stronger assumptions).

The proof of the other, easy, direction, under the same assumptions, is exactly the same as Hunziker's L^2-proof (see [3; 4; 8, vol. I]). Hunziker's proof uses only compactness of $V^l(-\Delta+1)^{-1}$, where Δ is the Laplacian in $\mathbf{R}^\nu$, and the Weyl criterion. We assume the compactness of $V^l(-\Delta+1)^{-1}$ on $L^p(\mathbf{R}^\nu)$ and the Weyl criterion, in our case, must be taken in form of Corollary 3.2 and the remark preceding it, both quoted at the end of Section 2.

(ii) Now we prove the statement under conditions (A) and (B). We begin with the difficult direction. Let $g = (\mathbf{1}-\Delta)^{-1}$. A result of [6, Proposition 3.1] implies that

(17) $$g^{1-\lambda}V_l g^\lambda \text{ is bounded on } L^{1/\lambda}(X).$$

Moreover, using an approximation argument one can show that

(18) $$g^{1-\lambda}j_a I_a g^\lambda \text{ is compact on } L^{1/\lambda}(X).$$

(17) implies that the operator H is defined on $g^{-1+\lambda}L^{1/\lambda}(X)$ with the domain $g^\lambda L^{1/\lambda}(X)$. It is easy to show that it is closed there. As in case (i) we obtain (now we use (18)):

(α) $\operatorname{ess\,spec}_{g^{-1+\lambda}L^{1/\lambda}}(H)$ is independent of λ
$\operatorname{disc\,spec}_{g^{-1+\lambda}L^{1/\lambda}}(H)$ is independent of λ.

(β) $(H-z)^{-1}$, $z\in\rho_{L^1}(H)$, maps continuously $g^{-1+\lambda}L^{1/\lambda}(X)$ into $g^\lambda L^{1/\lambda}(X)$.

Now, (β) implies that $(H-z)^{-1}$ is bounded on $L^p(X)$ into $\{f\in L^p(X) \mid Hf\in L^p(X)\}$, which, by condition (B), is $\mathrm{D}_{L^p}(H)$. Moreover, $(H-z)^{-1}$ is the inverse of $H-z$. Hence $\operatorname{spec}_{L^p}(H)\subset\operatorname{spec}_{L^1}(H)$. The second line in ($\alpha$), observation $\mathrm{D}_{g^{-1+\lambda}L^{1/\lambda}}(H) =$

$= g^{\lambda}L^{1/\lambda} \subset L^{1/\lambda}$ and condition (B) imply that

(19) $$\operatorname{disc\,spec}_{L^p}(H) \text{ is independent of } p.$$

This and our previous conclusion yield:

$$\operatorname{ess\,spec}_{L^p}(H) \subset \operatorname{ess\,spec}_{L^1}(H).$$

Due to part (i) with $p = 1$ and (19), $\operatorname{spec}_{L^1}(H)$ is given by the HVZ formula (6). Thus it remains to show the easy direction

(20) $$[\mu, \infty) \subset \operatorname{ess\,spec}_{L^p}(H), \quad \text{where } \mu = \min[\bigcup \operatorname{disc\,spec}_{L^2}(H^b) \cup \{0\}].$$

To demonstrate (20), let real $V_n^l \in C_0^\infty(\mathbf{R}^\nu)$ and $V_n^l(-\Delta + 1)^{-1} \to V^l(-\Delta + 1)^{-1}$ as $n \to \infty$ on $L^1(\mathbf{R}^\nu)$. Here, Δ is the Laplacian in $\mathbf{R}^\nu$. Let $V_n = \sum V_{l,n}$ and $H_n = T + V_n$. Then Theorems A.2.3, B.5.1 and B.10.1 of [14] together with equation (16) and the Lebesgue convergence theorem imply

(21) $$(H_n - \lambda)^{-1} \to (H - \lambda)^{-1} \quad \text{on } L^p(X)$$

for $\lambda < \min[\inf \sigma(H_n), \inf \sigma(H)]$. This implies that

(22) $$[\mu, \infty) \subset \bigcap_n \operatorname{spec}_{L^p}(H_n) \subset \operatorname{spec}_{L^p}(H).$$

Indeed, the first inclusion follows from the fact that by Part (i), $\operatorname{spec}_{L^p}(H_n)$ is independent of p and for $p = 2$ it is obviously true, due to (21). To demonstrate the second inclusion, let $a \in \rho_{L^p}(H)$ and $a \neq \lambda$. Then $(a - \lambda)^{-1} \in \rho_{L^p}[(H-\lambda)^{-1}]$, which implies, due to (21), that $(a - \lambda)^{-1} \in \rho_{L^p}[(H_n - \lambda)^{-1}]$ for n sufficiently large. The latter yields $a \in \rho_{L^p}(H_n)$ for the same n. Hence (22) follows. Equation (22) implies (20), which completes the proof of case (ii). ▨

d. Invariance of spectra of compact operators.

Finally we present an abstract result used in the proof of Theorem 6.

Theorem 13. [9, 10]. *Let X and Y be Banach spaces which are subspaces of some Hausdorff vector space with a weaker topology and such that $X \cap Y$ dense in X and in Y. Let K be a compact operator on X and on Y. Then the spectra of K and their multiplicities on these spaces are the same. Hence the eigenvectors of K on X and Y associated to non-zero eigenvalues belong also to $X \cap Y$.*

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Received September 21, 1983; revised January 10, 1984.

Note added in proof. K. Gustafson has noticed that the operator B in Lemma 3 should be closable (which was not stated in the preprint version of the paper) and has informed me about the following works containing a detailed discussion of the Weyl theorem:

GUSTAFSON, K.; WEIDMANN, J., On the essential spectrum, *J. Math. Anal. Appl.*, **25**(1969), 121–127.
GUSTAFSON, K., Weyl's theorem, in "*Linear Operators and Approximation*" Butzer et al. (eds.), *Internat. Ser. Numer. Math.*, **20**(1972), 80–93.

GUSTAFSON, K., On algebraic multiplicity, *Indiana Univ. Math. J.*, **25**(1976), 769–781.

From these references I have also learned that a result similar to our Theorem 5 was proven in:

SCHECHTER, M., Invariance of the essential spectrum, *Bull. Amer. Math. Soc.*, **71**(1965), 365–367.
SCHECHTER, M., On the essential spectrum of an arbitrary operator. I, *J. Math. Anal. Appl.*, **13**(1966) 205–215.

(see also

BROWDER, F. E., On the spectral theory of elliptic differential operators. I, *Math. Ann.*, **142**(1961), 22–130.).

Many references on other works on the Weyl theorem are given in the papers listed above.

It is a pleasure to thank K. Gustafson for his remark and information.