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ON FACTORIAL STATES OF OPERATOR ALGEBRAS. II

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1. INTRODUCTION

It was shown in [7,19] that the set P(A) of pure states of a (unital) C^*-algebra A (other than $A = \mathbf{C}$) is weak* dense in the state space S(A) if and only if A is both prime and antiliminal. More recently, it was shown in [2] that the set F(A) of factorial states is weak* dense in S(A) if and only if A is prime. Comparison of these results suggests that weak* density of P(A) in F(A) is related to antiliminality. On the other hand, abelian C^*-algebras, for which P(A) = F(A), must also be taken into consideration. It will be shown in Theorem 3.4 that these are essentially the only two cases — P(A) is weak* dense in F(A) if and only if A is an antiliminal extension of an abelian C^*-algebra.

Approximate factorial extensions of factorial states on C^*-subalgebras were also considered in [2], by means of the fact that the set $\mathrm{F}_\infty(A)$ of type I factorial states is always weak* dense in F(A). It will be shown in Section 5 how the arguments of [2] can be simplified by considering only the set $\mathrm{F}_\mathrm{f}(A)$ of states φ for which $\pi_\varphi(A)'$ is a *finite* type I factor. Both $\mathrm{F}_\infty(A)$ and $\mathrm{F}_\mathrm{f}(A)$ are described simply in terms of pure states in Section 2. In Section 4, it is shown, by methods parallel to those of Section 3, that the states for which $\pi_\varphi(A)'$ is a type I factor of bounded degree are weak* dense in F(A) if and only if A is an antiliminal extension of a subhomogeneous C^*-algebra.

Standard definitions and properties of C^*-algebras, as described in [6], will often be used without comment. Throughout, A will be a C^*-algebra (with or without unit), whose spectrum $\hat{A}$ is equipped with the Jacobson topology. The equivalence class in $\hat{A}$ of an irreducible representation π will be denoted by $[\pi]$.

For a subset E of A^*, $\bar{E}$ will be the weak* ($\sigma(A^*, A)$) closure of E in A^*. The sets of all, all pure, and all factorial, states of A will be denoted by S(A), P(A) and F(A) respectively. For a state φ, $(\mathscr{H}_\varphi, \pi_\varphi, \xi_\varphi)$ will be the Hilbert space, representation, and cyclic vector, associated with φ by the GNS construction.

If I is a (closed two-sided) ideal in A, then $(A/I)^\wedge$ will be identified with $\{[\pi] \in \hat{A} : \pi(I) = 0\}$ and $\hat{I}$ with $\hat{A} \setminus (A/I)^\wedge$ [6,3.2.1]. Similarly S(A/I) will be identified with

$\{\varphi \in S(A) : \varphi(I) = 0\}$ and $S(I)$ with $\{\varphi \in S(A) : \|\varphi|I\| = 1\}$. Then $P(A) = P(A/I) \cup P(I)$ [6, 2.11.8]. Furthermore, for φ in $F(A)$, the weak operator closure $\overline{\pi_\varphi(I)}$ of $\pi_\varphi(I)$ is an ideal in the factor $\pi_\varphi(A)''$, so either $\overline{\pi_\varphi(I)} = \pi_\varphi(A)''$ or $\pi_\varphi(I) = 0$. Thus $\pi_\varphi(A)$ is prime; also $F(A) = F(A/I) \cup F(I)$ (see [2]). Although the embedding of $S(I)$ in $S(A)$ is a $\sigma(I^*, I) - \sigma(A^*, A)$ homeomorphism, it is not uniformly continuous in general. However it should not cause confusion if the notation $\overline{S(I)}$, $\overline{P(I)}$, $\overline{F(I)}$ etc. is used for the $\sigma(A^*, A)$ closures of $S(I)$, $P(I)$, $F(I)$ etc. in A^*.

2. TYPE I FACTORIAL STATES

The first result gives a precise description of how type I factorial states arise. It overlaps with several results which are familiar in the literature (see for example [1, Proposition 2.3; 3, Proposition 2.4.27; 6, 5.4.11; 10, Theorem A]).

PROPOSITION 2.1. (i) *Let* φ *be a state of* A, *and suppose that* $\pi_\varphi(A)'$ *is a type* I_n *factor (where* $1 \leqslant n \leqslant \infty$*). Then* φ *is a* σ*-convex combination of* n *equivalent pure states of* A.

(ii) *Let* φ *be a* σ*-convex combination of equivalent pure states of* A, *so that there is an irreducible representation* π *of* A *on a Hilbert space* $\mathscr{H}$, *and a family* $\{\xi_\alpha : \alpha \in D\}$ *of vectors in* $\mathscr{H}$ *such that*

$$\varphi(a) = \sum_{\alpha \in D} \langle \pi(a)\xi_\alpha, \xi_\alpha \rangle \quad (a \in A).$$

Then $\pi_\varphi(A)'$ *is a type* I_d *factor, where* d *is the dimension of the linear span of* $\{\xi_\alpha : \alpha \in D\}$.

(iii) *Let* φ *be a proper* σ*-convex combination of pure states of* A, *not all of which are equivalent. Then* φ *is not factorial.*

Proof. (i) Let $\{e_\alpha : \alpha \in D\}$ be a maximal orthogonal family of minimal projections in $\pi_\varphi(A)'$, and $\xi_\alpha = e_\alpha \xi_\varphi$. Since ξ_φ is separating for $\pi_\varphi(A)'$, the vectors ξ_α are non-zero, and therefore D is at most countably infinite. The subrepresentations of π_φ on $e_\alpha \mathscr{H}_\varphi$ are equivalent irreducible representations. Thus if $\lambda_\alpha = \|\xi_\alpha\|^2$ and $\varphi_\alpha(a) = \lambda_\alpha^{-1} \langle \pi_\varphi(a)\xi_\alpha, \xi_\alpha \rangle$, then $\{\varphi_\alpha : \alpha \in D\}$ are equivalent pure states of A, and $\varphi = \sum_{\alpha \in D} \lambda_\alpha \varphi_\alpha$.

(ii) Let $\mathscr{H}_D$ be a Hilbert space with orthonormal basis $\{\eta_\alpha : \alpha \in D\}$, and

$$\mathscr{H}_1 = \mathscr{H} \otimes \mathscr{H}_D, \quad \pi_1(a) = \pi(a) \otimes 1, \quad \xi = \sum_{\alpha \in D} \xi_\alpha \otimes \eta_\alpha.$$

Then

$$\varphi(a) = \langle \pi_1(a)\xi, \xi \rangle$$

so π_φ is the subrepresentation of π_1 on the cyclic subspace $[\pi_1(A)\xi]$. Let $\{\xi'_\beta : \beta \in D'\}$ be an orthonormal basis of the linear span of $\{\xi_\alpha : \alpha \in D\}$, and let $\lambda_{\alpha\beta}$ be scalars with

$$\xi_\alpha = \sum_{\beta \in D'} \lambda_{\alpha\beta}\xi'_\beta \quad \sum_{\alpha,\beta} |\lambda_{\alpha\beta}|^2 = \sum_\alpha \|\xi_\alpha\|^2 = 1.$$

Let $\eta'_\beta = \sum_{\alpha \in D} \lambda_{\alpha\beta}\eta_\alpha$, and $\mathscr{H}'_D$ be the closed linear span of $\{\eta'_\beta : \beta \in D'\}$. Then $\xi = \sum_{\beta \in D'} \xi'_\beta \otimes \eta'_\beta$ and, by Kadison's Transitivity Theorem, $\mathscr{H}_\varphi = [\pi_1(A)\xi] = \mathscr{H} \otimes \mathscr{H}'_D$. The vectors η'_β are linearly independent, since if

$$\eta'_\gamma = \sum_{\beta \in D'} \mu_\beta \eta'_\beta$$

where $\mu_\gamma = 0$ and $\mu_\beta = 0$ for all except finitely many β, then

$$\lambda_{\alpha\gamma} = \sum_{\beta \in D'} \mu_\beta \lambda_{\alpha\beta}$$

so the linear span of $\{(\xi'_\beta + \mu_\beta\xi'_\gamma) : \beta \in D', \ \beta \neq \gamma\}$ contains each ξ_α and therefore contains ξ'_γ. This is a contradiction. Thus $\mathscr{H}'_D$ is of dimension d, and $\pi_\varphi(A)'$ is a factor of type I_d.

(iii) There are inequivalent pure states φ_1 and φ_2 with $\varphi_i \leqslant \lambda_i\varphi$ for some $\lambda_i > 0$. Let $\psi = (1/2)(\varphi_1 + \varphi_2)$. Since $\psi \leqslant (1/2)(\lambda_1 + \lambda_2)\varphi$, π_ψ is a subrepresentation of π_φ. It therefore suffices to show that π_ψ is not factorial.

There is an operator x in $\pi_\psi(A)'$, with $0 \leqslant x \leqslant 1$, such that

$$(1/2)\varphi_1(a) = \langle \pi_\psi(a)x\xi_\psi, \xi_\psi \rangle$$

$$(1/2)\,\varphi_2(a) = \langle \pi_\psi(a)\,(1 - x)\xi_\psi, \xi_\psi \rangle.$$

The positive linear functional ψ' defined by

$$\psi'(a) = \langle \pi_\psi(a)x(1 - x)\xi_\psi, \xi_\psi \rangle$$

is dominated both by φ_1 and by φ_2. Since φ_1 and φ_2 are pure and distinct, $\psi' = 0$, so $x(1 - x) = 0$, and x is a projection. Now π_{φ_1} and π_{φ_2} are the subrepresentations of π_ψ on $x\mathscr{H}_\psi$ and $(1 - x)\mathscr{H}_\psi$, repspectively, so that $\pi_\psi = \pi_{\varphi_1} \oplus \pi_{\varphi_2}$. Since π_{φ_1} and π_{φ_2} are disjoint and irreducible, $\pi_\psi(A)' = \pi_{\varphi_1}(A)' \oplus \pi_{\varphi_2}(A)' \simeq \mathbf{C}^2$.

The following notation can now be introduced, Proposition 2.1 giving the alternative definitions. Here k is either infinity or a finite positive integer.

$$\mathrm{F}_k(A) = \{\varphi \in \mathrm{S}(A) : \pi_\varphi(A)' \text{ is a type } \mathrm{I}_n \text{ factor, where } n \leqslant k\} =$$

$$= \left\{\sum_{i=1}^{k} \lambda_i \varphi_i : \lambda_i \geqslant 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ \varphi_i \text{ equivalent pure states}\right\} =$$

$$= \left\{\sum_{i=1}^{k} \varphi(a_i^* \cdot a_i) : \varphi \in \mathrm{P}(A),\ a_i \in A,\ \sum_{i=1}^{k} \varphi(a_i^* a_i) = 1\right\}$$

$$\mathrm{F}_{\mathrm{f}}(A) = \{\varphi \in \mathrm{S}(A) : \pi_\varphi(A)' \text{ is a finite type I factor}\} = \bigcup_{1 \leqslant k < \infty} \mathrm{F}_k(A).$$

Thus $\mathrm{F}_\infty(A)$ is the set of all states φ for which $\pi_\varphi(A)'$ is a type I factor. This set was denoted by $\mathrm{F}_{\mathrm{I}}(A)$ in [2].

If A is separable, all these sets are Borel subsets of $\mathrm{S}(A)$ [6, 7.3; 15, 5.7].

In general, the discussion above and in [6, 5.4.11] shows that, for φ in $\mathrm{F}_\infty(A)$, π_φ is quasi-equivalent to an irreducible representation $\tilde{\pi}_\varphi$, with $[\tilde{\pi}_\varphi]$ unique. Define $\theta : \mathrm{F}_\infty(A) \to \hat{A}$ by $\theta(\varphi) = [\tilde{\pi}_\varphi]$. If T is a closed subset of $\hat{A}$, there is an ideal I of A such that $T = (A/I)^\wedge$ and $\theta^{-1}(T) = \mathrm{S}(A/I) \cap \mathrm{F}_\infty(A)$. Thus $\theta^{-1}(T)$ is closed in $\mathrm{F}_\infty(A)$, so θ is continuous. If U is open in $\mathrm{F}_k(A)$, and

$$W = \left\{\varphi \in \mathrm{P}(A) : \sum_{i=1}^{k} \varphi(a_i^* \cdot a_i) \in U \text{ for some } a_i \text{ in } A\right\},$$

then W is open in $\mathrm{P}(A)$, and $\theta(U) = \theta(W)$ is open in $\hat{A}$ [6, 3.4.11; 15, 4.3.3]. Thus $\theta|\mathrm{F}_k(A)$ is open $(1 \leqslant k \leqslant \infty)$.

It is immediate, from the representation of states in $\mathrm{F}_\infty(A)$ as σ-convex combinations of equivalent pure states, that $\mathrm{F}_{\mathrm{f}}(A)$ is norm-dense in $\mathrm{F}_\infty(A)$. It was shown in [2, Corollary 3.4] that $\mathrm{F}_\infty(A)$ is weak* dense in $\mathrm{F}(A)$. The next result follows immediately from these facts, but a direct proof, using the method of [6, 11.2.4] is also given.

PROPOSITION 2.2. *For any C^*-algebra A, $\mathrm{F}_{\mathrm{f}}(A)$ is weak* dense in* $\mathrm{F}(A)$.

Proof. Let φ be a factorial state, and K be the kernel of π_φ. Then K is a prime ideal. It will be shown that $\mathrm{S}(A/K) \subset \overline{\mathrm{F}_{\mathrm{f}}(A/K)}$, from which the result follows.

By the Kreĭn-Milman Theorem, it suffices to show that $\varphi' \in \overline{\mathrm{F}_{\mathrm{f}}(A/K)}$ if $\varphi' = \sum_{i=1}^{n} \lambda_i \varphi_i$, where $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $\varphi_i \in \mathrm{P}(A/K)$. Let U be any convex weak* neighbourhood of 0 in A^*, and

$$V_i = \{[\pi_{\hat{\psi}}] : \psi \in \mathrm{P}(A),\ \psi - \varphi_i \in U\}.$$

Since $\psi \to [\pi_\psi]$ is an open map of $\mathrm{P}(A)$ into $\hat{A}$, $V_i = \hat{I}_i$ for some ideal I_i. Since $\varphi_i(K) = 0$ but $\varphi_i(I_i) \neq 0$, I_i is not contained in K. Since K is prime, $I_1 \cap \ldots \cap I_n$ is not contained in K. Let $\varphi_0 \in \mathrm{P}(I_1 \cap \ldots \cap I_n) \cap \mathrm{P}(A/K)$. Then $[\pi_{\varphi_0}] \in V_1 \cap \ldots \ldots \cap V_n$, so there are pure states ψ_i, equivalent to φ_0, such that $\psi_i - \varphi_i \in U$. Let $\psi = \sum_{i=1}^{n} \lambda_i \psi_i$, so that $\psi \in \mathrm{F}_n(A)$ and $\psi - \varphi' \in U$. This suffices to complete the proof.

3. APPROXIMATION OF FACTORIAL STATES BY PURE STATES

As indicated in Section 1, comparison of the results of [2, 7, 19] suggests that antiliminality is related to weak* density of $\mathrm{P}(A)$ in $\mathrm{F}(A)$. In this section, the exact relationship will be established, beginning with the sufficiency of antiliminality.

PROPOSITION 3.1. *Let A be an antiliminal C^*-algebra. Then* $\mathrm{P}(A)$ *is weak* dense in* $\mathrm{F}(A)$.

Proof. By Proposition 2.2, it suffices to show that if φ if a convex combination of equivalent pure states φ_i $(1 \leqslant i \leqslant n)$, then $\varphi \in \overline{\mathrm{P}(A)}$. But φ vanishes on the common kernel of each π_{φ_i}, so this assertion follows from [7, Lemma 5; 6, 11.2.3] — the assumption that A has a unit is not essential for those results.

The converse of Proposition 3.1 is false, since $\mathrm{P}(A) = \mathrm{F}(A)$ if A is abelian. But the two cases, of antiliminality and abelianness, essentially include all possibilities that $\mathrm{P}(A)$ is dense in $\mathrm{F}(A)$. This will be established in Theorem 3.4 after two lemmas.

LEMMA 3.2. *Suppose that* $\mathrm{P}(A)$ *is weak* dense in* $\mathrm{F}_2(A)$, *and that I is an ideal in A with continuous trace. Then I is abelian.*

Proof. If I is not abelian, there exists an irreducible representation of I on a Hilbert space of dimension greater than 1, and therefore there exists φ in $\mathrm{F}_2(\mathrm{I})\setminus\mathrm{P}(I)$ (Proposition 2.1). Now

$$\varphi \in \mathrm{F}_2(A) \subset \overline{\mathrm{P}(A)} \subset \overline{\mathrm{P}(I)} \cup \mathrm{S}(A/I).$$

Since $\varphi \in \mathrm{S}(I)$, $\varphi \in \overline{\mathrm{P}(I)}$. But $\mathrm{P}(I)$ is $\sigma(I^*, I)$ closed in $\mathrm{S}(I)$ [8, Theorem 6 and Remark on p. 601], so $\varphi \in \mathrm{P}(I)$. This is a contradiction.

LEMMA 3.3. *Suppose that* $\mathrm{P}(A)$ *is weak* dense in* $\mathrm{F}_2(A)$, *and that I is an abelian ideal in A. Then* $\mathrm{P}(A/I)$ *is weak* dense in* $\mathrm{F}_2(A/I)$.

Proof. By assumption,

$$F_2(A/I) \subset F_2(A) \subset \overline{P(A)} = \overline{P(A/I)} \cup \overline{P(I)}.$$

It therefore suffices to show that

$$F_2(A/I) \cap \overline{P(I)} \subset P(A/I).$$

Any state in $P(I)$ is multiplicative on A, and therefore the same is true for states in $\overline{P(I)}$. Hence $\overline{P(I)} \subset P(A)$, which is sufficient to complete the proof.

THEOREM 3.4. *For any C^*-algebra A, the following are equivalent:*

(i) $P(A)$ *is weak* dense in* $F(A)$,

(ii) $P(A)$ *is weak* dense in* $F_2(A)$,

(iii) *Either A is abelian, or there is an abelian ideal I such that A/I is antiliminal.*

Proof. (i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iii). Let I be the largest abelian ideal in A (this exists since the sum of abelian ideals is abelian), and J be an ideal in A containing I such that J/I has continuous trace. By Lemmas 3.3 and 3.2, J/I is abelian, so J is abelian. By maximality of I, $J = I$. Thus either $A = I$ or A/I is antiliminal.

(iii) $\Rightarrow$ (i). If A is abelian, $P(A) = F(A)$.

If I is an abelian ideal in A and A/I is antiliminal, then it follows from Proposition 3.1 that

$$F(A) = F(I) \cup F(A/I) \subset P(I) \cup \overline{P(A/I)} \subset \overline{P(A)}.$$

Since a prime C^*-algebra of dimension greater than one has no non-zero abelian ideal, one can recover from Theorem 3.4 and [2, Theorem 3.3] the result of [7,19] that P(A) is weak* dense in $S(A)$ if and only if A is both prime and antiliminal (or one-dimensional).

Glimm [8, Theorem 6] characterised those C^*-algebras A for which $P(A)$ is weak* closed in $S(A)$. It was shown in [2, Theorem 5.2] that $F(A)$ is weak* closed in $S(A)$ if and only if A is liminal and $\hat{A}$ is Hausdorff. One should make two observations about these results. Firstly, the arguments of [2, 8] apply equally to the condition that $\hat{A}$ is Hausdorff and the weaker condition that A has Hausdorff primitive ideal space — furthermore, if A is liminal, the conditions coincide. Secondly, using an approximate unit, it is easy to see that, for any C^*-algebra A,

$$\overline{F(A)} \subset \{\lambda\varphi : 0 \leqslant \lambda \leqslant 1, \quad \varphi \in \overline{F(A)} \cap S(A)\}$$

$$\overline{P(A)} \subset \{\lambda\varphi : 0 \leqslant \lambda \leqslant 1, \quad \varphi \in \overline{P(A)} \cap S(A)\}.$$

It follows immediately that condition (ii) in Theorem 3.5 below is equivalent to F(A) being weak* closed in S(A). The theorem shows that even if it is only assumed that the weak* closure of P(A) in S(A) is contained in F(A), then F(A) must already be weak* closed, but a direct proof of this does not seem to be available.

THEOREM 3.5. *For any C^*-algebra A, the following are equivalent:*

(i) *A is liminal and $\hat{A}$ is Hausdorff,*

(ii) $\overline{\mathrm{F}(A)} \subset \{\lambda\varphi : 0 \leqslant \lambda \leqslant 1,\ \varphi \in \mathrm{F}(A)\}$,

(iii) $\overline{\mathrm{P}(A)} \cap \mathrm{S}(A) \subset \mathrm{F}(A)$.

Proof. (i) $\Leftrightarrow$ (ii). [2, Theorem 5.2] (see also [17, Proposition 9]).

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Assume condition (iii), and suppose first that $\hat{A}$ is not Hausdorff. In the proof of [8, Theorem 6], Glimm showed that there exist inequivalent pure states φ_1 and φ_2 such that $(1/2)(\varphi_1 + \varphi_2) \in \overline{\mathrm{P}(A)}$. By Proposition 2.1 (iii), this contradicts (iii). Thus $\hat{A}$ is Hausdorff.

Let π be an irreducible representation of A on $\mathscr{H}$ with kernel P, and $\mathscr{K}$ be the C^*-algebra of compact operators on $\mathscr{H}$. Since $\hat{A}$ is Hausdorff, P is maximal, so either $\pi(A) = \mathscr{K}$ or $\pi(A) \cap \mathscr{K} = (0)$. If $\pi(A) \cap \mathscr{K} = (0)$, then it follows from [7, Theorem 2; 6, 11.2.1] (no assumption about a unit is needed) and (iii) that

$$\mathrm{S}(A/P) \subset \overline{\mathrm{P}(A/P)} \cap \mathrm{S}(A) \subset \overline{\mathrm{P}(A)} \cap \mathrm{S}(A) \subset \mathrm{F}(A).$$

Thus every state of $\pi(A)$ is factorial, so $\pi(A) = \mathscr{K}$ [2, Lemma 5.1]. This shows that A is liminal.

Wright [21, p. 578] has speculated hopefully that the criterion $\mathrm{F}(A) = \mathrm{S}(A)$ might be used to prove Naimark's conjecture that an (inseparable) C^*-algebra with only one irreducible representation is elementary. Similarly, the criterion that F(A) is weak* closed in S(A) might be relevant to the stronger conjecture that if $\hat{A}$ is Hausdorff, then A is liminal.

4. APPROXIMATION BY FACTORIAL STATES OF LOWER DEGREE

This section runs parallel to Section 3, with the role of P(A) now taken by $\mathrm{F}_k(A)$, where k is a fixed finite positive integer. The case $k = 1$ is just that considered in Section 3. Some of the arguments are a little more involved for $1 < k < \infty$.

Recall that a C^*-algebra A is said to be *k-subhomogeneous* if every irreducible representation of A is on a Hilbert space of dimension at most k.

LEMMA 4.1. *If A is k-subhomogeneous, then* $\mathrm{F}_k(A) = \mathrm{F}(A)$.

Proof. For φ in F(A), let I be the kernel of π_φ. Since I is a prime, hence primitive, ideal of the postliminal C^*-algebra A [11, Lemma 7.4], A/I is isomorphic to the algebra of $n \times n$ complex matrices, for some $n \leqslant k$. Hence $\varphi \in \mathrm{S}(A/I) = \mathrm{F}_k(A/I) \subset \mathrm{F}_k(A)$.

It follows from Proposition 2.1 (ii) that the converse of Lemma 4.1 is true — if $\mathrm{F}_k(A) = \mathrm{F}(A)$, then A is k-subhomogeneous. Similarly, $\mathrm{F}_f(A) = \mathrm{F}(A)$ if and only if every irreducible representation of A is on a finite-dimensional Hilbert space.

LEMMA 4.2. *If A has continuous trace, then* $\mathrm{F}_k(A)$ *is weak* closed in* S(A).

Proof. Let $\varphi \in \overline{\mathrm{F}_k(A)} \cap \mathrm{S}(A)$, so that φ is the weak* limit of a net of states of the form $\sum_{i=1}^{k} \lambda_i^\alpha \varphi_i^\alpha$ where $\lambda_i^\alpha \geqslant 0$, $\sum_{i=1}^{k} \lambda_i^\alpha = 1$, and $\{\varphi_i^\alpha : 1 \leqslant i \leqslant k\}$ are equivalent pure states. Passing to a subnet, it may be assumed that $\lambda_i^\alpha \to \lambda_i$, and $\varphi_i^\alpha \to \varphi_i$, where $\lambda_i \geqslant 0$, $\sum \lambda_i = 1$, $\varphi_i \geqslant 0$, $\|\varphi_i\| \leqslant 1$. Then $\varphi = \sum \lambda_i \varphi_i$. Let $Q = \{i : \lambda_i > 0\}$. Since $\|\varphi\| = 1$, $\varphi_i \in \mathrm{S}(A)$ for i in Q. Since P(A) is weak* closed in S(A) [8, Theorem 6], $\varphi_i \in \mathrm{P}(A)$ for i in Q. Since $\hat{A}$ is Hausdorff, and $\varphi \to [\pi_\varphi]$ is continuous, the equivalence relation is weak* closed in $\mathrm{P}(A) \times \mathrm{P}(A)$, so $\{\varphi_i : i \in Q\}$ are equivalent. By Proposition 2.1, $\varphi \in \mathrm{F}_k(A)$.

LEMMA 4.3. *Suppose that* $F_k(A)$ *is weak* dense in* $\mathrm{F}_{k+1}(A)$, *and that I is an ideal in A with continuous trace. Then I is k-subhomogeneous.*

Proof. If I is not k-subhomogeneous, there exists an irreducible representation of I on a Hilbert space of dimension greater than k, and therefore there exists φ in $F_{k+1}(I) \setminus F_k(I)$ (Proposition 2.1). Now

$$\varphi \in \mathrm{F}_{k+1}(A) \subset \overline{\mathrm{F}_k(A)} \subset \overline{\mathrm{F}}_k(I) \cup \mathrm{S}(A/I).$$

Since $\varphi \in \mathrm{S}(I)$, $\varphi \in \overline{\mathrm{F}_k(I)} \cap \mathrm{S}(I)$. By Lemma 4.2, $\varphi \in \mathrm{F}_k(I)$. This is a contradiction.

LEMMA 4.4. *Suppose that* $\mathrm{F}_k(A)$ *is weak* dense in* $\mathrm{F}_{k+1}(A)$, *and that I is a k-subhomogeneous ideal in A. Then* $\mathrm{F}_k(A/I)$ *is weak* dense in* $\mathrm{F}_{k+1}(A/I)$.

Proof. By assumption,

$$\mathrm{F}_{k+1}(A/I) \subset \mathrm{F}_{k+1}(A) \subset \overline{\mathrm{F}_k(A)} \subset \overline{\mathrm{F}_k(A/I)} \cup \overline{\mathrm{F}}_k(I).$$

It therefore suffices to show that

$$\mathrm{F}_{k+1}(A/I) \cap \overline{\mathrm{F}_k(I)} \subset \mathrm{F}_k(A/I).$$

Let $\varphi \in \mathrm{F}_{k+1}(A/I) \cap \overline{\mathrm{F}_k(I)}$, and φ_α be a net in $\mathrm{F}_k(I)$ converging to φ. Each φ_α is a convex combination of (at most) k equivalent pure states of I. Since I is k-subhomogeneous, there are integers $n_\alpha \leqslant k$, (irreducible) $*$-homomorphisms π_α of

A into the C^*-algebras $\mathscr{K}_{n_\alpha}$ of complex $n_\alpha \times n_\alpha$ matrices, and states ψ_α of $\mathscr{K}_{n_\alpha}$ such that $\varphi_\alpha = \psi_\alpha \circ \pi_\alpha$. It is possible to find a subnet for which n_α is constantly n and (using the finite-dimensionality of $\mathscr{K}_n$) $\pi_\alpha(a)$ converges to a limit $\pi(a)$ for each a, and ψ_α converges to some state ψ of $\mathscr{K}_n$. Then π is a $*$-homomorphism, and $\varphi = \psi \circ \pi$, so $\varphi \in \mathrm{F}_{k+1}(\pi(A)) = \mathrm{F}_k(\pi(A))$ (Lemma 4.1). Hence $\varphi \in \mathrm{F}_k(A/I)$.

THEOREM 4.5. *For any C^*-algebra A, the following are equivalent:*

(i) $\mathrm{F}_k(A)$ *is weak* dense in* $\mathrm{F}(A)$,

(ii) $\mathrm{F}_k(A)$ *is weak* dense in* $\mathrm{F}_{k+1}(A)$,

(iii) *Either A is k-subhomogeneous, or there is a k-subhomogeneous ideal I such that A/I is antiliminal.*

Proof. The proof is very similar to Theorem 3.4, Lemmas 4.3 and 4.4 replacing Lemmas 3.2 and 3.3.

COROLLARY 4.6. *Let k be a positive integer, and*

$$\mathrm{F}_k^0(A) = \mathrm{F}_k(A) \setminus \mathrm{F}_{k-1}(A) = \{\varphi \in \mathrm{S}(A) : \pi_\varphi(A)' \text{ is a type } \mathrm{I}_k \text{ factor}\}.$$

The following are equivalent:

(i) $\mathrm{F}_k^0(A)$ *is weak* dense in* $\mathrm{F}(A)$,

(ii) *A has no non-zero $(k-1)$-subhomogeneous ideal, and either A is k-subhomogeneous, or there is a k-subhomogeneous ideal I such that A/I is antiliminal.*

Proof. It suffices to show that $\mathrm{F}_{k-1}(A)$ has non-empty interior in $\mathrm{F}_k(A)$ if and only if A has a non-zero $(k-1)$-subhomogeneous ideal J. The corollary then follows immediately from Theorem 4.5.

If J exists, $\mathrm{F}_k(J) = \mathrm{F}_{k-1}(J)$ (Lemma 4.1), which is therefore contained in the interior of $\mathrm{F}_{k-1}(A)$ in $\mathrm{F}_k(A)$. Conversely, if $\mathrm{F}_{k-1}(A)$ has non-empty interior U in $\mathrm{F}_k(A)$, then $\theta(U)$ is open in $\hat{A}$ (see Section 2), so there is a non-zero ideal J such that $\theta(U) = \hat{J}$. Suppose that J is not $(k-1)$-subhomogeneous, so that there exist φ in U, an irreducible representation π of A on a Hilbert space $\mathscr{H}$ of dimension at least k, and vectors ξ_i $(1 \leqslant i \leqslant k-1)$ in $\mathscr{H}$ such that

$$\varphi(a) = \sum_{i=1}^{k-1} \langle \pi(a)\xi_i, \xi_i \rangle \quad (a \in A)$$

(Proposition 2.1). Let η_j $(1 \leqslant j \leqslant m)$ be fixed unit vectors in $\mathscr{H}$ such that $\{\xi_i\} \cup \{\eta_j\}$ spans a space of dimension k. For $\varepsilon > 0$, define

$$\varphi_\varepsilon(a) = (1 + m\varepsilon)^{-1} \left\{ \sum_{i=1}^{k-1} \langle \pi(a)\xi_i, \xi_i \rangle + \varepsilon \sum_{j=1}^{m} \langle \pi(a)\eta_j, \eta_j \rangle \right\}.$$

Then $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$, but $\varphi_\varepsilon \in \mathrm{F}_k^0(A)$ (Proposition 2.1). This contradicts the fact that φ is in the interior of $\mathrm{F}_{k-1}(A)$ in $\mathrm{F}_k(A)$.

5. EXTENSIONS OF FACTORIAL STATES

Now suppose that A is a C^*-subalgebra of some C^*-algebra B. A longstanding problem has been whether factorial states of A extend to factorial states of B. This has recently been solved affirmatively by Popa [16] and Longo [13] in the separable case. Part of the method had been introduced earlier by Sakai, giving positive results for (semi)nuclear C^*-algebras A, and for type I factorial states without restriction on A [4, 12, 20]. Further analysis of the construction in [4] shows that a type I factorial state can be extended to a type I factorial state (but there may also be factorial extensions of other types). However a simple argument for this is available, as in Proposition 5.1 with $k = \infty$. (In this section except where otherwise stated, k may be infinite.)

For a subset E of $\mathrm{S}(B)$, $E|A$ denotes the set of all restrictions $\varphi|A$ to A of states φ in E.

PROPOSITION 5.1. *Let A be a C^*-subalgebra of B. Then* $\mathrm{F}_k(A) \subset \mathrm{F}_k(B)|A$.

Proof. For any φ in $\mathrm{F}_k(A)$, Proposition 2.1 shows that there exist φ_0 in $\mathrm{P}(A)$ and a_i in A such that

$$\varphi(a) = \sum_{i=1}^{k} \varphi_0(a_i^* a a_i) \quad (a \in A).$$

Let ψ_0 be any pure state of B extending φ_0, and define

$$\psi(b) = \sum_{i=1}^{k} \psi_0(a_i^* b a_i) \quad (b \in B).$$

Then $\psi \in \mathrm{F}_k(B)$ and $\psi|A = \varphi$.

In the above proof, considering π_{φ_0} as a subrepresentation of $\pi_{\psi_0}|A$ and using again Proposition 2.1 (ii), one can see that if $\pi_\varphi(A)'$ is of type I_n ($1 \leqslant n \leqslant \infty$), then $\pi_\psi(B)'$ is also of type I_n.

There is another approach to Proposition 5.1 via the following lemma. Here $\mathscr{K}_k$ denotes the C^*-algebra of all compact operators on a separable Hilbert space $\mathscr{H}_k$ of dimension k. A state ψ of $A \otimes \mathscr{K}_k$ has a "restriction" $\psi|A$ to A given by

$$(\psi|A)(a) = \lim \psi(a \otimes e_n)$$

where $\{e_n\}$ is an approximate identity in $\mathscr{K}_k$ (see [9]).

LEMMA 5.2. *Let A be any C^*-algebra. Then* $\mathrm{F}_k(A) = \mathrm{P}(A \otimes \mathscr{K}_k)|A$.

Proof. Suppose that $\pi_\varphi(A)'$ is a factor of type I_n, where $1 \leqslant n \leqslant k$. By a result originally due to Murray and von Neumann [14], $\mathscr{H}_\varphi = \mathscr{H} \otimes \mathscr{H}_n$, $\pi_\varphi(a) = \pi(a) \otimes 1$, where π is an irreducible representation of A on $\mathscr{H}$. Let p_n be a projection of rank n in $\mathscr{K}_k$, so that $p_n\mathscr{K}_k p_n = \mathscr{K}_n$, and define

$$\psi(x) = \langle(\pi \otimes \iota_n)((1 \otimes p_n)x(1 \otimes p_n))\xi_\varphi, \xi_\varphi\rangle \quad (x \in A \otimes \mathscr{K}_k)$$

where ι_n is the identity representation of $\mathscr{K}_n$ on $\mathscr{H}_n$. Then $\psi|A = \varphi$ and elementary arguments show that $\psi \in \mathrm{P}(A \otimes \mathscr{K}_k)$.

Conversely, for ψ in $\mathrm{P}(A \otimes \mathscr{K}_k)$, let $\varphi = \psi|A$ and π_1 and π_2 be the restriction (in the sense of [9; 18, p. 204]) of π_ψ to A and $\mathscr{K}_k$ respectively. Then $\pi_1(A)''$ and $\pi_2(\mathscr{K}_k)''$ commute and generate the von Neumann algebra of all bounded linear operators on $\mathscr{H}_\psi$. Since $\pi_2(\mathscr{K}_k)''$ is a type I_k factor, $\pi_1(A)' = \pi_2(\mathscr{K}_k)''$ [14]. Since $\varphi(a) = \langle\pi_1(a)\xi_\psi, \xi_\psi\rangle$, π_φ is a subrepresentation of π_1, and therefore $\pi_\varphi(A)'$ is a type I_n factor for some $n \leqslant k$.

Now Proposition 5.1 follows from Lemma 5.2. For φ in $\mathrm{F}_k(A)$, there exists ψ in $\mathrm{P}(A \otimes \mathscr{K}_k)$ with $\psi|A = \varphi$. Then there exists $\tilde{\psi}$ in $\mathrm{P}(B \otimes \mathscr{K}_k)$ with $\tilde{\psi}|A\otimes\mathscr{K}_k = \psi$. If $\tilde{\varphi} = \tilde{\psi}|B$, then $\tilde{\varphi} \in F_k(B)$ and $\tilde{\varphi}|A = \varphi$.

As in [2, Theorem 4.4], it follows from Proposition 5.1 that, for φ in $\overline{\mathrm{F}_k(A)}$, there is an extension ψ in $\overline{\mathrm{F}_k(B)}$. Also it follows from Lemma 5.2 with $k = \infty$, and Proposition 2.2, that $\overline{\mathrm{F}(A)} = \overline{\mathrm{P}(A \otimes \tilde{\mathscr{K}}_\infty)}|A$, where $\tilde{\mathscr{K}}_\infty$ is the linear span of the compact and the scalar operators on $\mathscr{H}_\infty$. This relates the study of factorial state spaces to pure state spaces. For example, it is possible to simplify the proof of [2, Theorem 4.6 (1)] by reducing it to a study of tensor products $A \otimes \mathscr{K}_k$ for k finite (matrix algebras over A) rather than the more complicated tensor products and second duals of [2].

THEOREM 5.3. *Let A be a C^*-algebra, acting on a Hilbert space $\mathscr{H}$, containing the identity operator, and let $\bar{A}$ be the weak operator closure of A. Then* $\overline{\mathrm{F}_k(\bar{A})}|A = \overline{\mathrm{F}_k(A)}$ $(1 \leqslant k \leqslant \infty)$, *and hence* $\overline{\mathrm{F}(\bar{A})}|A = \overline{\mathrm{F}(A)}$.

Proof. By the above remark (with $B = \bar{A}$), $\overline{\mathrm{F}_k(A)} \subset \overline{\mathrm{F}_k(\bar{A})}|A$.

Conversely, suppose first that k is finite, $\varphi \in \mathrm{F}_k(\bar{A})$ and ψ is an extension of φ in $\mathrm{P}(\bar{A} \otimes \mathscr{K}_k)$ (Lemma 5.2). Regarding $A \otimes \mathscr{K}_k$ as acting on $\mathscr{H} \otimes \mathscr{H}_k$, $\overline{A \otimes \mathscr{K}_k} = \bar{A} \otimes \mathscr{K}_k$. Glimm [7, Theorem 5] showed that there is a net ψ_α in $\mathrm{P}(A \otimes \mathscr{K}_k)$ such that $\psi_\alpha(x) \to \psi(x)$ $(x \in A \otimes \mathscr{K}_k)$. Let $\varphi_\alpha = \psi_\alpha|A$. Then $\varphi_\alpha \in \mathrm{F}_k(A)$ and $\varphi_\alpha(a) \to \varphi(a)$ $(a \in A)$. Thus $\mathrm{F}_k(\bar{A})|A \subset \overline{\mathrm{F}_k(A)}$, and hence $\overline{\mathrm{F}_k(\bar{A})}|A \subset \overline{\mathrm{F}_k(A)}$.

The remainder of the theorem follows from the fact that

$$\overline{\mathrm{F}(\bar{A})} = \overline{\mathrm{F}_\infty(\bar{A})} = \Big(\bigcup_{1<k<\infty} \overline{\mathrm{F}_k(\bar{A})}\Big)^-, \quad \overline{\mathrm{F}(A)} = \overline{\mathrm{F}_\infty(A)}$$

(Proposition 2.2).

It was also shown in [2, Theorem 4.6(2)] that if A does not contain the identity operator on $[\pi(A)\mathscr{H}]$, then $\overline{F(\tilde{A})}|A = \overline{F(A)} \cup \{0\}$. Similarly $\overline{F_k(\tilde{A})}|A = \overline{F_k(A)} \cup \{0\}$. However, $0 \in \overline{P(A)} \subset \overline{F_k(A)}$ [6, 2.12.13], so $\overline{F_k(\tilde{A})}|A = \overline{F_k(A)}$.

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