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ON THE TOPOLOGICAL STABLE RANK OF IRRATIONAL ROTATION ALGEBRAS

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INTRODUCTION

In this paper we are concerned with the question whether the invertible elements in the irrational rotation algebras are dense or not, or to speak in the terminology of Rieffel's theory of the topological stable rank of C^*-algebras [3], is the topological stable rank of the irrational rotation algebras equal to 1 or 2 (cf. [3], 7.4). Using an elementary fact in continued fraction theory, namely there are irrational numbers α admitting an arbitrarily high order of approximation by rationals (in the sense of diophantine approximations), we shall show that there are irrational rotation algebras having a dense set of invertible elements. Actually, we shall show that for these irrational rotation algebras $\mathscr{A}_\alpha$ the invertible polynomials in the two canonical generators U and V of $\mathscr{A}_\alpha$ and their adjoints U^* and V^* are dense in the set of all polynomials. Since the spectra of those polynomials turn out to be very unstable depending on the rotation number α, we cannot expect the proof will provide us with explicit formulas for the spectra of polynomials. As an example for this instability we want to mention Hofstadter's empirical investigation of the spectrum of $U + V + U^* + V^*$ in [1]. Therefore, our method of proof will be based on general arguments and very rough estimates.

§ 1

For each $n \in \mathbf{N}$ we define inductively a function ψ_n on $\mathbf{N}$ as follows:

$$\psi_n(1) = n2^n;$$

$$\psi_n(k) = n2^n + n^4 \sum_{i=1}^{k-1} \psi(i), \quad k \geqslant 2.$$

Moreover, we define a function Φ on $\mathbf{N}$ by

$$\Phi(n) = (2^{n+5} n^7 \psi_n(n^4))^{-1}.$$

Continued fraction theory shows that there are irrational numbers $\alpha > 0$ such that

$$|\alpha - p_n/q_n| < \Phi(q_n), \tag{1.1}$$

where p_n/q_n (p_n and q_n are relatively prime integers) denotes the convergent of order n of α (cf. [2], proof of Theorem 22). Our main purpose is to prove the following theorem.

1.2. THEOREM. *For any irrational number $\alpha > 0$ satisfying* (1.1), *the invertible elements in the corresponding irrational rotation algebra $\mathcal{A}_\alpha$ are dense in $\mathcal{A}_\alpha$ with respect to the norm-topology.*

§ 2

Let $\alpha > 0$ be an irrational number. We set $\lambda = e^{2\pi i\alpha}$, and we fix two canonical generators U, V of $\mathcal{A}_\alpha$, i.e. U and V are unitaries in $\mathcal{A}_\alpha$ satisfying $UV = \lambda VU$. A polynomial (in U and V) is an element in $\mathcal{A}_\alpha$ of the form $\sum_{i,j\in\mathbf{Z}} a_{ij}U^iV^j$, where a_{ij} vanishes for all (i,j) in the complement of a finite subset of $\mathbf{Z}^2$. Since the set of all polynomials is dense in $\mathcal{A}_\alpha$, Theorem 1.2 is an immediate consequence of the following proposition, which tells us how to approximate arbitrary polynomials by invertible polynomials. (If $\mathcal{P}(x,y)$ is a rational function of the form $\sum_{i,j\in\mathbf{Z}} a_{ij}x^iy^j$ and A, B are invertible, not necessarily commuting operators, then we shall write $\mathcal{P}(A,B)$ for $\sum_{i,j\in\mathbf{Z}} a_{ij}A^iB^j$.)

2.1. PROPOSITION. *Assume that α has the property* (1.1) *and let $\mathcal{P}(U,V)$ be a polynomial in U and V. Then for any $0 < \varepsilon < 1$ there exist numbers $s, t \in [1-\varepsilon, 1+\varepsilon]$ such that the element $\mathcal{P}(sU, tV)$ is invertible.*

The proof of Proposition 2.1 is preceded by a short sequence of lemmas.

NOTATION 1. For any rational function in the variables x and y of the form $\mathcal{P}(x,y) = \sum_{i,j\in\mathbf{Z}} a_{ij}x^iy^j$ with complex coefficients a_{ij}, we call the finite subset $\{(i,j) | a_{ij}\neq 0\}$ of $\mathbf{Z}^2$ the support of $\mathcal{P}(x,y)$, and we denote it by $\mathrm{S}(\mathcal{P})$. By an extremal point of $\mathrm{S}(\mathcal{P})$ we mean an element in the plane which is an extremal point in the convex hull of $\mathrm{S}(\mathcal{P})$. Actually, such an extremal point belongs to $\mathrm{S}(\mathcal{P})$. We denote by $\hat{\mathrm{S}}(\mathcal{P})$ the intersection of $\mathbf{Z}^2$ and the convex hull of $\mathrm{S}(\mathcal{P})$.

2. For any positive integers p, q which are relatively prime, we denote by

$$\xi_{(p,q)} = e^{2\pi ip/q}$$

$$U_{(p,q)} = (u_{ij}^{(p,q)})_{1\leqslant i,j\leqslant q}, \quad \text{where } u_{ij}^{(p,q)} = \begin{cases} \xi_{(p,q)}^{i-1} & \text{if } i = j \\ 0 & \text{elsewhere} \end{cases}$$

$$V_q = (v_{ij}^{(q)})_{1\leqslant i,j\leqslant q}, \quad \text{where } v_{ij}^{(q)} = \begin{cases} 1 & \text{if } j = i+1 \text{ or } i = n,\ j = 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus, $U_{(p,q)}$ and V_q satisfy the following relations.

$$U_{(p,q)}^q = V_q^q = I_q \ (= \text{identity matrix})$$

$$U_{(p,q)}V_q = \xi_{(p,q)}V_qU_{(p,q)} .$$

2.2. LEMMA. *Let $p, q \in \mathbf{N}$ be relatively prime. Then for any polynomial $0 \neq \mathscr{P}(x, y) = \sum_{i,j\geqslant 0} a_{ij}x^iy^j$ there exists a polynomial $\mathscr{R}(x, y)$ with $\mathrm{S}(\mathscr{R}) \subseteq \hat{\mathrm{S}}(\mathscr{P})$, such that for all $x, y \in \mathbf{C}$*

$$\det(\mathscr{P}(xU_{(p,q)}, yV_q)) = \mathscr{R}(x^q, y^q).$$

Proof. For a given polynomial $\mathscr{P}(x, y)$ we consider the set K which is equal to the intersection of $\mathbf{Z}^2$ and the convex hull of the set $\{(qi, qj) \mid (i, j) \in \mathrm{S}(\mathscr{P})\}$. By definition of the determinant as a homogeneous polynomial in the entries of the the underlying matrix, we get the following representation,

$$\det(\mathscr{P}(xU_{(p,q)}, yV_q) = \sum_{(i,j)\in K} c_{ij}x^iy^j,$$

with properly chosen coefficients c_{ij}. Now, conjugating the matrix $\mathscr{P}(xU_{(p,q)}, yV_q)$ by all matrices of the form $U_{(p,q)}^kV_q^l$, $0 \leqslant k, l \leqslant q-1$, yields for any pair of q-th roots of unity ξ, η the identity

$$\sum_{(i,j)\in K} c_{ij}x^iy^j = \sum_{(i,j)\in K} c_{ij}(\xi x)^i(\eta y)^j; \quad x, y \in \mathbf{C},$$

hence $c_{ij} = \xi^i\eta^jc_{ij}$. This shows that $c_{ij} \neq 0$ only if $i, j \in q\mathbf{N}$.

REMARK. It is easy to see that if (i, j) is an extremal point of $\mathrm{S}(\mathscr{P})$ then $c_{qi,qj} = \pm a_{ij}^q$.

2.3. LEMMA. *Let $p, q \in \mathbf{N}$ be relatively prime, and let $\mathscr{P}(x, y)$ be a polynomial so that $\mathscr{P}(xU_{(p,q)}, yV_q)$ is invertible for all $x, y \in \mathbf{T}$ $(=\{z \in \mathbf{C} \mid |z| = 1\})$. Then there exists a (unique) polynomial $Q(x, y)$ such that*

$$\mathscr{P}(xU_{(p,q)}, yV_q)Q(xU_{(p,q)}, yV_q) = \det(\mathscr{P}(xU_{(p,q)}, yV_q)I_q ,$$

for each $x, y \in \mathbf{T}$.

Proof. An application of Cramer's rule shows us that for each $(i,j) \in (\mathbf{Z}^2)^+$ there exists a complex $q \times q$-matrix M_{ij} such that

$$\mathscr{P}(xU_{(p,q)}, yV_q)(\sum_{i,j\geqslant 0} M_{ij}x^iy^j) = \det(\mathscr{P}(xU_{(p,q)}, yV_q)I_q .$$

Since the matrices $U_{(p,q)}^iV_q^j$, $0 \leqslant i, j \leqslant q-1$, form a linear basis for all $q\times q$--matrices, we can represent the M_{ij} as linear combinations of these matrices, thus getting complex polynomials $Q_{ij}(x, y)$ $(0 \leqslant i, j \leqslant q-1)$ with

$$(2.4) \qquad \mathscr{P}(xU_{(p,q)}, yV_q)\left(\sum_{i,j=0}^{q-1} Q_{ij}(x, y)U^i_{(p,q)}V^j_q\right) = \det(\mathscr{P}(xU_{(p,q)}, yV_q))I_q$$

for all $x, y \in \mathbf{T}$. Moreover, the polynomials Q_{ij} are uniquely determined by the equation (2.4). Next, we conjugate both sides of (2.4) by all matrices of the form $U^k_{(p,q)}V^l_q$, $0 \leqslant k, l \leqslant q-1$, thus getting

$$\mathscr{P}(\xi xU_{(p,q)}, \eta yV_q)\left(\sum_{i,j=0}^{q-1} Q_{ij}(x, y)\xi^i\eta^j U^i_{(p,q)}V^j_q\right) = \det(\mathscr{P}(xU_{(p,q)}, yV_q)I_q,$$

for $x, y \in \mathbf{T}$, and for any pair ξ, η of q-th roots of unity. By Lemma 2.2 we have $\det(\mathscr{P}(\xi xU_{(p,q)}, \eta yV_q)) = \det(\mathscr{P}(xU_{(p,q)}, yV_q))$. Therefore, as the polynomials Q_{ij} are uniquely determined by (2.4) we obtain

$$Q_{ij}(\xi x, \eta y) = \xi^i\eta^j Q_{ij}(x, y), \quad x, y \in \mathbf{T},$$

for any pair of q-th roots of unity ξ, η. It follows from this condition that there exist polynomials $\tilde{Q}_{ij}(x, y)$ such that

$$Q_{ij}(x, y) = x^i y^j \tilde{Q}_{ij}(x^q, y^q).$$

We set $Q(x, y) = \sum_{i,j=0}^{q-1} Q_{ij}(x, y)$. Because of the last condition, and since $U^q_{(p,q)} = V^q_q = I_q$ holds, we get

$$\sum_{i,j=0}^{q-1} Q_{ij}(x, y)U^i_{(p,q)}V^j_q = Q(xU_{(p,q)}, yV_q),$$

thus accomplishing the proof.

2.5. LEMMA. *Let $p_1, p_2 \ldots$ and $q_1, q_2, \ldots$ be sequences of positive integers such that p_n and q_n are relatively prime, and let $\mathscr{P}(x, y) = \sum_{i,j\geqslant 0} a_{ij}x^iy^j$ be a non-zero polynomial. For each $n \in \mathbf{N}$ we choose a polynomial $\mathscr{R}_{(p_n,q_n)}(x, y) = \sum_{(i,j)\in\hat{S}(\mathscr{P})} c^{(p_n,q_n)}_{ij}x^iy^j$ with $\det(\mathscr{P}(xU_{(p_n,q_n)}, yV_{q_n})) = \mathscr{R}_{(p_n,q_n)}(x^{q_n}, y^{q_n})$ (cf. Lemma 2.2). Then there is a subsequence of $(p_1, q_1), (p_2, q_2), \ldots$ which, for convenience, we denote by $(p_1, q_1), (p_2, q_2), \ldots$ again, such that*

$$\lim_{n\to\infty} |c^{(p_n,q_n)}_{ij}|^{1/q_n} = c_{ij} < \infty \quad \textit{for } (i, j) \in \hat{S}(\mathscr{P});$$

and there is an element $(k, l) \in \hat{S}(\mathscr{P})$ *such that for each* $0 < \varepsilon < 1$ *there are numbers* $s, t \in [1-\varepsilon, 1+\varepsilon]$ *with*

$$c_{kl}s^k t^l > c_{ij}s^i t^j \quad \text{if } (i,j) \neq (k,l).$$

Proof. Each time, when we choose a subsequence of $(p_1, q_1), (p_2, q_2), \ldots$ we shall denote it by $(p_1, q_1), (p_2, q_2), \ldots$ again. First we choose a subsequence such that for some $(g, h) \in \hat{S}(\mathscr{P})$,

$$|c_{gh}^{(p_n,q_n)}| \geqslant |c_{ij}^{(p_n,q_n)}| \quad \text{for all } (i,j) \in \hat{S}(\mathscr{P}).$$

Since by our assumption $\mathscr{P}$ is non-zero, $|c_{gh}^{(p_n,q_n)}|$ is strictly positive. We set

$$\tilde{c}_{ij}^{(p_n,q_n)} = c_{ij}^{(p_n,q_n)}/c_{gh}^{(p_n,q_n)} .$$

Next, we choose a subsequence such that

$$\lim_{n\to\infty} |\tilde{c}_{ij}^{(p_n,q_n)}|^{1/q_n} = \tilde{c}_{ij} \quad \text{exists for all } (i,j) \in \hat{S}(\mathscr{P}).$$

Let $\Gamma = \{(i,j) \mid \tilde{c}_{ij} = 1\}$, and let (k, l) be an extremal point in the convex hull of Γ. Then for any given $0 < \varepsilon < 1$ we can find $s, t \in [1-\varepsilon, 1+\varepsilon]$ such that

$$s^k t^l > s^i t^j \quad \text{for } (i,j) \in \Gamma \setminus \{(k,l)\},$$

$$\tilde{c}_{ij}s^i t^j > \tilde{c}_{mn}s^m t^n \quad \text{for } (i,j) \in \Gamma,\ (m,n) \in \hat{S}(\mathscr{P}) \setminus \Gamma.$$

Now we get

$$|c_{kl}^{(p_n,q_n)}|^{1/q_n} s^k t^l \delta_{(p_n,q_n)} = |\det(\mathscr{P}(sU_{(p_n,q_n)}, tV_{q_n}))|^{1/q_n} \leqslant$$

$$\leqslant \|\mathscr{P}(sU_{(p_n,q_n)}, tV_{q_n})\| \leqslant \sum_{(i,j)\in\hat{S}(\mathscr{P})} |a_{ij}| s^i t^j,$$

where

$$\delta_{(p_n,q_n)} = |1 + \sum_{(i,j)\in\hat{S}(\mathscr{P})\setminus\{(k,l)\}} \tilde{c}_{ij}^{(p_n,q_n)}/\tilde{c}_{kl}^{(p_n,q_n)} s^{q_n(i-k)} t^{q_n(j-l)}|^{1/q_n}.$$

However, since $\lim_{n\to\infty} \delta_{(p_n,q_n)} = 1$, we get

$$\limsup_{n\to\infty} |c_{kl}^{(p_n,q_n)}|^{1/q_n} s^k t^l < \infty,$$

hence also,

$$\limsup_{n\to\infty} |c_{kl}^{(p_n,q_n)}|^{1/q_n} < \infty.$$

Therefore we can choose a subsequence such that

$$\lim_{n\to\infty} |c_{kl}^{(p_n,q_n)}|^{1/q_n} = c_{kl} < \infty.$$

If we set $c_{ij} = \tilde{c}_{ij} c_{kl}$, then all the other conditions listed in 2.5 are also satisfied.

Now we turn to the proof of Proposition 2.1.

Let α be a positive irrational number satisfying (1.1). In order to prove 2.1 it suffices to show that the statement of 2.1 is true for polynomials in U and V of the form $\sum_{i,j\geq 0} a_{ij} U^i V^j$. For, if $\mathscr{P}(U, V)$ is an arbitrary polynomial in U and V, then there exist $k, l \geq 0$ such that the support of the function $x^k y^l \mathscr{P}(x, y)$ is contained in $(\mathbf{Z}^2)^+$, and if $U^k V^l \mathscr{P}(U, V)$ is approximable by invertible elements, then the same is true for $\mathscr{P}(U, V)$. So, let us consider an arbitrary complex polynomial $\mathscr{P}(x, y) = \sum_{i,j\geq 0} a_{ij} x^i y^j$. Let p_n/q_n (p_n and q_n relatively prime) be the convergent of order n of α. By Lemma 2.2 and by Lemma 2.5 there exists for each $0 < \varepsilon < 1$ a subsequence of $(p_1, q_1), (p_2, q_2), \ldots$, which we shall denote by $(p_1, q_1), (p_2, q_2), \ldots$ again, and numbers $s, t \in [1-\varepsilon, 1+\varepsilon]$ such that (we set $\tilde{\mathscr{P}}(x, y) = \sum_{i,j\geq 0} \tilde{a}_{ij} x^i y^j$, where $\tilde{a}_{ij} = s^i t^j a_{ij}$) there exist polynomials

$$\mathscr{R}_{(p_n,q_n)}(x, y) = \sum_{(i,j)\in\hat{S}(\tilde{\mathscr{P}})} c_{ij}^{(p_n,q_n)} x^i y^j$$

with

$$\det(\tilde{\mathscr{P}}(xU_{(p_n,q_n)}, yV_{q_n})) = \mathscr{R}_{(p_n,q_n)}(x^{q_n}, y^{q_n}),$$

$$\lim_{n\to\infty} |c_{ij}^{(p_n,q_n)}|^{1/q_n} = c_{ij} < \infty, \quad (i, j) \in \hat{S}(\tilde{\mathscr{P}});$$

and there exists an element $(k, l) \in \hat{S}(\tilde{\mathscr{P}})$ such that

$$c_{kl} > c_{ij} \quad \text{for } (i, j) \in \hat{S}(\tilde{\mathscr{P}}) \setminus \{(k, l)\}.$$

We assume the polynomial $\tilde{\mathscr{P}}$ to be normalized, so that $c_{kl} = 1$. By Lemma 2.3 there are polynomials

$$Q_{(p_n,q_n)}(x, y) = \sum_{i,j\geq 0} b_{ij}^{(p_n,q_n)} x^i y^j$$

so that for each $n \in \mathbf{N}$

$$\tilde{\mathscr{P}}(xU_{(p_n,q_n)}, yV_{q_n}) Q_{(p_n,q_n)}(xU_{(p_n,q_n)}, yV_{q_n}) = \mathscr{R}_{(p_n,q_n)}(x^{q_n}, y^{q_n}) I_{q_n}; \quad x, y \in \mathbf{T}. \tag{2.6}$$

For notational reasons we set $\mathscr{R}_{(p_n,q_n)}(x^{q_n}, y^{q_n}) = \sum_{i,j\geq 0} \tilde{c}_{ij}^{(p_n,q_n)} x^i y^j$. For each $n \in \mathbf{N}$ the equation (2.6) is equivalent to the following linear system of scalar equations,

$$\sum_{(i,j)\in\hat{S}(\tilde{\mathscr{P}})} \tilde{a}_{ij} b_{g-i,h-j}^{(p_n,q_n)} \xi_{(p_n,q_n)}^{(i-g)j} = \tilde{c}_{gh}^{(p_n,q_n)}; \quad g, h \geq 0. \tag{2.7}$$

(For convenience we set $b_{ij}^{(p_n,q_n)} = 0$ for $(i,j) \in \mathbf{Z}^2 \setminus (\mathbf{Z}^2)^+$.) Now, there is an integer $m > 0$ such that for each $n \geqslant m$ the number q_n dominates the following numbers, $|\tilde{a}_{ij}|$, $|\tilde{a}_{ij}^{-1}|$ for $(i,j) \in \mathrm{S}(\tilde{\mathscr{P}})$; moreover $\hat{\mathrm{S}}(\tilde{\mathscr{P}}) \subseteq [0, q_n] \times [0, q_n]$, and

$$\sum_{(i,j) \in \hat{\mathrm{S}}(\tilde{\mathscr{P}}) \setminus \{(k,l)\}} |c_{ij}^{(p_n,q_n)} / c_{kl}^{(p_n,q_n)}| \leqslant 1/2, \quad 2^{-q_n} \leqslant |c_{kl}^{(p_n,q_n)}|.$$

We shall show that $\tilde{\mathscr{P}}(U, V) Q_{(p_n,q_n)}(U, V)$ is invertible for $n \geqslant m$. We denote by $\mathscr{S}_{(p_n,q_n)}(x, y) = \sum_{i,j \geqslant 0} a_{ij}^{(p_n,q_n)} x^i y^j$ the polynomial satisfying

$$\mathscr{S}_{(p_n,q_n)}(U, V) = \tilde{\mathscr{P}}(U, V) Q_{(p_n,q_n)}(U, V).$$

We get

$$d_{gh}^{(p_n,q_n)} = \sum_{(i,j) \in \hat{\mathrm{S}}(\tilde{\mathscr{P}})} \tilde{a}_{ij} b_{g-i,h-j}^{(p_n,q_n)} \lambda^{(i-g)j}.$$

We shall estimate the sum $\sum_{i,j \geqslant 0} |\tilde{c}_{ij}^{(p_n,q_n)} - d_{ij}^{(p_n,q_n)}|$. In order to do this, we need an estimate for the coefficients $b_{ij}^{(p_n,q_n)}$. Therefore we consider (2.7) as a system of linear equations in the variables $b_{ij}^{(p_n,q_n)}$. Since $\mathrm{S}(\mathscr{R}_{(p_n,q_n)}) \subseteq \hat{\mathrm{S}}(\tilde{\mathscr{P}})$, we get that $\mathrm{S}(Q_{(p_n,q_n)}) \subseteq$ $\subseteq$ convex hull of $\{(q_n i, q_n j) \mid (i,j) \in \hat{\mathrm{S}}(\tilde{\mathscr{P}})\} \subseteq [0, q_n^2] \times [0, q_n^2]$. Thus we know from the beginning that $b_{ij}^{(p_n,q_n)} = 0$ for $(i,j) \notin [0, q_n^2] \times [0, q_n^2]$. Now we can solve the system (2.7) by at most $q_n^2 \cdot q_n^2 = q_n^4$ steps, where in each step we compute another of the variables $b_{ij}^{(p_n,q_n)}$ by resolving a properly chosen equation of the system (2.7). So, basically what we do, is solving a system of linear equations whose underlying coefficient matrix is triangular. There are some equations left over, which we don't use for the resolution of (2.7). The absolute value of the variable we compute in the k-th step will be dominated by $\psi_{q_n}(k)$ (cf. Section 1; of course, this is a very rough estimate). Since the functions ψ_{q_n} are monotone increasing, we get

$$|b_{ij}^{(p_n,q_n)}| < \psi_{q_n}(q_n^4).$$

Next,

$$\sum_{g,h \geqslant 0} |\tilde{c}_{gh}^{(p_n,q_n)} - d_{gh}^{(p_n,q_n)}| \leqslant$$

$$\leqslant \sum_{g,h \in [0,q_n^2]} \sum_{(i,j) \in \hat{\mathrm{S}}(\tilde{\mathscr{P}})} |\tilde{a}_{ij}| \, |b_{ij}^{(p_n,q_n)}| \, |\xi_{(p_n,q_n)}^{(i-g)j} - \lambda^{(i-g)j}| \leqslant$$

$$\leqslant \sum_{g,h \in [0,q_n^2]} \sum_{i,j \in [0,q_n]} q_n \psi_{q_n}(q_n^4) 2\pi |i-g| \, j \, |p_n/q_n - \alpha| \leqslant$$

$$\leqslant 8 q_n^7 \psi(q_n^4) \Phi(q_n) \qquad \text{(by (1.1))},$$

whence,

$$\tilde{\mathscr{P}}(U, V)Q_{(p_n,q_n)}(U, V) =$$

$$= c_{kl}^{(p_n,q_n)}(U^{kq_n}V^{lq_n} + \sum_{(i,j)\in \hat{S}(\tilde{\mathscr{P}})\setminus\{(k,l)\}} c_{ij}^{(p_n,q_n)}/c_{kl}^{(p_n,q_n)}U^{iq_n}V^{jq_n}) + \mathscr{Z}(U, V),$$

where

$$\|\mathscr{Z}(U, V)\| \leqslant \sum_{i,j\geqslant 0} |\tilde{c}_{ij}^{(p_n,q_n)} - d_{ij}^{(p_n,q_n)}| \leqslant 8q_n^7\psi(q_n^4)\Phi(q_n) =$$

$$= 2^{-(q_n+2)} \leqslant (1/4)c_{kl}^{(p_n,q_n)}.$$

Since $\sum_{(i,j)\in \hat{S}(\tilde{\mathscr{P}})\setminus\{(k,l)\}} |c_{ij}^{(p_n,q_n)}/c_{kl}^{(p_n,q_n)}| \leqslant 1/2$, we conclude that $\tilde{\mathscr{P}}(U, V)Q_{(p_n,q_n)}(U, V)$ is invertible, thus $\tilde{\mathscr{P}}(U, V)$ is invertible, and this completes the proof of Proposition 2.1. Hence Theorem 1.2 is also proved.

REMARK. We haven't used the diophantine condition (1.1) until the very end of the proof of Proposition 2.1. Actually, all of the intermediate steps go through for an arbitrary rotation number also. For instance, let $\alpha = 1$. Then the rotation algebra associated with α is nothing but the commutative C^*-algebra of all continuous functions on $\mathbf{T}^2$, i.e. $\mathscr{A}_1 = C(\mathbf{T}^2)$. As an approximating sequence of rationals for α we could choose $\{(n-1)/n\}_{n\in\mathbf{N}}$. However, since the dimension of $\mathbf{T}^2$ is two, the topological stable rank of $C(\mathbf{T}^2)$ is also two ([3], 1.7), in particular the invertible elements of $C(\mathbf{T}^2)$ are not dense in $C(\mathbf{T}^2)$.

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