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ON EIGENVALUES IN THE ESSENTIAL SPECTRUM OF A TOEPLITZ OPERATOR

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INTRODUCTION

Let $\mathbf{T}$ be the unit circle in the complex plane, $\mathbf{C}$. For a bounded, measurable function φ on $\mathbf{T}$, define the Toeplitz operator T_φ on H^2 of the unit disk by

$$T_\varphi f = \mathbf{P}\,\varphi f$$

where $\mathbf{P}$ is the projection of L^2 on H^2.

If φ is continuous, the Fredholm theory of T_φ is well known [6, Chapter 7]. Indeed, $\rho_e(T_\varphi)$, the Fredholm resolvent set, is the complement of the curve $\varphi(\mathbf{T})$ and the index of $T_\varphi - \lambda I = T_{\varphi-\lambda}$, for $\lambda \notin \varphi(\mathbf{T})$, is $-\omega(\varphi, \lambda)$, the negative of the winding number of the curve $\varphi(\mathbf{T})$ about λ. In addition, the index $-\omega(\varphi, \lambda)$ is equal to the dimension of the kernel $\ker(T_\varphi - \lambda I)$, if $\omega(\varphi, \lambda) \leqslant 0$ and $\omega(\varphi, \lambda) = \dim\ker(T_\varphi^* - \bar{\lambda} I)$, if $\omega(\varphi, \lambda) \geqslant 0$.

The present paper is a report on an investigation of the dimension of $\ker(T_\varphi - \lambda I)$, when φ is continuous and λ lies *on* the curve $\varphi(\mathbf{T})$ (so that $T_\varphi - \lambda I$ is *not* a Fredholm operator.) Some previous work on the eigenspaces of non-Fredholm Toeplitz operators may be found in [10, 5, 3, 8]. Our work differs from that of these otherauthors in that we seek a description of eigendimension in terms of the geometrical properties of the curve $\varphi(\mathbf{T})$, modeled as closely as possible on the winding number characterization of $\dim\ker(T_\varphi - \lambda I)$, for $\lambda \notin \varphi(\mathbf{T})$. A test question for a geometricaltheory of eigendimension is the problem of whether $\dim\ker(T_\varphi - \lambda I) > 0$ can hold for $\lambda \in \partial\sigma(T_\varphi)$, the boundary of the spectrum of T_φ; [2], [9].

For our formula, we assume $\varphi(e^{it})$ has the form

$$(1) \qquad \varphi(e^{it}) = \prod_{j=1}^{n} (e^{-it} - e^{-i\theta_j})^{\alpha_j} h(e^{it}),$$

where $\alpha_j > 0$, $j = 1, \ldots, n$, and h is continuous and nonvanishing on $\mathbf{T}$. In Part 1, we prove that $\omega(h, 0)$, which determines $\dim\ker T_\varphi$, is characterized as follows (to be made precise in Section 1.1). Let Ω be a connected component of $\mathbf{C}\setminus\varphi(\mathbf{T})$, with $0 \in \partial\Omega$. If the boundary of Ω has positive inner angle at 0, we label as *negative*, the arcs $\lambda_j(t) = \varphi(e^{it})$ $(\theta_j - \varepsilon < t < \theta_j + \varepsilon)$, such that the two arcs of λ_j itself meet at positive angle at 0 and Ω lies on the right as λ_j is traversed through a neighborhood of 0. In this case,

$$\omega(h, 0) = \omega(\varphi, \Omega) + \sum_{j=1}^{n} [\alpha_j/2] + N \tag{2}$$

where N is the number of negative arcs, and $[\cdot]$ is the greatest integer function. If the boundary of Ω has inner angle 0 at 0, we must consider all arcs λ_j having a cusp at 0, with Ω in the 0-angle of the cusp. From this set, the number of negative arcs minus the number on nonnegative arcs must be added to the right side of (2).

In Part 2, we address the problem of boundary eigenvalues for T_φ. Based upon our characterization of $\omega(h, 0)$, it is not difficult to show that for φ satisfying (1), a point on the boundary *of a component of* $\rho(T_\varphi)$ cannot be an eigenvalue. However, as we show by example in Section 2.1, there may still be eigenvalues. The mild additional assumption that φ is C^1 with $\varphi' \neq 0$ at $\theta_1, \ldots, \theta_n$ (or, more generally, that for each j there is an m_j such that $\varphi \in C^{(m_j)}$ and $\varphi^{(m_j)}(e^{i\theta_j}) \neq 0$) is sufficient to imply that $\ker(T_\varphi - \lambda I) = \{0\}$, for all $\lambda \in \partial\sigma(T_\varphi)$.

1. GEOMETRIC FORMULA FOR DIM KER T_φ

1.1. THE FORMULA. We consider a Toeplitz operator with symbol φ having the form

$$\varphi(e^{it}) = \prod_{j=1}^{n} (e^{-it} - e^{-i\theta_j})^{\alpha_j} h(e^{it}), \tag{3}$$

where θ_j is real and $\alpha_j > 0$, $j = 1, \ldots, n$, and h is continuous and nonvanishing on $\mathbf{T}$. The function

$$p(e^{it}) = \prod_{j=1}^{n} (e^{it} - e^{i\theta_j})^{\alpha_j}$$

is outer and $\varphi = \bar{p}h$, so that $T_\varphi = T_{\bar{p}}T_h$. Since $T_{\bar{p}}$ is one-to-one, it follows that the kernels of T_φ and T_h are equal. By the Fredholm theory for Toeplitz operators,

$$\dim\ker T_h = -\,\omega\,(h, 0),$$

but it is not entirely clear how $\omega(h, 0)$ depends on φ. This is our first goal.

Suppose Ω is a (connected) component of $\mathbf{C}\setminus\varphi(\mathbf{T})$, with $0 \in \partial\Omega$, the boundary of Ω. For $\varepsilon > 0$, let $\Delta_\varepsilon = \{z\colon |z| \leqslant \varepsilon\}$, and let $I_1, \ldots, I_n$ be the components of $\varphi^{-1}(\Delta_\varepsilon)$ that contain $e^{i\theta_1}, \ldots, e^{i\theta_n}$, respectively. Each I_j is a closed arc on $\mathbf{T}$, with endpoints which we denote by e^{ia_j} and e^{ib_j}, $a_j < \theta_j < b_j$. It follows from (3) that

$$\arg\varphi(e^{i(\theta_j\pm 0)}) = \lim_{0<t\to 0} \arg\varphi(e^{i(\theta_j\pm t)})$$

exists. If we write

$$\alpha_j = 2n_j + \beta_j, \quad 0 \leqslant \beta_j < 2,$$

we have

$$\arg\varphi(e^{i(\theta_j+0)}) - \arg\varphi(e^{i(\theta_j-0)}) = \beta_j\pi, \quad \bmod 2\pi,$$

and hence

$$\arg\varphi(e^{ib_j}) - \arg\varphi(e^{ia_j}) = \beta_j\pi + \varepsilon_j, \quad \bmod 2\pi$$

where $\varepsilon_j \to 0$ as $\varepsilon \to 0$. For each such $\varepsilon > 0$, there is an ε_0, $0 < \varepsilon_0 \leqslant \varepsilon$, such that $\varphi^{-1}(\Delta_{\varepsilon_0}) \subseteq \cup I_j$. If a subscript ε denotes restriction to $\mathbf{T}\setminus \cup I_j$, we have $\varphi_\varepsilon = \bar{p}_\varepsilon h_\varepsilon$ and

$$\omega(\varphi_\varepsilon, 0) = \omega(\bar{p}_\varepsilon, 0) + \omega(h_\varepsilon, 0),$$

where the winding number of a continuous function on $\cup I_j$ is defined to be the sum of the net changes in argument on the I_j. A direct computation shows that

$$\lim_{\varepsilon\to 0} \omega(\bar{p}_\varepsilon, 0) = -\sum_{j=1}^{n} \alpha_j/2 = -\sum_{j=1}^{n} n_j - \sum_{j=1}^{n} \beta_j/2,$$

and since $\lim_{\varepsilon\to 0} \omega(h_\varepsilon, 0) = \omega(h, 0)$, we have

$$\omega(\varphi_\varepsilon, 0) = -\sum_{j=1}^{n} n_j - \sum_{j=1}^{n} \beta_j/2 + \omega(h, 0) + \delta, \tag{4}$$

where $\delta \to 0$, as $\varepsilon \to 0$.

We take $\varepsilon > 0$ sufficiently small that

(i) $(1/2)\sum_{j=1}^{n} |\varepsilon_j| + |\delta| < 1$,

(ii) $\Omega \cap \partial\Delta_\varepsilon \neq \varnothing$,

(iii) whenever $\beta_j \neq 0$, $\varphi(e^{is}) \neq \varphi(e^{it})$, for

$$a_j \leqslant s < \theta_j < t \leqslant b_j.$$

By (ii), we can find a curve Γ, lying in the interior of $\Omega \cap \Delta_\varepsilon$, except for the endpoints, which are at 0 and $a \in \Omega \cap \partial\Delta_\varepsilon$.

Let $\lambda_j = \varphi|_{I_j}$. We want to define what it means for λ_j to be a negative arc. The points $\lambda_j(e^{ia_j})$ and $\lambda_j(e^{ib_j})$ subtend two arcs in $\partial\Delta_\varepsilon$, exactly one of which, denoted γ_j, does not contain a. We orient γ_j by taking $\lambda_j(e^{ia_j})$ as its initial point, and $\lambda_j(e^{ib_j})$ as its terminal point. Thus we may refer to γ_j as being positively or negatively oriented (compared with the usual counter clockwise orientation of $\partial\Delta_\varepsilon$).

DEFINITION. If $\beta_j \neq 0$ and γ_j is negatively oriented we say λ_j is a *negative arc*.
If $\beta_i = 0$, γ_j is longer than its complementary arc on $\partial\Delta_\varepsilon$ and γ_j is negatively oriented we say that λ_j is a *negative arc*.

If $\beta_j = 0$, γ_j is longer than its complementary arc and γ_j is positively oriented, we say that λ_j is a *positive cusp*.

Let N be the number of negative arcs and let P be the number of positive cusps. We have

THEOREM 1. *If φ is given by* (3), *we have*

$$\omega(h, 0) = \omega(\varphi, \Omega) + \sum_{j=1}^{n} n_j + N - P. \tag{5}$$

REMARK 1. Although the designation of which arcs are negative and which cusps are positive may certainly depend upon the choice of the point a from $\Omega \cap \partial\Delta_\varepsilon$, (5) shows that $N - P$ is independent of a.

REMARK 2. The *Fredholm* index of a Toeplitz operator with continuous (nonvanishing) symbol is clearly invariant under an orientation preserving change of variable of the symbol on $\mathbf{T}$. Theorem 1 shows that if the symbol vanishes such a change of variable must at least be a C^1 (invertible) diffeomorphism to leave $\dim\ker T_\varphi$ invariant, as otherwise the $\{n_j\}$ might change.

Proof. In order to procede from (4), we define a function ψ on $\mathbf{T}$ to be equal to $\varphi(e^{it})$ on $\mathbf{T}\setminus \cup I_j$ and, on I_j, to be such that $\psi(e^{it})$ traverses γ_j from $\lambda_j(e^{ia_j})$ to $\lambda_j(e^{ib_j})$. Then ψ is defined and continuous on $\mathbf{T}$. The range of ψ is disjoint from the curve Γ that joins 0 to a, and so we have $\omega(\psi, 0) = \omega(\psi, a)$. Now we can homotopy each of the arcs $\lambda_j(e^{it})$ $(a_j \leqslant t \leqslant b_j)$ to γ_j, with fixed endpoints, the homotopy taking place in $\mathbf{C}\setminus\{a\}$, and hence $\omega(\psi, a) = \omega(\varphi, a)$. So we have

$$\omega(\varphi, a) = \omega(\psi, 0) = \omega(\varphi_\varepsilon, 0) + \sum_{j=1}^{n} \omega(\gamma_j, 0).$$

Each of the terms $\omega(\gamma_j, 0)$ is easily evaluated. Indeed, if $\beta_j \neq 0$, we have

$$\omega(\gamma_j,\ 0) = \beta_j/2 - 1 - \varepsilon_j/2,$$

if λ_j is a negative arc; and

$$\omega(\gamma_j,\ 0) = \beta_j/2 + \varepsilon_j/2,$$

otherwise. If $\beta_j = 0$,

$$\omega(\gamma_j,\ 0) = -1 - \varepsilon_j/2,$$

if λ_j is a negative arc;

$$\omega(\gamma_j,\ 0) = 1 + \varepsilon_j/2,$$

if λ_j is a positive cusp; and

$$\omega(\gamma_j,\ 0) = \pm\, \varepsilon_j/2,$$

otherwise. Adding up the $\omega(\gamma_j, 0)$, we get

$$\sum \omega(\gamma_j,\ 0) = \sum_{j=1}^{n} \beta_j/2 - N + P + \eta$$

where $|\eta| \leqslant \sum |\varepsilon_j|/2$. If we insert this into (4), we get

$$\omega(h, 0) = \omega(\varphi,\ a) + \sum_{j=1}^{n} n_j + N - P - \eta - \delta.$$

This shows that $\eta + \delta$ is an integer; but

$$|\eta + \delta| \leqslant |\eta| + \delta \leqslant \frac{1}{2} \sum_{j=1}^{n} |\varepsilon_j| + \delta < 1$$

by (i), so that $\eta + \delta = 0$. The proof is complete.

COROLLARY 1. *For φ satisfying* (3),

$$\dim \ker T_\varphi = 0 \quad \textit{if } \omega(\varphi,\ \Omega) + \sum n_j + N - P \geqslant 0;$$

otherwise,

$$\dim \ker T_\varphi = -\omega(\varphi,\ \Omega) - \sum n_j - N.$$

1.2. THE ADJOINT SYMBOL. After proving a theorem such as Theorem 1, it is normal to try to write down the dimension of the adjoint kernel, $\dim \ker T_\varphi^*$, and hence determine the index of T_φ. In this section, we discuss a difficulty with this program.

Suppose φ satisfies (3) and set

$$h_j(e^{it}) = (e^{-it} - e^{-i\theta_j})^{\alpha_j}.$$

Then

$$\overline{\varphi}(e^{it}) = \prod_{j=1}^{n} h_j(e^{it}) H(e^{it})$$

where

$$H(e^{it}) = \prod_{j=1}^{n} [\overline{h}_j(e^{it})/h_j(e^{it})]\overline{h}(e^{it}).$$

Now if α_j is not an integer, $\overline{h}_j(e^{it})/h_j(e^{it})$ has a jump discontinuity at $e^{i\theta_j}$, so the results of § 1.1 cannot be applied directly to calculate $\omega(H, 0)$. Actually, if no α_j has the form $l + 1/2$, where l is an integer, the theory of Toeplitz operators with piecewise continuous symbol [5,7] implies that T_H is Fredholm, and $\dim\ker T_H$ can be computed in terms of $\omega(h, 0)$ and $\{\alpha_j\}$, $j = 1,2, \ldots, n$. Using our formula for $\omega(h, 0)$, this case can be handled satisfactorily. But the next result shows that if $\alpha_j = l + 1/2$, for some integer l and some j, then serious difficulties arise and we cannot hope to determine $\omega(H, 0)$ from the geometry of $\varphi(\mathbf{T})$ alone.

PROPOSITION 1. *Suppose*

$$\varphi(e^{it}) = (e^{-it} - 1)^{1/2} h(e^{it}), \tag{6}$$

where h is continuous and nonvanishing and $\dim\ker T_{\overline{h}} = n > 0$, *then* $\dim\ker T_{\overline{\varphi}} \leqslant n$. *If $h \in \mathrm{Lip}\,\beta$, for some $\beta > 0$, then* $\dim\ker T_{\overline{\varphi}} = n - 1$. *On the other hand, there exists a (continuous, nonvanishing) h with* $\dim\ker T_{\overline{\varphi}} = n$.

Proof. The inequality follows by taking the conjugate of both sides of (6). Suppose $h \in \mathrm{Lip}\,\beta$, for some $\beta > 0$. Then we can write

$$\overline{h}(e^{it}) = g(e^{it})\, e^{-int} e^{iu(e^{it})}$$

where g, $1/g \in H^\infty$ and $u \in \mathrm{Lip}\,\beta$. Since u and its harmonic conjugate v lie in $\mathrm{Lip}\,\beta$, $F = (1/2)(u + iv)$ is bounded. Thus if $G = e^{iF}$, we have G, $1/G \in H^\infty$ and $e^{iu} = G\overline{G}$. Putting all this together, we see that

$$T_{\overline{\varphi}} = T_{1/\overline{G}} T_{e^{-int}(e^{it}-1)^{1/2}} T_{gG}$$

so that

$$\dim\ker T_{\overline{\varphi}} = \dim\ker[T_{e^{-int}} T_{(e^{it}-1)^{1/2}}]. \tag{7}$$

Now the kernel on the right side of (7) is

$$\{g \in H^2 : p = (z - 1)^{1/2} g \text{ is a polynomial of degree } n - 1\}.$$

It is immediate that dim ker $T_{\bar{\varphi}} = n - 1$, since the set of all p appearing in the above kernel is exactly the set of polynomials of degree $\leqslant n - 1$ with $p(1) = 0$.

Now we prove the last assertion of the proposition. By conformal mapping onto an unbounded domain, it is possible to find a function u, continuous on the circle, such that its conjugate v satisfies

$$v(e^{it}) \leqslant - \left|\log|e^{it} - 1|\right|^{1/2}.$$

If we let $F = (1/2)(u + iv)$, then it is easily checked that $e^{-iF} \in H^\infty$ and

$$G(z) = e^{-iF(z)}(e^{iz} - 1)^{-1/2} \in H^2.$$

Let $\varphi(e^{it}) = (e^{-it} - 1)^{1/2} e^{int} e^{-iu}$. We see that ker $T_{\bar{\varphi}}$ is

$$\{Gp : p \text{ is a polynomial of degree at most } n - 1\}$$

and hence dim ker $T_{\bar{\varphi}} = n$.

2. BOUNDARY EIGENVALUES

2.1. EXAMPLES. In this section, we preface our results on Toeplitz operators *without* boundary eigenvalues by some examples of Toeplitz operators *with* boundary eigenvalues.

First we give a simple example of a continuous symbol φ such that T_φ has a boundary eigenvalue. Let

$$\varphi(e^{it}) = |e^{it} - 1|^\alpha e^{-it} \quad 0 < \alpha < 1.$$

Clearly $\varphi(1) = 0$, and, since φ takes no positive real values, 0 is on the boundary of the spectrum of T_φ. Write

$$\varphi(e^{it}) = (e^{-it} - 1)^{-\alpha/2} e^{-it}(e^{it} - 1)^{\alpha/2}.$$

Now since $0 < \alpha < 1$, $g = (e^{it} - 1)^{\alpha/2} \in H^2$ and clearly $T_\varphi g = 0$. When $\alpha = 1/2$, this is essentially the same example given recently by Clancey [2].

We can use the same ideas to give an example of a continuous symbol φ such that T_φ has a boundary eigenvalue of infinite multiplicity. Let

$$\psi(z) = \exp\{-(1 + z)/(1 - z)\}$$

denote the "atomic" inner function and $\{z_k\}$ the set of points such that $\psi(z_k) = 1$. Then $|z_k| = 1$ and $z_k \to 1$, as $k \to \infty$. It is not difficult to see that if $\sum \alpha_k < 1/2$, then the product

$$g(z) = (z - 1)^{\alpha_0} \prod_{k=1}^{\infty} (z - z_k)^{\alpha_k}$$

converges to a function that is holomorphic in $|z| < 1$ and continuous in $|z| \leqslant 1$. Further $g(z_k) = 0$, $k = 1, 2, \ldots$, and $1/g \in H^2$. We define $\varphi(e^{it}) = |g(e^{it})|^2\bar{\psi}(e^{it})$. As before, φ is continuous, $\varphi(1) = 0$ and φ takes no positive real values; hence, 0 is on the boundary of the spectrum of T_φ. Now the kernel of $T_{\bar{\psi}}$ is $(\psi H^2)^\perp$. Let $h \in H^\infty \cap (\psi H^2)^\perp$. Then $h/g \in H^2$ and

$$T_\varphi(h/g) = T_{\bar{g}} T_{\bar{\psi}} T_g h/g = 0.$$

Now $(\psi H^2)^\perp$ is infinite dimensional and $H^\infty \cap (\psi H^2)^\perp$ is dense in $(\psi H^2)^\perp$, from which it follows that $\dim \ker T_\varphi = \infty$.

Although less directly related to our main theorems, we also give here an example of a *coanalytic* Toeplitz operator with a boundary eigenvalue. Suppose, indeed, that $\varphi = \bar{g}$, with $g \in H^\infty$. Then the spectrum of T_φ is the closure of the set $\{\overline{g(z)} : |z| < 1\}$. If we factor $g = \psi h$, where ψ is inner and h is an outer function, then the kernel of T_φ is $(\psi H^2)^\perp$. In other words, $\bar{\lambda}$ is a boundary eigenvalue for T_φ if and only if λ is on the boundary of $\bar{g}(U)$, U the unit disk, and $g - \lambda$ has a nontrivial inner factor. Theorem 1 of [1] shows that this situation can occur. In fact, if V is a bounded, connected, open set and $g: U \to V$ is the universal covering map, then for $\lambda \in \partial V$, $g - \lambda$ has a nontrivial inner factor if and only if λ is an irregular point for the Dirichlet problem for V. Now it is well known that there is a bounded, connected, open set V such that $0 \in \partial V$ and 0 is an irregular point. This shows that a coanalytic T_φ may have a boundary eigenvalue. It also follows from [1] that the set of all boundary eigenvalues for a coanalytic Toeplitz operator must have logarithmic capacity zero.

2.2. BOUNDARY EIGENVALUE THEOREMS. In this section, we apply Theorem 1 to obtain two theorems and one example about boundary eigenvalues for T_φ, where φ has the form (3).

THEOREM 2. *If* 0 *lies on the boundary of a component of the resolvent set of* T_φ, *then* 0 *is not an eigenvalue of* T_φ.

Proof. We may apply Theorem 1 with $\omega(\varphi, \Omega) = 0$. We have

$$\omega(h, 0) = \sum_{j=1}^{n} n_j + N - P \geqslant \sum_{j=1}^{n} n_j - P.$$

Now if λ_j is a positive cusp, then, in particular, $\beta_j = 0$, and hence $n_j \geqslant 1$, since $\alpha_j \neq 0$. It follows that $\sum n_j - P \geqslant 0$, and the theorem follows from Corollary 1.

Of course the spectrum of T_φ may have boundary points that do not lie on the boundary of a component of the resolvent set. The next example shows that such points *may* be eigenvalues for a symbol φ satisfying (3).

EXAMPLE. *A Toeplitz operator* T_φ, *with* φ *satisfying* (3), *may have* 0 *as a boundary eigenvalue.*

Choose a real sequence $a_1 = \pi/2 > a_2 > \ldots \to 0$, such that if $\delta_n = a_n - a_{n+1}$, then $\delta_n/a_{n+1} \to 0$ (for example, choose $a_n = \pi/(2n)$). Let $t_n = (1/2)(a_n + a_{n+1})$ denote the midpoint of $[a_{n+1}, a_n]$. Define $\varphi(e^{it})$ on $[0, \pi/2 = a_1]$ by $\varphi(1) = 0$ and

$$(8)\quad \varphi(e^{it}) = \begin{cases} t & \text{on } (a_{n+1}, a_{n+1} + \delta_n/4) \\ t_n - \frac{1}{4}\delta_n \exp\frac{4i\pi}{\delta_n}\left[t - a_{n+1} - \frac{3}{4}\delta_n\right] & \text{on } [a_{n+1}+\delta_n/4, a_{n+1}+3\delta_n/4) \\ a_n - 3(a_n - t) & \text{on } [a_{n+1} + 3\delta_n/4, a_n]. \end{cases}$$

Then for $t \in [a_{n+1}, a_n]$,

$$|\varphi(e^{it}) - t_n| \leqslant \frac{1}{2}\delta_n,$$

$$|\varphi(e^{it}) - t| \leqslant |\varphi(e^{it}) - t_n| + |t_n - t| \leqslant \delta_n$$

so that

$$\left|\frac{\varphi(e^{it})}{t} - 1\right| < \frac{\delta_n}{t} < \frac{\delta_n}{a_{n+1}} \to 0, \quad n \to \infty.$$

Now extend $\varphi(e^{it})$ to $[-\pi, \pi]$ by defining

$$(9)\qquad \varphi(e^{it}) = \begin{cases} t & \text{on } [-\pi/2, 0) \\ \frac{1}{2}\pi e^{3i(\pi/2-t)} & \text{on } [\pi/2, 3\pi/2). \end{cases}$$

Then $\varphi(e^{it})/t \to 1$ as $t \to 0$, so that

$$\varphi(e^{it}) = (e^{-it} - 1)h(e^{it})$$

with h continuous and nonzero.

If λ is in the lower half plane, in the interior of the disk $\Delta_{\pi/2}$ and at a distance $> \delta_n/4$ from t_n, for each n, then, by (8) and (9),

$$\omega(\varphi, \lambda) = -2.$$

Since there is only one arc λ_1, Corollary 1 shows that

$$(10)\qquad \dim\ker T_\varphi \geqslant 1$$

(in fact it is easy to see that equality holds in (10)). But $0 \in \partial\mathrm{sp}\, T_\varphi$. Indeed, for $\lambda = t_n + \varepsilon i$ $(0 < \delta_n/4)$ the curve $\varphi(e^{it})$ $(\pi/2 \leqslant t \leqslant 7\pi/6)$ has winding number -1 about λ; the curve $\varphi(e^{it})$ $(a_{n+1} + \delta_n/4 \leqslant t \leqslant a_{n+1} + 3\delta_n/4)$ has winding number $+1$ about λ; and the rest of $\varphi(e^{it})$ has winding number 0 about λ. Therefore 0 is a boundary eigenvalue of T_φ. In the next theorem, we rule out selfintersection by the arcs of the λ_j terminating at 0, which, in the last example, made 0 a boundary point of sp T_φ. The additional assumption elliminates boundary eigenvalues.

THEOREM 3. *Suppose φ satisfies (3) and suppose further that there exists $\eta > 0$ such that if $\theta_j - \eta \leqslant t < s < \theta_j$ or if $\theta_j \leqslant s < t < \theta_j + \eta$, then $\varphi(e^{is}) \neq \varphi(e^{it})$ $(1 \leqslant j \leqslant n)$. Then 0 is not a boundary eigenvalue of T_φ.*

Proof. Pick

$$\varepsilon < \inf\{|\varphi(e^{it})| : |t - \theta_j| \geqslant \eta, \text{ for all } j\}.$$

If $\lambda_1, \ldots, \lambda_n$ satisfy the requirements of § 1.1 for this ε (λ_j is the arc of $\varphi(\mathbf{T})$ such that $I_j = \lambda_j^{-1}(\Delta_\varepsilon)$ is the connected component of $e^{i\theta_j}$ in $\varphi^{-1}(\Delta_\varepsilon)$), we can further assume that λ_j is free from selfintersections (even between the two parts of λ_j terminating at 0), unless $\beta_j = 0$. Since, from (3), the limits

$$\lim_{t \to \theta_j \pm} \arg \varphi(e^{it})$$

exist for each j, we can also assume (by reducing ε, if necessary) that there are two sectors Λ_1 and Λ_2 in Δ_ε such that Λ_1 contains a sequence $\{\delta_n\} \subset \rho(T_\varphi)$, $\delta_n \to 0$; and such that the interior of Λ_2 lies in $\rho_e(T_\varphi)$.

Pick $\varepsilon_1 \leqslant \varepsilon$, so that $\varphi^{-1}(\Delta_{\varepsilon_1}) \subset \cup I_j$. Let e^{ia_j} and e^{ib_j} be the endpoints, on the unit circle, of the smallest (connected) arc containing $I_j \cap \varphi^{-1}(\Delta_{\varepsilon_1})$.

Pick $\varepsilon_2 < \varepsilon_1$ such that a point $\delta = \delta_n$ (in $\rho(T_\varphi)$) lies in $\Delta_{\varepsilon_1} \setminus \Delta_{\varepsilon_2}$. Let e^{ic_j} and e^{id_j} be the endpoints of the connected component of $\varphi^{-1}(\Delta_{\varepsilon_2})$ containing $e^{i\theta_j}$, chosen so that the arc of λ_j joining e^{ia_j} to e^{ic_j} [resp. joining e^{ib_j} to e^{id_j}] does not pass through the origin. Thus if an arc $\lambda_j(e^{it})$, as $t \to \theta_j$ $(t < \theta_j)$, enters Δ_{ε_1} first at e^{ia_j}, it enters Δ_{ε_2} for the last time before passing through 0 at e^{ic_j}, then passes through 0, then reenters $\Delta_{\varepsilon_1} \setminus \Delta_{\varepsilon_2}$ at e^{id_j} and leaves Δ_{ε_1} for the last time at e^{ib_j}.

In $\Delta_\varepsilon \setminus [\Lambda_2 \cup \{\delta\}]$, homotopy the arc of λ_j joining e^{ia_j} and e^{ic_j}, with fixed endpoints, to a simple arc lying in $\Delta_{\varepsilon_1} \setminus \Delta_{\varepsilon_2}$. Do the same with the arc of λ_j joining e^{ib_j} and e^{id_j} $j = 1, \ldots, n$. In this way, we can arrange a homotopy in $\Delta_\varepsilon \setminus [\Lambda_2 \cup \cup \{\delta\}]$ between φ and $\tilde{\varphi}$ so that an arc γ of $|z| = |\delta|$ joining δ to a point of Λ_2 crosses $p + q$ simple arcs of $\tilde{\varphi}$, p of which leave Λ_2 to the right as they cross γ and q of which leave Λ_2 to the left as they cross γ, according to the orientation of $\tilde{\varphi}$. It is easy to see that if, as we pass between two regions of $\mathbf{C} \setminus \tilde{\varphi}(\mathbf{T})$, we cross a simple arc of $\tilde{\varphi}$, then the winding number of $\tilde{\varphi}$ increases by ± 1, according to the orientation of the simple arc. It follows that

$$0 = \omega(\tilde{\varphi}, \delta) = \omega(\tilde{\varphi}, \Lambda_2) + p - q = \omega(\varphi, \Lambda_2) + p - q$$

and, by Theorem 1,

$$\omega(h, 0) = \omega(\varphi, \Lambda_2) + \sum n_j + N = q - p + \sum n_j + N.$$

(There can be no positive cusps, since Λ_2 is a sector with vertex at 0.)

If $q \geqslant p$, this proves $\omega(h, 0) \geqslant 0$. If $p > q$, there are at least $p-q$ arcs λ_j which are either negative with respect to Λ_2 or have $\beta_j = 0$ (since arcs with $\beta_j > 0$ are nonintersecting in Δ_ε). If $\beta_j = 0$, we have $n_j > 0$ (since $\alpha_j \neq 0$), so that

$$\sum n_j + N \geqslant p - q.$$

In any case, $\omega(h, 0) \geqslant 0$ and the proof is complete by Corollary 1.

The following corollary implies that T_φ cannot have boundary eigenvalues for a class of φ including $\varphi \in C^1$, with $\varphi' \neq 0$ (generalizing Wang [9], Theorem 1) and φ rational (generalizing Clark and Morrel [4], Lemma 2.2).

COROLLARY 2. *Suppose φ is continuous on* $\mathbf{T}$, *vanishes at* $e^{i\theta_1}, \ldots, e^{i\theta_n}$, *and in a neighborhood of each θ_j, $\varphi \in C^{(m_j)}$*, with $\varphi^{(m_j)}(e^{i\theta_j}) \neq 0$. *Then* 0 *is not a boundary eigenvalue of T_φ.*

Proof. Clearly (3) is satisfied with $\alpha_j = m_j$, $j = 1, \ldots, n$. To show Theorem 3 is applicable, we prove: *if $\varphi \in C^{(k)}$, in a neighborhood of* $e^{i\theta}$, *and if there are sequences* $\{s_\nu\}$, $\{r_\nu\}$, *with*

$$r_1 > s_1 > r_2 > s_2 > \ldots > \theta, \quad r_\nu \to \theta$$

$$\varphi(e^{ir_\nu}) = \varphi(e^{is_\nu}),$$

then $\varphi^{(k)}(e^{i\theta}) = 0$. This will suffice to prove the corollary.

Write $\varphi(e^{it}) = \varphi_1(t) + i\varphi_2(t)$, with φ_1, φ_2 real valued. Then $\varphi_1, \varphi_2 \in C^{(k)}$ and $\varphi_i(r_\nu) = \varphi_i(s_\nu)$, for $i = 1,2$ and $\nu = 1,2, \ldots$. By the mean value theorem, $\varphi_1'(t_\nu) = 0$, for some $t_\nu \in (s_\nu, r_\nu)$. By induction, $\varphi_1^{(k)}(u_\nu) = 0$, for some sequence $u_\nu \to \theta$. This proves $\varphi_1^{(k)}(\theta) = 0$. Similarly, $\varphi_2^{(k)}(\theta) = 0$, and the claim follows.

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