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NON-UNICELLULAR STRICTLY CYCLIC QUASI-NILPOTENT SHIFTS ON BANACH SPACES

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1. INTRODUCTION

The sequence of unit vectors $\{e_n\}_0^\infty$ in the Banach space X is a *normalized M-basis* if its span is dense and if there is a total sequence of linear functionals $\{e_n^*\}_0^\infty$ for which $e_n^*(e_m) = \delta_{n,m}$. The sequence $\{e_n^*\}$ is uniquely determined by the above conditions. The usual basis of ℓ^p, $1 \leqslant p < \infty$, is a normalized M-basis, and it is easy to construct normalized M-bases on any separable Banach space [8, Proposition 1.f.3, p. 43]. Suppose that $\{w_n\}_0^\infty$ is a sequence of non-zero scalars. The bounded operator T on X is a *weighted shift* with *weights* $\{w_{n+1}/w_n\}_0^\infty$ with respect to the normalized M-basis $\{e_n\}$ if $T(w_n e_n) = w_{n+1}e_{n+1}$ for all $n \geqslant 0$. On ℓ^p, the sequence $\{w_n\}$ determines such a shift if and only if the sequence of weights is bounded, but for completely arbitrary normalized M-bases it may be necessary to require that $\sum (|w_{n+1}|/|w_n|)\|e_n^*\|$ converge.

It has long been recognized [5], [9], [10] that a convenient way to study weighted shifts and their invariant subspaces is to replace the space X by a Banach space B of formal power series, called the *space of power series determined by* $\{w_n\}$, in such a way that the shift T is represented by multiplication by the indeterminate z. Explicitly we identify z^n with $w_n e_n$ and, more generally, we identify x in X with $\sum ((e_n^*(x))/w_n)z^n$ in B (see [5, pp. 19–20] for a more detailed description of this identification). When $X = \ell^p$ with the usual basis, the space of power series determined by $\{w_n\}$ is just the weighted ℓ^p space $\ell^p(w_n)$ of all power series $f = \sum_0^\infty \lambda_n z^n$ for which the norm $\|f\| = \|f\|_p = \left[\sum_0^\infty |\lambda_n w_n|^p\right]^{1/p}$ is finite. The weighted shift is said to be *strictly cyclic* when the space B of power series determined by $\{w_n\}$ is an algebra. (Our use of the term "strictly cyclic weighted shift" is equivalent to the usual meaning of the term in operator theory [7], [10, Proposition 31, p. 94], [4, pp. 83–85]).

Any subalgebra B of the algebra of formal power series $C[[z]]$ is a *Banach algebra of power series* if it is a Banach space under a norm for which the algebra of polynomials is a dense subspace and if the coefficient functionals $\sum \lambda_n z^n \to \lambda_k$ are all continuous, so that B is a Banach algebra under an equivalent norm [4, p. 1]. This terminology differs in one respect from that in the first author's earlier papers. Here we always assume that our fundamental spaces and algebras of power series contain the constants, while in the earlier papers these spaces do not. In particular, what we call B in this paper was $B^{\#}$ in earlier papers (B in earlier papers is B_1 in the present paper).

When T is strictly cyclic, then the closed invariant subspaces of the operator of multiplication by z, which correspond to the invariant subspaces of T, are precisely the closed ideals of B. Hence T is unicellular precisely when the only closed non-zero ideals of B are the *standard ideals* $B_k = \mathrm{cl}(Bz^k) = \left\{f = \sum_0^\infty \lambda_n z^n \in B : \lambda_0 = \lambda_1 \ldots = \lambda_{k-1} = 0\right\}$ for k a non-negative integer. Clearly, if all closed ideals are standard, then B_1 is the radical of B (which is equivalent to T being quasinilpotent or to $\lim(w_n)^{1/n} = 0$ [5, Theorem 3.5, p. 24]). Considerable recent research (see [1] for example) has been devoted to the problem of determining whether there exist non-unicellular strictly cyclic quasinilpotent shifts [10, Question 19, p. 105] or, equivalently, whether there exist non-standard ideals in Banach algebras of power series B with radical B_1 [1, Problem 10, p. 462]. Recently the second author has constructed such non-unicellular shifts on ℓ^1 [12] and on ℓ^p [13]. In this paper we show that every normalized M-basis has a non-unicellular strictly cyclic quasinilpotent shift and we sharpen the results in [13] on such shifts in ℓ^p. We also show that it is even possible to construct such shifts so that B_1 is the closed algebra generated by each series $f = \sum_1^\infty \lambda_n z^n$ in B with $\lambda_1 \neq 0$. This result is the reverse of the second author's construction [11] of algebras $\ell^1(w_n)$ with no non-standard ideals but with non-standard algebras.

2. BACKGROUND RESULTS

In this section we state, in the form in which we will need them, results and terminology from some of our earlier papers. The non-zero power series $f = \sum_0^\infty \lambda_n z^n$ in the Banach algebra of power series B is said to have *order* k if its first non-zero coefficient is λ_k. Also f is said to be *standard* or *non-standard* according to whether the principal closed ideal $\mathrm{cl}(Bf)$ is standard or non-standard.

Some of the results of this paper can be summarized in operator-theoretic language in the following theorem:

THEOREM 2.1. *Suppose that $\{e_n\}$ is a normalized M-basis of the Banach space X. Then there exists a sequence $\{w_n\}$ of positive numbers such that the shift with weights $\{w_{n+1}/w_n\}_0^\infty$ is a non-unicellular strictly cyclic quasinilpotent shift and such that the space of power series determined by $\{w_n\}$ is generated, as a closed algebra with identity, by each of its series of order* 1.

All the results in Sections 3 and 4 will be stated in power series form; Theorem 2.1 is a special case of Theorems 3.2 and 4.3, which are proved below. One goal of the results in Sections 3 and 4 is to attempt to find non-standard series of smallest possible order.

We say that the sequence of non-zero scalars $\{w_n\}_0^\infty$ is a *radical weight* if $w_{n+m}/(w_n w_m)$ is bounded and if $\lim w_n^{1/n} = 0$. It is well-known and easy to prove (cf. [2, p. 645]) that $\ell^1(w_n)$ is a radical algebra with identity adjoined if and only if $\{w_n\}$ is a radical weight. We say that $\{w_n\}$ is a *non-standard weight* if $\{w_n\}$ is a radical weight with all $w_n > 0$ and if there is a non-standard power series of order 1 in $\ell^1(w_n)$. (The condition $w_n > 0$ is just a technical convenience, since $\ell^1(w_n) = \ell^1(|w_n|)$.)

The starting point for all our constructions in this paper is the following result [13, Theorem 3.3.1] of the second author.

THEOREM 2.2. *If $\{r_n\}_0^\infty$ is a sequence of positive numbers, then there exists a non-standard weight $\{w_n\}$ with $(w_{n+1}/w_n) \leqslant r_n$ for all n.*

For each space B of power series and each non-negative integer j, we define (cf. [6, p. 286]) $S_{-j}(B)=\{f \in C[[z]] : fz^j \in B\}$. Notice that we always have $S_{-j}(\ell^p(w_n)) = \ell^p(w_{n+j})$. The following result will play a central role in our treatment of subalgebras. It is a specialization of [4, Theorem 6.1, p. 30] to $\ell^p(w_n)$ using [4, Lemma 4.8 (A), p. 24] and the remarks in [4, p. 37].

THEOREM 2.3. *Suppose that $1 \leqslant p < \infty$ and that $\{w_n\}_0^\infty$ is a sequence of non-zero scalars. If $S_{-2}(\ell^p(w_n)) = \ell^p(w_{n+2})$ is an algebra and if $\{(nw_{n+1})/(w_n)\}$ is bounded, then $\ell^p(w_n)$ is generated, as a Banach algebra with identity, by each of its series of order* 1.

Actually [4, Theorem 6.1, p. 30] shows that if $f = \lambda_1 z + \sum_2^\infty \lambda_n z^n$ is a series of order 1 in $\ell^p(w_n)$, then there is an automorphism of $\ell^p(w_n)$ which takes $\lambda_1 z$ to f. Though the statement of our results in Section 4 will only say that f generates the algebra involved, in fact in each case there will be such an automorphism taking $\lambda_1 z$ to f. We could still have f generate the algebra under somewhat weaker hypotheses [4, pp. 40—41].

3. NON-UNICELLULAR SHIFTS AND NON-STANDARD SERIES

All of our constructions of non-standard series start with the existence of a non-standard series in some weighted ℓ^1-space, guaranteed by Theorem 2.2, and define from this series a new non-standard series in an algebra constructed to have appropriate properties. The construction of this new series is facilitated by the following lemma:

LEMMA 3.1. *Suppose that B and R are Banach algebras of power series with $Rz^k \subseteq B \subseteq R$. If f is a non-standard series in R, then fz^k is a non-standard series in B.*

Proof. Let I be the closed ideal generated by f in R. It follows from the definition of Banach algebra of power series and from the closed graph theorem that B is continuously embedded in R. Hence $I \cap B$ is a closed ideal in B and contains fz^k. If fz^k were standard, then $I \cap B$, and hence I, would contain a power of z. It is easy to see that this would imply that I was a standard ideal (see for instance [3, Lemma 3.2, p. 173]). This completes the proof.

The next theorem proves the existence of non-unicellular strictly cyclic quasinilpotent shifts with respect to any normalized M-basis.

THEOREM 3.2. *If $\{e_n\}$ is a normalized M-basis of the Banach space X, then there is a sequence $\{w_n\}$ of positive scalars for which the space of power series determined by $\{w_n\}$ is a radical algebra with identity adjoined and which has a non-standard element of order* 3.

Proof. By Theorem 2.2, we can find a sequence $\{w_n\}$ for which $\{w_{n+2}\}$ is a non-standard weight and $\sum(w_{n+1}/w_n)\|e_n^*\|$ converges. Let B be the space of power series determined by $\{w_n\}$. We always have $\ell^1(w_n) \subseteq B$. Also, since the map which takes the power series f in B to the coefficient of z^n has norm $\|e_n^*\|/w_n$, our choice of $\{w_n\}$ gives $B \subseteq S_{-1}(\ell^1(w_n))$. Once we show that B is an algebra it will follow from Lemma 3.1 that B has a non-standard series of order 3; any series fz^2, with f non-standard of order 1 in $\ell^1(w_{n+2}) = S_{-2}(\ell^1(w_n))$, will do.

We now show that B is an algebra. Suppose that f and g belong to B, and let $f = \lambda + zf_0$ and $g = \mu + zg_0$, where λ and μ are scalars. Then zf_0 and zg_0 belong to $S_{-1}(\ell^1(w_n))$, so that f_0 and g_0 belong to $S_{-2}(\ell^1(w_n)) = \ell^1(w_{n+2})$ which is an algebra by assumption. Hence $f_0g_0 \in \ell^1(w_{n+2})$ and therefore $fg = \lambda\mu + \lambda zg_0 + \mu zf_0 + z^2(f_0g_0)$ belongs to B. So B is an algebra, and the proof is complete.

The proof that B is an algebra in the above theorem is a special case of the proof of [2, Lemma 3.6, p. 648]. Using the full strength of the argument of [2, Lemma 3.6, p. 648], we could show, exactly as in the proof of Theorem 3.1 above, that if $\{w_{n+2k}\}$ is a non-standard weight and if $\sum(w_{n+k}/w_n)\|e_n^*\|$ converges, then B has a non-standard series of order $2k+1$ (namely fz^{2k}, where f is a non-standard series of order 1 in $\ell^1(w_{n+2k})$). This is the result proved for $\ell^p(w_n)$ by the second author in [13, Proposition 2.3 and Theorem 3.3.1]. We now prove a sharper result for $\ell^p(w_n)$ providing a non-standard series of lower order. We start with a standard lemma (cf. [6, Theorem 1′, p. 283]) about weighted ℓ^p-spaces. We give a proof which will be valid for some sequence spaces more general than ℓ^p.

LEMMA 3.3. *Suppose that $\ell^1(w_n)$ is an algebra. If f belongs to $\ell^1(w_n)$ and $g \in \ell^p(w_n)$, then fg belongs to $\ell^p(w_n)$.*

Proof. If $w_{n+k} \leqslant M w_n w_k$ for all n and k and if g is in $\ell^p(w_n)$, then an easy calculation shows that $\|z^k g\|_p \leqslant M w_k \|g\|_p$ for all k. Hence if $f = \sum \lambda_n z^n$ is in $\ell^1(w_n)$, we have

$$\|fg\|_p = \|\sum \lambda_n z^n g\|_p \leqslant \sum |\lambda_n|\, \|z^n g\|_p \leqslant M \|f\|_1 \|g\|_p .$$

THEOREM 3.4. *Suppose that $1 \leqslant p < \infty$ and that q is the Hölder conjugate of p. If $\{w_{n+k}\}$ is a non-standard weight and if $\{w_{n+k}/w_n\}$ belongs to ℓ^q, then $\ell^p(w_n)$ is a radical algebra with identity adjoined and contains a non-standard series of order $k+1$.*

Before we start the proof, note the existence of $\{w_n\}$ with the indicated properties is guaranteed by Theorem 2.2.

Proof of Theorem 3.4. It follows from the fact that $\{w_{n+k}/w_n\}$ belongs to ℓ^q that we have $\ell^1(w_n) \subseteq \ell^p(w_n) \subseteq S_{-k}(\ell^1(w_n)) = \ell^1(w_{n+k})$. Thus once we show that $\ell^p(w_n)$ is an algebra, it will follow from Lemma 3.1 that fz^k is a non-standard series of order $k+1$ in $\ell^p(w_n)$ whenever f is a non-standard series of order 1 in $\ell^1(w_{n+k})$.

That $\ell^p(w_n)$ is an algebra is [6, Theorem 7′, p. 287]. Since the details of the proof are omitted in [6], we include them here. Suppose that f and g are series in $\ell^p(w_n)$, and let $g = h + z^k g_0$, where h is a polynomial. Since $\ell^p(w_n)$ is a vector space closed under multiplication by z, we have fh belongs to $\ell^p(w_n)$. Also f belongs to the algebra $\ell^1(w_{n+k})$ and g_0 belongs to $S_{-k}(\ell^p(w_n)) = \ell^p(w_{n+k})$, so it follows from Lemma 3.3 that fg_0 belongs to $S_{-k}(\ell^p(w_n))$. Hence $fg = z^k(fg_0)$ belongs to $\ell^p(w_n)$, so the proof is complete.

Specializing to $k = 1$, we obtain:

COROLLARY 3.5. *There exists a sequence $\{w_n\}$ of positive numbers for which, for all $1 \leqslant p < \infty$, $\ell^p(w_n)$ is a radical algebra with identity adjoined and contains a non-standard series of order 2.*

Proof. By Theorem 2.2, there is a sequence $\{w_n\}$ for which $\{w_{n+1}\}$ is a non-standard weight and for which $\{nw_{n+1}/w_n\}$ is bounded. Such a sequence satisfies the hypotheses of Theorem 3.4 for $k = 1$ and for each p such that $1 \leqslant p < \infty$.

The results in Theorem 3.4 and Corollary 3.5, which are sharper than the results for arbitrary bases, depend only on the inequality $\|z^k g\|_p \leqslant M w_k \|g\|_p$ in the proof of Lemma 3.3. The same is true for Theorem 4.2 in the next section. Thus these sharper results hold for most other classical sequence spaces. They would hold, for instance, whenever $\{e_n\}$ is a normalized unconditional basis and the simple unilateral shift which maps e_n to e_{n+1} is a power-bounded operator (see [8, Proposition 1.c.6, p. 18]).

4. GENERATORS OF CLOSED SUBALGEBRAS

We say that the Banach algebra of power series B *has no non-standard subalgebras* if B is the smallest closed algebra with identity containing f whenever f is a series of order 1 in B. In this section we construct Banach algebras of power series with non-standard ideals but no non-standard subalgebras. We consider ℓ^1, ℓ^p, and arbitrary spaces separately because we find non-standard series of different orders in each of these three cases.

THEOREM 4.1. *There is a radical weight $\{w_n\}$ for which $\ell^1(w_n)$ has a non-standard series of order* 3 *but no non-standard subalgebras.*

Proof. By Theorem 2.2, there is a sequence $\{w_n\}$ for which $\{w_{n+2}\}$ is a non--standard weight for which (nw_{n+1}/w_n) is bounded. It is easy to see, either by a direct calculation or as a special case of [2, p. 656], that $\{w_n\}$ is a radical weight. It follows from Lemma 3.1 that $\ell^1(w_n)$ has a non-standard series of order 3, and it follows from Theorem 2.3 that $\ell^1(w_n)$ has no non-standard subalgebras. This completes the proof.

THEOREM 4.2. *There is a sequence $\{w_n\}$ of positive numbers for which, for all $1 < p < \infty$, $\ell^p(w_n)$ is a radical algebra with identity adjoined and with a non--standard series of order* 4 *but with no non-standard subalgebras.*

Proof. By Theorem 2.2 we can find a sequence $\{w_n\}$ for which $\{w_{n+3}\}$ is a non-standard weight and $\{nw_{n+1}/w_n\}$ is bounded. Since $\{1/n\}_1^\infty$ belongs to all ℓ^p, it follows from Hölder's inequality that all $\ell^p(w_{n+2}) \subseteq S_{-1}(\ell^1(w_{n+2}))$. Hence it follows from [6, Theorem 7′, p. 287], which is proved in the proof of Theorem 3.4 above, that $\ell^p(w_{n+2}) = S_{-2}(\ell^p(w_n))$ is an algebra. It therefore follows from Theorem 2.3 that $\ell^p(w_n)$ has no non-standard subalgebras. Also, since $\ell^1(w_n) \subseteq \ell^p(w_n) \subseteq$ $\subseteq S_{-1}(\ell^1(w_n)) \subseteq S_{-3}(\ell^1(w_n)) = \ell^1(w_{n+3})$, it follows from Lemma 3.1 that $\ell^p(w_n)$ has a non-standard series of order 4. This completes the proof.

THEOREM 4.3. *If $\{e_n\}$ is a normalized M-basis of the Banach space X, then there is a sequence $\{w_n\}$ of positive numbers for which the space of power series determined by $\{w_n\}$ has a non-standard series of order* 5 *but no non-standard subalgebras.*

Proof. By Theorem 2.2, there is a sequence $\{w_n\}$ for which $\{w_{n+4}\}$ is a non--standard weight and for which $\sum n(w_{n+1}/w_n)\|e_n^*\|$ converges. As in the proof of Theorem 3.2, we have $\ell^1(w_n) \subseteq B \subseteq S_{-1}(\ell^1(w_n))$ and B is an algebra. Also $S_{-2}(B)$ is a Banach space of power series in which the norm of z^n is w_{n+2} [4, Definition 2.4, p. 8] and we have $S_{-2}(B) \subseteq S_{-1}(\ell^1(w_{n+2}))$. Since $S_{-2}(\ell^1(w_{n+2})) = \ell^1(w_{n+4})$ is an algebra, it follows from [2, Lemma 3.6, p. 648], or from the proof in Theorem 3.2 above, that $S_{-2}(B)$ is an algebra. It therefore follows from [4, Theorem 6.1, p. 30], together with [4, Lemma 4.8 (c), p. 24] and the remarks on [4, p. 37], that B has no non-standard subalgebras. On the other hand $\ell^1(w_n) \subseteq B \subseteq S_{-4}(\ell^1(w_n)) = \ell^1(w_{n+4})$; so B has a non-standard series of order 5 by Lemma 3.1. This completes the proof.

It would be nice to improve the results in this paper by finding non-standard elements of smaller order. When B is a Banach algebra of power series with radical B_1 all series of order 0 must be standard, but so far we have non-standard series of order 1 only for $\ell^1(w_n)$ [12]. When B has no non-standard subalgebras all its series of order 1 must be standard, but even when $B = \ell^1(w_n)$ we were only able to find non-standard series of order 3. We conjecture that there are $\ell^1(w_n)$ with no non--standard subalgebras but with non-standard series of order 2.

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