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EXISTENCE AND COMPLETENESS OF THE WAVE OPERATORS FOR DISSIPATIVE HYPERBOLIC SYSTEMS

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1. INTRODUCTION

In the recent years scattering theory has been extensively studied both for the Schrödinger equation and the classical hyperbolic systems ([1] – [4], [10]— — [23]). A new elegant method, treating the Schrödinger equation, has been developed by Enss in [2]—[4]. On the other hand, Lax and Phillips studied in [10] — — [13] the scattering for the classical hyperbolic systems by using an abstract approach. In this theory a central role is played by two spaces D_+ and D_- with the properties:

$$
(1.1)\qquad \begin{cases} \text{i)} & U_0(t)D_\pm \subset D_\pm \quad \text{for } t \gtrless 0,\\ \text{ii)} & \bigcap_t U_0(t)D_\pm = \{0\},\\ \text{iii)} & \overline{\bigcup_t U_0(t)D_\pm} = \mathscr{H}_0 . \end{cases}
$$

Here $\{U_0(t)\}$ is a group of unitary operators acting on a Hilbert space $\mathscr{H}_0$. The perturbed system is described by a semigroup $\{V(t)\,;\, t \geqslant 0\}$ of contraction operators, acting on a Hilbert space $\mathscr{H} \subset \mathscr{H}_0$. In order to build a scattering theory, Lax and Phillips assumed in [12] the following properties:

$$
(1.2)\qquad \begin{cases} D_\pm \subset \mathscr{H},\\ U_0(-t)\,V(t)d = d \quad \text{for } d \in D_+,\ t \geqslant 0,\\ U_0(t)\,V^*(t)\,d = d \quad \text{for } d \in D_-,\ t \geqslant 0. \end{cases}
$$

Let P_+ denote the orthogonal projection onto the orthogonal complement of D_+ in $\mathscr{H}$. The condition

$$
(1.3)\qquad \lim_{t\to\infty} P_+ V(t) f = 0 \quad \text{for } f \in \mathscr{H}
$$

is one of the basic assumptions in [12] and it leads to the main difficulties in the applications of the abstract theory.

The proof of (1.3) is given by Lax and Phillips in [12] only for the wave equation in the exterior of a bounded obstacle $K \subset \mathbf{R}^n$, $n \geqslant 2$. The investigation of the case when n is even is rather technical. Recently Petkov considered in [17] the mixed problems for dissipative hyperbolic systems, provided n odd. He proved the existence of the wave operator W, introduced below, without applying the abstract approach in [12]. However, the proof in [17] is based essentially on the strong Huyghens principle fulfilled for odd space dimensions. Moreover, in [17] some restrictions on the point spectrum of the generator G of $V(t)$ has been imposed.

The main purpose of this work is to prove the existence of the wave operator W for dissipative hyperbolic systems in the exterior of a bounded obstacle $K \subset \mathbf{R}^n$, $n \geqslant 3$, without any other restrictions on n. For this purpose we propose a suitable adaptation of Enss method in order to study the mixed boundary-valued problems for dissipative hyperbolic systems.

We shall describe our assumptions which are close to those given in [17]. Let $\Omega \subset \mathbf{R}^n$ be an open domain with bounded complement and smooth, connected, compact boundary $\partial\Omega$. Consider the operator

$$G = \left(\sum_{j=1}^{n} A_j(x)\partial_{x_j}\right) + B(x),$$

where $A_j(x)$, $B(x)$ are $(m \times m)$ matrices with elements in $C^\infty(\bar{\Omega})$. We assume the following properties:

(H$_1$)
a) $A_j(x)$ are symmetric,

b) we have $A_j(x) = A_j^0$, $B(x) = 0$ for $|x| \geqslant \rho_0$,

c) the matrix

$$A(x, \xi) = \sum_{j=1}^{n} A_j(x)\xi_j$$

has simple eigenvalues $\tau_k(x, \xi) \neq 0$ for $k = 1, 2, \ldots, m$,

$$\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \setminus \{0\}, \quad x \in \bar{\Omega}.$$

The above conditions implies that m is even. The operator G is a perturbation of the operator $G_0 = \sum_{j=1}^{n} A_j^0 \partial_{x_j}$, which is the generator of a group of unitary operators $U_0(t)$ on the Hilbert space $\mathscr{H}_0 = [L^2(\mathbf{R}^n)]^m$ of vector-valued functions. The solution to the Cauchy problem

$$(1.4)\qquad \begin{cases}(\partial_t - G_0)u = 0,\\ u(0,x) = f(x)\end{cases}$$

is described by $u(t, x) = U_0(t)f$.

In order to introduce the perturbed semigroup, consider a smooth $(m/2 \times m)$ matrix $A(x)$ and the boundary problem

$$(1.5)\qquad \begin{cases}(\partial_t - G)u = 0 \quad \text{on } (0, \infty) \times \Omega,\\ u \in \operatorname{Ker} A(x) \quad \text{on } (0, \infty) \times \partial\Omega,\\ u(0, x) = f(x).\end{cases}$$

We impose the following assumptions:

$$(\mathrm{H}_2)\qquad \begin{cases}\text{a) } B(x) + B^*(x) - \sum_{j=1}^{n} \partial_{x_j} A_j(x) \leqslant 0 \quad \text{for } x \in \bar{\Omega},\\ \text{b) } \operatorname{rank} A(x) = m/2 \quad \text{for } x \in \partial\Omega,\\ \text{c) } \langle u, A(x, \nu(x))u\rangle \leqslant 0 \quad \text{for } u \in \operatorname{Ker} A(x),\ x \in \partial\Omega.\end{cases}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{C}^m$, a^* denotes the adjoint matrix to a and

$$A(x, \nu(x)) = \sum_{j=1}^{n} A_j(x)\nu_j(x),$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ is the unit normal at $x \in \partial\Omega$ pointed into $K = \mathbf{R}^n \setminus \Omega$.

We shall denote by $\mathscr{H}$ the Hilbert space $[L^2(\Omega)]^m$ of vector-valued functions with the scalar product

$$(u, v) = \int_{\Omega} \langle u(x), v(x)\rangle \, dx$$

and the norm $\|\cdot\|$. A dense domain for G in $\mathscr{H}$ can be defined using the closure in the graph norm $\|u\| + \|Gu\|$ of smooth functions $u(x)$ satisfying the boundary condition $A(x)u = 0$. This domain will be denoted by $\mathscr{D}(G)$. Using the results of Lax and Phillips in [9], we conclude that the operator G is a maximal dissipative one and generates a semigroup of contraction operators $\{V(t)\,;\, t \geqslant 0\}$ on $\mathscr{H}$. Denoting by $\mathscr{H}_b^{\perp}$ the orthogonal complement of the linear space $\mathscr{H}_b$ spanned by the eigenvectors of G, having eigenvalues on the imaginary axis, we are able to introduce the following assumption

(H_3) For $f \in \mathscr{D}(G) \cap \mathscr{H}_b^{\perp}$ we have $\|\partial_x u\| \leqslant C(\|Gu\| + \|u\|)$.

Our main result is:

THEOREM 1.1. *Suppose the assumptions* $(H_1)-(H_3)$ *fulfilled and* $f \in \mathscr{H}_b^{\perp}$. *Then there exists a sequence* $\{t_\nu\}_{\nu=1}^{\infty}$, $t_\nu \nearrow \infty$, *such that*

$$\lim_{\nu\to\infty} \sup_{s\geq 0} \|(V(s) - U_0(s))\, V(t_\nu)f\| = 0.$$

The above result enables us to obtain the following analogue of the condition (1.3).

COROLLARY 1.2. *Suppose the assumptions* $(H_1)-(H_3)$ *fulfilled. Then*

$$\lim_{t\to\infty} P_+ V(t)f = 0 \quad \text{for} \quad f \in \mathscr{H}_b^{\perp}. \tag{1.6}$$

Furthermore, we can prove the existence of the wave operator

$$W = \text{s-}\lim_{t\to\infty} U_0(-t)V(t).$$

COROLLARY 1.3. *Suppose the assumptions* $(H_1)-(H_3)$ *fulfilled. Then the limit*

$$W(f) = \lim_{t\to\infty} U_0(-t)\, V(t)f$$

exists for $f \in \mathscr{H}_b^{\perp}$.

Using the properties (1.2), we conclude that the wave operators

$$\begin{cases} \Omega_+ = \text{s-}\lim_{t\to\infty} V(t)J_0U_0(-t) \\ \Omega_- = \text{s-}\lim_{t\to\infty} V^*(t)J_0U_0(t) \end{cases} \tag{1.7}$$

exist (see [12], [17]). Here J_0 is the orthogonal projection from $\mathscr{H}_0$ onto $\mathscr{H}$.

The second problem, discussed in this work, is related to the completeness of the wave operators $\Omega_\pm$. Similar problem was treated by Neidhardt in [15], [16]. His approach is based on the dilation theory of contraction semigroups. An important role in our approach is played by the space

$$\mathscr{H}_\infty^- = \{f \in H_b^{\perp}\ ;\ \lim_{t\to\infty} V(t)f = 0\}. \tag{1.8}$$

The space $\mathscr{H}_\infty^-$ is connected with the existence of disappearing solutions, which is a typical phenomenon for dissipative systems. More precisely, if $f \in \mathscr{H}_\infty^- \cap D_+^{\perp}$ then $u(t,x) = V(t)f$ is a disappearing solution, that is there exists $T_0 > 0$ such that $V(t)f = 0$ for $t \geq T_0$. This result has been obtained by the author in [5] and it shows that $\mathscr{H}_\infty^-$ is nonempty for some boundary conditions.

Since the generator G_0 of the group $\{U_0(t)\}$ has an absolutely continuous spectrum and the wave operators exist in our case, one can obtain the following inclusion (see Section 6):

$$\operatorname{Im} \Omega_- \subset \mathscr{H}_b^\perp \ominus \mathscr{H}_\infty^- . \tag{1.9}$$

This fact enables us to determine the completeness of the wave operator Ω_- by the equality

$$\overline{\operatorname{Im} \Omega_-} = \mathscr{H}_b^\perp \ominus \mathscr{H}_\infty^- . \tag{1.10}$$

Note that $\mathscr{H}_\infty^- = \{0\}$, when $\{V(t)\}$ is a group of unitary operators. Therefore the equality (1.10) leads to the well known definition of the completeness of the wave operators, used in the literature (see [2], [21], [22]).

THEOREM 1.4. *Suppose the assumptions* $(H_1)-(H_3)$ *fulfilled. Then the wave operator* Ω_- *is complete*, i.e. (1.10) *holds.*

REMARK. If one replaces the assumption (H_3) by the dual estimate

$$\|\partial_x f\| \leqslant C(\|G(f)\| + \|f\|) \quad \text{for } f \in \mathscr{D}(G^*) \cap \mathscr{H}_b^\perp, \tag{H_3^*}$$

it is possible to obtain the property

$$\lim_{t\to\infty} P_- V^*(t) f = 0 \quad \text{for } f \in \mathscr{H}_b^\perp .$$

Thus we can define the completeness of the wave operator Ω_+ by the equality

$$\overline{\operatorname{Im} \Omega_+} = \mathscr{H}_b^\perp \ominus \mathscr{H}_\infty^+ .$$

Note that P_- is the orthogonal projection onto the orthogonal complement of D_- and

$$\mathscr{H}_\infty^+ = \{ f \in \mathscr{H}_b^\perp \ ; \ \lim_{t\to\infty} V^*(t) f = 0\}.$$

Our approach is based on the application of the Enss method to the mixed problems for dissipative hyperbolic systems. A basic assumption in the Enss theory is the condition

$$\int_0^\infty \|(G-G_0)(G_0-1)^{-1}\chi(|x| \geqslant R)\|\, dR < \infty, \tag{1.11}$$

where $\chi(|x| \geqslant R)$ is the characteristic function of the set $\{x; |x| \geqslant R\}$. The fact that we treat a domain with boundary makes some trouble in the formulation of the analogue of (1.11), since the operators G and G_0 act on different spaces. In order to overcome this difficulty, we propose the following condition

$$\lim_{R\to\infty} \|((G-1)^{-1}-(G_0-1)^{-1})\chi(|x|\geq R)\| = 0. \tag{1.12}$$

This property is based on the Huyghens principle for first order symmetric strictly hyperbolic systems (see Section 2). The novelty of our approach are the arguments based on the following equality

$$\begin{aligned}(U_0(s)-V(s))h &= -V(s)\varphi h + \varphi U_0(s)h - \\ &- \int_0^s V(s-\sigma)(G(1-\varphi)-(1-\varphi)G_0)\,U_0(\sigma)h\,d\sigma,\end{aligned} \tag{1.13}$$

where $h \in \mathscr{H}$, $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi(x) = 1$ for $|x| \leqslant \rho_0$ (see(H_1)) and $\varphi(x) = 0$ for x sufficiently large. This equality allows one to investigate the influence of the boundary and to overcome the difficulties connected with the domains of G and G_0.

Following closely the approach of Simon [22] and using the property (1.12), one can reduce the proof of Theorem 1.1 to the verification of a suitable estimate of $(U_0(s) - V(s))h$. The equality (1.13) allows us to apply the stationary phase method related to the action of the unperturbed group $\{U_0(t)\}$ on the outgoing and incoming parts of the Enss decomposition. The needed estimates are obtained by using a suitable representation of the group $\{U_0(t)\}$.

Finally, we shall sketch the plan of the work. Some preliminary results are given in Section 2. We study the basic points of the Enss method in Sections 3 and 4. The main tool for the investigation of the incoming and outgoing components, appearing in the Enss decomposition is the stationary phase method, discussed in Section 4. We prove in Section 5 the result given in Theorem 1.1. From this result we obtain in Section 6 Corollaries 1.2 and 1.3. Moreover, we prove the completeness of the wave operator Ω_- in the last Section 6.

I wish to express my gratitude to Vesselin Petkov for his suggestion to adapt the Enss method to the classical scattering and for his valuable advices during the preparation of this work.

2. PRELIMINARIES

The kernel of our considerations is the following form of RAGE theorem, obtained by Simon in [22]:

THEOREM 2.1. *Let G be the generator of a contraction semigroup $\{V(t); t \geqslant 0\}$ on a Hilbert space $\mathscr{H}$. Denote by $\mathscr{H}_b^\perp$ the orthogonal complement of the space $\mathscr{H}_b$*

spanned by the eigenvectors of the operator G with eigenvalues on the imaginary axis. Suppose that L is a bounded operator and $L(G-1)^{-1}$ *is a compact one. Then*

$$\lim_{T\to\infty} T^{-1}\int_0^T \|LV(t)f\|^2\,dt = 0 \quad \textit{for } f\in\mathcal{H}_b^{\perp}.$$

From this result we shall obtain a weak convergence of a sequence of the form $V(t_\nu)f$, which will be very important to us in the sequel. More precisely, we have the following:

LEMMA 2.2. *Let* G, $V(t)$, $\mathcal{H}$, $\mathcal{H}_b^{\perp}$ *be as in Theorem 2.1 and* $\langle\cdot,\cdot\rangle$ *be the scalar product in* $\mathcal{H}$. *Then there exists a sequence* $t_\nu\nearrow\infty$, *such that*

$$\lim_{\nu\to\infty}\langle V(t_\nu)f, g\rangle = 0 \quad \textit{for } f\in\mathcal{H}_b^{\perp},\ g\in\mathcal{H}.$$

Proof. Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthogonal basis in the Hilbert space $\mathcal{H}_b^{\perp}$. Consider the following compact operator

$$L(f) = \sum_{k=1}^{\infty}\langle f,\varphi_k\rangle\varphi_k 2^{-k}.$$

Using Theorem 2.1, one can find a sequence $t_\nu\nearrow\infty$, such that

$$\lim_{\nu\to\infty}\langle V(t_\nu)\varphi_j,\varphi_k\rangle = 0$$

for every $j, k = 1,2,3,\ldots$. Since $\{V(t)\}$ is a contraction semigroup, we conclude that $\lim_{\nu\to\infty}\langle V(t_\nu)f, g\rangle = 0$ for every $f, g\in\mathcal{H}_b^{\perp}$. On the other hand, $\mathcal{H}_b^{\perp}$ is left invariant by $V(t)$, according to the results in [22]. Hence, $\lim_{\nu\to\infty}\langle V(t_\nu)f, g\rangle = 0$ for $f\in\mathcal{H}_b^{\perp}$, $g\in\mathcal{H}$. This completes the proof of the lemma.

The second important tool will be the application of domain of dependence argument for the unperturbed system. Let the eigenvalues of the matrix $A(-\xi) = -\sum_{j=1}^{n} A^0_j\xi_j$ be ordered as follows

$$\tau_1(\xi) > \tau_2(\xi) > \ldots > \tau_m(\xi)$$

for $\xi = (\xi_1,\ldots,\xi_n)\in\mathbf{R}^n\setminus 0$. The corresponding unit eigenvectors will be denoted by $r_1(\xi),\ldots,r_m(\xi)$. We note that $r_j(\xi)$ and $\tau_j(\xi)$ are smooth functions in the domain $\mathbf{R}^n\setminus 0$. Moreover, $r_j(\xi)$ and $\tau_j(\xi)$ are homogeneous functions of degree 0 and 1 respectively. Using the Euler equality $\tau_j(\xi) = \langle \xi, \nabla\tau_j(\xi)\rangle$ and the assumption (H_1), we conclude that $\nabla\tau_j(\xi)$ are homogeneous functions of degree 0 and $\nabla\tau_j(\xi) \neq 0$ for $\xi \in \mathbf{R}^n \setminus 0$. Hence, we can choose the number $c_{\min}$ so small that

$$|\nabla\tau_j(\xi)| \geq c_{\min} > 0. \tag{2.1}$$

Set $c_{\max} = \max_{1\leq j\leq m} (\max_{\xi\in S^{n-1}} \tau_j(\xi))$. Define the energy of the solution $u(t,x)$ to the Cauchy problem (1.4) in the sphere $\{x ; |x| \leq R\}$ by the integral

$$E(t, R) = \int_{|x|\leq R} |u(t,x)|^2 \, dx.$$

Using an integration by parts (see [10] for more details), we obtain the inequality

$$E(t, R - tc_{\max}) \leq E(0, R).$$

This estimate leads to the following:

PROPOSITION 2.3. *Suppose that $f(x) \in \mathscr{H}_0$ and $f(x) = 0$ for $|x| \leq R$. Then the solution $u(t,x)$ to the problem (1.4) vanishes for $|x| \leq R - tc_{\max}$.*

The above proposition enables one to obtain the following decay at infinity:

PROPOSITION 2.4. *Let $k \geq 1$ be a fixed integer. Then*

$$\lim_{R\to\infty} \|[(G-1)^{-k} - (G_0-1)^{-k}]\chi(|x| \geq R)\| = 0.$$

Proof. Let ρ be chosen so large that $K \subset \{x; |x| \leq \rho\}$ and $\rho \geq \rho_0$ (see assumption (H_1)). Given any $f \in \mathscr{H}_0$, we have the property

$$\chi(|x| \geq R) f \in \mathscr{H} \quad \text{for } R \geq \rho.$$

Hence, we can apply the operator $(G-1)^{-k}$ to this element. Using the resolvent equality

$$(G-1)^{-k} = (-1)^k/(k-1)! \int_0^\infty V(t) t^{k-1} \exp(-t) \, dt,$$

we obtain the equality

$$\begin{aligned}&[(G-1)^{-k} - (G_0-1)^{-k}]\chi(|x| \geq R) f = \\ &= (-1)^k/(k-1)! \int_0^\infty t^{k-1} \exp(-t) [V(t) - U_0(t)] \chi(|x| \geq R) f \, dt.\end{aligned} \tag{2.2}$$

On the other hand, we have $U_0(t)\chi(|x| \geqslant R)f = 0$ for $|x| \leqslant \rho$ and $0 \leqslant t \leqslant (R-\rho)/c_{\max}$, according to Proposition 2.3. This fact leads to the property

$$[V(t) - U_0(t)]\,\chi(|x| \geqslant R)f = 0 \quad \text{for } R \geqslant \rho,\ 0 \leqslant t \leqslant (R-\rho)/c_{\max}.$$

Combining the above property and the equality (2.2), we obtain the estimate

$$\|[(G-1)^{-k} - (G_0-1)^{-k}]\,\chi(|x| \geqslant R)f\| \leqslant$$
$$\leqslant 2(R-\rho)^{k-1} \exp\left(-(R-\rho)/c_{\max}\right) \|f\| / [(k-1)!\, c_{\max}^{k-1}].$$

Choosing $R \to \infty$, from this inequality we obtain the needed property

$$\lim_{R\to\infty} \|[(G-1)^{-k} - (G_0-1)^{-k}]\,\chi(|x| \geqslant R)\| = 0.$$

This proves the proposition.

3. THE ENSS DECOMPOSITION

Given a function $\varphi(x)$ in the Schwartz space $S(\mathbf{R}^n)$, we can write the Fourier transform of $\varphi(x)$ in the form

$$\hat{\varphi}(\xi) = \int \exp\left(-i\langle x, \xi\rangle\right) \varphi(x)\, dx.$$

First, we shall construct a function $f(x) \in S(\mathbf{R}^n)$ with the properties

$$\text{(3.1)} \qquad \begin{cases} \text{a) } f(x) \geqslant 0, \\ \text{b) } f(0) = 1, \\ \text{c) } \operatorname{supp}_{\xi} \hat{f}(\xi) \subset \{\xi ; |\xi| \leqslant 1\}. \end{cases}$$

In order to find such a function, choose a function $f_1(x) \in S(\mathbf{R}^n)$ with $f_1 \not\equiv 0$ and $\operatorname{supp}_{\xi} \hat{f}_1(\xi) \subset \{\xi ; |\xi| \leqslant 1/2\}$. Set $f_2(x) = |f_1(x)|^2$. Then a) and c) are fulfilled. Moreover, the relations

$$f_2(0) = (2\pi)^{-n} \int \hat{f}_1(\xi)\bar{\hat{f}}_1(-\xi)\, d\xi = (2\pi)^{-n} \int |\hat{f}_1(\xi)|^2\, d\xi > 0$$

show that the conditions (3.1) are fulfilled if $f(x) = f_2(x)/f_2(0)$.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$ and a positive real number M, consider the function

$$\text{(3.2)} \qquad f_\alpha(x) = M^{-n} \int f((x-y)/M)\chi_\alpha(y)\, dy,$$

where χ_α is the characteristic function of the unit cube in $\mathbf{R}^n$ centered at α. Thus, we consider the following partition of the unity in the space of the variables $x \in \mathbf{R}^n : 1 = \sum_\alpha f_\alpha(x)$. In order to construct a suitable decomposition of the dual space with variables $\xi \in \mathbf{R}^n$, we define three smooth functions $g_1(s)$, $g_{in}(s)$ and $g_{out}(s)$ on $\mathbf{R}$ with the properties

$$(3.3)\qquad \begin{cases} g_1(s) = 0 \quad \text{for } |s| \leqslant 1, \quad g_1(s) = 1 \quad \text{for } |s| \geqslant 2, \\ g_{in}(s) = 0 \quad \text{for } s \geqslant 1/2, \quad g_{in}(s) = 1 \quad \text{for } s \leqslant -1/2, \\ g_{out}(s) = 1 - g_{in}(s). \end{cases}$$

Let M be an arbitrary positive number and $\varphi(x)$ be the vector-valued function with elements in $S(\mathbf{R}^n)$. An important role in the Enss method is played by the following pseudodifferential operators:

$$G_{ex}^{\alpha,M}(\varphi) = (2\pi)^{-n} \sum_{j=1}^{n} \int \exp(i\langle x, \xi\rangle)\, g_1(M/|\xi|) g_1(M|\xi|/2) \times$$

$$\times g_{ex}(M\langle \nabla\tau_j(\xi), \alpha/|\alpha|\rangle)\langle \varphi_j(\xi), r_j(\xi)\rangle r_j(\xi)\, d\xi,$$

where ex = out or in, $\alpha \in \mathbf{Z}^n$. Set

$$(3.4)\qquad \begin{aligned} P_{out}^{\nu,M} &= \sum_{|\alpha| \geqslant \nu} G_{out}^{\alpha,M} f_\alpha(x), \\ P_{in}^{\nu,M} &= \sum_{|\alpha| \geqslant \nu} G_{in}^{\alpha,M} f_\alpha(x). \end{aligned}$$

Given any $\Phi \in \mathscr{H}_0$, the elements $P_{out}^{\nu,M}(\Phi)$ and $P_{in}^{\nu,M}(\Phi)$ are called outgoing and incoming states (see Enss [2] — [4]). A basic point in the Enss method is the analysis of the propagation of these states under the unperturbed operator $U_0(t)$. This problem is discussed in the next section.

4. PROPAGATION OF THE OUTGOING AND INCOMING STATES UNDER THE ACTION OF THE UNPERTURBED GROUP $\{U_0(t)\}$

The main tool for the investigation of the outgoing and incoming states is the following:

PROPOSITION 4.1. *There exists $\delta_0 > 0$, depending only on the matrices $A_1, \ldots, A_n$, such that the following inequalities are fulfilled:*

$$(4.1)\qquad \|\chi(|x| \leqslant \delta_0(\nu + s))\, U_0(s)\, P_{out}^{\nu,M}\| \leqslant C_N(\nu + s)^{-N},$$

$$(4.2)\qquad \|\chi(|x| \leqslant \delta_0(\nu + s))\, U_0^*(s)\, P_{in}^{\nu,M*}\| \leqslant C_N(\nu + s)^{-N}$$

for $t \geqslant 0$, $\nu \geqslant 1$ with some constant C_N independent of ν and t.

Proof. Using the Fourier transform, we can represent the solution $U_0(t)f$ to the Cauchy problem (1.4) in the form

$$U_0(t)f = (2\pi)^{-n}\int \sum_{j=1}^{m} \exp(i(\langle x, \xi\rangle - t\tau_j(\xi)))\langle \hat{f}(\xi), r_j(\xi)\rangle r_j(\xi)\,d\xi.$$

It is not difficult to compute, by exploiting the above equality, the exact expression of $U_0(s)P_{out}^{\nu,M}(\Phi)$, where the pseudodifferential operators $P_{out}^{\nu,M}$ are determined according to the equalities (3.4). We find

$$U_0(s)P_{out}^{\nu,M}(\Phi) = \sum_{|\alpha|\geq\nu}\sum_{j=1}^{m}\int \exp(i(\langle x, \xi\rangle - s\tau_j(\xi)))F_{out}^{\alpha,j}(\xi)\langle \widehat{f_\alpha\Phi}(\xi), r_j(\xi)\rangle\,d\xi, \tag{4.3}$$

where

$$F_{out}^{\alpha,j}(\xi) = (2\pi)^{-n}g_1(M/|\xi|)g_1(M|\xi|/2)\times \times g_{out}(M\langle \nabla\tau_j(\xi), \alpha/|\alpha|\rangle)r_j(\xi). \tag{4.4}$$

Denote by $f*g$ the convolution of the functions f and g, i.e.

$$(f*g)(x) = \int f(x-y)g(y)\,dy.$$

Then we have the relation

$$\widehat{f_\alpha\Phi}(\xi) = \exp(-i\langle\alpha, \xi\rangle)(2\pi)^{-n}(\hat{f}_0 * \hat{\Phi}_\alpha)(\xi), \tag{4.5}$$

where $f_0(x) = M^{-n}(f(\cdot/M)*\chi_0)$, χ_0 is the characteristic function of the unit cube centered at 0 and $\hat{\Phi}_\alpha(\eta) = \exp(i\langle\alpha, \eta\rangle)\hat{\Phi}(\eta)$. From (4.3) and (4.5) we obtain the equality

$$U_0(s)P_{out}^{\nu,M}(\Phi)(2\pi)^n = \sum_{|\alpha|\geq\nu}\sum_{j=1}^{m}\int \exp(i(\langle x-\alpha, \xi\rangle - s\tau_j(\xi)))\,F_{out}^{\alpha,j}(\xi)\langle(\hat{f}_0 * \hat{\Phi}_\alpha), r_j(\xi)\rangle\,d\xi. \tag{4.6}$$

Our choice of the function g_{out} implies that

$$\langle \nabla\tau_j(\xi), \alpha\rangle 2M \geq -|\alpha|, \quad \text{when } g_{out}(M\langle\nabla\tau_j(\xi), \alpha/|\alpha|\rangle)\neq 0.$$

Using the inequality (2.1) we obtain the estimate

$$\|\alpha + s\nabla\tau_j(\xi)\|^2 = |\alpha|^2 + s^2\|\nabla\tau_j(\xi)\|^2 + 2s\langle\nabla\tau_j(\xi), \alpha\rangle \geq$$
$$\geq |\alpha|^2 + s^2 c_{min}^2 - (s^2 + |\alpha|^2)/(2M).$$

Choosing $M > (2\min(1, c^2_{\min}))^{-1}$, we can arrange the inequality

$$|\alpha - s\nabla\tau_j(\xi)| \geqslant 2\delta_0(s + |\alpha|), \tag{4.7}$$

where $\delta_0^2 = [\min(1, c^2_{\min}) - (2M)^{-1}]/8$. Since the inequality (4.7) guarantees that

$$|x - \alpha - s\nabla\tau_j(\xi)| \geqslant \delta_0(s + |\alpha|) \quad \text{for } |x| \leqslant \delta_0(s + |\alpha|), \tag{4.8}$$

we can apply the stationary phase method. More precisely, consider the differential operator $L_\alpha(\xi, \partial_\xi)$ determined by the equality

$$L^*_\alpha(\xi, \partial_\xi) = \langle (x - \alpha - s\nabla\tau_j(\xi))/|x - \alpha - s\nabla\tau_j(\xi)|^2, \partial_\xi \rangle.$$

Here L^* denotes the adjoint operator to L. Integrating by parts in (4.6), we obtain

$$\begin{aligned} U_0(s)\, P^{\nu,M}_{\text{out}}(\Phi) = (2\pi)^{-n} \sum_{|\alpha| \geqslant \nu} \sum_{j=1}^{m} \int & \exp(\mathrm{i}(\langle x - \alpha, \xi\rangle - s\tau_j(\xi))) \times \\ & \times L^N(F^{\alpha,j}_{\text{out}}(\xi)\langle (\hat{f}_0 * \hat{\Phi}_\alpha)(\xi), r_j(\xi)\rangle)\, \mathrm{d}\xi. \end{aligned} \tag{4.9}$$

The derivatives of $F^{\alpha,j}_{\text{out}}(\xi)$ can be uniformly bounded, when M is a fixed number. The same is valid for the derivatives of $\hat{f}_0(\xi)$, since $f_0 = M^{-n}(f(\cdot/M) * \chi_0)$. The coefficients of the operator $L^N_\alpha(\xi, \partial_\xi)$ can be bounded above by $C_N(s+|\alpha|)^{-N}$, according to the property (4.8). By using the classical Young's inequality we derive from (4.9) that the following estimate is fulfilled

$$|U_0(s)\, P^{\nu,M}_{\text{out}}(\Phi)| \leqslant C_N(s + \nu)^{-N}\|\hat{\Phi}_\alpha\| \quad \text{for } |x| \leqslant \delta_0(s + \nu).$$

Since $\|\hat{\Phi}_\alpha\| = (2\pi)^n\|\Phi\|$, we obtain the inequality (4.1).

We turn to the proof of the inequality (4.2). The starting point will be the following equality, obtained in a similar manner as (4.9):

$$\begin{aligned} U_0^*(s)\, P^{\nu,M*}_{\text{in}}(\Phi) = \sum_{|\alpha| \geqslant \nu} \sum_{j=1}^{m} \int & \exp(\mathrm{i}(\langle x - \alpha, \xi\rangle + s\tau_j(\xi))) \times \\ & \times (\hat{f}_0 * F^{\alpha,j}_{\text{in}})(\xi)\, \mathrm{d}\xi, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} F^{\alpha,j}_{\text{in}}(\xi) = (2\pi)^{-2n}\langle \hat{\Phi}(\xi), r_j(\xi)\rangle\, r_j(\xi) \exp(\mathrm{i}\langle \alpha, \xi\rangle) \times \\ \times\, g_1(M/|\xi|)\, g_1(M|\xi|/2)\, g_{\text{in}}(M\langle \nabla\tau_j(\xi), \alpha/|\alpha|\rangle). \end{aligned}$$

Since $\hat{f}_0(\xi) = \hat{f}(M\xi)\,\hat{\chi}_0(\xi)$ and f satisfies the properties (3.1), we have $\operatorname{supp}_\xi f_0(\xi) \subset$ $\subset \{\xi\,;\, |\xi| < M^{-1}\}$. Moreover, we have the inclusion

$$\operatorname{supp}_\xi F_{\text{in}}^{\alpha,j}(\xi) \subset \{\xi\,;\, 2M < |\xi| < M\}.$$

Applying the well known property $\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v)$, we obtain the inclusion

$$\operatorname{supp}_\xi ((\hat{f}_0\hat{\chi}_0) * (F_{\text{in}}^{\alpha,j}))(\xi) \subset \{\xi\,;\, 1/M < |\xi| < 2M\}.$$

This fact implies that the integration in (4.10) is taken on a compact, disjoint from the unique singular point $\xi = 0$ of the vector-valued functions $r_j(\xi)$. Using the arguments given above in the proof of (4.1), we obtain the needed inequality. This completes the proof of the proposition.

5. PROOF OF THEOREM 1.1

It is easy to prove that the set

$$V = \{(G - 1)^{-2}G(\Phi)\,;\, \Phi \in \mathscr{D}(G) \cap \mathscr{H}_b^{\perp}\}$$

is dense in $\mathscr{H}_b^{\perp}$ (see Lemma 2, § 9 in [22]). Indeed, assuming $h \perp V$ in $\mathscr{H}_b^{\perp}$ and using the equality $(G - m)^{-2}G = (G - m)^{-2}(G - 1)^2(G - 1)^{-2}G$, we obtain the relation $\langle h, m^2 \cdot (G - m)^{-2}G\Phi\rangle = 0$ for $m > 0$. Taking $m \nearrow \infty$, we get $\langle h, G\Phi\rangle = 0$ for $\Phi \in \mathscr{D}(G) \cap \mathscr{H}_b^{\perp}$. Consequently $G^*h = 0$. But this fact implies that $h \in \mathscr{H}_b$ and hence $h = 0$. So, we can assume without loose of generality that $f =$ $= (G - 1)^{-2}G(\Phi)$ with $\Phi \in \mathscr{D}(G) \cap \mathscr{H}_b^{\perp}$. Setting $\Phi_\nu = V(t_\nu)\Phi$, $f_\nu = V(t_\nu)f$, we shall use the following decomposition, introduced by Simon in [22]

$$(G - 1)^{-2}G(\Phi_\nu) = \Phi_{\nu,M}^{(1)} + \Phi_{\nu,M}^{(2)} + \Phi_{\nu,M}^{\text{out}} + \Phi_{\nu,M}^{\text{in}},$$

where

$$\text{(5.1)}\qquad \begin{cases} \Phi_{\nu,M}^{(1)} = (F(G) - F(G_0))(\Phi_\nu) + \sum\limits_{|\alpha| \leqslant \nu} f_\alpha F(G_0)P_M(G_0)(\Phi_\nu), \\ \Phi_{\nu,M}^{(2)} = F(G_0)\cdot(1 - P_M(G_0))(\Phi_\nu), \\ \Phi_{\nu,M}^{\text{out}} = P_{\text{out}}^{\nu,M}(F(G_0)P_M(G_0)\Phi_\nu), \\ \Phi_{\nu,M}^{\text{in}} = P_{\text{in}}^{\nu,M}(F(G_0)P_M(G_0)\Phi_\nu). \end{cases}$$

Here we set $F(G) = (G - 1)^{-2}G$, $F(G_0) = (G_0 - 1)^{-2}G_0$ and $P_M(G_0) =$ $= (2\pi)^{-1}\int U_0(t)\hat{P}_M(t)\,\mathrm{d}t$, where $\hat{P}_M(s)$ is a smooth function with a compact support in **R** determined as follows

$$P_M(s) = g_1(M \cdot s^{-1}) g_1(M \cdot s),$$

$$g_1(s) \in C^\infty,$$

$$g_1(s) = 0 \quad \text{for } s \leqslant 1,$$

$$g_1(s) = 1 \quad \text{for } s \geqslant 2.$$

The operators $P_{\text{out}}^{\nu,M}$ and $P_{\text{in}}^{\nu,M}$ are determined in Section 3.

Let $M > 1$ be fixed. First, we shall verify the property

$$\lim_{\nu\to\infty} \Phi_{\nu,M}^{(1)} = 0. \tag{5.2}$$

In order to do this, consider the inequality

$$\|(F(G) - F(G_0))\Phi_\nu\| \leqslant C(\|(F(G) - F(G_0))\chi(|x| \geqslant \nu)\| + \|\chi(|x| \leqslant \nu)V(t_\nu)\Phi\|). \tag{5.3}$$

The second term $\|\chi(|x| \leqslant \nu)V(t_\nu)\Phi\|$ tends to 0, if we choose a suitable sequence $\{t_\nu\}$. Indeed, let $\nu \geqslant 1$ be a fixed integer and $\{t_k\}$ be a sequence with $\lim_{k\to\infty} t_k = \infty$. Using the coercive estimate (H_3), one can obtain that the sequence $\{V(t_k)\Phi\}$ is bounded in the Sobolev space $\mathscr{H}^1_{\text{loc}}(\mathbf{R}^n)$. Applying the Rellich's compactness theorem and the fact that $\{V(t_k)\Phi\}$ tends weakly to 0, according to Lemma 2.2, we find an integer k_ν, so that

$$\|\chi(|x| \leqslant \nu)V(t_{k_\nu})\Phi\| \leqslant 1/\nu.$$

Hence, we can choose a suitable sequence $\{t_\nu\}$ with the property

$$\lim_{\nu\to\infty} \chi(|x| \leqslant \nu)V(t_\nu)\Phi = 0. \tag{5.4}$$

The first term in the right hand-side of the inequality (5.3) tends to 0, according to Proposition 2.4. From this fact and (5.4) we conclude that the left hand-side of (5.3) tends also to 0.

Now we shall estimate the term $I_\nu = \sum_{|\alpha|\leqslant\nu} f_\alpha(x)F(G_0)P_M(G_0)$. Our starting point is the inequality

$$\sup_x |x-\alpha|^N |f_\alpha(x)| \leqslant C_N \quad \text{for } N = 1, 2, 3, \dots .$$

Here the constant C_N is independent of ν. The above inequality implies that I_ν can be estimated above by

$$\nu C\|\chi(|x| \leqslant 2\nu)F(G_0)P_M(G_0)\Phi_\nu\| + C\|\Phi_\nu\|/\nu,$$

where C is a constant independent of ν. The coercive estimate (H_3) and the inequality

$$\|\partial_x f\| \leqslant C(\|G_0 f\| + \|f\|)$$

for the elliptic operator $G_0 = \sum_{j=1}^{n} A_j^0 \partial_{x_j}$ show that the sequence $\{F(G_0)P_M(G_0)V(t_\nu)\Phi\}$ is bounded in $H^1_{loc}(\mathbf{R}^n)$. Applying the Rellich's compactness theorem and the weak convergence to 0 of the sequence $\{V(t_\nu)\Phi\}$ in $\mathscr{H}_0$, we can arrange the property

$$\lim_{\nu\to\infty} \nu\|\chi(|x| \leqslant 2\nu)F(G_0)P_M(G_0)V(t_\nu)\Phi\| = 0.$$

Having in mind the above arguments, we obtain that $\lim_{\nu\to\infty} I_\nu = 0$. Combining this fact and the inequality (5.4), we complete the proof of (5.2).

For the second term $\Phi^{(2)}_{\nu,M}$ in (5.1) we have the representation formula

$$\Phi^{(2)}_{\nu,M} = (2\pi)^{-1}\int U_0(t)\hat{q}_M(t)\,\mathrm{d}t,$$

where $q_M(s) = \mathrm{i}(s - \mathrm{i})^{-2} s g_1(M/s) g_1(Ms)$. This fact leads to the inequality

$$\|\Phi^{(2)}_{\nu,M}\| \leqslant \varepsilon(M), \tag{5.5}$$

where

$$\varepsilon(M) = \max_{s\in\mathbf{R}} |(s - \mathrm{i})^{-2} s g_1(M/s) g_1(Ms)|$$

tends to 0 as M tends to infinity.

The main tool for the investigation of the last two terms $\Phi^{\text{out}}_{\nu,M}$ and $\Phi^{\text{in}}_{\nu,M}$ will be the equality

$$\begin{aligned}(U_0(s) - V(s))h) &= -V(s)\varphi h + \varphi U_0(s)h - \\ &- \int_0^s V(s-\sigma)(G(1-\varphi) - (1-\varphi)G_0)U_0(\sigma)\,\mathrm{d}\sigma,\end{aligned} \tag{5.6}$$

where $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$, $\varphi(x) = 1$ for $x \in K$ or $|x| \leqslant \rho_0$ and $\varphi(x) = 0$ for x sufficiently large. Setting $\psi = -G(1-\varphi) + (1-\varphi)G_0$, we conclude from the assumption (H_1) that ψ is a matrix-valued function with elements in $C_0^\infty(\mathbf{R}^n)$. We shall take advantage of (5.6) by writing the inequality

$$\|(U_0(s) - V(s))\Phi^{\text{out}}_{\nu,M}\| \leqslant \|\varphi\Phi_{\nu,M}\| + \|\varphi U_0(s)\Phi^{\text{out}}_{\nu,M}\| + $$

$$+ \int_0^s \|\psi U_0(\sigma)\Phi^{\text{out}}_{\nu,M}\|\,\mathrm{d}\sigma.$$

The representation formula (5.1) together with Proposition 4.1 and the above estimate yield the property

$$(5.7)\qquad \lim_{\nu\to\infty}\max_{s\geqslant 0}\|(U_0(s)-V(s))\Phi^{\mathrm{out}}_{\nu,M}\| = 0.$$

We estimate the term $\Phi^{\mathrm{in}}_{\nu,M}$ following closely the approach in § 9 of [22]. First we choose the sequence $\{t_\nu\}$ so that $\lim_{\nu\to\infty}\varphi(x)V(t_\nu)\Phi = 0$ for $M = 1,2,3,\dots$. Then the equality

$$(5.8)\qquad \lim_{\nu\to\infty} P^{\nu,M}_{\mathrm{in}}(F(G_0)P_M(G_0)(1-\varphi)V(t_\nu)\Phi) = 0$$

follows from the properties

$$(5.9)\qquad \lim_{\nu\to\infty} P^{\nu,M}_{\mathrm{in}}(U_0(t_\nu))g = 0 \quad \text{for } g\in\mathscr{H}_0,$$

$$(5.10)\qquad \lim_{\nu\to\infty} P^{\nu,M}_{in}(F(G_0)P_M(G_0)(1-\varphi)V(t_\nu) - U_0(t_\nu)F(G_0)P_M(G_0))\Phi = 0.$$

The property (5.10) is equivalent to

$$(5.11)\qquad \lim_{\nu\to\infty}\|(V^*(t_\nu)(1-\varphi)P^*_M(G_0)F^*(G_0) - P^*_M(G_0)F^*(G_0)U^*_0(t_\nu))P^{\nu,M*}_{\mathrm{in}}\| = 0.$$

Now we need the following variant of the equality (5.6)

$$\begin{aligned}(V^*(t_\nu)(1-\varphi)P^*_M(G_0)F^*(G_0) &- P^*_M(G_0)F^*(G_0)U^*_0(t_\nu))h = \\ &= -\varphi P^*_M(G_0)F^*(G_0)U^*_0(t_\nu)h - \\ &- \int_0^{t_\nu} V^*(t_\nu - s)\psi P^*_M(G_0)F^*(G_0)U^*_0(s)h\,ds,\end{aligned}$$

where $\psi = -G^*(1-\varphi) + (1-\varphi)G^*_0$ is a matrix-valued function with elements in $C^\infty_0(\mathbf{R}^n)$. Hence, for the proof of (5.10) it is sufficient to verify that

$$\begin{aligned}\lim_{\nu\to\infty}\Bigg(\int_0^\infty &\|\psi P^*_M(G_0)F^*(G_0)U^*_0(\sigma)P^{\nu,M*}_{\mathrm{in}}\|\,d\sigma + \\ &+ \max_{\sigma\geqslant 0}\|\varphi P^*_M(G_0)F^*(G_0)U^*_0(\sigma)P^{\nu,M*}_{\mathrm{in}}\|\Bigg) = 0.\end{aligned}$$

We see that this property follows from Proposition 4.1. The above observation shows that the equality (5.10) holds. In order to verify (5.9) we choose $g\in C^\infty_0(\mathbf{R}^n)$

with $\text{supp}(g) \subset \{x ; |x| \leqslant R\}$. Then (5.9) is a direct consequence from the equality

$$P_{\text{in}}^{\nu,M} U_0(t_\nu) \chi(|x| \leqslant R) = \chi(|x| \leqslant R) U_0^*(t_\nu) P_{\text{in}}^{\nu,M*}$$

and Proposition 4.1. Combining (5.9), (5.10) and our choice of the sequence $\{t_\nu\}$, we find the equality

$$\lim_{\nu \to \infty} \Phi_{\nu,M}^{\text{in}} = 0.$$

From this property, (5.2), (5.5) and (5.7) we obtain the estimate

$$\sup_{s \geqslant 0} \|(U_0(s) - V(s)) V(t_\nu) f\| \leqslant \varepsilon(M) + J_{\nu,M} ,$$

where $\lim_{\nu \to \infty} J_{\nu,M} = 0$ for $M = 1, 2, 3, \ldots$ and $\lim_{M \to \infty} \varepsilon(M) = 0$. This completes the proof of Theorem 1.1.

6. COMPLETENESS OF THE WAVE OPERATORS

First in this section we turn to the

Proof of Corollary 1.2. Let $f \in \mathscr{H}_b^\perp$ and the sequence $\{t_\nu\}$ be chosen according to Theorem 1.1, so that

(6.1) $$\lim_{\nu \to \infty} \sup_{s \geqslant 0} \|(U_0(s) - V(s)) V(t_\nu) f\| = 0.$$

Consider the equality

(6.2) $$U_0(-s-t)V(s+t) - U_0(-t)V(t) = U_0(-s-t)(V(s) - U_0(s))V(t).$$

Combining (6.1) and (6.2), we conclude that the norm

$$\|U_0(-t_\nu)V(t_\nu)f - U_0(t_{\nu+k})V(t_{\nu+k})f\|$$

is sufficiently small, when $k \geqslant 0$ and ν is sufficiently large. Hence, $\lim_{\nu \to \infty} U_0(-t_\nu)V(t_\nu)f = g$ exists and

$$\|U_0(-t_\nu)\, V(t_\nu) f - g\| = \|V(t_\nu) f - U_0(t_\nu) g\|$$

tends to 0 when $\nu \to \infty$. Since P_+ is a projection, we obtain the property

(6.3) $$\lim_{\nu \to \infty} \|P_+ V(t_\nu) f - P_+ U_0(t_\nu) g\| = 0.$$

On the other hand, it is easy to obtain that $\lim_{t \to \infty} P_+ U_0(t) g = 0$ for $g \in \mathcal{H}_0$. Indeed, the equalities (1.1) (iii) show that we can assume $g = U_0(T)d$ for some $d \in D_+$. Choosing $t > -T$, we get $U_0(s)d \in D_+$ and $P_+ U_0(s)d = 0$ for $s \geqslant t + T$. This observation and (6.3) yield $\lim_{\nu \to \infty} P_+ V(t_\nu) f = 0$. Now we can apply the well-known (see [10], [12]) semigroup property

$$P_+ V(t) P_+ V(s) f = P_+ V(s + t) f \quad \text{for } f \in \mathcal{H},\ t, s \geqslant 0.$$

Thus, we obtain that the function $\alpha(t) = \|P_+ V(t) f\|$ is a nonincreasing one. This fact and the property $\lim_{\nu \to \infty} P_+ V(t_\nu) f = 0$ imply that $\alpha(t)$ tends to 0 when $t \to \infty$ This proves the corollary.

We shall only sketch the proof of Corollary 1.3, since the existence of the wave operator W is established by Lax and Phillips in [12], provided the property (1.3) is fulfilled.

Proof of Corollary 1.3. Let $f \in \mathcal{H}_b^\perp$ and

$$V(t) f = d(t) + P_+ V(t) f,$$

where $d(t) \in D_+$. Using the equality (6.2), we obtain the inequality

$$\|U_0(-s-t) V(s+t) f - U_0(-t) V(t) f\| \leqslant \|(V(s) - U_0(s)) d(t)\| + 2\|P_+ V(t) f\|.$$

Note that $(V(s) - U_0(s)) d(t) = 0$, according to the property (1.2). Applying Corollary 1.2, we conclude from the above arguments that the norm $\|U_0(-s-t) V(s+t) f - - U_0(-t) V(t) f\|$ is sufficiently small when $s \geqslant 0$ and t is a sufficiently large number. This completes the proof of the existence of the wave operator

$$W = \operatorname{s-lim}_{t \to \infty} U_0(-t) V(t).$$

Next we turn to the analysis of the completeness of the wave operators

$$\Omega_+ = \operatorname{s-lim}_{t \to \infty} V(t) J_0 U_0(-t),$$

(6.4)

$$\Omega_- = \operatorname{s-lim}_{t \to \infty} V^*(t) J_0 U_0(t).$$

We start with the property

(6.5)
$$\operatorname{Im} \Omega_- \subset \mathcal{H}_b^\perp \ominus \mathcal{H}_\infty^\perp,$$

marked into the introduction. Let $f \in \mathcal{H}_0$ and $\Psi \in \mathcal{D}(G)$ be an eigenvector of the operator G with eigenvalue on the imaginary axis, i.e. $G\Psi = iE\Psi$ for some $E \in \mathbf{R}$.

This assumption implies that $V(t)\Psi = \exp(iEt)\Psi$. Then we have the equalities

$$\langle \Omega_- f, \Psi \rangle = \lim_{t\to\infty} \langle V^*(t) J_0 U_0(t) f, \Psi \rangle = \lim_{t\to\infty} \langle U_0(t) f, V(t)\Psi \rangle =$$

$$= \lim_{t\to\infty} \exp(iEt) \langle U_0(t) f, \Psi \rangle.$$

Since the spectrum of G_0 is absolutely continuous, applying the spectral theorem, we get $\langle \Omega_- f, \Psi \rangle = \lim_{t\to\infty} \exp(iEt) \int \exp(-its)\varkappa(s)\,ds$, where $\varkappa(s) \in L^1(\mathbf{R})$. Now the Riemann-Lebesgue lemma yields $\Psi \perp \operatorname{Im}\Omega_-$. This observation shows that $\operatorname{Im}\Omega_- \subset \mathscr{H}_b^\perp$. Let Φ be chosen in $\mathscr{H}_\infty^-$, i.e. $\lim_{t\to\infty} \|V(t)\Phi\| = 0$. Then the equality

$$\langle \Omega_- f, \Phi \rangle = \lim_{t\to\infty} \langle J_0 U_0(t) f, V(t)\Phi \rangle$$

leads to the needed inclusion (6.5).

Finally, we turn to the following

Proof of Theorem 1.4. Consider the restriction of the operator

$$\Omega_- W = \text{s-}\lim_{t\to\infty} V^*(t) V(t) \tag{6.6}$$

on the Hilbert space $H_- = \mathscr{H}_b^\perp \ominus \mathscr{H}_\infty^-$. The inclusion (6.5) implies that $\operatorname{Im}\Omega_- W \subset H_-$. It is easy to check the property $\operatorname{Ker}\Omega_- W = 0$. Indeed, assuming $f \in \operatorname{Ker}\Omega_- W$, from (6.6) we obtain the relations $0 = \langle \Omega_- W f, f \rangle = \lim_{t\to\infty} \|V(t) f\|^2$. From the fact that $f \perp \mathscr{H}_\infty^-$ we are going to the equality $f = 0$. On the other hand, the operator $\Omega_- W$ is a strong limit of selfadjoint contraction operators, according to (6.6). Hence $\Omega_- W$ is a selfadjoint contraction operator with trivial kernel. This property implies that $\operatorname{Im}\Omega_- W$ covers the Hilbert space $H_- = \mathscr{H}_b^\perp \ominus \mathscr{H}_b^-$. Combining this fact and the inclusion (6.5), we obtain the equality

$$\overline{\operatorname{Im}\Omega_-} = \mathscr{H}_b^\perp \ominus \mathscr{H}_\infty^\perp .$$

This completes the proof of Theorem 1.4.

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