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FACTORING TRACE-CLASS OPERATOR-VALUED FUNCTIONS WITH APPLICATIONS TO THE CLASS $\mathbf{A}_{\aleph_0}$

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, let $\mathscr{A}_T$ denote the *dual algebra* generated by T, i.e., the smallest subalgebra of $\mathscr{L}(\mathscr{H})$ that contains T and $I_{\mathscr{H}}$ and is closed in the ultraweak operator topology. Moreover, let Q_T denote the quotient space $C_1(\mathscr{H})/{}^{\perp}\mathscr{A}_T$, where $C_1(\mathscr{H})$ denotes the trace-class ideal in $\mathscr{L}(\mathscr{H})$ under the trace norm and ${}^{\perp}\mathscr{A}_T$ denotes the preannihilator of $\mathscr{A}_T$ in $C_1(\mathscr{H})$. One knows that $\mathscr{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = \mathrm{tr}(AL), \quad A \in \mathscr{A}_T, \ L \in C_1(\mathscr{H}),$$

where $[L]$ denotes the image of L in Q_T. If x and y are vectors in $\mathscr{H}$, we write, as usual, $x \otimes y$ for the rank-one operator in $C_1(\mathscr{H})$ defined by

$$(x \otimes y)(u) = \langle u, y\rangle x, \quad u \in \mathscr{H}.$$

Then, of course, $[x \otimes y] \in Q_T$, and it is easy to see that

$$(2) \qquad \langle A, [x \otimes y]\rangle = \langle Ax, y\rangle, \quad A \in \mathscr{A}_T, \ x, y \in \mathscr{H}.$$

In a similar vein, if $\mathbf{T}$ denotes the unit circle in $\mathbf{C}$, we denote by $L^p = L^p(\mathbf{T})$, $1 \leqslant p < \infty$, the usual Banach spaces of Lebesgue p-integrable functions on $\mathbf{T}$ and by $L^\infty = L^\infty(\mathbf{T})$ the Banach space of essentially bounded measurable functions on $\mathbf{T}$. One knows that L^∞ is the dual space of L^1 under the pairing

$$(3) \qquad \langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})g(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t, \quad f \in L^\infty, \ g \in L^1.$$

Furthermore if for $1 \leqslant p \leqslant \infty$ we denote by $H^p = H^p(\mathbf{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish, then one knows that H^∞ is a weak*-closed subspace of L^∞ and that the preannihilator ${}^\perp(H^\infty)$ of H^∞ in L^1 is the space H_0^1 consisting of those functions g in H^1 whose analytic extension $\hat{g}$ to $\mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ satisfies $\hat{g}(0) = 0$. It follows easily that H^∞ is the dual space of L^1/H_0^1 under the pairing

$$\langle f, [g] \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) g(e^{it}) \, dt, \quad f \in H^\infty, \ g \in L^1. \tag{4}$$

If T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ (i.e., a contraction whose unitary part is either absent or absolutely continuous), then the pairs of spaces $\{\mathscr{A}_T, Q_T\}$ and $\{H^\infty, L^1/H_0^1\}$ are related by the Sz.-Nagy—Foiaş functional calculus $\Phi_T: \mathscr{A}_T \to H^\infty$ defined by

$$\Phi_T(f) = f(T), \quad f \in H^\infty.$$

The mapping Φ_T is a norm-decreasing, weak*-continuous, algebra homomorphism, and the range of Φ_T is weak* dense in $\mathscr{A}_T$ (cf. [9, Theorem 3.2]). It therefore follows from general principles that there exists a bounded, linear, one-to-one map $\varphi_T: Q_T \to L^1/H_0^1$ such that $\Phi_T = \varphi_T^*$.

In [4, I], we defined the class $\mathbf{A} = \mathbf{A}(\mathscr{H})$ to be the set of all absolutely continuous contractions in $\mathscr{L}(\mathscr{H})$ such that Φ_T is an isometry. In this case one knows (cf. [9]) that Φ_T is a weak* homeomorphism of H^∞ onto $\mathscr{A}_T$ and φ_T is an isometry of Q_T onto L^1/H_0^1. Furthermore, if n is any cardinal number with $1 \leqslant n \leqslant \aleph_0$, we also defined in [4, I] the class $\mathbf{A}_n = \mathbf{A}_n(\mathscr{H})$ to consist of all those T in $\mathbf{A}$ with the property that for every system $\{[L_{ij}]\}_{0 \leqslant i, j < n}$ of elements of Q_T, there exist sequences of vectors $\{x_i\}_{0 \leqslant i < n}$ and $\{y_j\}_{0 \leqslant j < n}$ from $\mathscr{H}$ such that

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leqslant i, j < n. \tag{5}$$

Operators in the class $\mathbf{A}_{\aleph_0}$ have a rich dilation theory, which is expounded in [4, I], and they also have huge invariant-subspace lattices, which enabled us to establish in [4, II] that such operators are *reflexive*. Thus it is of considerable interest to find additional classes of operators contained in $\mathbf{A}_{\aleph_0}$, and this is one of the main functions of this paper. We show that the (soon to be defined) classes $(\mathrm{BCP})_\theta$, $0 \leqslant \theta < 1$, which arise naturally from many different perspectives, are contained in $\mathbf{A}_{\aleph_0}$. This leads to a better understanding of the class $\mathbf{A}_{\aleph_0}$ itself and shows, in particular, that $\mathbf{A}_{\aleph_0}$ contains many contractions whose spectra coincide with the unit circle.

But to locate additional operators in $\mathbf{A}_{\aleph_0}$ is by no means the only purpose of this paper. We also, with the help of the notion of the functional model of a contraction as developed in [21], continue to expand the scope of [2], carrying to fruition two new ideas. The first is to solve systems of equations in the space $L^1(\mathbf{T})$ instead of in a quotient space $Q_T \cong L^1/H_0^1$, and the second is yet another generalization — to solve systems of equations in $L^1(\sigma)$, where σ is any measurable subset of $\mathbf{T}$. We are confident, in view of [24], that these results will be pertinent to research on the invariant subspace problem for contractions with spectral radius one.

The following lemma, which we prove in § 3, contains the essence of the idea that allows us to consider systems of equations in $L^1(\mathbf{T})$ and $L^1(\sigma)$ in place of Q_T.

LEMMA 1.1. *If T is any completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$, and x and y are any vectors in $\mathcal{H}$, then there exists a unique function in $L^1(\mathbf{T})$ (depending, of course, on T, x, and y), to be denoted by $x\cdot y$, whose sequence of Fourier coefficients $\{c_n(x\cdot y)\}_{n=-\infty}^{\infty}$ is given by*

$$c_{-n}(x\cdot y) = \langle T^n x, y\rangle, \quad c_n(x\cdot y) = \langle T^{*n}x, y\rangle, \quad n = 0, 1, 2, \ldots . \tag{6}$$

Moreover, the expression $x\cdot y$ is linear in x, conjugate linear in y, and satisfies

$$\varphi_T([x\otimes y]) = [x\cdot y], \tag{7}$$

where, of course, $[x\otimes y]$ is the coset in Q_T of $x\otimes y$ and $[x\cdot y]$ is the coset in L^1/H_0^1 of $x\cdot y$.

Thus, given a completely nonunitary contraction T in $A(\mathcal{H})$, by making use of this lemma we may and do consider systems of equation in $L^1(\mathbf{T})$ of the form

$$x_i\cdot y_j = f_{ij}, \quad 0 \leqslant i,j < n, \tag{8}$$

where $\{f_{ij}\}_{0\leqslant i,j<n}$ is an arbitrary $n\times n$ array of functions in L^1, and, of course, if $\{x_i\}_{0\leqslant i<n}$ and $\{y_j\}_{0\leqslant j<n}$ are sequences of vectors from $\mathcal{H}$ that solve (8), then these sequences also solve

$$[x_i\otimes y_j] = \varphi_T^{-1}([f_{ij}]), \quad 0 \leqslant i,j < n, \tag{9}$$

which is equivalent to (5), so $T\in \mathbf{A}_n$. Carrying this idea one step further, if σ is a measurable subset of $\mathbf{T}$, we may replace (8) by the weaker system of equations

$$(x_i\cdot y_j)|\sigma = f_{ij}|\sigma, \quad 0 \leqslant i, j < n, \tag{10}$$

and try to solve this system (under a weaker hypothesis on the operator T). Theorems of this nature will be found in § 6.

Finally, the manner in which we accomplish these objectives of solving systems of the form (5), (8), and (10) is to prove some factorization theorems for certain trace-class operator-valued functions $F\colon \sigma \to C_1(\mathcal{H})$. These results show that we can write $F = Y^*X$ where X and Y are Hilbert-Schmidt operator-valued functions which also satisfy additional conditions relevant to T. These factorization theorems are interesting in their own right, and we expect to demonstrate their utility in network theory and realization theory in future papers.

2. PRELIMINARIES: $(\mathrm{BCP})_{\theta,\sigma}$ OPERATORS

In this section we define and say a few words about the classes of operators $(\mathrm{BCP})_{\theta,\sigma} \subset \mathscr{L}(\mathscr{H})$, where $0 \leqslant \theta < 1$ and σ is a measurable subset of $\mathbf{T}$, since it is for certain of these classes that our ability to solve systems of equations of the form (8) or (10) will be demonstrated in § 6. With σ as indicated, we say that a set $\Lambda \subset \mathbf{D}$ is *dominating for* σ if almost every point of σ is a non-tangential limit of a sequence of points from Λ.

For any completely nonunitary contraction T in $\mathscr{L}(\mathscr{H})$ and for any μ in $\mathbf{D}$, let us write T_μ for the Möbius transform

$$T_\mu = (T - \mu I)(I - \bar{\mu} T)^{-1}. \tag{11}$$

One knows that each T_μ is a completely nonunitary contraction along with T (cf. [21, p. 14]), and for every θ, $0 \leqslant \theta \leqslant 1$, we define the sets

$$\begin{aligned} L_\theta(T) &= \{\mu \in \mathbf{D} : \inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leqslant \theta\}, \quad \text{and} \\ R_\theta(T) &= \{\mu \in \mathbf{D} : \inf \sigma_e((T_\mu T_\mu^*)^{1/2}) \leqslant \theta\}. \end{aligned} \tag{12}$$

Clearly, for any such T, $L_1(T) = R_1(T) = \mathbf{D}$, so our interest focuses on the interval $0 \leqslant \theta < 1$. Furthermore it is easy to see from (12) that $\mu \in L_0(T)$ if and only if $\mu \in \sigma_{le}(T)$, the left essential spectrum of T. Hence it follows from (11) and (12) that

$$L_0(T) = \sigma_{le}(T) \subset L_\theta(T), \quad 0 < \theta \leqslant 1,$$

and similarly that

$$R_0(T) = \sigma_{re}(T) \subset R_\theta(T), \quad 0 < \theta \leqslant 1.$$

Putting these relations together, we deduce that

$$\sigma_e(T) = L_0(T) \cup R_0(T) \subset L_\theta(T) \cup R_\theta(T), \quad 0 < \theta \leqslant 1. \tag{13}$$

Thus we define, for every θ with $0 \leqslant \theta < 1$ and for every measurable subset σ of $\mathbf{T}$, the class $(\mathrm{BCP})_{\theta,\sigma}$ to be the set of all completely nonunitary contractions T in $\mathscr{L}(\mathscr{H})$ for which the set $L_\theta \cup R_\theta$ is *dominating for* σ. For brevity we write $(\mathrm{BCP})_\theta$ for the class $(\mathrm{BCP})_{\theta,\mathbf{T}}$. We observe from (13) and [19] that $(\mathrm{BCP})_0 = (\mathrm{BCP})$, and, moreover, we show in § 7 (Example 7.2) that the (obviously increasing) family

$\{(\mathrm{BCP})_\theta\}_{0\leqslant\theta<1}$ is *strictly* increasing. To give the reader some intuition as to what it means for a completely nonunitary contraction T to belong to $(\mathrm{BCP})_\theta$, we introduce the sets $\zeta_\theta(T)$, defined by Apostol in [1]:

$$(14)\quad \zeta_\theta(T) = (\sigma_e(T)\cap \mathbf{D})\cup\{\mu\in\mathbf{D}\setminus\sigma_e(T):\theta\,\|(\pi(T)-\mu I)^{-1}\|\geqslant 1/(1-|\mu|)\},$$

where, as usual, π denotes the projection of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra. We will show in § 7 that

$$(15)\qquad \zeta_\theta(T)\subset L_\theta(T)\cup R_\theta(T),\quad 0\leqslant\theta<1,$$

so if $\zeta_\theta(T)$ is dominating for $\mathbf{T}$, i.e., if it is the case that the essential resolvent of T grows rapidly near sufficiently many points of $\mathbf{T}$, then T will belong to $(\mathrm{BCP})_\theta$. It turns out, in fact, that there exist operators T in $(\mathrm{BCP})_\theta$, $0<\theta<1$, such that $\sigma(T)\cap\mathbf{D}=\emptyset$ (see Example 7.2).

3. PRELIMINARIES: FUNCTIONAL MODELS

In this section we review briefly various preliminary material that will be needed in the sequel. In particular, we recall some facts about the functional model of a contraction operator, as developed in [21], because such models will play a major role in what follows.

Let $\mathscr{K}$ and $\mathscr{N}$ be separable, complex Hilbert spaces, and let $\mathscr{L}(\mathscr{K},\mathscr{N})$ denote the Banach space of all bounded linear operators T mapping $\mathscr{K}$ into $\mathscr{N}$. Let $C_\infty(\mathscr{K},\mathscr{N})$ denote the subspace of $\mathscr{L}(\mathscr{K},\mathscr{N})$ consisting of all compact operators, and for each p, $1\leqslant p<\infty$, let $C_p(\mathscr{K},\mathscr{N})$ denote the Schatten p-class of all those T in $C_\infty(\mathscr{K},\mathscr{N})$ for which

$$\|T\|_p=\Big(\sum_j|\lambda_j|^p\Big)^{1/p}<+\infty,$$

where $\{\lambda_j\}$ is the sequence of eigenvalues of $(T^*T)^{1/2}$ repeated according to their multiplicities. We will use from [11] various properties of the Banach spaces $C_p(\mathscr{K},\mathscr{N})$. Particularly important will be the *trace class* $C_1(\mathscr{K},\mathscr{N})$ and the *Hilbert-Schmidt class* $C_2(\mathscr{K},\mathscr{N})$. We recall that if S, $T\in C_2(\mathscr{K},\mathscr{N})$, then $T^*S\in C_1(\mathscr{K},\mathscr{K})$ and $\|T^*S\|_1\leqslant\|T\|_2\,\|S\|_2$; cf. [11, p. 104]. Furthermore, if $\{e_j\}$ and $\{f_k\}$ are any orthonormal bases of $\mathscr{K}$ and $\mathscr{N}$, respectively, then

$$\|T\|_2=\Big(\sum_j\|Te_j\|^2\Big)^{1/2}$$

for every T in $C_2(\mathscr{K},\mathscr{N})$ and

$$\|T\|_1\leqslant\sum_{j,k}|\langle Te_j,f_k\rangle|$$

for every T in $C_1(\mathscr{K},\mathscr{N})$; cf. [11, p. 111]. We write $C_p(\mathscr{K})$ for $C_p(\mathscr{K},\mathscr{K})$ and recall also that each $C_p(\mathscr{K})$, $1\leqslant p<\infty$, is a selfadjoint ideal in $\mathscr{L}(\mathscr{K})$ that is a Banach space under the norm $\|\cdot\|_p$; moreover, the map $T\to T^*$ preserves the norms $\|\cdot\|_p$.

We turn now to a discussion of functional models for contractions. It is well-known from [21] that every completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$ is unitarily equivalent to an essentially unique functional model. In order to describe functional models, recall that if $\mathcal{E}$ is any separable (complex) Banach space, and σ is any (Lebesgue) measurable subset of $\mathbf{T}$, then the Banach spaces $L^p(\sigma, \mathcal{E})$, consisting of measurable, p-integrable, $\mathcal{E}$-valued functions are defined for every p, $1 \leqslant p \leqslant \infty$ (cf. [6, Chapter 17]). We will write simply $L^p(\mathcal{E})$ for $L^p(\mathbf{T}, \mathcal{E})$. The *Hardy space* $H^p(\mathcal{E})$, $1 \leqslant p \leqslant \infty$, is the subspace of $L^p(\mathcal{E})$ consisting of those functions f in $L^p(\mathcal{E})$ whose Fourier coefficients

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(e^{it})\,dt$$

vanish for all $n < 0$.

Now let $\mathcal{K}$ and $\mathcal{N}$ be separable Hilbert spaces. We denote by $L^\infty(\mathcal{L}(\mathcal{K}, \mathcal{N}))$ the Banach space consisting of all essentially bounded, strongly measurable functions Φ from $\mathbf{T}$ into $\mathcal{L}(\mathcal{K}, \mathcal{N})$ under the norm

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{\mathbf{T}} \|\Phi(e^{it})\|.$$

(A function $\Phi\colon \mathbf{T} \to \mathcal{L}(\mathcal{K}, \mathcal{N})$ is *strongly measurable* if the function $\Phi(e^{it})k$ is measurable for every vector k in $\mathcal{K}$.) The space $H^\infty(\mathcal{L}(\mathcal{K}, \mathcal{N}))$ consists of all functions Φ in $L^\infty(\mathcal{L}(\mathcal{K}, \mathcal{N}))$ whose Fourier coefficients $c_n(\Phi)$ vanish for all $n < 0$.

Given such a Φ in $L^\infty(\mathcal{L}(\mathcal{K}, \mathcal{N}))$, we define the *(generalized) Laurent operator* $M_\Phi\colon L^2(\mathcal{K}) \to L^2(\mathcal{N})$ by the formula

$$(M_\Phi k)(e^{it}) = \Phi(e^{it})k(e^{it}), \quad k \in L^2(\mathcal{K}).$$

It is easy to see that M_Φ is bounded and satisfies $\|M_\Phi\| = \|\Phi\|_\infty$. Moreover, we define the *(generalized) Toeplitz operator* $T_\Phi\colon H^2(\mathcal{K}) \to H^2(\mathcal{N})$ associated with Φ by the formula

$$T_\Phi h = P_{H^2(\mathcal{N})} M_\Phi h, \quad h \in H^2(\mathcal{K}),$$

where, here and henceforth, we employ the familiar notation $P_{\mathcal{M}}$ for the (orthogonal) projection whose range is the subspace $\mathcal{M}$.

Turning now to the business of introducing the functional model of a contraction operator, let $\mathcal{F}$ and $\mathcal{F}_*$ be separable, complex, Hilbert spaces. A *contractive analytic function* is any triple $\{\mathcal{F}, \mathcal{F}_*, \Theta\}$ where Θ is an element of $H^\infty(\mathcal{L}(\mathcal{F}, \mathcal{F}_*))$ such that $\|\Theta\|_\infty \leqslant 1$. Given such a Θ in $H^\infty(\mathcal{L}(\mathcal{F}, \mathcal{F}_*))$, it is easy to see that M_Θ maps $H^2(\mathcal{F})$ into $H^2(\mathcal{F}_*)$ and hence that $T_\Theta = M_\Theta | H^2(\mathcal{F})$.

If $\{\mathcal{F}, \mathcal{F}_*, \Theta\}$ is any contractive analytic function, we define the function $\Delta = \Delta_\Theta$ in $L^\infty(\mathcal{L}(\mathcal{F}))$ by setting

$$\Delta(e^{it}) = (I_{\mathcal{F}} - \Theta(e^{it})^*\Theta(e^{it}))^{1/2}, \quad e^{it} \in \mathbf{T}.$$

(The strong measurability of Δ is a consequence of the strong measurability of Θ. Throughout the paper we write "$e^{it} \in \mathbf{T}$" to mean "for almost all $e^{it} \in \mathbf{T}$".) This permits us to define the Hilbert space (of functions)

$$\mathscr{K}_+ = \mathscr{K}_+(\Theta) = H^2(\mathscr{F}_*) \oplus \{M_\Delta L^2(\mathscr{F})\}^- \tag{16}$$

which is a subspace, of course, of $L^2(\mathscr{F}_*) \oplus L^2(\mathscr{F})$. Since $L^2(\mathscr{F}_*) \oplus L^2(\mathscr{F})$ is canonically isomorphic to $L^2(\mathscr{F}_* \oplus \mathscr{F})$, it is sometimes useful to regard $\mathscr{K}_+$ as a subspace of this latter Hilbert space, and therefore as a Hilbert space of functions $u: \mathbf{T} \to \mathscr{F}_* \oplus \mathscr{F}$. Observe that $\mathscr{K}_+$ is obviously invariant under the unitary operator $U = M_\Lambda$ on $L^2(\mathscr{F}_* \oplus \mathscr{F})$, where Λ is the function in $H^\infty(\mathscr{L}(\mathscr{F}_* \oplus \mathscr{F}))$ defined by $\Lambda(e^{it}) = e^{it} I_{\mathscr{F}_* \oplus \mathscr{F}}$. Finally, a calculation shows that the mapping $u \to T_\Theta u \oplus$ $\oplus\, M_\Delta u$ of $H^2(\mathscr{F})$ into $\mathscr{K}_+$ is an isometry, so

$$\mathscr{G} = \mathscr{G}(\Theta) = \{T_\Theta u \oplus M_\Delta u : u \in H^2(\mathscr{F})\} \tag{17}$$

is a subspace of $\mathscr{K}_+$, and another calculation shows that $\mathscr{G}$ is invariant under U.

Thus, given any contractive analytic function $\{\mathscr{F}, \mathscr{F}_*, \Theta\}$, we may define the *functional model Hilbert space*

$$\mathscr{H}(\Theta) = \mathscr{K}_+(\Theta) \ominus \mathscr{G}(\Theta) \tag{18}$$

associated with $\{\mathscr{F}, \mathscr{F}_*, \Theta\}$, and also the *functional model operator*

$$S(\Theta) = P_{\mathscr{H}(\Theta)} M_\Lambda | \mathscr{H}(\Theta) \tag{19}$$

acting on $\mathscr{H}(\Theta)$. Since $\mathscr{H}(\Theta)$ is a semi-invariant subspace for M_Λ, we have

$$\begin{aligned} S(\Theta)^n &= P_{\mathscr{H}(\Theta)} M_\Lambda^n | \mathscr{H}(\Theta), \text{ and} \\ S(\Theta)^{*n} &= P_{\mathscr{H}(\Theta)} M_\Lambda^{-n} | \mathscr{H}(\Theta) \end{aligned} \tag{20}$$

for every positive integer n.

The model operator $S(\Theta)$ is a completely nonunitary contraction and every completely nonunitary contraction T in $\mathscr{L}(\mathscr{H})$ is unitarily equivalent to a certain model operator [21, p. 248]. In order to describe the model operator associated with such a T, we recall from [21, Chapter V] that every function Θ in $H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{F}_*))$ has an analytic extension $\hat{\Theta}: \mathbf{D} \to \mathscr{L}(\mathscr{F}, \mathscr{F}_*)$ defined, for example, by the strong Poisson integral

$$\hat{\Theta}(re^{i\tau}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\tau - t) + r^2} \Theta(e^{it})\,dt, \quad 0 \leqslant r < 1,\ e^{i\tau} \in \mathbf{T}.$$

Moreover, $\hat{\Theta}$ is bounded on $\mathbf{D}$ and satisfies $\|\hat{\Theta}\|_\infty = \sup_{\lambda \in \mathbf{D}} \|\hat{\Theta}(\lambda)\| = \|\Theta\|_\infty$, so that if $\{\mathscr{F}, \mathscr{F}_*, \Theta\}$ is a contractive analytic function, then $\|\hat{\Theta}\|_\infty \leqslant 1$. Furthermore, if one begins with any bounded analytic function $\hat{\Theta}: \mathbf{D} \to \mathscr{L}(\mathscr{F}, \mathscr{F}_*)$, then $\hat{\Theta}$ has strong radial limits $\Theta(e^{it}) = \lim_{r \nearrow 1} \hat{\Theta}(re^{it})$ almost everywhere on $\mathbf{T}$, $\Theta \in H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{F}_*))$, $\|\Theta\|_\infty = \|\hat{\Theta}\|_\infty$, and the Poisson integral of Θ is $\hat{\Theta}$. Thus a function Θ in $H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{F}_*))$ can, equivalently, be given by specifying a bounded analytic function $\hat{\Theta}: \mathbf{D} \to \mathscr{L}(\mathscr{F}, \mathscr{F}_*)$.

Now suppose that T is a given contraction in $\mathscr{L}(\mathscr{H})$, write $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$, and define the subspaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ of $\mathscr{H}$ to be the closures of the ranges of D_T and D_{T^*}, respectively. It is easy to see that

(21) $$TD_T = D_{T^*}T,$$

and therefore that $T\mathscr{D}_T \subset \mathscr{D}_{T^*}$. Moreover, a calculation shows that the analytic function $\hat{\Theta}_T$ defined on $\mathbf{D}$ by

(22) $$\hat{\Theta}_T(\lambda) = \{-T + \lambda D_{T^*}(I - \lambda T^*)^{-1}D_T\}|\mathscr{D}_T, \quad \lambda \in \mathbf{D},$$

satisfies $\|\hat{\Theta}_T\|_\infty \leqslant 1$ (cf. [21, p. 238]), so the associated boundary function Θ_T belongs to $H^\infty(\mathscr{D}_T, \mathscr{D}_{T^*})$, and we call the contractive analytic function $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T\}$ the *characteristic function* of T. The point of this construction of Θ_T is that one knows from [21, Chapter VI] that T is completely nonunitary if and only if T is unitarily equivalent to the model operator $S(\Theta_T)$ arising from this contractive analytic function.

Thus, in what follows we will study a given completely nonunitary contraction T in $\mathscr{L}(\mathscr{H})$ by working with the (unitarily equivalent) model operator $S(\Theta_T)$ acting on the model space $\mathscr{H}(\Theta_T)$.

The first order of business is to provide a proof of Lemma 1.1.

Proof of Lemma 1.1. Let T be a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$, and let $\Theta = \Theta_T$ be the characteristic function of T, so that T is unitarily equivalent to the model operator $S(\Theta)$ acting on the Hilbert space $\mathscr{H}(\Theta) \subset L^2(\mathscr{F}_* \oplus \mathscr{F})$ via a unitary operator $W: \mathscr{H} \to \mathscr{H}(\Theta)$. For any vectors x and y in $\mathscr{H}$, write $x(\cdot) = Wx$, $y(\cdot) = Wy$, and define

(23) $$(x \cdot y)(e^{it}) = \langle x(e^{it}), y(e^{it})\rangle_{\mathscr{F}_* \oplus \mathscr{F}}, \quad e^{it} \in \mathbf{T}.$$

That $x \cdot y \in L^1$ follows easily from the inequality

$$|\langle x(e^{it}), y(e^{it})\rangle| \leqslant \frac{1}{2}(\|x(e^{it})\|^2 + \|y(e^{it})\|^2),$$

valid almost everywhere on $\mathbf{T}$, and that the expression $x \cdot y$ is linear in x and conjugate linear in y is immediate from (23). To establish that the Fourier coefficients $c_n(x\cdot y)$ satisfy (6), we compute, using (20):

$$\langle T^n x, y\rangle = \langle (WT^nW^*)Wx, Wy\rangle = \langle S(\Theta)^n x(\cdot), y(\cdot)\rangle =$$

$$(24)\qquad = \langle P_{\mathscr{H}(\Theta)} M_A^n x(\cdot), y(\cdot)\rangle = \langle M_A^n x(\cdot), y(\cdot)\rangle =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \langle e^{int}x(e^{it}), y(e^{it})\rangle_{\mathscr{F}_*\oplus\mathscr{F}}\, dt = c_{-n}(x\cdot y), \qquad n = 0, 1, 2, \ldots,$$

and a similar computation shows that $\langle T^{*n}x, y\rangle = c_n(x\cdot y)$ for $n = 1, 2, \ldots$. The uniqueness of the function $x\cdot y$ in L^1 satisfying (6) is obvious, so that it remains only to establish (7). From (24) and (2) we obtain

$$\langle e^{int}, x\cdot y\rangle = \langle T^n x, y\rangle = \langle \Phi_T(e^{int}), [x\otimes y]\rangle =$$

$$= \langle e^{int}, \varphi_T([x\otimes y])\rangle, \qquad n = 0, 1, 2, \ldots,$$

and since the polynomials are weak*-dense in H^∞, (7) follows immediately.

We next note some elementary relations between T and its characteristic function. From (21) it is immediate that

$$(25)\qquad T\mathscr{D}_T^\perp = T(\ker D_T) \subset \ker D_{T^*} = \mathscr{D}_{T^*}^\perp,$$

and a trivial calculation shows that $T|\mathscr{D}_T^\perp$ is an isometry of $\mathscr{D}_T^\perp$ onto $\mathscr{D}_{T^*}^\perp$. Since we also have $T\mathscr{D}_T \subset \mathscr{D}_{T^*}$ from (21) and $T|\mathscr{D}_T = -\hat{\Theta}_T(0)$ from (22), we conclude that

$$(26)\qquad T^*T = \hat{\Theta}_T(0)^*\hat{\Theta}_T(0) \oplus I_{\mathscr{H}\ominus\mathscr{D}_T}, \qquad TT^* = \hat{\Theta}_T(0)\hat{\Theta}_T(0)^* \oplus I_{\mathscr{H}\ominus\mathscr{D}_{T^*}}.$$

This, in turn, yields

$$(27)\qquad \begin{aligned} &\inf \sigma_e((T^*T)^{1/2}) = \inf \sigma_e((\hat{\Theta}_T(0)^*\hat{\Theta}_T(0))^{1/2}), \text{ and} \\ &\inf \sigma_e((TT^*)^{1/2}) = \inf \sigma_e((\hat{\Theta}_T(0)\hat{\Theta}_T(0)^*)^{1/2}), \end{aligned}$$

where we set $\inf \sigma_e(A) = 1$ whenever A is a positive contraction acting on a finite dimensional Hilbert space.

Furthermore, if $\mu \in \mathbf{D}$ and T_μ is the Möbius transform in (11), then one knows from [21, pp. 14, 240] that each T_μ is a completely nonunitary contraction since T is, and that the characteristic function of T_μ coincides with the contractive analytic function $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_\mu\}$, where

$$(28)\qquad \hat{\Theta}_\mu(\lambda) = \hat{\Theta}_T\left(\frac{\lambda+\mu}{1+\bar{\mu}\lambda}\right), \qquad \lambda\in\mathbf{D},\ \mu\in\mathbf{D}.$$

Thus, applying (27) and (28) to the operator T_μ, we obtain

$$\inf \sigma_e((T_\mu^* T_\mu)^{1/2}) = \inf \sigma_e((\hat{\Theta}_T(\mu)^* \hat{\Theta}_T(\mu))^{1/2}) \text{ and}$$

(29)

$$\inf \sigma_e((T_\mu T_\mu^*)^{1/2}) = \inf \sigma_e((\hat{\Theta}_T(\mu)\hat{\Theta}_T(\mu)^*)^{1/2}).$$

These relationships will play a significant role in Section 5.

We continue now to try to motivate the discussion in Sections 5 and 6 by noting that if we have fixed a model Hilbert space $\mathscr{H}(\Theta)$ and a model operator $T = S(\Theta)$, then the system of equations (8) with $n = \aleph_0$ takes the form

$$f_{ij}(e^{it}) = (x_i \cdot y_j)(e^{it}) = \langle x_i(e^{it}), y_j(e^{it})\rangle_{\mathscr{F}_* \oplus \mathscr{F}}, \quad i, j \in \mathbf{N}, \quad e^{it} \in \mathbf{T}. \tag{30}$$

Let $\mathscr{N}$ be a Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbf{N}}$. Then the system (30) can be rewritten, at least formally, in the form

$$F(e^{it}) = Y(e^{it})^* X(e^{it}), \tag{31}$$

where $X(e^{it})$ and $Y(e^{it})$ are linear transformations from $\mathscr{N}$ into $\mathscr{F}_* \oplus \mathscr{F}$ that satisfy $X(e^{it})e_j = x_j(e^{it})$, $Y(e^{it})e_j = y_j(e^{it})$, $j \in \mathbf{N}$, and $F(e^{it})$ is the linear transformation on $\mathscr{N}$ whose matrix with respect to the basis $\{e_j\}_{j\in\mathbf{N}}$ is given by $[f_{ji}(e^{it})]_{i,j\in\mathbf{N}}$. Therefore we will consider the equation (31), where X and Y are elements of a space of functions defined on $\mathbf{T}$ and taking values in $\mathscr{L}(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F})$, while F is a function defined on (some measurable subset σ of) $\mathbf{T}$ and taking values in $\mathscr{L}(\mathscr{N})$. For technical reasons we will take X and Y to belong to the Hilbert space $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ and F to belong to the space $L^1(\sigma, C_1(\mathscr{N}))$. Of course, we want our solutions X and Y of (31) to eventually provide families of vectors $\{x_i\}_{i\in\mathbf{N}}$ and $\{y_j\}_{j\in\mathbf{N}}$ in $\mathscr{H}(\Theta)$, and to this end, we say that an element X of $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ is *$\mathscr{H}(\Theta)$-oriented* if the function Xa in $L^2(\mathscr{F}_* \oplus \mathscr{F})$ defined by $(Xa)(e^{it}) = X(e^{it})a$ belongs to $\mathscr{H}(\Theta)$ for each a in $\mathscr{N}$.

LEMMA 3.1. *The set of all $\mathscr{H}(\Theta)$-oriented functions in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ forms a closed subspace.*

Proof. The lemma obviously follows from the inequality

$$\|Xa\|_{L^2(\mathscr{F}_* \oplus \mathscr{F})} \leq \|a\|_{\mathscr{N}} \cdot \|X\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))}, \quad a \in \mathscr{N},$$

which itself follows from the various definitions and the fact that $\|A\| \leq \|A\|_2$ for every operator A in $C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F})$.

LEMMA 3.2. *For every pair X and Y in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$, the function Y^*X defined by $(Y^*X)(e^{it}) = Y(e^{it})^*X(e^{it})$, $e^{it} \in \mathbf{T}$, belongs to $L^1(C_1(\mathcal{N}))$ and satisfies*

$$\|Y^*X\|_{L^1(C_1(\mathcal{N}))} \leq \|X\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} \cdot \|Y\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))}. \tag{32}$$

Moreover, if $Y = X$, (32) becomes an equality.

Proof. The lemma follows from a straightforward computation using the definitions, the inequality

$$\|Y^*(e^{it})X(e^{it})\|_1 \leq \|X(e^{it})\|_2 \|Y(e^{it})\|_2$$

(which becomes an equality if $Y = X$), and the Schwarz inequality.

In order to study the system of simultaneous equations

$$[f_{ij}] = [x_i \cdot y_j], \quad f_{ij} \in L^1, \ i, j \in \mathbf{N}, \tag{33}$$

in L^1/H_0^1, we will also need the subspace $H_0^1(C_1(\mathcal{N}))$ of $L^1(C_1(\mathcal{N}))$ which consists of those functions F in $L^1(C_1(\mathcal{N}))$ for which the Fourier coefficients $c_n(F)$ of F vanish for all $n \leq 0$. In line with our previous notation, we denote by $(L^1/H_0^1)(C_1(\mathcal{N}))$ the quotient space $L^1(C_1(\mathcal{N}))/H_0^1(C_1(\mathcal{N}))$ and by $[F]$ the coset in $(L^1/H_0^1)(C_1(\mathcal{N}))$ of the function F in $L^1(C_1(\mathcal{N}))$.

4. SOME SPECIAL ELEMENTS OF $\mathcal{H}(\Theta)$ AND $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$

In this section we construct some particular elements of the spaces $\mathcal{H}(\Theta)$ and $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ that will play a basic role in § 5. The operator T will always be assumed to be a completely nonunitary contraction that is a model operator $T = S(\Theta)$, acting on a model Hilbert space $\mathcal{H}(\Theta)$. We continue to use in this section all of the notation and facts concerning $S(\Theta)$ that were developed in Section 3. Before we begin our program, we make one final comment in the way of motivation for what follows. In [9] a large role was played by the elements $[C_\lambda]$ in Q_T that satisfy

$$\langle f(T), [C_\lambda]\rangle = \hat{f}(\lambda), \quad f \in H^\infty, \ \lambda \in \mathbf{D}. \tag{34}$$

Moreover, one of the first lemmas in [9] showed that the $[C_\lambda]$ could be approximated arbitrarily closely in Q_T by elements of the form $[x \otimes x]$, where x is a unit vector in $\mathcal{H}$. Recall from [19] that when $T \in$ (BCP), then $\Phi_T\colon H^\infty \to \mathcal{A}_T$ and $\varphi_T\colon Q_T \to L^1/H_0^1$ are isometric isomorphisms. Thus it follows immediately from the Poisson integral formula that $\varphi([C_\mu]) = [|p_\mu^2|]$, where p_μ is the H^2-function defined by

$$p_\mu(e^{it}) = (1 - |\mu|^2)^{1/2}(1 - \bar{\mu}e^{it})^{-1}, \quad \mu \in \mathbf{D}. \tag{35}$$

Indeed,

$$\langle f, \varphi_T([C_\mu])\rangle = \langle f(T), [C_\mu]\rangle = \hat{f}(\mu) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\mu|^2}{|\mu - e^{it}|^2} f(e^{it})\,dt = \langle f, [|p_\mu^2|]\rangle, \quad f \in H^\infty.$$

Therefore it is reasonable that the first step in our program will be to show that the functions $|p_\mu^2|$ can be approximated in L^1 by functions of the form $x \cdot x$ for certain vectors x in $\mathscr{H}(\Theta)$. To this end, we introduce, for every function f in H^2 and every a_* in $\mathscr{F}_*$, the function $f \cdot a_*$ in $H^2(\mathscr{F}_*)$ defined by

$$(f \cdot a_*)(e^{it}) = f(e^{it})a_*, \quad e^{it} \in \mathbf{T}.$$

It is immediate from this definition and (23) that if we set $x = p_\mu \cdot a_* \oplus 0$ in $L^2(\mathscr{F}_* \oplus \mathscr{F})$, then $x \cdot x = |p_\mu^2|$ whenever a_* is a unit vector, but the problem is that, in general, $x \notin \mathscr{H}(\Theta)$. Thus we introduce the functions $p_\mu \# a_*$ defined by

$$p_\mu \# a_* = P_{\mathscr{H}(\Theta)}(p_\mu \cdot a_* \oplus 0), \quad \mu \in \mathbf{D},\ a_* \in \mathscr{F}_*. \tag{36}$$

In order to compute an explicit formula for these functions, we need a lemma. Recall that it is an easy consequence of the Cauchy integral formula (cf. [20, p. 235]) that

$$(1 - |\mu|^2)^{1/2}\hat{g}(\mu) = \langle g, p_\mu\rangle_{H^2}, \quad \mu \in \mathbf{D},\ g \in H^2. \tag{37}$$

LEMMA 4.1. *If Φ is any function in $H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{F}_*))$, then*

$$T_\Phi^*(p_\mu \cdot a_*) = p_\mu \cdot (\hat{\Phi}(\mu)^* a_*), \quad \mu \in \mathbf{D},\ a_* \in \mathscr{F}_*. \tag{38}$$

Proof. For every u in $H^2(\mathscr{F})$, we have

$$\langle u, T_\Phi^*(p_\mu \cdot a_*)\rangle_{H^2(\mathscr{F})} = \langle T_\Phi u, p_\mu \cdot a_*\rangle_{H^2(\mathscr{F}_*)} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \langle \Phi(e^{it})u(e^{it}), p_\mu(e^{it})a_*\rangle_{\mathscr{F}_*}\,dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \overline{p_\mu(e^{it})}\langle \Phi(e^{it})u(e^{it}), a_*\rangle_{\mathscr{F}_*}\,dt =$$

$$= (1 - |\mu|^2)^{1/2}\langle \hat{\Phi}(\mu)\hat{u}(\mu), a_*\rangle_{\mathscr{F}_*}$$

by virtue of (37) and the easily derived fact that the function $\lambda \to \langle \hat{\Phi}(\lambda)\hat{u}(\lambda), a_*\rangle$ is analytic on $\mathbf{D}$ and has as boundary function the function $e^{it} \to \langle \Phi(e^{it})u(e^{it}), a_*\rangle$

in H^2. Thus, continuing along the same lines, we have

$$(1-|\mu|^2)^{1/2}\langle\hat{\Phi}(\mu)\hat{u}(\mu),\ a_*\rangle_{\mathscr{F}_*}=(1-|\mu|^2)^{1/2}\langle\hat{u}(\mu),\ \hat{\Phi}(\mu)^*a_*\rangle_{\mathscr{F}_*}=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\overline{p_\mu(e^{it})}\langle u(e^{it}),\ \hat{\Phi}(\mu)^*a_*\rangle_{\mathscr{F}_*}\,dt=\langle u,\ p_\mu\cdot(\hat{\Phi}(\mu)^*a_*)\rangle_{H^2(\mathscr{F})},$$

which proves (38).

Returning now to the derivation of an explicit formula for $p_\mu \# a_*$, we note that if $a_*\in\mathscr{F}_*$, then

$$(p_\mu\cdot a_*)\oplus 0\in\mathscr{K}_+(\Theta)=H^2(\mathscr{F}_*)\oplus\{M_\Delta L^2(\mathscr{F})\}^-,\quad \mu\in\mathbf{D},$$

and thus by virtue of (18),

$$p_\mu \# a_*=(p_\mu\cdot a_*)\oplus 0-P_{\mathscr{G}(\Theta)}((p_\mu\cdot a_*)\oplus 0), \tag{39}$$

where $\mathscr{G}(\Theta)$ is as in (17). If we denote by $V\colon H^2(\mathscr{F})\to\mathscr{K}_+(\Theta)$ the isometry of $H^2(\mathscr{F})$ onto $\mathscr{G}(\Theta)$ defined by $Vu=T_\Theta u\oplus M_\Delta u$, then V can be regarded as the column matrix

$$\begin{pmatrix}T_\Theta\\ M_\Delta\end{pmatrix},$$

and it follows easily from this and (38) that

$$P_{\mathscr{G}(\Theta)}((p_\mu\cdot a_*)\oplus 0)=VV^*((p_\mu\cdot a_*)\oplus 0)=V(T_\Theta^*(p_\mu\cdot a_*))=$$

$$=V(p_\mu\cdot(\hat{\Theta}(\mu)^*a_*))=T_\Theta(p_\mu\cdot(\hat{\Theta}(\mu)^*a_*))\oplus M_\Delta(p_\mu\cdot(\hat{\Theta}(\mu)^*a_*)).$$

Putting this together with (39) gives us the desired formula.

LEMMA 4.2. *If* $\mu\in\mathbf{D}$, $a_*\in\mathscr{F}_*$, *and* $p_\mu \# a_*$ *is defined as in* (36), *then for almost all* $e^{it}\in\mathbf{T}$, *we have*

$$(p_\mu \# a_*)(e^{it})=p_\mu(e^{it})\{[a_*-\Theta(e^{it})\hat{\Theta}(\mu)^*a_*)]\oplus[-\Delta(e^{it})\hat{\Theta}(\mu)^*a_*]\}. \tag{40}$$

In the same vein, if we define the functions $p_\mu\nabla a$ in $L^2(\mathscr{F}_*\oplus\mathscr{F})$ by

$$(p_\mu\nabla a)(e^{it})=e^{-it}\overline{p_\mu(e^{it})}(\Theta(e^{it})a\oplus\Delta(e^{it})a),\quad e^{it}\in\mathbf{T},\ \mu\in\mathbf{D},\ a\in\mathscr{F}, \tag{41}$$

then also $(p_\mu\nabla a)\cdot(p_\mu\nabla a)=|p_\mu^2|$ whenever a is a unit vector in $\mathscr{F}$. Therefore, as before, we are interested in obtaining an explicit formula for the elements

$$p_\mu\,\Box\,a=P_{\mathscr{H}(\Theta)}(p_\mu\nabla a). \tag{42}$$

An easy calculation shows that the $p_\mu \nabla a$ are orthogonal to $\mathcal{G}(\Theta)$ in $L^2(\mathcal{F}_* \oplus \mathcal{F})$, and thus we have from (18) that

$$p_\mu \square a = P_{\mathcal{K}_-(\Theta)}(p_\mu \nabla a).$$

Furthermore, since the function $k_{\mu,a}(e^{it}) = \Delta(e^{it})(e^{-it}\overline{p_\mu(e^{it})}a)$ clearly belongs to $\{M_\Delta L^2(\mathcal{F})\}^-$, it is immediate that

$$P_{\mathcal{K}_+(\Theta)}(p_\mu \nabla a) = (P_{H^2(\mathcal{F}_*)} h_{\mu,a}) \oplus k_{\mu,a}$$

where

$$h_{\mu,a}(e^{it}) = e^{-it}\overline{p_\mu(e^{it})}\Theta(e^{it})a =$$

$$= (1 - |\mu|^2)^{1/2} \left\{\left(\frac{1}{e^{it} - \mu}\right)\Theta(e^{it})a\right\} =$$

$$= (1 - |\mu|^2)^{1/2} \left\{\left(\frac{\Theta(e^{it}) - \hat{\Theta}(\mu)}{e^{it} - \mu}\right)a + \left(\frac{1}{e^{it} - \mu}\right)\hat{\Theta}(\mu)a\right\}.$$

Since the first function in this last expression clearly belongs to $H^2(\mathcal{F}_*)$ and the second function can be seen to be orthogonal to $H^2(\mathcal{F}_*)$ by a calculation, we have established the following formula.

LEMMA 4.3. *For any fixed μ in $\mathbf{D}$ and a in $\mathcal{F}$, we have, for almost all $e^{it} \in \mathbf{T}$,*

$$(43) \qquad (p_\mu \square a)(e^{it}) = e^{-it}\overline{p_\mu(e^{it})}\{(\Theta(e^{it})a - \hat{\Theta}(\mu)a) \oplus \Delta(e^{it})a\}.$$

Since in this paper we are interested in solving *systems* of equations of the form (5), instead of a single equation, the functions $p_\mu \# a_*$ and $p_\mu \square a$ in $\mathcal{H}(\Theta)$ are not sufficient for our purposes. Therefore using formulas (40) and (43) we introduce two analogous families of functions in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$, where $\mathcal{N}$ is an arbitrary separable Hilbert space.

For every A_* in $C_2(\mathcal{N}, \mathcal{F}_*)$ and every μ in $\mathbf{D}$ we define the function $p_\mu \# A_*$ by setting

$$(44) \quad (p_\mu \# A_*)(e^{it}) = p_\mu(e^{it})\{[A_* - \Theta(e^{it})\hat{\Theta}(\mu)^* A_*] \oplus [-\Delta(e^{it})\hat{\Theta}(\mu)^* A_*]\}.$$

LEMMA 4.4. *For every μ in $\mathbf{D}$ and A_* in $C_2(\mathcal{N}, \mathcal{F}_*)$, the function $p_\mu \# A_*$ is an $\mathcal{H}(\Theta)$-oriented element of $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ that satisfies*

$$(45) \qquad \|p_\mu \# A_*\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} \le \|A_*\|_2.$$

Proof. By virtue of Lemma 4.2, for every a in $\mathcal{N}$ the function $(p_\mu \# A_*)a$ coincides with the projection onto $\mathcal{H}(\Theta)$ of $p_\mu \cdot (A_* a) \oplus 0$, so it will follow that $p_\mu \# A_*$ is $\mathcal{H}(\Theta)$-oriented once we know that $p_\mu \# A_* \in L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$. Now suppose that $\{e_j\}_{j\in J}$ is an orthonormal basis of $\mathcal{N}$. Then

$$\|p_\mu \# A_*\|^2_{L^2(C_2(\mathcal{N}, \mathcal{F}_*\oplus\mathcal{F}))} = \frac{1}{2\pi}\int_0^{2\pi} \|(p_\mu \# A_*)(e^{it})\|_2^2 \, dt =$$

$$(46) \qquad = \sum_{j\in J} \frac{1}{2\pi}\int_0^{2\pi} \|(p_\mu \# A_*)(e^{it})e_j\|^2_{\mathcal{F}_*\oplus\mathcal{F}} \, dt = \sum_{j\in J} \|(p_\mu \# A_*)e_j\|^2_{\mathcal{H}(\Theta)} =$$

$$= \sum_{j\in J} \|P_{\mathcal{H}(\Theta)}(p_\mu \cdot (A_* e_j) \oplus 0)\|^2_{\mathcal{H}(\Theta)} \leqslant \sum_{j\in J} \|p_\mu \cdot (A_* e_j) \oplus 0\|^2_{L^2(\mathcal{F}_*\oplus\mathcal{F})} \leqslant$$

$$\leqslant \sum_{j\in J} \|p_\mu\|^2_{H^2} \|A_* e_j\|^2_{\mathcal{F}_*} = \sum_{j\in J} \|A_* e_j\|^2_{\mathcal{F}_*} = \|A_*\|_2^2 .$$

These inequalities show, at the same time, that $p_\mu \# A_* \in L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ and that (45) is valid, thus completing the proof.

The second family of functions to be introduced is defined as follows. For every A in $C_2(\mathcal{N}, \mathcal{F})$ and for every μ in $\mathbf{D}$, we set

$$(47) \qquad (p_\mu \square A)(e^{it}) = e^{-it}\overline{p_\mu(e^{it})}\{[\Theta(e^{it})A - \hat{\Theta}(\mu)A] \oplus \Delta(e^{it})A\}, \qquad e^{it} \in \mathbf{T}.$$

The following lemma is the counterpart of Lemma 4.4 for these functions.

LEMMA 4.5. *For every μ in $\mathbf{D}$ and A in $C_2(\mathcal{N}, \mathcal{F})$, the function $p_\mu \square A$ is an $\mathcal{H}(\Theta)$-oriented element of $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ that satisfies*

$$\|p_\mu \square A\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_*\oplus\mathcal{F}))} \leqslant \|A\|_2 .$$

Proof. By virtue of Lemma 4.3, for every a in $\mathcal{N}$ the function $(p_\mu \square A)a$ coincides with the projection onto $\mathcal{H}(\Theta)$ of $p_\mu \square (Aa)$, so the proof can be concluded by the following sequence of inequalities similar to (46):

$$\|p_\mu \square A\|^2_{L^2(C_2(\mathcal{N},\mathcal{F}_*\oplus\mathcal{F}))} = \sum_{j\in J} \|P_{\mathcal{H}(\Theta)}(p_\mu \nabla (Ae_j))\|^2_{\mathcal{H}(\Theta)} \leqslant$$

$$\leqslant \sum_{j\in J} \|p_\mu \nabla (Ae_j)\|^2_{L^2(\mathcal{F}_*\oplus\mathcal{F})} =$$

$$= \sum_{j\in J} \frac{1}{2\pi}\int_0^{2\pi} \|e^{-it}\overline{p_\mu(e^{it})}[\Theta(e^{it})Ae_j \oplus \Delta(e^{it})Ae_j]\|^2_{\mathcal{F}_*\oplus\mathcal{F}} \, dt =$$

$$= \sum_{j\in J} \frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2 \|Ae_j\|^2_{\mathcal{F}} \, dt = \sum_{j\in J} \|Ae_j\|^2_{\mathcal{F}} = \|A\|_2^2 ,$$

where, just as in the proof of Lemma 4.4, $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{N}$.

We conclude this section with the following result, which is similar to Lemma 4.3 of [9].

LEMMA 4.6. *Suppose* $\mu \in \mathbf{D}$ *and* $Z \in L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$. *Then for every bounded (in* $\|\cdot\|_2$*) sequence* $\{A_{*n}\}_{n=1}^{\infty}$ *in* $C_2(\mathcal{N}, \mathcal{F}_*)$ *such that* $\{A_{*n}^*\}_{n=1}^{\infty}$ *converges to zero in the strong operator topology, we have*

$$(48)\qquad \lim_{n\to\infty} \|Z^*(p_\mu \,\#\, A_{*n})\|_{L^1(C_1(\mathcal{N}))} = \lim_{n\to\infty} \|(p_\mu \,\#\, A_{*n})^* Z\|_{L^1(C_1(\mathcal{N}))} = 0.$$

Furthermore, if $\{A_n\}$ *is any bounded (in* $\|\cdot\|_2$*) sequence in* $C_2(\mathcal{N}, \mathcal{F})$ *such that* $\{A_n^*\}_{n=1}^{\infty}$ *converges to zero in the strong operator topology, then*

$$(49)\qquad \lim_{n\to\infty} \|Z^*(p_\mu \,\square\, A_n)\|_{L^1(C_1(\mathcal{N}))} = \lim_{n\to\infty} \|(p_\mu \,\square\, A_n)^* Z\|_{L^1(C_1(\mathcal{N}))} = 0.$$

Proof. The proofs of (48) and (49) are almost identical, so we only prove (48). Furthermore the equality of the two limits in (48) is obvious, because

$$(Z^*(p_\mu \,\#\, A_{*n}))(e^{it})^* = ((p_\mu \,\#\, A_{*n})^* Z)(e^{it})$$

for $e^{it} \in \mathbf{T}$. Suppose now that we have shown that for $e^{it} \in \mathbf{T}$,

$$(50)\qquad \lim_n \|Z(e^{it})^*(p_\mu \,\#\, A_{*n})(e^{it})\|_1 = 0.$$

Then, since from (44) we have

$$\|Z(e^{it})^*(p_\mu \,\#\, A_{*n})(e^{it})\|_1 \leqslant \|Z(e^{it})\|_2 3|p_\mu(e^{it})|\,\|A_{*n}\|_2$$

for each fixed e^{it}, we obtain

$$\lim_n \|Z^*(p_\mu \,\#\, A_{*n})\|_{L^1(C_1(\mathcal{N}))} = 0$$

from the Lebesgue dominated convergence theorem. Thus, to complete the proof, it suffices to prove (50). But (50) follows from the following general fact, whose proof we sketch below: If W and $\{C_n\}_{n=1}^{\infty}$ are Hilbert-Schmidt operators such that the sequence $\{\|C_n\|_2\}_{n=1}^{\infty}$ is bounded and $\{C_n^*\}_{n=1}^{\infty}$ converges strongly to zero, then $\|WC_n\|_1 \to 0$. (Sketch of proof: If W is the rank-one operator $W = x \otimes y$, then

$$\|WC_n\|_1 = \|(x \otimes y)C_n\|_1 = \|x \otimes (C_n^* y)\|_1 = \|x\| \cdot \|C_n^* y\| \to 0.$$

Since the result is true for rank-one operators, it is true for all finite-rank operators, and this set is dense in the space of Hilbert-Schmidt operators in the Hilbert--Schmidt norm. Now use the fact that $\|WC_n\|_1 \leqslant \|W\|_2\|C_n\|_2 \leqslant \|W\|_2 (\sup\|C_n\|_2)$ to conclude the proof.)

5. SOME NEAR-FACTORIZATION LEMMAS

In this section we establish matricial versions of the approximation lemmas common to papers in this area. The operator T will always be a completely nonunitary contraction that is a model operator $T = S(\Theta)$ acting on a model Hilbert space $\mathscr{H}(\Theta)$, so the results of the preceding sections are applicable. Throughout this section $\mathscr{N}$ will denote a separable, but otherwise arbitrary, complex Hilbert space and σ a measurable subset of $\mathbf{T}$. If $f \in L^1(\sigma)$ and $X \in C_1(\mathscr{N})$, then we denote by fX the element of $L^1(\sigma, C_1(\mathscr{N}))$ defined by setting $(fX)(e^{it}) = f(e^{it})X$ for $e^{it} \in \sigma$. If $F \in L^1(C_1(\mathscr{N}))$, we write $[F]$ for the coset of F in the quotient space $(L^1/H_0^1)(C_1(\mathscr{N})) = L^1(C_1(\mathscr{N}))/H_0^1(C_1(\mathscr{N}))$.

The first step in our factorization technique consists of an "operator-valued" version of the usual device of approximating the functions $|p_\mu^2|$ by products of the form $x \cdot y$ with $x, y \in \mathscr{H}(\Theta)$; cf. [2, Lemma 2.2] and [9, Lemma 4.3].

LEMMA 5.1. *Suppose* $\mu \in \mathbf{D}$, $0 \leqslant \theta < 1$, *and* $A_*, B_* \in C_2(\mathscr{N}, \mathscr{F}_*)$. *If* $\|\hat{\Theta}(\mu)^* B_* x\|_{\mathscr{F}} \leqslant \theta \|B_* x\|_{\mathscr{F}_*}$ *for all x in $\mathscr{N}$, then*

$$\|[|p_\mu^2| B_*^* A_* - (p_\mu \# B_*)^*(p_\mu \# A_*)]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} \leqslant \theta \|A_*\|_2 \|B_*\|_2. \tag{51}$$

If, in addition, we have $\|\hat{\Theta}(\mu)^* A_* x\|_{\mathscr{F}} \leqslant \theta \|A_* x\|_{\mathscr{F}_*}$ *for all x in $\mathscr{N}$, then*

$$\||p_\mu^2| B_*^* A_* - (p_\mu \# B_*)^*(p_\mu \# A_*)\|_{L^1(C_1(\mathscr{N}))} \leqslant 2\theta \|A_*\|_2 \|B_*\|_2. \tag{52}$$

Proof. It follows easily from (44) that if $a_* \oplus a \in \mathscr{F}_* \oplus \mathscr{F}$, then, for almost all $e^{it} \in \mathbf{T}$,

$$\{(p_\mu \# B_*)(e^{it})^*\}(a_* \oplus a) = \overline{p_\mu(e^{it})}\{(B_*^* - B_*^* \hat{\Theta}(\mu)\Theta(e^{it})^*)a_* - B_*^* \hat{\Theta}(\mu)\Delta(e^{it})a\}.$$

Thus, using the definition in Lemma 3.2 and the identity

$$\Theta(e^{it})^*\Theta(e^{it}) + \Delta(e^{it})^*\Delta(e^{it}) = I_{\mathscr{F}}, \quad e^{it} \in \mathbf{T},$$

we get

$$\begin{aligned} &\{|p_\mu^2| B_*^* A_* - (p_\mu \# B_*)^*(p_\mu \# A_*)\}(e^{it}) = \\ &= |p_\mu(e^{it})|^2\{B_*^* A_* - [B_*^* - B_*^* \hat{\Theta}(\mu)\Theta(e^{it})^*][A_* - \Theta(e^{it})\hat{\Theta}(\mu)^* A_*] - \\ &\qquad -[-B_*^* \hat{\Theta}(\mu)\Delta(e^{it})][-\Delta(e^{it})\hat{\Theta}(\mu)^* A_*]\} = \\ &= |p_\mu(e^{it})|^2\{B_*^* \hat{\Theta}(\mu)\Theta(e^{it})^* A_* + B_*^* \Theta(e^{it})\hat{\Theta}(\mu)^* A_* - B_*^* \hat{\Theta}(\mu)\hat{\Theta}(\mu)^* A_*\} = \\ &= |p_\mu(e^{it})|^2 B_*^* \hat{\Theta}(\mu)\Theta(e^{it})^* A_* + |p_\mu(e^{it})^2|\{B_*^* \Theta(e^{it}) - B_*^* \hat{\Theta}(\mu)\}\hat{\Theta}(\mu)^* A_*. \end{aligned} \tag{53}$$

Furthermore the second term on the right in the last equality can be written as

$$p_\mu(e^{it})^2\{B_*^*\Theta(e^{it}) - B_*^*\hat{\Theta}(\mu)\}\hat{\Theta}(\mu)^*A_* =$$

(54)

$$= \{p_\mu(e^{it})\Theta(e^{it})^*B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^*B_*\}^*p_\mu(e^{it})\hat{\Theta}(\mu)^*A_*,$$

and we know from (38) that for every x in $\mathscr{N}$,

$$p_\mu(e^{it})\Theta(e^{it})^*B_*x - p_\mu(e^{it})\hat{\Theta}(\mu)^*B_*x =$$

$$= \{M_\Theta^*(p_\mu\cdot B_*x) - T_\Theta^*(p_\mu\cdot B_*x)\}(e^{it}) =$$

(55)

$$\{M_\Theta^*(p_\mu\cdot B_*x) - P_{\mathscr{H}^2(\mathscr{F})}M_\Theta^*(p_\mu\cdot B_*x)\}(e^{it}) =$$

$$= \{P_{L^2(\mathscr{F})\ominus H^2(\mathscr{F})}(M_\Theta^*(p_\mu\cdot B_*x))\}(e^{it}),$$

which shows, by virtue of the way one computes Fourier coefficients, that the function

$$e^{it} \to p_\mu(e^{it})\Theta(e^{it})^*B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^*B_*$$

belongs to $L^2(C_2(\mathscr{N}, \mathscr{F})) \ominus H^2(C_2(\mathscr{N}, \mathscr{F}))$. It follows that the function

$$e^{it} \to \{p_\mu(e^{it})\Theta(e^{it})^*B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^*B_*\}^*$$

belongs to $H_0^2(C_2(\mathscr{F}, \mathscr{N}))$, and therefore the product in (54) belongs to $H_0^1(C_1(\mathscr{N}))$, since the function

$$e^{it} \to p_\mu(e^{it})\hat{\Theta}(\mu)^*A_*$$

clearly belongs to $H^2(C_2(\mathscr{N}, \mathscr{F}))$. This means that we may neglect the function in (54) in computing the $(L^1/H_0^1)(C_1(\mathscr{N}))$ norm of the function in (53), and thus we have

$$\|[|p_\mu^2|B_*^*A_* - (p_\mu \# B_*)^*(p_\mu \# A_*)]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} \leqslant$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \||p_\mu(e^{it})|^2B_*^*\hat{\Theta}(\mu)\Theta(e^{it})^*A_*\|_{C_1(\mathscr{N})}\,dt \leqslant$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2\|\hat{\Theta}(\mu)^*B_*\|_2\|A_*\|_2\,dt =$$

$$= \|\hat{\Theta}(\mu)^*B_*\|_2\|A_*\|_2.$$

If $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{N}$, then, by hypothesis,

$$(56)\qquad \|\hat{\Theta}(\mu)^* B_*\|_2^2 = \sum_{j\in J} \|\Theta(\mu)^* B_* e_j\|_{\mathcal{F}}^2 \leqslant \sum_{j\in J} \theta^2 \|B_* e_j\|_{\mathcal{F}_*}^2 = \theta^2 \|B_*\|_2^2,$$

so (51) is proved. To prove (52), we must also estimate the norm of the H_0^1-function in (54). We do this using the Schwarz inequality and the inequality $\|Y^*X\|_1 \leqslant \|Y\|_2\|X\|_2$:

$$\frac{1}{2\pi}\int_0^{2\pi} \|\{p_\mu(e^{it})\Theta(e^{it})^* B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^* B_*\}^* p_\mu(e^{it})\hat{\Theta}(\mu)^* A_*\|_{C_1(\mathcal{N})}\, dt \leqslant$$

$$(57)\qquad \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} \|p_\mu(e^{it})\Theta(e^{it})^* B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^* B_*\|_{C_2(\mathcal{N},\mathcal{F})}^2\, dt\right)^{1/2} \times$$

$$\times \left(\frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2 \|\hat{\Theta}(\mu)^* A_*\|_{C_2(\mathcal{N},\mathcal{F})}^2\, dt\right)^{1/2}.$$

The second factor equals $\|\hat{\Theta}(\mu)^* A_*\|_2$, and a calculation like that in (54), using the hypothesis, shows that

$$\|\hat{\Theta}(\mu)^* A_*\|_2 \leqslant \theta\|A_*\|_2.$$

To estimate the first factor on the right hand side of (57) we choose an orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{N}$, and we compute using (55):

$$\frac{1}{2\pi}\int_0^{2\pi} \|p_\mu(e^{it})\Theta(e^{it})^* B_* - p_\mu(e^{it})\hat{\Theta}(\mu)^* B_*\|_{C_2(\mathcal{N},\mathcal{F})}^2\, dt =$$

$$= \sum_{j\in J} \frac{1}{2\pi}\int_0^{2\pi} \|p_\mu(e^{it})\Theta(e^{it})^* B_* e_j - p_\mu(e^{it})\hat{\Theta}(\mu)^* B_* e_j\|_{\mathcal{F}}^2\, dt =$$

$$= \sum_{j\in J} \|P_{L^2(\mathcal{F})\ominus H^2(\mathcal{F})}(M_\Theta^*(p_\mu \cdot B_* e_j))\|_{L^2(\mathcal{F})}^2 \leqslant$$

$$\leqslant \sum_{j\in J} \|p_\mu \cdot B_* e_j\|_{L^2(\mathcal{F}_*)}^2 = \sum_{j\in J} \|B_* e_j\|_{\mathcal{F}_*}^2 = \|B_*\|_2^2.$$

Thus the H_0^1-function in (54) has norm less than or equal to $\theta\|A_*\|_2\|B_*\|_2$, and (52) now follows from the decomposition in (53).

The following lemma is the analog of Lemma 5.1 for the functions $p_\mu \,\square\, A$.

LEMMA 5.2. *Suppose $\mu \in \mathbf{D}$, $0 \leqslant \theta < 1$, and $A, B \in C_2(\mathcal{N}, \mathcal{F})$. If $\|\hat{\Theta}(\mu)Ax\|_{\mathcal{F}_*} \leqslant \theta\|Ax\|_{\mathcal{F}}$ for all x in $\mathcal{N}$, then*

$$(58)\qquad \|[|p_\mu^2| B^* A - (p_\mu \,\square\, B)^*(p_\mu \,\square\, A)]\|_{(L^1/H_0^1)(C_1(\mathcal{N}))} \leqslant \theta\|A\|_2\|B\|_2.$$

If, in addition, we have $\|\hat{\Theta}(\mu)Bx\|_{\mathscr{F}_*} \leqslant \theta \|Bx\|_{\mathscr{F}}$ *for all* x *in* $\mathscr{N}$, *then*

$$(59) \qquad \||p_\mu^2| B^*A - (p_\mu \square B)^*(p_\mu \square A)\|_{L^1(C_1(\mathscr{N}))} \leqslant 2\theta \|A\|_2 \|B\|_2.$$

Proof. It follows easily from (47) that for every $a_* \oplus a \in \mathscr{F}_* \oplus \mathscr{F}$ and almost all $e^{it} \in \mathbf{T}$, we have

$$\{(p_\mu \square B)(e^{it})^*\}(a_* \oplus a) = e^{it}p_\mu(e^{it})\{[B^*\Theta(e^{it})^* - B^*\hat{\Theta}(\mu)^*]a_* + B^*\Delta(e^{it})a\}.$$

Thus, as in the previous proof, we obtain

$$(60) \quad \begin{aligned} &\{|p_\mu^2| B^*A - (p_\mu \square B)^*(p_\mu \square A)\}(e^{it}) = \\ &= |p_\mu(e^{it})|^2\{B^*A - [B^*\Theta(e^{it})^* - B^*\hat{\Theta}(\mu)^*][\Theta(e^{it})A - \hat{\Theta}(\mu)A] - B^*\Delta(e^{it})^2A\} = \\ &= |p_\mu(e^{it})|^2\{B^*\hat{\Theta}(\mu)^*\Theta(e^{it})A + B^*\Theta(e^{it})^*\hat{\Theta}(\mu)A - B^*\hat{\Theta}(\mu)^*\hat{\Theta}(\mu)A\} = \\ &= |p_\mu(e^{it})|^2 B^*\Theta(e^{it})^*\hat{\Theta}(\mu)A + |p_\mu(e^{it})|^2 B^*\hat{\Theta}(\mu)^*[\Theta(e^{it})A - \hat{\Theta}(\mu)A]. \end{aligned}$$

The second term on the right in (60) can be written as

$$(61) \quad \begin{aligned} &|p_\mu(e^{it})|^2 B^*\hat{\Theta}(\mu)^*[\Theta(e^{it})A - \hat{\Theta}(\mu)A] = \\ &= [e^{it}p_\mu(e^{it})B^*\hat{\Theta}(\mu)^*](1 - |\mu|^2)^{1/2} \left[\frac{\Theta(e^{it}) - \hat{\Theta}(\mu)}{e^{it} - \mu}\right] A. \end{aligned}$$

Since the function $e^{it} \to e^{it}p_\mu(e^{it})B^*\hat{\Theta}(\mu)^*$ clearly belongs to $H_0^2(C_2(\mathscr{F}_*, \mathscr{N}))$, while the function $e^{it} \to \left(\dfrac{\Theta(e^{it}) - \hat{\Theta}(\mu)}{e^{it} - \mu}\right)A \in H^2(C_2(\mathscr{N}, \mathscr{F}_*))$, it follows that the function in (61) belongs to $H_0^1(C_1(\mathscr{N}))$. Thus we may neglect that function in the computation of the $(L^1/H_0^1)(C_1(\mathscr{N}))$ norm of the function in (60). Therefore

$$\begin{aligned} &\|[|p_\mu^2| B^*A - (p_\mu \square B)^*(p_\mu \square A)]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} \leqslant \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2 \|B^*\Theta(e^{it})^*\hat{\Theta}(\mu)A\|_{C_1(\mathscr{N})}\,dt \leqslant \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2 \|B\|_2 \|\hat{\Theta}(\mu)A\|_2\,dt = \|B\|_2 \|\hat{\Theta}(\mu)A\|_2. \end{aligned}$$

A computation analogous to (56) and using the hypothesis shows that $\|\hat{\Theta}(\mu)A\|_2 \leqslant \theta\|A\|_2$, and therefore (58) is proved. To prove (59), we notice that the function

$$e^{it} \to (1 - |\mu|^2)^{1/2} \left[\frac{\Theta(e^{it}) - \hat{\Theta}(\mu)}{e^{it} - \mu}\right] Aa$$

coincides with the projection onto $H^2(\mathscr{F}_*)$ of the function

$$h_{\mu,Aa}(e^{it}) = (e^{-it})\overline{p_\mu(e^{it})}\Theta(e^{it})Aa$$

(cf. the proof of Lemma 4.3). To complete the proof it suffices to obtain a sufficiently good upper bound on the L^1-norm of the function factored in (61), which we now do, using the Schwarz inequality:

$$\frac{1}{2\pi}\int_0^{2\pi} \| p_\mu(e^{it})|^2 B^*\hat{\Theta}(\mu)^*[\Theta(e^{it})A - \hat{\Theta}(\mu)A]\|_{C_1(\mathscr{N})}\,dt \leqslant$$

(62)

$$\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |p_\mu(e^{it})|^2 \|\hat{\Theta}(\mu)B\|_2^2\,dt\right)^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} \left\|(1-|\mu|^2)^{1/2}\left[\frac{\Theta(e^{it})-\hat{\Theta}(\mu)}{e^{it}-\mu}\right]A\right\|_2^2 dt\right)^{1/2}.$$

The first factor equals $\|\hat{\Theta}(\mu)B\|_2$ and a calculation like that in (56), using the hypothesis, shows that

$$\|\hat{\Theta}(\mu)B\|_2 \leqslant \theta\|B\|_2.$$

To estimate the second factor on the right hand side of (62), we choose an orthonormal basis $\{e_j\}_{j\in J}$ of $\mathscr{N}$, and we compute:

$$\frac{1}{2\pi}\int_0^{2\pi} \left\|(1-|\mu|^2)^{1/2}\left[\frac{\Theta(e^{it})-\hat{\Theta}(\mu)}{e^{it}-\mu}\right]A\right\|_2^2 dt =$$

$$= \sum_{j\in J}\frac{1}{2\pi}\int_0^{2\pi} \left\|(1-|\mu|^2)^{1/2}\left[\frac{\Theta(e^{it})-\hat{\Theta}(\mu)}{e^{it}-\mu}\right]Ae_j\right\|_{\mathscr{F}_*}^2 dt =$$

$$= \sum_{j\in J}\|P_{H^2(\mathscr{F}_*)}h_{\mu,Ae_j}\|_{L^2(\mathscr{F}_*)}^2 \leqslant \sum_{j\in J}\|h_{\mu,Ae_j}\|_{L^2(\mathscr{F}_*)}^2 \leqslant$$

$$\leqslant \sum_{j\in J}\|Ae_j\|^2 = \|A\|_2^2.$$

Thus the H_0^1-function in (61) has norm less than or equal to $\theta\|A\|_2\|B\|_2$, and (59) follows.

If σ is any measurable subset of $\mathbf{T}$ and $f\in L^1(C_1(\mathscr{N}))$, then we write $f\,|\,\sigma$ for the restriction of the function f to σ, so, of course, $f\,|\,\sigma \in L^1(\sigma, C_1(\mathscr{N}))$.

Lemmas 5.1 and 5.2 point the way to the version of [2, Lemma 1.2] appropriate to our context.

LEMMA 5.3. *Let σ be any measurable subset of $\mathbf{T}$ and let Λ be a subset of $\mathbf{D}$ that is dominating for σ. Then the closed absolutely convex hull of the set*

$$\{(|p_\mu^2|\,C)|\sigma : \mu\in\Lambda,\ C\in C_1(\mathscr{N}),\ \|C\|_1\leqslant 1\}$$

coincides with the unit ball in $L^1(\sigma, C_1(\mathscr{N}))$.

Proof. Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of finite-rank projections in $\mathcal{L}(\mathcal{N})$ converging strongly to $I_{\mathcal{N}}$, let $f \in L^1(\sigma, C_1(\mathcal{N}))$, and define the sequence $\{g_n\} \subset L^1(\sigma, C_1(\mathcal{N}))$ by setting

$$g_n(e^{it}) = P_n f(e^{it}) P_n, \quad e^{it} \in \sigma, \ n \in \mathbf{N}.$$

Obviously

$$\|g_n\|_{L^1(\sigma, C_1(\mathcal{N}))} \leqslant \|f\|_{L^1(\sigma, C_1(\mathcal{N}))}, \quad n \in \mathbf{N},$$

and it is immediate from the Lebesgue bounded convergence theorem and the fact that $\|P_n X P_n - X\|_1 \to 0$ for every X in $C_1(\mathcal{N})$ that $\|g_n - f\|_{L^1(\sigma, C_1(\mathcal{N}))} \to 0$. Since the g_n may be regarded as elements of $L^1(\sigma, C_1(P_n\mathcal{N}))$, it suffices to prove the lemma under the additional hypothesis that $\mathcal{N}$ is finite-dimensional. In this case it is known (and clear) that the dual space of $L^1(\sigma, C_1(\mathcal{N}))$ can be identified with $L^\infty(\sigma, \mathcal{L}(\mathcal{N}))$ under the pairing

$$\langle \Phi, f \rangle = \frac{1}{2\pi} \int_\sigma \operatorname{tr}(\Phi(e^{it}) f(e^{it}))\, dt, \quad \Phi \in L^\infty(\sigma, \mathcal{L}(\mathcal{N})),\ f \in L^1(\sigma, C_1(\mathcal{N})).$$

It is an easy consequence of the Hahn-Banach theorem (cf. [9, Proposition 2.8] that to complete the proof it is enough to show that

$$\|\Phi\|_{L^\infty(\sigma, \mathcal{L}(\mathcal{N}))} = \sup\{|\langle \Phi, (|p_\mu^2|C)|\sigma \rangle| : \mu \in \Lambda, C \in C_1(\mathcal{N}), \|C\|_1 \leqslant 1\}$$

for every Φ in $L^\infty(\sigma, \mathcal{L}(\mathcal{N}))$. We obviously have for every fixed C in $C_1(\mathcal{N})$

$$\langle \Phi, (|p_\mu^2|C)|\sigma \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |\mu|^2}{|e^{it} - \mu|^2} \operatorname{tr}(\Phi_1(e^{it})C)\, dt,$$

where the function Φ_1 coincides with Φ on σ and $\Phi_1 \equiv 0$ on $\mathbf{T} \setminus \sigma$. Thus $\langle \Phi, (|p_\mu^2|C)|\sigma \rangle$ coincides with the Poisson integral of the function $h : e^{it} \to \operatorname{tr}(\Phi_1(e^{it})C)$ evaluated at μ, that is, $\hat{h}(\mu)$. Since, by Fatou's theorem (cf. [21, p. 185]), $h(e^{it})$ coincides almost everywhere on $\mathbf{T}$ with the nontangential limit of $\hat{h}$, we see that

$$\begin{aligned} \sup\{|\langle \Phi, (|p_\mu^2|C)|\sigma \rangle| : \mu \in \Lambda\} &= \sup_{\mu \in \Lambda} |\hat{h}(\mu)| = \\ (63) \qquad &= \operatorname*{ess\,sup}_{\mathbf{T}} |h(e^{it})| = \operatorname*{ess\,sup}_{\sigma} |\operatorname{tr}(\Phi(e^{it})C)|. \end{aligned}$$

Therefore, to complete the proof, we show that

$$(64) \qquad \sup\{\operatorname*{ess\,sup}_{\sigma} |\operatorname{tr}(\Phi(e^{it})C)| : C \in C_1(\mathcal{N}), \|C\|_1 \leqslant 1\} = \|\Phi\|_{L^\infty(\sigma, \mathcal{L}(\mathcal{N}))}.$$

Since $|\operatorname{tr}(\Phi(e^{it})C)| \leqslant \|\Phi(e^{it})C\|_1 \leqslant \|\Phi(e^{it})\| \|C\|_1$, it is obvious that the left hand side of (64) is less than or equal to the right hand side. To prove the opposite inequality, we choose a countable dense set $\{C_n\}_{n=1}^{\infty}$ in the unit ball of $C_1(\mathcal{N})$, and for each $n \in \mathbf{N}$ we choose a subset E_n of σ of measure zero such that

$$\operatorname*{ess\,sup}_{\sigma} |\operatorname{tr}(\Phi(e^{it})C_n)| = \sup_{\sigma \setminus E_n} |\operatorname{tr}(\Phi(e^{it})C_n)|.$$

Thus the left side of (64) is greater than or equal to

$$\sup_{n\in N}\{\sup_{\sigma\setminus E_n}|\mathrm{tr}(\Phi(e^{it})C_n)|\} \geqslant \sup_{n\in N}\{\sup_{\sigma\setminus(\cup_n E_n)}|\mathrm{tr}(\Phi(e^{it})C_n)|\} =$$

$$= \sup_{\sigma\setminus(\cup_n E_n)}\{\sup_{n\in N}|\mathrm{tr}(\Phi(e^{it})C_n)|\} = \sup_{\sigma\setminus(\cup_n E_n)}\|\Phi(e^{it})\| \geqslant \|\Phi\|_{L^\infty(\sigma, \mathscr{L}(\mathscr{N}))},$$

where we have used the fact that $\mathscr{L}(\mathscr{N})$ is the dual space of $C_1(\mathscr{N})$ under the pairing $\langle A, B\rangle = \mathrm{tr}(AB)$; thus, the proof is complete.

REMARK 5.4. If the hypotheses of Lemma 5.3 are satisfied, one can deduce easily from either the conclusion and the proof of this lemma, or by using the fact that the operators of finite rank in the unit ball of $C_1(\mathscr{N})$ are dense (in $\|\cdot\|_1$) in the unit ball, that, given f in $L^1(\sigma, C_1(\mathscr{N}))$, there exist points $\mu_1, \ldots, \mu_m$ in A and finite-rank operators $C_1, \ldots, C_m$ in $C_1(\mathscr{N})$ such that

$$(65)\qquad \Big\|f - \sum_{j=1}^{m}(|p_{\mu_j}^2|C_j)|\sigma\Big\|_{L^1(\sigma, C_1(\mathscr{N}))} < \varepsilon, \quad \sum_{j=1}^{m}\|C_j\|_1 \leqslant \|f\|_{L^1(\sigma, C_1(\mathscr{N}))}.$$

Furthermore, if $\delta > 0$ and (65) is applied to the function $(1-\delta)f$ in place of f, hen we infer the existence of sequences $\mu_1, \ldots, \mu_m$ in A and finite rank operators $C_1, \ldots, C_m$ in $C_1(\mathscr{N})$ such that

$$(66)\qquad \begin{aligned} &\Big\|f - \sum_{j=1}^{m}(|p_{\mu_j}^2|C_j)|\sigma\Big\|_{L^1(\sigma, C_1(\mathscr{N}))} < \delta(1 + \|f\|_{L^1(\sigma, C_1(\mathscr{N}))}), \text{ and} \\ &\sum_{j=1}^{m}\|C_j\|_1 \leqslant (1-\delta)\|f\|_{L^1(\sigma, C_1(\mathscr{N}))}. \end{aligned}$$

From properties of the quotient map, we have the following obvious corollary of Lemma 5.3.

COROLLARY 5.5. *If $A \subset \mathbf{D}$ is a dominating set for $\mathbf{T}$, then the closed absolutely convex hull of the set*

$$\{[|p_\mu^2|C] : \mu \in A,\ C \in C_1(\mathscr{N}),\ \|C\|_1 \leqslant 1\} \subset (L^1/H_0^1)(C_1(\mathscr{N}))$$

coincides with the unit ball of $(L^1/H_0^1)(C_1(\mathscr{N}))$.

The following proposition is an analog, in our setting, of [2, Lemmas 2.1 and 2.3]. The reader who is familiar with the earlier work in this area ([9], for example) will note that unlike in [9], where $\sigma_{le}(T)$ played a dominant role, this proposition treats left and right spectral behavior symmetrically. It is also this proposition that allows us to avoid the hypothesis, in our main theorem, that either the sequence $\{T^n\}_{n=1}^\infty$ or the sequence $\{T^{*n}\}_{n=1}^\infty$ (for $T = S(\Theta)$) tends strongly to zero. We remark that it is an easy consequence of the polar decomposition and the spectral theorem that if A is any operator in $\mathscr{L}(\mathscr{H}, \mathscr{K})$ such that

$$\inf \sigma_e((A^*A)^{1/2}) \leqslant \theta < 1,$$

then for every $\varepsilon > 0$, there is an infinite dimensional subspace $\mathcal{M}_\varepsilon$ of $\mathcal{H}$ such that

$$\|Ax\| \leqslant (\theta + \varepsilon)\|x\|, \quad x \in \mathcal{M}_\varepsilon.$$

PROPOSITION 5.6. *Suppose $T = S(\Theta)$ is any completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$, $0 \leqslant \theta < 1$, and μ belongs to the set $L_\theta \cup R_\theta$ in* (12). *Then for every C in $C_1(\mathcal{N})$ and every positive ε, there exist sequences $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ (depending on μ and C) of $\mathcal{H}(\Theta)$-oriented functions in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ with the following properties:*

(i) $\| |p_\mu^2| C - Y_n^* X_n \|_{L^1(C_1(\mathcal{N}))} \leqslant (2\theta + \varepsilon)\|C\|_1$,

(ii) $\| [|p_\mu^2| C - Y_n^* X_n] \|_{(L^1/H_0^1)(C_1(\mathcal{N}))} \leqslant (\theta + \varepsilon)\|C\|_1$,

(iii) $\lim_n \|Z^* X_n\|_{L^1(C_1(\mathcal{N}))} = \lim_n \|Z^* Y_n\|_{L^1(C_1(\mathcal{N}))} = 0$

for every Z in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$, and

(iv) $\|X_n\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} \leqslant \|C\|_1^{1/2}$, $\quad \|Y_n\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} \leqslant \|C\|_1^{1/2}$

for all $n \in \mathbf{N}$.

Proof. We recall from (12) and (29) that $\mu \in R_\theta$ if and only if $\inf \sigma_e((\hat{\Theta})(\mu)\hat{\Theta}(\mu)^*)^{1/2}) \leqslant \theta$ and since $\theta < 1$, it follows from the remark following (27) that $\dim \mathcal{F}_* = \aleph_0$. Thus, if $\mu \in R_\theta$, there exists an infinite dimensional subspace $\mathcal{G}_*$ of $\mathcal{F}_*$ such that $\|\hat{\Theta}(\mu)^* x_*\| \leqslant (\theta + \varepsilon/2)\|x_*\|$ for all $x_* \in \mathcal{G}_*$. We choose a sequence $\{V_n\}_{n=1}^\infty$ of isometries in $\mathcal{L}(\mathcal{N}, \mathcal{G}_*)$ with pairwise orthogonal ranges. If $C = UP$ is the polar decomposition of C ($P = (C^*C)^{1/2}$), we define $A_{*n} = V_n P^{1/2}$ and $B_{*n} = V_n P^{1/2} U^*$ for $n \in \mathbf{N}$. Clearly then $C = B_{*n}^* A_{*n}$ and $\|B_{*n}\|_2 = \|A_{*n}\|_2 = \|C\|_1^{1/2}$ for $n \in \mathbf{N}$. We now define $X_n = p_\mu \# A_{*n}$ and $Y_n = p_\mu \# B_{*n}$ for $n \in \mathbf{N}$. Since A_{*n} and B_{*n} have ranges contained in $\mathcal{G}_*$, we have

$$\|\hat{\Theta}(\mu)^* A_{*n} x\| \leqslant \left(\theta + \frac{\varepsilon}{2}\right)\|A_{*n} x\|$$

and

$$\|\hat{\Theta}(\mu)^* B_{*n} x\| \leqslant \left(\theta + \frac{\varepsilon}{2}\right)\|B_{*n} x\|, \quad x \in \mathcal{N},\ n \in \mathbf{N},$$

and therefore (i) and (ii) follow immediately from Lemma 5.1. That (iv) holds is a consequence of Lemma 4.4. To prove (iii), we observe that since the ranges of the isometries V_n are pairwise orthogonal, the sequences $\{A_{*n}^*\}_{n=1}^\infty$ and $\{B_{*n}^*\}_{n=1}^\infty$ converge to zero in the strong operator topology on $\mathcal{L}(\mathcal{F}_*, \mathcal{N})$, and (iii) follows immediately from Lemma 4.6.

Now we turn to the case that $\mu \in L_\theta$. As above, there exists an infinite dimensional subspace $\mathcal{G}$ of $\mathcal{F}$ such that $\|\hat{\Theta}(\mu)x\| \leqslant \left(\theta + \frac{\varepsilon}{2}\right)\|x\|$ for all x in $\mathcal{G}$. We choose a sequence $\{W_n\}_{n=1}^\infty$ of isometries in $\mathcal{L}(\mathcal{N}, \mathcal{G})$ with pairwise orthogonal ranges.

In this case we set $A_n = W_n P^{1/2}$, $B_n = W_n P^{1/2} U^*$, $X_n = p_\mu \square A_n$, and $Y_n = p_\mu \square B_n$ for all $n \in \mathbf{N}$, and, as above, (i) — (iv) follow immediately from the definitions and Lemmas 5.2, 4.5, and 4.6.

6. THE FACTORIZATION THEOREMS

In this section, we finally prove our basic factorization theorems and derive several important corollaries. The following proposition is the counterpart in our development of [2, Lemma 2.4].

PROPOSITION 6.1. *Suppose $\varepsilon > 0$, σ is a measurable subset of* $\mathbf{T}$, *and $T = S(\Theta)$ is any completely nonunitary contraction such that for some θ, $0 \leqslant \theta < 1$, $T \in (\mathrm{BCP})_{\theta,\sigma}$. Suppose, moreover, that X and Y are $\mathscr{H}(\Theta)$-oriented functions in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ and $F \in L^1(\sigma, C_1(\mathscr{N}))$. Then there exist $\mathscr{H}(\Theta)$-oriented functions X' and Y' in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ such that*

$$\|F - (Y'^*X')|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))} \leqslant (2\theta + \varepsilon)\|F - (Y^*X)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))},$$

$$\|X - X'\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathscr{N}))}, \tag{67}$$

and

$$\|Y - Y'\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathscr{N}))}.$$

Proof. If $F(e^{it}) = (Y^*X)(e^{it})$ almost everywhere on σ, we can take $X' = X$ and $Y' = Y$. Thus we may suppose that $\omega = \|F - (Y^*X)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))} \neq 0$. Choose $\delta > 0$ such that $\delta(1 + \omega) < \varepsilon\omega/2$. It follows from Remark 5.5 that there exist points $\mu_1, \ldots, \mu_m$ in $L_\theta \cup R_\theta$ and operators $C_1, \ldots, C_m$ in $C_1(\mathscr{N})$ such that

$$\|F - (Y^*X)|\sigma - \sum_{j=1}^{m} (|p^2_{\mu_j}|C_j)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))} \leqslant \varepsilon\omega/2 \tag{68}$$

and

$$\sum_{j=1}^{m} \|C_j\|_1 \leqslant (1 - \delta)\omega. \tag{69}$$

We choose now $\eta > 0$ such that $m(m-1)\eta < \delta\omega$ and $[\omega + m(m+1)]\eta < \varepsilon\omega/2$, and define by a finite induction procedure $\mathscr{H}(\Theta)$-oriented functions $R_1, \ldots, R_m$ and $S_1, \ldots, S_m$ in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ such that the following inequalities are satisfied:

$$\||p^2_{\mu_j}|C_j - S_j^* R_j\|_{L^1(C_1(\mathscr{N}))} < (2\theta + \eta)\|C_j\|_1, \quad 1 \leqslant j \leqslant m, \tag{70}$$

$$\|R_j\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|C_j\|_1^{1/2}, \quad \|S_j\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|C_j\|_1^{1/2}, \quad 1 \leqslant j \leqslant m, \tag{71}$$

$$\begin{gathered} \|S_j^* X\|_{L^1(C_1(\mathscr{N}))} < \eta, \quad \|Y^* R_j\|_{L^1(C_1(\mathscr{N}))} < \eta, \quad \|S_j^* R_k\| < \eta, \quad 1 \leqslant j, k \leqslant m, j \neq k, \\ \|R_j^* R_k\|_{L^1(C_1(\mathscr{N}))} < \eta, \quad \|S_j^* S_k\|_{L^1(C_1(\mathscr{N}))} < \eta, \quad 1 \leqslant j, k \leqslant m, j \neq k. \end{gathered} \tag{72}$$

Indeed, suppose that $1 \leqslant p \leqslant m$ and that the functions R_j and S_j have been chosen to satisfy (70), (71), and (72) for $j < p$. By Proposition 5.6 (with η in place of ε), there exist sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ of $\mathscr{H}(\Theta)$-valued functions in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ satisfying conditions (i) — (iv) of that proposition for $\mu = \mu_p$ and $C = C_p$. It follows from condition (iii) that if we set $R_p = X_n$ and $S_p = Y_n$ for n sufficiently large, then the inequalities (72) will be satisfied for $1 \leqslant j, k \leqslant p$. Since for this choice of R_p and S_p, (70) and (71) follow for $j = p$ from conditions (i) and (iv), the induction is complete.

We define now $X' = X + \sum_{j=1}^{m} R_j$, $Y' = Y + \sum_{j=1}^{m} S_j$, and note first that

$$\|X' - X\|^2_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} = \|(X' - X)^*(X' - X)\|_{L^1(C_1(\mathscr{N}))} \leqslant$$

$$\leqslant \sum_{j=1}^{m} \|R_j^* R_j\|_{L^1(C_1(\mathscr{N}))} + \sum_{j \neq k} \|R_j^* R_k\|_{L^1(C_1(\mathscr{N}))} \leqslant$$

$$\leqslant \sum_{j=1}^{m} \|R_j\|^2_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} + m(m-1)\eta \leqslant$$

$$\leqslant \sum_{j=1}^{m} \|C_j\|_1 + m(m-1)\eta \leqslant (1-\delta)\omega + m(m-1)\eta,$$

by virtue of (72), (71), and (69). Therefore

$$\|X' - X\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \omega^{1/2}$$

by virtue of the way η was chosen, and exactly the same computation shows that also $\|Y' - Y\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \omega^{1/2}$. On the other hand,

$$\|F - (Y'^* X')|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))} \leqslant \|F - (Y^* X)|\sigma - \sum_{j=1}^{m} (|p^2_{\mu_j}| C_j)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))} +$$

$$+ \sum_{j=1}^{m} \|p^2_{\mu_j}| C_j - S_j^* R_j\|_{L^1(C_1(\mathscr{N}))} + \sum_{j=1}^{m} \|Y^* R_j\|_{L^1(C_1(\mathscr{N}))} +$$

$$+ \sum_{j=1}^{m} \|S_j^* X\|_{L^1(C_1(\mathscr{N}))} + \sum_{j \neq k} \|S_j^* R_k\|_{L^1(C_1(\mathscr{N}))} \leqslant$$

$$\leqslant \varepsilon\omega/2 + (2\theta + \eta) \sum_{j=1}^{m} \|C_j\|_1 + m\eta + m\eta + m(m-1)\eta \leqslant$$

$$\leqslant \varepsilon\omega/2 + (2\theta + \eta)\omega + m(m+1)\eta$$

by virtue of (68), (70), (72), and (69), and therefore the inequalities (67) follow from our choice of η.

The proof of the following proposition is almost identical to that of the above proposition, but it makes use of the estimate (ii) in Proposition 5.6 instead of (i).

PROPOSITION 6.2. *Suppose* $\varepsilon > 0$ *and* $T = S(\Theta)$ *belongs to the class* $(\mathrm{BCP})_\theta$ *for some* θ *satisfying* $0 \leqslant \theta < 1$. *Suppose, moreover, that* X *and* Y *are* $\mathscr{H}(\Theta)$*-oriented functions in* $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ *and* $[F] \in (L^1/H_0^1)(C_1(\mathscr{N}))$. *Then there exist* $\mathscr{H}(\Theta)$*-oriented functions* X' *and* Y' *in* $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ *such that*

$$\|[F] - [Y'^*X']\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} \leqslant (\theta + \varepsilon)\|[F] - [Y^*X]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))},$$

$$\|X' - X\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|[F] - [Y^*X]\|^{1/2}_{(L^1/H_0^1)(C_1(\mathscr{N}))},$$

and

$$\|Y' - Y\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant \|[F] - [Y^*X]\|^{1/2}_{(L^1/H_0^1)(C_1(\mathscr{N}))}.$$

Proof. As in the proof of Proposition 6.1, we may suppose that $\omega = \|[F] - [Y^*X]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} \neq 0$. Choose $\delta > 0$ such that $\delta^{1/2}(1 + (1 + \delta^{1/2})\omega) < \varepsilon\omega/2$. Next choose a function $G \in L^1(C_1(\mathscr{N}))$ such that $[G] = [F] - [Y^*X]$ and $\|G\|_{L^1(C_1(\mathscr{N}))} < (1 + \delta^{1/2})\omega$. It follows now from Remark 5.4 (with $\delta^{1/2}$ in place of δ) that there exist points $\mu_1, \ldots, \mu_m$ in $L_\theta \cup R_\theta$ and operators $C_1, \ldots, C_m$ in $C_1(\mathscr{N})$ such that

$$\left\| G - \sum_{j=1}^{m} |p_{\mu_j}^2| C_j \right\|_{L^1(C_1(\mathscr{N}))} < \delta^{1/2}(1 + \|G\|_{L^1(C_1(\mathscr{N}))})$$

and

$$\sum_{j=1}^{m} \|C_j\|_1 \leqslant (1 - \delta^{1/2})\|G\|_{L^1(C_1(\mathscr{N}))} \leqslant (1 - \delta)\omega.$$

It follows from our choice of δ that we have

$$\left\| [F] - [Y^*X] - \sum_{j=1}^{m} [|p_{\mu_j}^2| C_j] \right\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} < \varepsilon\omega/2.$$

We have therefore obtained the analogs of (68) and (69) for the space $(L^1/H_0^1)(C_1(\mathscr{N}))$, and from this point the proof proceeds like that of the preceding proposition, with the only change being that (70) is replaced by the estimate

$$\|[|p_{\mu_j}^2| C_j] - [S_j^* R_j]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))} < (\theta + \eta)\|C_j\|_1.$$

Thus no more need be said about the proof.

It will be worthwhile for future applications to know that the X' and Y' that arose in the conclusion of Proposition 6.1 can be controlled on $\mathbf{T} \setminus \sigma$, and that is the purpose of the following corollary. The notation $\|f\|_{L^i(\tau,\mathscr{E})}$, where $\tau \subset \mathbf{T}$, means, of course, $\|f|\tau\|_{L^i(\tau,\mathscr{E})}$.

COROLLARY 6.3. *The functions X' and Y' provided by Proposition 6.1 can be chosen such that*

$$\|X' - X\|_{L^2(\mathbf{T}\setminus\sigma,\, C_2(\mathscr{N},\, \mathscr{F}_*\oplus\mathscr{F}))} < \varepsilon,$$

and

$$\|Y' - Y\|_{L^2(\mathbf{T}\setminus\sigma,\, C_2(\mathscr{N},\, \mathscr{F}_*\oplus\mathscr{F}))} < \varepsilon.$$

Proof. If $A \subset \mathbf{D}$ is dominating for the subset $\sigma \subset \mathbf{T}$ and $\alpha > 0$, then the set $A_\alpha = \{\mu \in A : \||p_\mu^2|\|_{L^1(\mathbf{T}\setminus\sigma)} < \alpha\}$ is also dominating for σ. Indeed, it suffices to consider the function

$$\hat{u}(\mu) = \frac{1}{2\pi} \int_{\mathbf{T}\setminus\sigma} |p_\mu(e^{it})|^2 \, dt, \quad \mu \in \mathbf{D}, \tag{73}$$

which is the Poisson integral of the characteristic function $u = \chi_{\mathbf{T}\setminus\sigma}$ of $\mathbf{T}\setminus\sigma$. Thus $\chi_{\mathbf{T}\setminus\sigma}$ coincides almost everywhere with the nontangential limit of $\hat{u}$. If $e^{it} \in \sigma$, $0 = \chi_{\mathbf{T}\setminus\sigma}(e^{it})$ is the nontangential limit of $\hat{u}$ at e^{it}, and $\{\mu_n\}_{n=1}^\infty$ is a sequence of points in A converging nontangentially to e^{it}, then $|\hat{u}(\mu_n)| < \alpha$ for n sufficiently arge, so $\mu_n \in A_\alpha$ for such n. Thus, in the proof of Proposition 6.1, the points $\mu_1, \ldots, \mu_m$ can be chosen so that $\||p^2_{\mu_j}|\|_{L^1(\mathbf{T}\setminus\sigma)} < \alpha$. It is easily seen from (40), (43), and the definition of the functions $\{R_j\}_{j=1}^m$ in that same proof that

$$\|R_j(e^{it})\|_2 \leqslant 3|p_{\mu_j}(e^{it})|\, \|C_j\|_1^{1/2}, \quad e^{it} \in \mathbf{T},\ 1 \leqslant j \leqslant m.$$

Therefore

$$\|R_j^*R_j\|_{L^1(\mathbf{T}\setminus\sigma,\, C_1(\mathscr{N}))} \leqslant 9\||p^2_{\mu_j}|\|_{L^1(\mathbf{T}\setminus\sigma)}\|C_j\|_1 \leqslant 9\alpha\|C_j\|_1, \quad 1 \leqslant j \leqslant m, \tag{74}$$

so we obtain from (74), (72), and (69) the following estimate:

$$\begin{aligned}\|X' - X\|^2_{L^2(\mathbf{T}\setminus\sigma,\, C_2(\mathscr{N},\, \mathscr{F}_*\oplus\mathscr{F}))} &= \|(X'-X)^*(X'-X)\|_{L^1(\mathbf{T}\setminus\sigma,\, C_1(\mathscr{N}))} \leqslant \\ &\leqslant \sum_{j=1}^m \|R_j^*R_j\|_{L^1(\mathbf{T}\setminus\sigma,\, C_1(\mathscr{N}))} + \sum_{j\neq k} \|R_j^*R_k\|_{L^1(C_1(\mathscr{N}))} \leqslant \\ &\leqslant 9\alpha \sum_{j=1}^m \|C_j\|_1 + m(m-1)\eta \leqslant 9\,\alpha\omega + m(m-1)\eta,\end{aligned} \tag{75}$$

and the rightmost member in (75) is less than ε if α and η are chosen sufficiently small. Since a similar estimate holds for $\|Y' - Y\|_{L^2(\mathbf{T}\setminus\sigma,\, C_2(\mathscr{N},\, \mathscr{F}_*\oplus\mathscr{F}))}$, the proof is complete.

REMARK 6.4. It will be important for future applications of the techniques in this paper to notice that in Propositions 6.1 and 6.2, if one were given initially a finite set $Z_1, \ldots, Z_k$ in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ and $\eta > 0$, then X' and Y' defined in those proofs could have been chosen so that

$$\|Z_i^*(X' - X)\|_{L^1(C_1(\mathcal{N}))} < \eta \quad \text{and} \quad \|Z_i^*(Y' - Y)\|_{L^1(C_1(\mathcal{N}))} < \eta, \quad 1 \leqslant i \leqslant k.$$

Indeed, it follows from Proposition 4.6 that the functions $R_1, \ldots, R_m$ and $S_1, \ldots, S_m$ can be chosen so that

$$\|Z_i^* R_j\|_{L^1(C_1(\mathcal{N}))} < \eta/m, \quad \|Z_i^* S_j\|_{L^1(C_1(\mathcal{N}))} < \eta/m, \quad 1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant m.$$

We are finally prepared to prove our basic matricial factorization theorems. Note that with the choice $\sigma = \mathbf{T}$ the first theorem applies to $(\mathrm{BCP})_\theta$-operators.

THEOREM 6.5. *Suppose* $0 \leqslant \theta < 1/2$, $0 < \varepsilon < 1 - 2\theta$, *and* σ *is a measurable subset of* $\mathbf{T}$. *Suppose also that* $T = S(\Theta)$ *is a completely nonunitary contraction such that* $T \in (\mathrm{BCP})_{\theta,\sigma}$, $F \in L^1(\sigma, C_1(\mathcal{N}))$, *and* X *and* Y *are given* $\mathcal{H}(\Theta)$*-oriented functions in* $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$. *Then there exist* $\mathcal{H}(\Theta)$*-oriented functions* X' *and* Y' *in* $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ *such that* $F(e^{it}) = Y'(e^{it})^* X'(e^{it})$ *almost everywhere on* σ, *while*

$$(76)\qquad \begin{aligned} \|X' - X\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} &\leqslant (1-(2\theta+\varepsilon)^{1/2})^{-1} \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathcal{N}))}, \\ \|Y' - Y\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} &\leqslant (1-(2\theta+\varepsilon)^{1/2})^{-1} \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathcal{N}))}. \end{aligned}$$

Furthermore, the set of all $\mathcal{H}(\Theta)$*-oriented functions* X'' *in* $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ *for which there exists an* $\mathcal{H}(\Theta)$*-oriented function* Y'' *in* $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ *satisfying* $F(e^{it}) = Y''(e^{it})^* X''(e^{it})$ *almost everywhere on* σ *is dense in the subspace of all* $\mathcal{H}(\Theta)$*-oriented functions.*

Proof. We shall construct sequences $\{X_n\}_{n=0}^\infty$ and $\{Y_n\}_{n=0}^\infty$ of $\mathcal{H}(\Theta)$-oriented functions in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ such that $X_0 = X$, $Y_0 = Y$, and

$$(77)\qquad \|F - (Y_n^* X_n)|\sigma\|_{L^1(\sigma, C_1(\mathcal{N}))} \leqslant (2\theta + \varepsilon)^n \|F - (Y^*X)|\sigma\|_{L^1(\sigma, C_1(\mathcal{N}))},$$

$$(78)\qquad \begin{aligned} \|X_{n+1} - X_n\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} &\leqslant (2\theta + \varepsilon)^{n/2} \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathcal{N}))}, \quad \text{and} \\ \|Y_{n+1} - Y_n\|_{L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} &\leqslant (2\theta + \varepsilon)^{n/2} \|F - (Y^*X)|\sigma\|^{1/2}_{L^1(\sigma, C_1(\mathcal{N}))} \end{aligned}$$

for all $n \geqslant 0$. Indeed, the inequality (77) is obviously satisfied for $n = 0$; suppose now that $\mathscr{H}(\Theta)$-oriented functions $\{X_n\}_{n=0}^k$ and $\{Y_n\}_{n=0}^k$ have been chosen so that (77) is satisfied for $0 \leqslant n \leqslant k$ and (78) is satisfied for $0 \leqslant n \leqslant k-1$. An application of Proposition 6.1 with $F-(Y_k^* X_k)|\sigma$, X_k, Y_k in place of F, X, Y yields $\mathscr{H}(\Theta)$-oriented functions X_{k+1} and Y_{k+1} such that (77) is valid for $n=k+1$ and (78) is valid for $n=k$. Thus, by induction, sequences $\{X_n\}_{n=0}^\infty$ and $\{Y_n\}_{n=0}^\infty$ exist having the desired properties. It follows from (78) that these sequences are Cauchy in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$, and thus converge in norm to functions X' and Y'. respectively, which must be $\mathscr{H}(\Theta)$-oriented by Lemma 3.1. It is obvious from the continuity of the product $(X, Y) \to Y^*X$ and (76) that $F=(Y'^*X')|\sigma$ in $L^1(\sigma, C_1(\mathscr{N}))$. To see that (76) is satisfied, we compute

$$\|X'-X\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} = \left\| \sum_{n=0}^{\infty} (X_{n+1}-X_n) \right\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant$$

$$\leqslant \left[\sum_{n=0}^{\infty} (2\theta+\varepsilon)^{n/2} \right] \|F-(Y^*X)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))}^{1/2} \leqslant$$

$$\leqslant (1-(2\theta+\varepsilon)^{1/2})^{-1} \|F-(Y^*X)|\sigma\|_{L^1(\sigma, C_1(\mathscr{N}))}^{1/2}.$$

A similar computation for $\|Y'-Y\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))}$ completes the proof of (76).

To prove the last statement of the theorem, let X^0 be an arbitrary $\mathscr{H}(\Theta)$-oriented function. An application of what has already been proved with $X=\tau X^0$, $\tau > 0$, and $Y=0$ yields the existence of $\mathscr{H}(\Theta)$-oriented functions X_τ and Y_τ such that $F(e^{it}) = Y_\tau(e^{it})^* X_\tau(e^{it})$ almost everywhere on σ and such that

$$\|X_\tau - \tau X^0\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant [1-(2\theta+\varepsilon)^{1/2}]^{-1} \|F\|_{L^1(\sigma, C_1(\mathscr{N}))}^{1/2}.$$

Thus

$$\lim_{\tau\to\infty} \|\tau^{-1}X_\tau - X^0\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} = 0,$$

and since $F=((\tau Y_\tau)^*(\tau^{-1}X_\tau))|\sigma$ in $L^1(\sigma, C_1(\mathscr{N}))$, the proof is complete.

The following theorem follows from Proposition 6.2 exactly as the preceding one followed from Proposition 6.1, so no proof need be given.

THEOREM 6.6. *Suppose* $0 \leqslant \theta < 1$, $0 < \varepsilon < 1-\theta$, *and* $T=S(\Theta)$ *is an operator in* $(\mathrm{BCP})_\theta$. *If* $[F] \in (L^1/H_0^1)(C_1(\mathscr{N}))$ *and* X, Y *are* $\mathscr{H}(\Theta)$*-oriented functions in* $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$, *then there exist* $\mathscr{H}(\Theta)$*-oriented functions* X' *and* Y' *in* $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ *such that* $[F]=[Y'^*X']$ *and*

$$\|X'-X\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant [1-(\theta+\varepsilon)^{1/2}]^{-1} \|[F]-[Y^*X]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))}^{1/2},$$

(79)

$$\|Y'-Y\|_{L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))} \leqslant [1-(\theta+\varepsilon)^{1/2}]^{-1} \|[F]-[Y^*X]\|_{(L^1/H_0^1)(C_1(\mathscr{N}))}^{1/2}.$$

Furthermore, the set of all $\mathcal{H}(\Theta)$-oriented functions X'' in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ for which there exists an $\mathcal{H}(\Theta)$-oriented function Y'' in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ satisfying $[F] = [Y''^* X'']$ is dense in the subspace of all $\mathcal{H}(\Theta)$-oriented functions.

REMARK 6.7. It is important for future applications to observe that if one were given initially a finite set $Z_1, \ldots, Z_k$ of functions in $L^2(C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))$ in Theorem 6.5 or 6.6, then the resulting functions X' and Y' could be chosen to have the additional properties

$$\|Z_i^*(X' - X)\|_{L^1(C_1(\mathcal{N}))} < \varepsilon, \quad 1 \leqslant i \leqslant k,$$

and

$$\|Z_i^*(Y' - Y)\|_{L^1(C_1(\mathcal{N}))} < \varepsilon, \quad 1 \leqslant i \leqslant k.$$

To see this, note from Remark 6.4 that we may impose the additional conditions

$$\|Z_j^*(X_{n+1} - X_n)\|_{L^1(C_1(\mathcal{N}))} < 2^{-n-1}\varepsilon, \quad 1 \leqslant i \leqslant k,$$

and

$$\|Z_j^*(Y_{n+1} - Y_n)\|_{L^1(C_1(\mathcal{N}))} < 2^{-n-1}\varepsilon, \quad 1 \leqslant i \leqslant k,$$

in the proofs of Theorem 6.5 and 6.6.

REMARK 6.8. It is also useful for future applications of Theorem 6.5 to observe that the resulting functions X' and Y' can be chosen to have the additional properties

$$\|X' - X\|_{L^2(\mathbf{T}\setminus\sigma, C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} < \varepsilon,$$

and

$$\|Y' - Y\|_{L^2(\mathbf{T}\setminus\sigma, C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} < \varepsilon.$$

To see this, note from Corollary 6.3 that we may impose the additional conditions

$$\|X_{n+1} - X_n\|_{L^2(\mathbf{T}\setminus\sigma, C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} < 2^{-n-1}\varepsilon,$$

and

$$\|Y_{n+1} - Y_n\|_{L^2(\mathbf{T}\setminus\sigma, C_2(\mathcal{N}, \mathcal{F}_* \oplus \mathcal{F}))} < 2^{-n-1}\varepsilon$$

in the proof of Theorem 6.5.

The following corollary shows, in particular, that when $T = S(\Theta) \in (\mathrm{BCP})_\theta$ for some θ, $0 \leqslant \theta < 1/2$, one can solve finite systems of simultaneous equations of the form (8).

COROLLARY 6.9. *Suppose $0 \leqslant \theta < 1/2$, $0 < \alpha$, $n \in \mathbf{N}$, and σ is a measurable subset of $\mathbf{T}$. Suppose also that $T = S(\Theta)$ is a completely nonunitary contraction such that $T \in (\mathrm{BCP})_{\theta,\sigma}$. Let $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^n$ be given sequences of vectors in $\mathscr{H}(\Theta)$ and $\{f_{ij}\}_{i,j=1}^n$ be a doubly indexed sequence of functions in $L^1(\sigma)$ satisfying*

$$\|f_{ij} - (x_i \cdot y_j)|\sigma\|_{L^1(\sigma)} < \alpha, \quad 1 \leqslant i, j \leqslant n. \tag{80}$$

Then there exist sequences $\{x_i'\}_{i=1}^n$ and $\{y_j'\}_{j=1}^n$ in $\mathscr{H}(\Theta)$ such that for almost every $e^{it} \in \sigma$,

$$f_{ij}(e^{it}) = (x_i' \cdot y_j')(e^{it}), \quad 1 \leqslant i, j \leqslant n, \tag{81}$$

and

$$\|x_i - x_i'\| \leqslant (1 - (2\theta)^{1/2})^{-1} n\alpha^{1/2}, \quad \|y_j - y_j'\| \leqslant (1 - (2\theta)^{1/2})^{-1} n\alpha^{1/2}, \quad 1 \leqslant i, j \leqslant n. \tag{82}$$

Proof. Let $\mathscr{N}$ be a Hilbert space with orthonormal basis $\{e_j\}_{j=1}^n$, and define functions X, Y in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ and F in $L^1(C_1(\mathscr{N}))$ by setting, for $e^{it} \in \mathbf{T}$,

$$X(e^{it})e_j = x_j(e^{it}), \quad Y(e^{it})e_j = y_j(e^{it}), \quad 1 \leqslant j \leqslant n,$$

and

$$F(e^{it})e_j = \sum_{k=1}^n f_{jk}(e^{it})e_k, \quad 1 \leqslant j, k \leqslant n.$$

The computation

$$\langle (Y^*X)(e^{it})e_j, e_m\rangle = \langle X(e^{it})e_j, Y(e^{it})e_m\rangle = \langle x_j(e^{it}), y_m(e^{it})\rangle = (x_j \cdot y_m)(e^{it}) =$$

$$= \left\langle \sum_{k=1}^n (x_j \cdot y_k)(e^{it})e_k, e_m \right\rangle, \quad e^{it} \in \mathbf{T}, \quad 1 \leqslant j, m \leqslant n,$$

proves that

$$(Y^*X)(e^{it})e_j = \sum_{k=1}^n (x_j \cdot y_k)(e^{it})e_k$$

almost everywhere on $\mathbf{T}$, and therefore that

$$(F - (Y^*X)|\sigma)(e^{it})e_j = \sum_{k=1}^n [f_{jk}(e^{it}) - (x_j \cdot y_k)(e^{it})]e_k, \quad 1 \leqslant j, k \leqslant n, \tag{83}$$

almost everywhere on σ. Thus, using the fact [11, p. 111] that the trace-norm of a matrixis less than or equal to the sum of the moduli of the matrix entries, we obtain from (83) and (80) the upper bound

$$\|F - (Y^*X)\sigma\|_{L^1(\sigma, C_1(\mathcal{N}))} = \frac{1}{2\pi}\int_\sigma \|(F - (Y^*X)\sigma)(e^{it})\|_1 \, dt \leqslant$$

(84)

$$\leqslant \frac{1}{2\pi}\int_\sigma \sum_{j,k=1}^{n} |f_{jk}(e^{it}) - (x_j \cdot y_k)(e^{it})| \, dt = \sum_{j,k=1}^{n} \|f_{jk} - (x_j \cdot y_k)\sigma\|_{L^1(\sigma)} < n^2\alpha' < n^2\alpha,$$

where $\alpha' < \alpha$ is chosen sufficiently close to α. Consequently, given $\varepsilon > 0$, it follows from Theorem 6.5 that there exist $\mathscr{H}(\Theta)$-oriented functions X' and Y' in $L^2(C_2(\mathcal{N}, \mathscr{F}_* \oplus \mathscr{F}))$ satisfying (76) and $F(e^{it}) = Y'(e^{it})^* X'(e^{it})$ almost everywhere on σ. Also, (76) combined with (84) yields

$$\|X' - X\|_{L^2(C_2(\mathcal{N}, \mathscr{F}_* \oplus \mathscr{F}))} (1 - (2\theta + \varepsilon)^{1/2})^{-1} n(\alpha')^{1/2} \tag{85}$$

and the same bound for $\|Y' - Y\|_{L^2(C_2(\mathcal{N}, \mathscr{F}_* \oplus \mathscr{F}))}$. If we define

$$x_j'(e^{it}) = X'(e^{it})e_j, \quad y_j'(e^{it}) = Y'(e^{it})e_j, \quad e^{it} \in \mathbf{T}, \ 1 \leqslant j \leqslant n,$$

then it follows from (83) (with X and Y replaced by X' and Y') that (81) is valid almost everywhere on σ, and the corollary follows because

$$(1 - (2\theta + \varepsilon)^{1/2})^{-1} n(\alpha')^{1/2} < (1 - (2\theta)^{1/2})^{-1} n\alpha^{1/2}$$

if ε is chosen small enough.

REMARK 6.10. If follows easily from Remark 6.8 that given $\varepsilon > 0$, the sequences $\{x_i'\}_{i=1}^n$ and $\{y_j'\}_{j=1}^n$ of vectors appearing in (81) and (82) can be chosen so that $\|x_i'\|_{L^2(\mathbf{T}\setminus\sigma, \mathscr{F}_* \oplus \mathscr{F})} < \varepsilon$, $1 \leqslant i \leqslant n$, and similarly for the $\|y_j'\|_{L^2(\mathbf{T}\setminus\sigma, \mathscr{F}_* \oplus \mathscr{F})}$.

The proof of the following corollary is almost identical to the proof of Corollary 6.9, except that it uses Theorem 6.6 instead of Theorem 6.5, and is therefore omitted.

COROLLARY 6.11. *Suppose* $0 \leqslant \theta < 1$, $0 < \alpha$, $n \in \mathbf{N}$, *and* $T = S(\Theta) \in (\mathrm{BCP})_\theta$. *Let* $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^n$ *be given sequences of vectors in* $\mathscr{H}(\Theta)$ *and* $\{[f_{ij}]\}_{i,j=1}^n$ *be a doubly indexed sequence of elements of* L^1/H_0^1 *satisfying*

$$\|[f_{ij}] - [x_i \cdot y_j]\|_{L^1/H_0^1} < \alpha, \quad 1 \leqslant i, j \leqslant n.$$

Then there exist sequences $\{x_i'\}_{i=1}^n$ *and* $\{y_j'\}_{j=1}^n$ *in* $\mathscr{H}(\Theta)$ *such that*

$$[f_{ij}] = [x_i' \cdot y_j'], \quad 1 \leqslant i, j \leqslant n,$$

and

$$\|x_i - x_i'\| < (1 - \theta^{1/2})^{-1} n\alpha^{1/2}, \quad \|y_j - y_j'\| < (1 - \theta^{1/2})^{-1} n\alpha^{1/2}, \quad 1 \leqslant i, j \leqslant n.$$

The following corollary shows, as promised in the introduction, that for $(\mathrm{BCP})_\theta$-operators T, the functional calculus Φ_T in (2) is an isometry, i.e., $T \in \mathbf{A}$.

COROLLARY 6.12. *Let T be a* $(\mathrm{BCP})_\theta$ *operator for some* θ, $0 \leqslant \theta < 1$. *Then the maps* $\Phi_T \colon H^\infty \to \mathscr{A}_T$ *and* $\varphi_T \colon Q_T \to L^1/H_0^1$ *are isometric and surjective.*

Proof. As noted in the introduction, it suffices to show that Φ_T is isometric, and we may suppose that $T = S(\Theta)$ is a model operator. It follows from the preceding corollary (with $n = 1$) that for every μ in $\mathbf{D}$, there exist vectors x_μ and y_μ in $\mathscr{H}(\Theta)$ with the property that $[x_\mu \cdot y_\mu] = [|p_\mu^2|]$. Thus, from (2), (7), (35), and the Poisson integral formula, we obtain

$$\begin{aligned} &\langle f(T)x_\mu, y_\mu\rangle = \langle f(T), [x_\mu \otimes y_\mu]\rangle = \langle \Phi_T(f), [x_\mu \otimes y_\mu]\rangle = \\ (86) \qquad &= \langle f, \varphi_T([x_\mu \otimes y_\mu])\rangle = \langle f, [x_\mu \cdot y_\mu]\rangle = \langle f, |p_\mu^2|\rangle = \hat{f}(\mu), \quad f \in H^\infty. \end{aligned}$$

If we apply (86) to the function f^n, we obtain

$$|\hat{f}(\mu)|^n \leqslant \|f(T)\|^n \|x_\mu\| \|y_\mu\|, \quad f \in H^\infty, \ n \in \mathbf{N},$$

$$|\hat{f}(\mu)| \leqslant \|f(T)\| \|x_\mu\|^{1/n} \|y_\mu\|^{1/n}, \quad f \in H^\infty, \ n \in \mathbf{N},$$

and hence $\|f\|_\infty \leqslant \|f(T)\|$, which completes the proof.

The following corollary, which is just a rephrasing of Corollary 6.11 using Corollary 6.12 to identify the spaces L^1/H_0^1 and $Q(T)$ when T is a $(\mathrm{BCP})_\theta$-operator, was the essential tool used in [3] to prove that (BCP)-operators are reflexive.

COROLLARY 6.13. *Suppose* $0 \leqslant \theta < 1$, $0 < \alpha$, $n \in \mathbf{N}$, *and T is a* $(\mathrm{BCP})_\theta$-*operator in* $\mathscr{L}(\mathscr{H})$. *If* $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^n$ *are sequences of vectors in* $\mathscr{H}$ *and* $\{[L_{ij}]\}_{i,j=1}^n$ *is a doubly indexed sequence of elements of* Q_T *satisfying*

$$\|[L_{ij}] - [x_i \otimes y_j]\|_{Q(T)} < \alpha, \quad 1 \leqslant i, j \leqslant n,$$

then there exist sequences $\{x_i'\}_{i=1}^n$ *and* $\{y_j'\}_{j=1}^n$ *of vectors in* $\mathscr{H}$ *such that*

$$[L_{ij}] = [x_i' \otimes y_j'], \quad 1 \leqslant i, j \leqslant n,$$

and

$$\|x_i - x_i'\| < (1 - \theta^{1/2})^{-1} n\alpha^{1/2}, \quad \|y_j - y_j'\| < (1 - \theta^{1/2})^{-1} n\alpha^{1/2}, \quad 1 \leqslant i, j \leqslant n.$$

REMARK 6.14. It follows from Remark 6.7 that if, in Corollaries 6.9, 6.11, and 6.13, one were given initially a positive number ε and a finite set $z_1, \ldots, z_m$ of elements of $\mathscr{H}$ (or $\mathscr{H}(\Theta)$), then the resulting sequences $\{x_i'\}_{i=1}^n$ and $\{y_j'\}_{j=1}^n$ could be chosen to have the additional properties:

$$\|(x_i' - x_i)\cdot z_k\|_{L^1} < \varepsilon, \quad \|(y_j' - y_j)\cdot z_k\|_{L^1} < \varepsilon, \quad 1 \leqslant i, j \leqslant n, \ 1 \leqslant k \leqslant m.$$

The following corollary shows, in particular, as mentioned in the introduction, that infinite systems of equations of the form (10) can be solved for $(\mathrm{BCP})_{\theta,\sigma}$-operators when $0 \leqslant \theta < 1/2$.

COROLLARY 6.15. *Suppose $0 \leqslant \theta < 1/2$, σ is a measurable subset of* $\mathbf{T}$, *and $T = S(\Theta)$ is a completely nonunitary contraction such that $T \in (\mathrm{BCP})_{\theta,\sigma}$. If $\{f_{ij}\}_{i,j=1}^\infty$ is any infinite doubly indexed sequence of functions in $L^1(\sigma)$, then there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ in $\mathscr{H}(\Theta)$ such that $f_{ij} = (x_i \cdot y_j)|\sigma$ in $L^1(\sigma)$ for $1 \leqslant i,$ $< \infty$.*

Proof. Let $\mathscr{N}$ be a Hilbert space with an orthonormal basis $\{e_j\}_{j=1}^\infty$, and choose inductively sequences $\{a_i\}_{i=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ of positive numbers such that

$$a_i b_j \|f_{ij}\|_{L^1(\sigma)} < 2^{-(i+j)}, \quad 1 \leqslant i, j < \infty. \tag{87}$$

(Having chosen $a_1, b_1, \ldots, a_k, b_k$ so that (87) is satisfied for $1 \leqslant i, j \leqslant k$, choose a_{k+1} so that the required inequalities are satisfied for $i = k+1$, $j \leqslant k$, and then choose b_{k+1} so that the required inequalities are satisfied for $j = k+1$, $i \leqslant k+1$.) Then, since $\sum_{i,j=1}^\infty a_i b_j \|f_{ij}\|_{L^1(\sigma)} < +\infty$, it follows from the Lebesgue monotone convergence theorem that the series $\sum_{i,j=1}^\infty a_i b_j |f_{ij}(\mathrm{e}^{\mathrm{i}t})|$ converges at almost all $\mathrm{e}^{\mathrm{i}t} \in \sigma$, and for such $\mathrm{e}^{\mathrm{i}t}$, we can define an operator $F(\mathrm{e}^{\mathrm{i}t}) \in \mathscr{L}(\mathscr{N})$ by setting

$$F(\mathrm{e}^{\mathrm{i}t}) e_j = \sum_{k=1}^\infty a_j b_k f_{jk}(\mathrm{e}^{\mathrm{i}t})\, e_k, \quad 1 \leqslant j, k < \infty. \tag{88}$$

Since the matrix for $F(\mathrm{e}^{\mathrm{i}t})$ relative to the basis $\{e_j\}_{j=1}^\infty$ has absolutely summable entries, $F(\mathrm{e}^{\mathrm{i}t})$ is bounded, and, in fact, belongs to $C_1(\mathscr{N})$. The function $F\colon \sigma \to C_1(\mathscr{N})$ is obviously measurable from (88), and

$$\|F\|_{L^1(\sigma, C_1(\mathscr{N}))} = \frac{1}{2\pi}\int_\sigma \|F(\mathrm{e}^{\mathrm{i}t})\|_1 \,\mathrm{d}t \leqslant \frac{1}{2\pi}\int_\sigma \sum_{i,j=1}^\infty a_i b_j |f_{ij}(\mathrm{e}^{\mathrm{i}t})| \,\mathrm{d}t \leqslant$$

$$\leqslant \sum_{i,j=1}^\infty a_i b_j \|f_{ij}\|_{L^1(\sigma)} \leqslant \sum_{i,j=1}^\infty 2^{-(i+j)} = 1.$$

Therefore, by Theorem 6.5 (with $X \equiv Y \equiv 0$), there exist $\mathscr{H}(\Theta)$-oriented functions X' and Y' in $L^2(C_2(\mathscr{N}, \mathscr{F}_* \oplus \mathscr{F}))$ such that $F(e^{it}) = Y'(e^{it})^* X'(e^{it})$ for almost all $e^{it} \in \sigma$. We define

$$x_i(e^{it}) = (1/a_i)X'(e^{it})e_i, \quad y_j = (1/b_j)Y'(e^{it})e_j, \quad e^{it} \in \mathbf{T},\ 1 \leqslant i, j < \infty.$$

The vectors $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ belong to $\mathscr{H}(\Theta)$, and a computation similar to that in the proof of Corollary 6.9 shows that $f_{ij}(e^{it}) = (x_i \cdot y_j)(e^{it})$ for almost all $e^{it} \in \sigma$, so the proof is complete.

The next corollary shows that all $(\mathrm{BCP})_\theta$-operators, $0 \leqslant \theta < 1$, belong to the class $\mathbf{A}_{\aleph_0}$ in the terminology of [4], and thus to them we may apply our dilation theory in [4].

COROLLARY 6.16. *Suppose $0 \leqslant \theta < 1$, T is a $(\mathrm{BCP})_\theta$-operator in $\mathscr{L}(\mathscr{H})$, and $\{[L_{ij}]\}_{i,j=1}^\infty$ is a doubly indexed sequence of elements of Q_T. Then there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of vectors in $\mathscr{H}$ such that*

$$[L_{ij}] = [x_i \otimes y_j], \quad 1 \leqslant i, j < \infty. \tag{89}$$

Thus, in particular, $\bigcup_{0 \leqslant \theta < 1} (\mathrm{BCP})_\theta \subset \mathbf{A}_{\aleph_0}$.

Proof. The argument proceeds just like that of the preceding corollary, using Theorem 6.6 in place of Theorem 6.5, and employs the fact that for $T \in (\mathrm{BCP})_\theta$, $0 \leqslant \theta < 1$, we know from Corollary 6.12 that $T \in \mathbf{A}$. Thus no more need be said about it.

The following result is an immediate consequence of Corollary 6.16 and [4, II].

COROLLARY 6.17. *Every operator T in* $\bigcup_{0 \leqslant \theta < 1} (\mathrm{BCP})_\theta$ *is reflexive.*

7. CONCLUDING REMARKS

We begin this concluding section with a comparison between the results in §6 and the work of Apostol [1].

The following proposition shows that Corollary 6.17 covers the operators considered in [1].

PROPOSITION 7.1. *Suppose T is an arbitrary contraction in $\mathscr{L}(\mathscr{H})$. Then for every θ satisfying $0 \leqslant \theta < 1$,*

$$\zeta_\theta(T) \subset L_\theta(T) \cup R_\theta(T) \subset \zeta_{\left(\frac{2\theta}{1-\theta}\right)}(T). \tag{90}$$

Proof. It is an elementary fact from the theory of Banach algebras that

(91) $$\sigma_e((T_\mu^* T_\mu)^{1/2})\setminus(0) = \sigma_e((T_\mu T_\mu^*)^{1/2})\setminus(0), \quad \mu \in \mathbf{D},$$

where T_μ is as in (11), and hence

$$L_\theta \setminus \sigma_e(T) = R_\theta(T)\setminus\sigma_e(T), \quad 0 \leqslant \theta < 1,$$

or, equivalently,

(92) $$L_\theta(T) \cup (\sigma_e(T) \cap \mathbf{D}) = R_\theta(T) \cup (\sigma_e(T) \cap \mathbf{D}) = L_\theta(T) \cup R_\theta(T), \quad 0 \leqslant \theta < 1.$$

Furthermore, if $\mu \in \mathbf{D}$ and $\mu \notin \sigma_e(T)$, then an easy consequence of (11) is that

(93) $$\|(\pi(T) - \mu I)^{-1}\| \leqslant \|(I - \bar{\mu}T)^{-1}\| \|\pi(T_\mu)^{-1}\| \leqslant \leqslant \frac{1}{(1 - |\mu|)(\inf \sigma_e(T_\mu^* T_\mu)^{1/2}))}$$

since

(94) $$\|(I - \bar{\mu}T)^{-1}\| \leqslant 1/(1 - |\mu|), \quad \|\pi(T_\mu)^{-1}\| = 1/\inf \sigma_e((T_\mu^* T_\mu)^{1/2}).$$

Therefore from (14) and (93) we see that

$$\zeta_\theta(T)\setminus\sigma_e(T) \subset L_\theta(T), \quad 0 \leqslant \theta < 1,$$

so we conclude from (92) that

(95) $$\zeta_\theta(T) \subset L_\theta(T) \cup R_\theta(T). \quad 0 \leqslant \theta < 1.$$

In the opposite direction, if $\mu \in \mathbf{D}\setminus\sigma_e(T)$, then from (11) we have

$$\pi(T_\mu)^{-1} = (\pi(T - \mu I))^{-1}\pi(I - \bar{\mu}T) =$$

$$= (\pi(T - \mu I))^{-1}\pi((1 - |\mu|^2)I - \bar{\mu}(T - \mu I)) =$$

$$= (1 - |\mu|^2)(\pi(T - \mu I))^{-1} - \bar{\mu}\pi(I),$$

so that

(96) $$\|\pi(T_\mu)^{-1}\| \leqslant 2(1 - |\mu|)\|(\pi(T - \mu I))^{-1}\| + 1.$$

If $\mu \in (L_\theta(T) \cup R_\theta(T))\setminus\sigma_e(T)$, then, by virtue of (91) and (94), we have

$$1/\theta \leqslant \|\pi(T_\mu)^{-1}\|,$$

so we infer from (96) that

$$\left(\frac{1-\theta}{2\theta}\right)\left(\frac{1}{1-\mu}\right) = \frac{1}{2}\left(\frac{1}{\theta}-1\right)\left(\frac{1}{1-\mu}\right) \leqslant (\pi(T-\mu I))^{-1}.$$

This implies the second inclusion in (91), so the proof is complete.

We note, finally, that if T is a completely nonunitary contraction such that $\sigma(T) \supset \mathbf{T}$, then one knows from standard spectral theory that $\zeta_\theta(T) = \mathbf{D}$ for $\theta \geqslant 1$, so the second inclusion in (91) is nontrivial only for $\theta < 1/3$.

We now give an example which, at the same time, shows, as promised in § 2, that the family $\{(\mathrm{BCP})_\theta\}_{0\leqslant\theta<1}$ is strictly increasing, and that there exist $(\mathrm{BCP})_\theta$-operators whose spectra equal $\mathbf{T}$.

EXAMPLE 7.2. Fix θ, $0 \leqslant \theta < 1$, and denote by Θ the function in $H^\infty(\mathscr{L}(\mathscr{H}))$ defined by $\hat{\Theta}(\lambda) = \theta 1_{\mathscr{H}}$, $\lambda \in \mathbf{D}$. It is obvious that $\{\mathscr{H}, \mathscr{H}, \Theta\}$ is a contractive analytic function, and so we may set $T = S(\Theta)$. It is obvious from (29) that $L_\theta(T) = R_\theta(T) = \mathbf{D}$ while $L_{\theta'}(T) = R_{\theta'}(T) = \emptyset$ if $0 \leqslant \theta' < \theta$, and so T belongs to $(\mathrm{BCP})_\theta$ but not to $(\mathrm{BCP})_{\theta'}$ for $\theta' < \theta$. This shows that the family $\{(\mathrm{BCP})_\theta\}_{0\leqslant\theta<1}$ is strictly increasing.

It is also clear from the characterization of the spectrum of an operator in terms of its characteristic function [21, Theorem VI.4.1] that the spectrum of the operator $T = S(\Theta)$ constructed above coincides with $\mathbf{T}$. (It can easily be seen that T is a bilateral weighted shift operator of infinite multiplicity with weight sequence $\{\ldots 1_{\mathscr{H}}, 1_{\mathscr{H}}, \theta 1_{\mathscr{H}}, 1_{\mathscr{H}}, 1_{\mathscr{H}}, \ldots\}$, and T is obviously similar to the unweighted shift of infinite multiplicity.)

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