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ABELIAN OPERATOR ALGEBRAS AND TENSOR PRODUCTS

JON KRAUS

One of the central results in the theory of tensor products of von Neumann algebras is Tomita's commutation formula: if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, then

$$\mathcal{M}' \overline{\otimes} \mathcal{N}' = (\mathcal{M} \overline{\otimes} \mathcal{N})'. \tag{1}$$

It was observed in [15] that if we let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the projection lattices of $\mathcal{M}$ and $\mathcal{N}$ respectively, then (1) can be rewritten as

$$\operatorname{alg}\mathcal{L}_1 \overline{\otimes} \operatorname{alg}\mathcal{L}_2 = \operatorname{alg}(\mathcal{L}_1 \otimes \mathcal{L}_2). \tag{2}$$

This version of Tomita's formula makes sense for any pair of reflexive algebras $\operatorname{alg}\mathcal{L}_1$ and $\operatorname{alg}\mathcal{L}_2$. It remains an open question whether the tensor product formula (2) is valid when $\operatorname{alg}\mathcal{L}_1$ and $\operatorname{alg}\mathcal{L}_2$ are arbitrary reflexive algebras. However, (2) has been verified in a number of special cases [15, 17, 19, 20, 21, 22, 23]. In particular, it is known that if $\mathcal{L}_1$ is a commutative subspace lattice that is either completely distributive [23] or finite width [19], then (2) is valid for $\mathcal{L}_1$ and any subspace lattice $\mathcal{L}_2$. One of the main results of this paper is that if T is a subnormal operator acting on a Hilbert space $\mathcal{H}$, or if T is a (BCP)-operator on $\mathcal{H}$ (or more generally if $T \in \mathbf{A}_{\aleph_0}(\mathcal{H})$), then (2) is valid when $\mathcal{L}_1 = \operatorname{lat}(T)$ and $\mathcal{L}_2$ is any subspace lattice.

The proofs of the results concerning the tensor product formula (2) in [19, 22, 23] and this paper all make use of slice maps. If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, and φ is in the predual $\mathcal{M}_*$ of $\mathcal{M}$, then the right slice map R_φ (see, e.g., [33]) is the unique σ-weakly continuous linear map from $\mathcal{M} \overline{\otimes} \mathcal{N} \to \mathcal{N}$ such that

$$R_\varphi(A \otimes B) = \varphi(A)B, \quad A \in \mathcal{M},\ B \in \mathcal{N}.$$

The left slice maps $L_\psi : \mathcal{M} \overline{\otimes} \mathcal{N} \to \mathcal{M}$, $\psi \in \mathcal{N}_*$, are similarly defined. A σ-weakly closed subspace $\mathcal{S}$ of $B(\mathcal{H})$ (the algebra of bounded operators on $\mathcal{H}$) is said to have *Property* S_σ [22] if whenever $\mathcal{T}$ is a σ-weakly closed subspace of a von Neumann algebra $\mathcal{N}$, we have

$$\{A \in \mathcal{S} \overline{\otimes} \mathcal{N} : R_\varphi(A) \in \mathcal{T} \text{ for all } \varphi \in \mathcal{M}_*\} = \mathcal{S} \overline{\otimes} \mathcal{T}.$$

It is shown in [22] that if alg $\mathcal{L}_1$ has Property S_σ, then (2) is valid for any subspace lattice $\mathcal{L}_2$. Thus if every σ-weakly closed subspace $\mathcal{S}$ has Property S_σ then the tensor product formula (2) is valid for all pairs of subspace lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. In Section 2 of this paper we prove the rather surprising fact that the converse is true. Moreover, we show that if there is a σ-weakly closed subspace of $B(\mathcal{H})$ without Property S_σ, then there is an abelian reflexive subalgebra of $B(\mathcal{H})$ without Property S_σ. It seems very likely that there are σ-weakly closed subspaces without Property S_σ, and hence that there are σ-weakly closed abelian algebras without Property S_σ. We have not been able to produce such an algebra, but we are able to show that certain important classes of σ-weakly closed abelian algebras do have Property S_σ. This both narrows the search for an algebra without Property S_σ and provides new information about these classes of algebras. In Section 3 we prove (as a special case of a more general result) that $H^\infty(\mathbf{R})$ has Property S_σ. In Section 4 we show that if T is either a subnormal operator on $\mathcal{H}$ or is in $A_{\aleph_0}(\mathcal{H})$, then $\mathcal{A}(T)$ (the σ-weakly closed subalgebra of $B(\mathcal{H})$ generated by T and I) has Property S_σ. Since it is known that for such operators we have $\mathcal{A}(T) = \operatorname{alg}\operatorname{lat}(T)$, this implies the result about $\operatorname{lat}(T)$ mentioned above.

1. PRELIMINARIES

In this paper we will always assume that Hilbert spaces are separable. A sublattice $\mathcal{L}$ of the projection lattice of $B(\mathcal{H})$ is said to be a *subspace lattice* if it contains 0 and I and is strongly closed. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a *commutative subspace lattice*. If $\mathcal{L}$ is a subspace lattice, alg $\mathcal{L}$ denotes the set of operators in $B(\mathcal{H})$ that leave the (ranges of the) projections in $\mathcal{L}$ invariant. If $\mathcal{S}$ is a subset of $B(\mathcal{H})$, then lat $\mathcal{S}$, the set of projections left invariant by the elements of $\mathcal{S}$, is a subspace lattice. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is *reflexive* if $\mathcal{A} = \operatorname{alg}\operatorname{lat}\mathcal{A}$. Note that the reflexive algebras are precisely the algebras of the form alg $\mathcal{L}$, where $\mathcal{L}$ is a subspace lattice. If $\mathcal{L}_1 \subset B(\mathcal{H}_1)$ and $\mathcal{L}_2 \subset B(\mathcal{H}_2)$ are subspace lattices, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the subspace lattice (in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$) generated by $\{P_1 \otimes P_2 \colon P_i \in \mathcal{L}_i,\ i = 1, 2\}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $\mathcal{S} \subset B(\mathcal{H})$ and $\mathcal{T} \subset B(\mathcal{K})$ be σ-weakly closed subspaces. Then $\mathcal{S} \overline{\otimes} \mathcal{T}$ denotes the σ-weakly closed linear span of $\{S \otimes T : S \in \mathcal{S} \text{ and } T \in \mathcal{T}\}$, and we set

$$F(\mathcal{S}, \mathcal{T}) = \{A \in B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}) : R_\varphi(A) \in \mathcal{T} \text{ for all } \varphi \in B(\mathcal{H})_* \\ \text{and } L_\psi(A) \in \mathcal{S} \text{ for all } \psi \in B(\mathcal{K})_*\}.$$

(It is easily checked that we can replace $B(\mathscr{H})$ and $B(\mathscr{K})$ in the definition of $F(\mathscr{S}, \mathscr{T})$ by any von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$ containing $\mathscr{S}$ and $\mathscr{T}$ [22, Remark 1.2].) Note that it is immediate from the definition that $\mathscr{S} \mathbin{\overline{\otimes}} \mathscr{T} \subset F(\mathscr{S}, \mathscr{T})$. It is an open question whether the subspace tensor product formula

$$F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \mathbin{\overline{\otimes}} \mathscr{T} \tag{3}$$

is always valid. Tomiyama proved in [33] that (3) is valid when $\mathscr{S}$ and $\mathscr{T}$ are von Neumann algebras. His proof uses Tomita's commutation theorem, and, in fact, Tomita's theorem is equivalent to the validity of (3) for all von Neumann algebras. Hence (3) can be looked at as a very general version of Tomita's commutation formula.

A σ-weakly closed subspace $\mathscr{S}$ has Property S_σ if and only if $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \mathbin{\overline{\otimes}} \mathscr{T}$ for all σ-weakly closed subspaces $\mathscr{T}$ [22, Remark 1.5]. It is shown in [22, Theorem 1.9] that every semidiscrete von Neumann algebra (and hence every type I von Neumann algebra [13, Proposition 3.5]) has Property S_σ. It is also shown that $\mathscr{R}(F_2)$ (the regular group von Neumann algebra of the free group on two generators) has Property S_σ [22, Theorem 1.18]. A von Neumann algebra is said to have the CCAP (completely contractive approximation property) [16] if there is a net $\{\Phi_\alpha\}$ of σ-weakly continuous, completely contractive maps of finite rank from $\mathscr{M}$ to $\mathscr{M}$ such that the net $\{\Phi_\alpha(A)\}$ converges σ-weakly to A for all A in $\mathscr{M}$. (A linear map $\Phi: \mathscr{M} \to \mathscr{M}$ is completely contractive if $\sup_n \|\Phi_n\| \leq 1$, where $\Phi_n = \Phi \otimes 1_n$ is the map from $\mathscr{M} \otimes \mathscr{M}_n(\mathbf{C})$ to itself given by $\Phi_n([A_{ij}]) = [\Phi(A_{ij})]$.) By definition [13], semidiscrete von Neumann algebras satisfy a stronger property than the CCAP (the maps Φ_α can be chosen so that Φ_α is completely positive and $\Phi_\alpha(I) = I$), and De Cannière and Haagerup proved in [12] that $\mathscr{R}(F_2)$ has the CCAP. (Haagerup has classified the discrete groups G for which $\mathscr{R}(G)$ has the CCAP [16].) Hence the next result generalizes Theorems 1.9 and 1.18 in [22].

THEOREM 1.1. *Let $\mathscr{M}$ be a von Neumann algebra with the CCAP. Then $\mathscr{M}$ has Property* S_σ.

Proof. Since $\mathscr{M}$ has the CCAP, an argument similar to that preceding Proposition 3.1 in [13] shows that the identity map on $\mathscr{M}_*$ can be approximated in the topology of simple norm convergence by finite-rank linear maps from $\mathscr{M}_*$ to $\mathscr{M}_*$ whose adjoints are completely contractive maps (from $\mathscr{M}$ to $\mathscr{M}$). Using this fact and [22, Lemma 1.17], the proof of Theorem 1.9 in [22] can be easily modified to show that $\mathscr{M}$ has Property S_σ. ▨

It is quite possible that all von Neumann algebras have Property S_σ. However, at present there are no examples known of von Neumann algebras with Property S_σ which do not have the CCAP. In particular, it is an open question whether $\mathscr{R}(\mathrm{SL}(3, \mathbf{Z}))$ (which does not have the CCAP [16]) has Property S_σ.

2. THE EQUIVALENCE OF THE SUBSPACE TENSOR PRODUCT FORMULA AND THE REFLEXIVE ALGEBRA TENSOR PRODUCT FORMULA

In this section we prove two results which together imply that the subspace tensor product formula (3) is equivalent to the reflexive algebra tensor product formula (2) in the sense that (3) is valid for all pairs of σ-weakly closed subspace $\mathscr{S}$ and $\mathscr{T}$ if and only if (2) is valid for all pairs of reflexive algebras $\mathrm{alg}\,\mathscr{L}_1$ and $\mathrm{alg}\,\mathscr{L}_2$.

THEOREM 2.1. *Suppose there is a σ-weakly closed subspace of $B(\mathscr{H})$ which does not have Property S_σ. Then there is an abelian reflexive subalgebra of $B(\mathscr{H})$ which does not have Property S_σ.*

Proof. If $\mathscr{H}$ were finite dimensional, then every subspace of $B(\mathscr{H})$ would have Property S_σ [22, Proposition 1.7]. Hence $\mathscr{H}$ is infinite dimensional. Let $\mathscr{S}$ be a σ-weakly closed subspace of $B(\mathscr{H})$ without Property S_σ. Let $\mathscr{B}$ be the set of all operators on $\mathscr{H} \oplus \mathscr{H}$ which admit a matrix representation of the form $\begin{pmatrix} \lambda I & S \\ 0 & \lambda I \end{pmatrix}$, where $\lambda \in \mathbf{C}$ and $S \in \mathscr{S}$. Then it is easily checked that $\mathscr{B}$ is an abelian σ-weakly closed algebra. Suppose that $\mathscr{B}$ has Property S_σ, and let P be the projection $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Then $P^{\perp}\mathscr{B}P = \left\{\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} : S \in \mathscr{S}\right\}$ also has Property S_σ [22, Proposition 1.10]. Let $\mathscr{H}_2$ be the two-dimensional Hilbert space with basis $\{e_1, e_2\}$ and define U from $\mathscr{H} \oplus \mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}_2$ by

$$U(x \oplus y) = x \otimes e_1 + y \otimes e_2 .$$

Then U is unitary and $U(P^{\perp}\mathscr{B}P)U^* = \mathscr{S} \overline{\otimes} \mathbf{C}E_{12}$, where $E_{12} \in B(\mathscr{H}_2)$ is the operator that maps e_2 to e_1 and e_1 to 0. Hence $\mathscr{S} \overline{\otimes} \mathbf{C}E_{12}$ has Property S_σ and so $\mathscr{S}$ has Property S_σ [22, Proposition 1.15]. But this is a contradiction, and thus $\mathscr{B}$ does not have Property S_σ. Let $\mathscr{A} = \mathscr{B} \overline{\otimes} \mathbf{C}I$, where I denotes the identity operator on $\mathscr{H}$. Then $\mathscr{A}$ does not have Property S_σ [22, Proposition 1.15]. Since $\mathscr{H}$ is infinite dimensional, $B(\mathscr{H} \oplus \mathscr{H}) \overline{\otimes} \mathbf{C}I$ has a separating vector, and so $\mathscr{A}$ is reflexive. (More generally, every σ-weakly closed unital subalgebra of a von Neumann algebra with a separating vector is reflexive [26, Theorem 3.5].) Let V be any unitary map from $(\mathscr{H} \oplus \mathscr{H}) \otimes \mathscr{H}$ onto $\mathscr{H}$. Then $V\mathscr{A}V^*$ is an abelian reflexive subalgebra of $B(\mathscr{H})$ without Property S_σ. ▨

THEOREM 2.2. *Suppose $\mathscr{A}$ is a reflexive subalgebra of $B(\mathscr{H})$ without Property S_σ, and let $\mathscr{L}_1 = \mathrm{lat}\,\mathscr{A}$. Then there is a subspace lattice $\mathscr{L}_2$ such that* $\mathrm{alg}\,\mathscr{L}_1 \overline{\otimes} \mathrm{alg}\,\mathscr{L}_2 \neq \mathrm{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$.

Proof. Since $\mathscr{A}$ does not have Property S_σ, we can find a Hilbert space $\mathscr{K}$ and a σ-weakly closed subspace $\mathscr{S}$ of $B(\mathscr{K})$ such that $\mathscr{A} \overline{\otimes} \mathscr{S} \neq F(\mathscr{A}, \mathscr{S})$. Let $\{E_{ij}\}_{i,j=1,2}$ be the usual basis of matrix units for the 2×2 matrix algebra $M_2(\mathbf{C})$.

Let $\mathscr{C} = \{\lambda I \otimes (E_{11} + E_{22}) + S \otimes E_{12} : \lambda \in \mathbf{C}, S \in \mathscr{S}\}$, where I is the identity operator on $\mathscr{K}$. $\left(\text{Note that } \mathscr{C} \text{ can be naturally identified with } \left\{\begin{pmatrix} \lambda I & S \\ 0 & \lambda I \end{pmatrix} : \lambda \in \mathbf{C}, S \in \mathscr{S}\right\}.\right)$ Then $\mathscr{C}$ is a σ-weakly closed unital subalgebra of $B(\mathscr{K}) \mathbin{\overline{\otimes}} M_2(\mathbf{C})$. Let $\mathscr{B} = \mathscr{C} \mathbin{\overline{\otimes}} \mathbf{C}I$. Since $F(\mathscr{A},\mathscr{S}) \neq \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{S}$, $\mathscr{K}$ must be infinite dimensional [22, Proposition 1.7], and so $\mathscr{B}$ is a reflexive algebra [26, Theorem 3.5]. Define a map Φ from $B(\mathscr{H}) \mathbin{\overline{\otimes}} B(\mathscr{K})$ to $(B(\mathscr{H}) \mathbin{\overline{\otimes}} B(\mathscr{K})) \mathbin{\overline{\otimes}} M_2(\mathbf{C}) \mathbin{\overline{\otimes}} B(\mathscr{K})$ by $\Phi(T) = T \otimes E_{12} \otimes I$. Let $\mathscr{N} = B(\mathscr{K}) \mathbin{\overline{\otimes}} M_2(\mathbf{C}) \mathbin{\overline{\otimes}} B(\mathscr{K})$. For $\varphi \in B(\mathscr{H})_*$, let R_φ denote the right slice map from $B(\mathscr{H}) \mathbin{\overline{\otimes}} B(\mathscr{K}) \to B(\mathscr{K})$ associated with φ, and let $\tilde{R}_\varphi$ denote the right slice map from $B(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{N} \to \mathscr{N}$ associated with φ. Then it is easily checked that

$$\tilde{R}_\varphi(\Phi(T)) = R_\varphi(T) \otimes E_{12} \otimes I, \tag{4}$$

where we make the obvious identification between $(B(\mathscr{H}) \mathbin{\overline{\otimes}} B(\mathscr{K})) \mathbin{\overline{\otimes}} M_2(\mathbf{C}) \mathbin{\overline{\otimes}} B(\mathscr{K})$ and $B(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{N}$. Now let T be an operator in $F(\mathscr{A}, \mathscr{S})$ which is not in $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{S}$. It follows from (4) that $\Phi(T) \in F(B(\mathscr{H}), \mathscr{B})$. Moreover, since $B(\mathscr{K})$ has Property S_σ, $T \in \mathscr{A} \mathbin{\overline{\otimes}} B(\mathscr{K})$, and so $\Phi(T) \in \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{N}$. Hence $\Phi(T) \in F(\mathscr{A}, \mathscr{B})$. However, $\Phi(T)$ is not in $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$. For it is easily seen that $T \notin \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{S}$ implies that $T \otimes E_{12} \notin \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{C}$, which in turn implies that $\Phi(T) = T \otimes E_{12} \otimes I \notin \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$. Hence $F(\mathscr{A}, \mathscr{B}) \neq \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$. Let $\mathscr{L}_2 = \operatorname{lat}\mathscr{B}$. Then $F(\mathscr{A}, \mathscr{B}) = F(\operatorname{alg}\mathscr{L}_1, \operatorname{alg}\mathscr{L}_2) = \operatorname{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$ (see [22], p. 372), and so $\operatorname{alg}\mathscr{L}_1 \mathbin{\overline{\otimes}} \operatorname{alg}\mathscr{L}_2 = \mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B} \neq F(\mathscr{A}, \mathscr{B}) = \operatorname{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$. ▨

COROLLARY 2.3. *Let $\mathscr{M}$ be a von Neumann algebra without Property S_σ, and let $\mathscr{L}_1$ be the projection lattice of $\mathscr{M}$. Then there is a subspace lattice $\mathscr{L}_2$ such that* $\operatorname{alg}\mathscr{L}_1 \mathbin{\overline{\otimes}} \operatorname{alg}\mathscr{L}_2 \neq \operatorname{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$.

Proof. Since $\mathscr{M}$ does not have Property S_σ, $\mathscr{M}' = \operatorname{alg}\mathscr{L}_1$ also does not have Property S_σ [22, Proposition 1.16]. Hence we can apply Theorem 2.2 to get the desired result. ▨

REMARK 2.4. Suppose $\mathscr{A}$ is a reflexive algebra without Property S_σ, and $\mathscr{B}$ is the algebra constructed in the proof of Theorem 2.2. Then $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B} = (\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{C}) \mathbin{\overline{\otimes}} \mathbf{C}I$ is reflexive by Theorem 3.5 in [26]. Thus, if there is a σ-weakly closed subspace without Property S_σ, then there is a pair of reflexive algebras $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$ is reflexive but $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B} \neq \operatorname{alg}(\operatorname{lat}\mathscr{A} \otimes \operatorname{lat}\mathscr{B})$.

3. ALGEBRAS OF ANALYTIC OPERATORS WITH PROPERTY S_σ

In this section, G will denote a locally compact abelian group with dual group Γ and Haar measure m. We will write $L^p(G)$ in place of $L^p(m)$, $1 \leqslant p \leqslant \infty$. We will often identify the functions in $L^\infty(G)$ with the associated multiplication operators on $L^2(G)$. For $\gamma \in \Gamma$, e_γ denotes the function in $L^\infty(G)$ defined by $e_\gamma(g) = (g, \gamma)$, where $g, \gamma \to (g, \gamma)$ is the dual pairing of G and Γ. For f in $L^1(G)$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\gamma) = \int_G f(g)(g, \gamma)\,\mathrm{d}m(g).$$

If $\mathcal{M}$ is a von Neumann algebra and $\alpha: g \to \alpha_g$ is a homomorphism of G into the group of $*$-automorphisms of $\mathcal{M}$ such that all of the maps $g \to \alpha_g(A)$ $(A \in \mathcal{M})$ are σ-weakly continuous, the triple $(\mathcal{M}, G, \alpha)$ is said to be a W^*-dynamical system. If $E \subset \Gamma$, we denote the spectral subspace of α associated with E by $\mathcal{M}^\alpha(E)$. Spectral subspaces have proved to be a powerful tool in the study of W^*-dynamical systems. We refer the reader to [1, 27, 31] for the definition and properties of spectral subspaces.

If $(\mathcal{M}, G_1, \alpha)$ and $(\mathcal{N}, G_2, \beta)$ are W^*-dynamical systems, the W^*-dynamical system $(\mathcal{M} \overline{\otimes} \mathcal{N}, G_1 \times G_2, \alpha \otimes \beta)$ is defined by setting $(\alpha \otimes \beta)_{(g_1, g_2)} = \alpha_{g_1} \otimes \beta_{g_2}$ $(g_1 \in G_1, g_2 \in G_2)$, where $\alpha_{g_1} \otimes \beta_{g_2}$ is the unique $*$-automorphism of $\mathcal{M} \overline{\otimes} \mathcal{N}$ such that $(\alpha_{g_1} \otimes \beta_{g_2})(A \otimes B) = \alpha_{g_1}(A) \otimes \beta_{g_2}(B)$ $(A \in \mathcal{M}, B \in \mathcal{N})$. Let Γ_i denote the dual group of G_i, $i = 1, 2$. It is shown in [22] that if E is a subset of Γ_1 such that $\mathcal{M}^\alpha(E)$ has Property S_σ, then

$$(5) \qquad (\mathcal{M} \overline{\otimes} \mathcal{N})^{\alpha \otimes \beta}(E \times F) = \mathcal{M}^\alpha(E) \overline{\otimes} \mathcal{N}^\beta(F)$$

for all $F \subset \Gamma_2$. We will show in this section that certain spectral subspaces associated with positive semigroups have Property S_σ. A subset Σ of Γ is a *positive semigroup* if it satisfies

(i) $\Sigma + \Sigma \subset \Sigma$,

(ii) $\Sigma \cap (-\Sigma) = \{0\}$ and

(iii) $\Sigma = \overline{\operatorname{int} \Sigma}$ (i.e., Σ is the closure of its interior).

If we define a binary relation on Γ by $\gamma \geqslant \lambda$ if and only if $\gamma - \lambda \in \Sigma$, then $\geqslant$ is a partial order and $\Sigma = \{\gamma : \gamma \geqslant 0\}$. If $(\mathcal{M}, G, \alpha)$ is a W^*-dynamical system, $\mathcal{M}^\alpha(\Sigma)$ is a σ-weakly closed unital subalgebra of $\mathcal{M}$, referred to as the algebra of *analytic operators* in $\mathcal{M}$ (relative to α) [25]. When $G = \mathbf{R}$ (so $\Gamma = \mathbf{R}$) and $\Sigma = [0, \infty)$, an operator A in $\mathcal{M}$ is analytic relative to α if and only if all of the maps $t \to \varphi(\alpha_t(A))$ $(\varphi \in \mathcal{M}_*)$ are in $H^\infty(\mathbf{R})$ [25]. If G_1 and G_2 are locally compact abelian groups with dual groups Γ_1 and Γ_2, and if $\Sigma_i \subset \Gamma_i$, $i = 1, 2$, are positive semigroups, then $\Sigma_1 \times \Sigma_2$ is a positive semigroup in the dual group $\Gamma_1 \times \Gamma_2$ of $G_1 \times G_2$. Hence if $(\mathcal{M}, G_1, \alpha)$ and $(\mathcal{N}, G_2, \beta)$ are W^*-dynamical systems and $\mathcal{M}^\alpha(\Sigma_1)$ has Property S_σ, then if follows from (5) (with $E = \Sigma_1$ and $F = \Sigma_2$) that the tensor product of the algebra of analytic operators in $\mathcal{M}$ (relative to α) with the algebra of analytic operators in $\mathcal{N}$ (relative to β) is the algebra of analytic operators in $\mathcal{M} \overline{\otimes} \mathcal{N}$ (relative to $\alpha \otimes \beta$).

If G is a locally compact abelian group, we can define a unitary representation U of G on $L^2(G)$ by the formula $(U_g f)(h) = f(h - g)$, $(f \in L^2(G), g, h \in G)$. If we let $\alpha = \operatorname{ad} U$ (i.e., $\alpha_g(A) = U_g A U_g^*$ for $A \in B(L^2(G))$ and $g \in G$), then $(B(L^2(G)), G, \alpha)$ is a W^*-dynamical system. We refer to α as the *translation group* of $B(L^2(G))$. If $\mathcal{M}$ is a von Neumann algebra on $L^2(G)$ that is translation-invariant (i.e., $\alpha_g(\mathcal{M}) = \mathcal{M}$ for all g in G), then $(\mathcal{M}, G, \alpha)$ is a W^*-dynamical system, where the restriction of α_g to $\mathcal{M}$ is again denoted by α_g. We will show below (Theorem 3.2) that if $\mathcal{M}$ contains $L^\infty(G)$, then $\mathcal{M}^\alpha(\Sigma)$ has Property S_σ for any positive semigroup $\Sigma \subset \Gamma$. When $G = \mathbf{R}$, $\Sigma = [0, \infty)$, and $\mathcal{M} = L^\infty(\mathbf{R})$, $\mathcal{M}^\alpha(\Sigma) = H^\infty(\mathbf{R})$ (see Example 3.3 below), so Theorem 3.2 implies that $H^\infty(\mathbf{R})$ has Property S_σ.

In order to prove Theorem 3.2 we need the following lemma, which is also used in the proof of Theorem 3.6.

LEMMA 3.1. *Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system, and let $\mathcal{T}$ be a σ-weakly closed subspace of a von Neumann algebra $\mathcal{N}$. Let $X = \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{T}$, and for g in G, let β_g be the restriction to X of $\alpha_g \otimes 1$, where 1 is the identity automorphism of $\mathcal{N}$. For $E \subset \Gamma$, let $X^\beta(E)$ denote the spectral subspace of β associated with E. Then for any open subset V of Γ we have*

$$X^\beta(V) = \mathcal{M}^\alpha(V) \mathbin{\overline{\otimes}} \mathcal{T}. \tag{6}$$

If $\mathcal{M}$ has Property S_σ, then for any closed subset E of Γ we have

$$X^\beta(E) = F(\mathcal{M}^\alpha(E), \mathcal{T}). \tag{7}$$

Proof. Let $\mathcal{R} = \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{N}$, and let $X_* = \mathcal{R}_*/X^\perp$, where $X^\perp = \{\varphi \in \mathcal{R}_* : \varphi(X) = \{0\}\}$. Then X can be identified with the dual space of X_* in a natural way, and the $\sigma(X, X_*)$ topology on X coincides with the relative σ-weak topology on X. Hence, if we view X_* as the space of $\sigma(X, X_*)$-continuous linear functionals on X, then the pair (X, X_*) satisfies condition (1.1) in [27]. Moreover, $g \to \beta_g$ is a $\sigma(X, X_*)$-continuous representation of G on X by $\sigma(X, X_*)$-continuous isometries. Thus we can define the spectral subspace $X^\beta(E)$ for any subset E of Γ [27, Definition 2.1.1]. By definition these spectral subspaces are $\sigma(X, X_*)$-closed. If V is an open subset of Γ, then by Proposition 2.3.3 (ii) in [27] we have that $\mathcal{M}^\alpha(V)$ is the σ-weakly closed linear span of $\{\alpha(f)(A) : A \in \mathcal{M}, f \in L^1(G), \text{ and } \operatorname{supp}\hat{f} \subset V\}$ and $X^\beta(V)$ is the $\sigma(X, X_*)$-closed linear span of $\{\beta(f)(B) : B \in X, f \in L^1(G), \text{ and } \operatorname{supp}\hat{f} \subset V\}$. Moreover, it is easily verified that $\beta(f)(A \otimes T) = \alpha(f)(A) \otimes T$ whenever $A \in \mathcal{M}$, $T \in \mathcal{T}$ and $f \in L^1(G)$. It is immediate from these facts that $X^\beta(V) = \mathcal{M}^\alpha(V) \mathbin{\overline{\otimes}} \mathcal{T}$.

Next suppose that $\mathcal{M}$ has Property S_σ and E is a closed subset of Γ. Since $\mathcal{M}$ has Property S_σ, $F(\mathcal{M}, \mathcal{T}) = \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{T} = X$. Furthermore, it is shown in the proof of Theorem 2.6 in [22] that if we set $\beta_g = \alpha_g \otimes 1$ on $\mathcal{R}$, then $\mathcal{R}^\beta(E) = F(\mathcal{M}^\alpha(E), \mathcal{N})$. Hence $X^\beta(E) = \mathcal{R}^\beta(E) \cap X = F(\mathcal{M}^\alpha(E), \mathcal{N}) \cap F(\mathcal{M}, \mathcal{T}) = F(\mathcal{M}^\alpha(E), \mathcal{T})$. ▨

THEOREM 3.2. *Let $\mathcal{M}$ be a von Neumann algebra on $L^2(G)$, and let α be the translation group of $B(L^2(G))$. If $\mathcal{M}$ is translation-invariant and contains $L^\infty(G)$, then $\mathcal{M}^\alpha(\Sigma)$ has Property S_σ for any positive semigroup Σ in Γ.*

Proof. Let Σ be a positive semigroup in Γ, and let $\mathcal{T}$ be a σ-weakly closed subspace of a von Neumann algebra $\mathcal{N}$. It suffices to show that $F(\mathcal{M}^\alpha(\Sigma), \mathcal{T}) = \mathcal{M}^\alpha(\Sigma) \overline{\otimes} \mathcal{T}$. Let $\mathcal{R} = \mathcal{M} \overline{\otimes} \mathcal{N}$, and for g in G, let $\beta_g = \alpha_g \otimes 1$, where 1 is the identity automorphism of $\mathcal{N}$. Let $X = \mathcal{M} \overline{\otimes} \mathcal{T}$, and let β_g also denote the restriction of β_g to X. For $\gamma \in \Gamma$, set $E_\gamma = e_\gamma \otimes I$, where I is the identity operator in $\mathcal{N}$. Since α is the translation group, $\beta_g(E_\gamma) = \alpha_g(e_\gamma) \otimes I = (g, \gamma)e_\gamma \otimes I = (g, \gamma)E_\gamma$ for all g in G and γ in Γ. Hence $E_\gamma \in \mathcal{R}^\beta(\gamma) \equiv \mathcal{R}^\beta(\{\gamma\})$ for all γ in Γ [27, Lemma 2.3.8 (iv)]. Thus if $A \in X^\beta(\Sigma) = \mathcal{R}^\beta(\Sigma) \cap X$, then $AE_\gamma \in \mathcal{R}^\beta(\Sigma)\mathcal{R}^\beta(\gamma) \subset \mathcal{R}^\beta(\Sigma + \gamma)$ [27, Lemma 3.2.1]. But $AE_\gamma \in X$ (since $e_\gamma \in \mathcal{M}$), so $AE_\gamma \in X^\beta(\Sigma + \gamma)$ for all γ in Γ. Let $\Phi(\gamma) = AE_\gamma$. Then it is easily checked that Φ is continuous with respect to the σ-weak topology on $\mathcal{R}$, and so with respect to the $\sigma(X, X_*)$ topology on X. Since $\Phi(0) = A$ and $0 \in \overline{\operatorname{int} \Sigma}$, A is in the $\sigma(X, X_*)$-closure of $\{\Phi(\gamma) : \gamma \in \operatorname{int} \Sigma\}$. But $\gamma \in \operatorname{int} \Sigma$ implies $\Sigma + \gamma \subset \operatorname{int} \Sigma$, so $A \in X^\beta(\operatorname{int} \Sigma)$. Hence $X^\beta(\Sigma) \subset X^\beta(\operatorname{int} \Sigma)$. The reverse inclusion is trivially true, so $X^\beta(\Sigma) = X^\beta(\operatorname{int} \Sigma)$. A similar argument shows that $\mathcal{M}^\alpha(\Sigma) = \mathcal{M}^\alpha(\operatorname{int} \Sigma)$. Moreover, since $\mathcal{M}$ contains $L^\infty(G)$, $\mathcal{M}'$ is abelian, and so $\mathcal{M}$ is type I, and thus $\mathcal{M}$ has Property S_σ. Hence we can apply Lemma 3.1 to conclude that

$$F(\mathcal{M}^\alpha(\Sigma), \mathcal{T}) = X^\beta(\Sigma) = X^\beta(\operatorname{int} \Sigma) = \mathcal{M}^\alpha(\operatorname{int} \Sigma) \overline{\otimes} \mathcal{T} = \mathcal{M}^\alpha(\Sigma) \overline{\otimes} \mathcal{T}.$$

▨

EXAMPLE 3.3. Let $\mathcal{M} = L^\infty(G)$, and let α be the translation group of $B(L^2(G))$. Then $\alpha_g(A)(h) = A(h + g)$ $(A \in \mathcal{M}, g, h \in G)$. Hence if $f \in L^1(G)$ and $A \in \mathcal{M}$, then for each h in G we have

$$[\alpha(f)(A)](h) = \int_G f(g)\alpha_g(A)(h)\, dm(g) = \int_G f(g)A(h + g)\, dm(g). \tag{8}$$

Moreover, if F is a closed subset of Γ, then

$$\mathcal{M}^\alpha(F) = \{A \in \mathcal{M} : \alpha(f)(A) = 0 \text{ for all } f \in I_0(F)\}, \tag{9}$$

where $I_0(F)$ denotes the closure in $L^1(G)$ of $\{f \in L^1(G) : \hat{f}$ vanishes on a neighborhood of $F\}$ [27, Lemma 2.3.6]. Since $I_0(F)$ is translation-invariant, it follows from (8) and (9) that $\mathcal{M}^\alpha(F) = \{A \in \mathcal{M} : \int_G A(g)f(g)\, dm(g) = 0$ for all $f \in I_0(F)\} \equiv I_0(F)^\perp$.

Let $I(F) = \{f \in L^1(G) : \hat{f}$ vanishes on $F\}$. Then the hull $\{\gamma \in \Gamma : \hat{f}(\gamma) = 0$ for all $f \in I(F)\}$ of $I(F)$ equals F, and $I(F)$ is the largest closed ideal in $L^1(G)$ with this property. The smallest closed ideal in $L^1(G)$ with hull equal to F is $I_0(F)$ [29, 7.2.5]. If $I_0(F) = I(F)$, then F is said to be an S-set (or set of spectral synthesis). (For a discussion of the relation of S-sets to spectral synthesis in $L^\infty(G)$ see Section 7.8 of [29].) Let $\mathrm{W}(F)$ denote the σ-weakly closed (= w*-closed) linear span in $\mathcal{M}$ of $\{e_\gamma : \gamma \in F\}$. Then $\mathrm{W}(L) = I(F)^\perp$. Hence $\mathcal{M}^\alpha(F) = \mathrm{W}(F)$ if and only if F is an S-set. Now let Σ be a positive semigroup in Γ. Then Σ is an S-set [29, 7.5.6], and so it follows from Theorem 3.2 that $\mathrm{W}(\Sigma) = \mathcal{M}^\alpha(\Sigma)$ has Property S_σ. In particular, taking $G = \mathbf{R}$ and $\Sigma = [0, \infty)$, we conclude that $H^\infty(\mathbf{R})$ has Property S_σ. If we let G be the circle group $\mathbf{T} = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ (so $\Gamma = \mathbf{Z}$) and let $\Sigma = \{0, 1, 2, \ldots\}$, we get that $\mathcal{M}^\alpha(\Sigma) = H^\infty(\mathbf{T})$ has Property S_σ. (This also follows from Theorem 2.2 in [22].) We will use this fact several times in the next section.

A natural question is whether we can replace Σ with an arbitrary S-set in the statement of Theorem 3.2. In particular, does $\mathrm{W}(F)$ have Property S_σ for every S-set F? This last question is related to an unsolved problem in harmonic analysis. The problem is: if G_1 and G_2 are locally compact abelian groups with dual groups Γ_1 and Γ_2 and if $F_1 \subset \Gamma_1$ and $F_2 \subset \Gamma_2$ are S-sets, is $F_1 \times F_2$ an S-set? The answer is known to be yes if either $A(F_1)$ or $A(F_2)$ has the approximation property. (See [18] and Theorem 1.5.1 in [34].) However, it is not known whether $A(F)$ has the approximation property for every S-set F, so the general problem remains open. The next proposition gives the relation between this problem and Property S_σ.

PROPOSITION 3.4. *Let G_1 and G_2 be locally compact abelian groups with dual groups Γ_1 and Γ_2. Suppose $F_1 \subset \Gamma_1$ and $F_2 \subset \Gamma_2$ are* S-*sets, and suppose* $\mathrm{W}(F_1)$ *has Property* S_σ. *Then $F_1 \times F_2$ is an* S-*set.*

Proof. Let $\mathcal{M} = L^\infty(G_1)$, $\mathcal{N} = L^\infty(G_2)$ and $\mathcal{R} = L^\infty(G_1 \times G_2)$. Denote the translation groups of $B(L^2(G_1))$, $B(L^2(G_2))$ and $B(L^2(G_1 \times G_2))$ by α, β, and δ respectively. Then $\mathrm{W}(F_1) = \mathcal{M}^\alpha(F_1)$, $\mathrm{W}(F_2) = \mathcal{N}^\beta(F_2)$, and, since $\mathrm{W}(F_1)$ has Property S_σ, $\mathcal{M}^\alpha(F_1) \overline{\otimes} \mathcal{N}^\beta(F_2) = (\mathcal{M} \overline{\otimes} \mathcal{N})^{\alpha\otimes\beta}(F_1 \times F_2)$. Let $V: L^2(G_1) \otimes L^2(G_2) \to L^2(G_1 \times G_2)$ be the unitary map which is defined on elementary tensors by

$$V(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1) f_2(g_2) \quad (f_i \in L^1(G_i),\ g_i \in G_i,\ i = 1, 2).$$

Then

$$\mathcal{R}^\delta(F_1 \times F_2) = V((\mathcal{M} \overline{\otimes} \mathcal{N})^{\alpha\otimes\beta}(F_1 \times F_2))V^* =$$

$$= V(\mathrm{W}(F_1) \overline{\otimes} \mathrm{W}(F_2))V^* = \mathrm{W}(F_1 \times F_2).$$

Hence $F_1 \times F_2$ is an S-set. ▨

REMARK 3.5. It follows from Proposition 3.4 and Theorems 2.1 and 2.2 that if there is a pair of S-sets whose product is not an S-set, then there is a pair of subspace lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ for which the tensor product formula (2) fails to hold.

Froelich has obtained a better, more direct result in [14]. To each closed subset F of Γ Froelich associates, in a natural way, a commutative subspace lattice, which we will denote by $\mathscr{L}(F)$. He proves [14, Theorem 6.9] that if G is separable and if F_1 and F_2 are S-sets in Γ, then $F_1 \times F_2$ is an S-set if and only if formula (2) holds when $\mathscr{L}_i = \mathscr{L}(F_i)$, $i = 1, 2$. Hence if there is a pair of S-sets in Γ whose product is not an S-set, then there is a pair of *commutative* subspace lattices for which (2) fails to hold. (If $\mathscr{A}$ is reflexive, then lat$\mathscr{A}$ is commutative if and only if $\mathscr{A}$ contains a m.a.s.a. . Hence the reflexive algebras constructed in the proof of Theorem 2.1 never have commutative subspace lattices.)

The next result generalizes Theorem 3.5 in [23]. It also implies (in combination with Theorem 4.2.3 in [25]) Theorem 3.1 in [23].

THEOREM 3.6. *Let* $(\mathscr{M}, \mathbf{R}, \alpha)$ *be a* W^**-dynamical system and suppose* $\mathscr{M}$ *and* $\mathscr{M}^\alpha(0)$ *have Property* S_σ. *Then* $\mathscr{M}^\alpha([0, \infty))$ *has Property* S_σ.

Proof. Let $\mathscr{T}$ be a σ-weakly closed subspace of a von Neumann algebra $\mathscr{N}$. It suffices to show that $F(\mathscr{M}^\alpha([0, \infty)), \mathscr{T}) = \mathscr{M}^\alpha([0, \infty)) \overline{\otimes} \mathscr{T}$. Let X and β be as in the statement of Lemma 3.1 (with $G = \mathbf{R}$). Let $\mathscr{R} = \mathscr{M} \overline{\otimes} \mathscr{N}$, and let $X_* = \mathscr{R}_*/X^\perp$, as in the proof of Lemma 3.1. Then $t \to \beta_t$ is a $\sigma(X, X_*)$-continuous representation of $\mathbf{R}$ on X by $\sigma(X, X_*)$-continuous isometries. Moreover, for each $A \in X$ the orbit $\{\beta_t(A) : t \in \mathbf{R}\}$ is bounded, and so is relatively $\sigma(X, X_*)$-compact by Alaoglu's Theorem. Hence it follows from Lemma 3.6 and 3.8 in [35] that $X^\beta([0, \infty)) = X^\beta(\{0\} \cup \cup (0, \infty))$ is the $\sigma(X, X_*)$-closure of $X^\beta(0) + X^\beta((0, \infty))$. Since $\mathscr{M}$ has Property S_σ, $X^\beta([0, \infty)) = F(\mathscr{M}^\alpha([0, \infty)), \mathscr{T})$ and $X^\beta(0) = F(\mathscr{M}^\alpha(0), \mathscr{T})$ by Lemma 3.1. Since $\mathscr{M}^\alpha(0)$ has Property S_σ, $F(\mathscr{M}^\alpha(0), \mathscr{T}) = \mathscr{M}^\alpha(0) \overline{\otimes} \mathscr{T}$. Furthermore, $X^\beta((0, \infty)) = \mathscr{M}^\alpha((0, \infty)) \overline{\otimes} \mathscr{T}$ by Lemma 3.1. Hence $X^\beta(0) + X^\beta((0, \infty)) \subset \mathscr{M}^\alpha([0, \infty)) \overline{\otimes} \mathscr{T}$, and so $F(\mathscr{M}^\alpha([0, \infty)), \mathscr{T}) \subset \mathscr{M}^\alpha([0, \infty)) \overline{\otimes} \mathscr{T}$. But the reverse inclusion is always valid, so $F(\mathscr{M}^\alpha([0, \infty)), \mathscr{T}) = \mathscr{M}^\alpha([0, \infty)) \overline{\otimes} \mathscr{T}$. ▨

REMARK 3.7. Let $(\mathscr{M}, \mathbf{R}, \alpha)$ be a W^*-dynamical system and suppose $\mathscr{M}$ has Property S_σ. If there is a normal conditional expectation from $\mathscr{M}$ onto $\mathscr{M}^\alpha(0)$, then $\mathscr{M}^\alpha(0)$ also has Property S_σ [22, Proposition 1.19]. Such an expectation exists, in particular, if α is the modular automorphism group of $\mathscr{M}$ associated with some faithful, normal, strictly semifinite weight on $\mathscr{M}$ [31, 10.9], or if $\mathscr{M}$ is a finite von Neumann algebra and α is arbitrary [31, 10.6]. Thus if $\mathscr{M}$ has the CCAP, then in either of these cases $\mathscr{M}^\alpha([0, \infty))$ has Property S_σ.

4. SINGLY GENERATED ALGEBRAS WITH PROPERTY S_σ

If $T \in B(\mathscr{H})$, let $\mathscr{W}(T)$ denote the weakly closed subalgebra of $B(\mathscr{H})$ generated by T and I. Then $\mathscr{W}(T)$ is the closure in the weak operator topology of $\mathscr{A}(T)$, which is in turn the closure in the σ-weak topology of the set of polynomials in T. An operator T is said to be *reflexive* if $\mathscr{W}(T)$ is reflexive. Olin and Thomson showed in [28] that if $S \in B(\mathscr{H})$ is a subnormal operator (i.e., the restriction of a normal operator to an invariant subspace), then S is reflexive and $\mathscr{A}(S) = \mathscr{W}(S)$. Thus $\mathscr{A}(S)$ is a reflexive algebra. In the first part of this section we prove the following result about $\mathscr{A}(S)$.

THEOREM 4.1. *Let $S \in B(\mathscr{H})$ be a subnormal operator. Then $\mathscr{A}(S)$ has Property S_σ.*

In proving Theorem 4.1 we will repeatedly use two results concerning completely bounded maps. If $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebra, if $\mathscr{S}$ is a subspace of $\mathscr{A}$, and if $\Phi: \mathscr{S} \to \mathscr{B}$ is a bounded linear map, we denote the map $\Phi \otimes 1_n: \mathscr{S} \otimes M_n(\mathbf{C}) \to \mathscr{B} \otimes M_n(\mathbf{C})$ by Φ_n, and say that Φ is *completely bounded* if $\sup_n \|\Phi_n\|$ is finite. The first result we will use is that if $\mathscr{B}$ is abelian, then every bounded linear map from $\mathscr{S}$ to $\mathscr{B}$ is completely bounded [24, Lemma 1]. The second result we will use is the following proposition.

PROPOSITION 4.2. *Let $\mathscr{S}_1 \subset B(\mathscr{H}_1)$ and $\mathscr{S}_2 \subset B(\mathscr{H}_2)$ be σ-weakly closed subspaces. Suppose that $\mathscr{S}_1$ has Property S_σ and that there is a linear isomorphism Φ from $\mathscr{S}_1$ onto $\mathscr{S}_2$ such that both Φ and Φ^{-1} are σ-weakly continuous and completely bounded. Then $\mathscr{S}_2$ has Property S_σ.*

Proof. Let $\mathscr{T}$ be a σ-weakly closed subspace of a von Neumann algebra $\mathscr{N}$, and let A be an operator in $\mathscr{S}_2 \overline{\otimes} \mathscr{N}$ such that $R_\varphi(A) \in \mathscr{T}$ for all $\varphi \in B(\mathscr{H}_2)_*$. It suffices to show that $A \in \mathscr{S}_2 \overline{\otimes} \mathscr{T}$. Let $\Psi = \Phi^{-1}$. Then since Φ and Ψ are σ-weakly continuous completely bounded maps, it follows from a straightforward modification of the proof of Lemma 1.5 in [12] that there are (unique) σ-weakly continuous linear maps $\Phi \otimes 1: \mathscr{S}_1 \overline{\otimes} \mathscr{N} \to \mathscr{S}_2 \overline{\otimes} \mathscr{N}$ and $\Psi \otimes 1: \mathscr{S}_2 \overline{\otimes} \mathscr{N} \to \mathscr{S}_1 \overline{\otimes} \mathscr{N}$ such that

$$(\Phi \otimes 1)(C \otimes D) = \Phi(C) \otimes D \qquad C \in \mathscr{S}_1,\ D \in \mathscr{N},$$

and

$$(\Psi \otimes 1)(C \otimes D) = \Psi(C) \otimes D \qquad C \in \mathscr{S}_2,\ D \in \mathscr{N}.$$

Let $B = (\Psi \otimes 1)(A)$. Then $B \in \mathscr{S}_1 \overline{\otimes} \mathscr{N}$ and $R_\varphi(B) = R_{\varphi \circ \Psi}(A) \in \mathscr{T}$ for all $\varphi \in B(\mathscr{H}_1)_*$. But $\mathscr{S}_1$ has Property S_σ, so $B \in \mathscr{S}_1 \overline{\otimes} \mathscr{T}$. Hence $A = (\Phi \otimes 1)(B)$ is in $(\Phi \otimes 1)(\mathscr{S}_1 \overline{\otimes} \mathscr{T}) = \mathscr{S}_2 \overline{\otimes} \mathscr{T}$, and so $\mathscr{S}_2$ has Property S_σ. ▨

Now suppose that $S \in B(\mathscr{H})$ is a subnormal operator, and let N be its minimal normal extension, defined on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. (We refer the reader to [10] for the general theory of subnormal operators.) Let $W^*(N)$ denote the von Neumann algebra generated by N. For $A \in \mathscr{A}(N)$, let $\Phi(A)$ be the restriction of A to $\mathscr{H}$. Then by Theorem 2.1 in [11], Φ is a σ-weakly bicontinuous isometric isomorphism of $\mathscr{A}(N)$ onto $\mathscr{A}(S)$. Moreover, Φ is obviously completely bounded, and Φ^{-1} is completely bounded since the range of Φ^{-1} is contained in the abelian

C^*-algebra $W^*(N)$. Hence it follows from Proposition 4.2 that to prove Theorem 4.1 it suffices to show that $\mathscr{A}(N)$ has Property S_σ whenever N is a normal operator. If N is normal, then by the spectral theorem there is a measure μ on the spectrum of N and a $*$-isomorphism Φ of $L^\infty(\mu)$ onto $W^*(N)$ such that $\Phi(z) = N$ and $\Phi(1) = I$. (In this section, the word measure will always denote a finite, positive, regular Borel measure on $\mathbf{C}$. If μ is a measure, we will view the elements of $L^\infty(\mu)$ as multiplication operators on $L^2(\mu)$, so that the σ-weak topology on $L^\infty(\mu)$ is just the w*-topology. If $f \in L^\infty(\mu)$, $\|f\|_\mu$ will denote the essential supremum norm of f.) The map Φ is σ-weakly bicontinuous and both Φ and Φ^{-1} are completely bounded since $W^*(N)$ and $L^\infty(\mu)$ are both abelian C^*-algebras. Let $P^\infty(\mu)$ denote the σ-weak closure of the polynomials in $L^\infty(\mu)$. Then Φ maps $P^\infty(\mu)$ onto $\mathscr{A}(N)$. Hence it follows from Proposition 4.2 that to prove Theorem 4.1 it suffices to prove the following result.

PROPOSITION 4.3. *Let μ be a compactly supported measure on* $\mathbf{C}$. *Then $P^\infty(\mu)$ has Property* S_σ.

In [30] Sarason gave a characterization of $P^\infty(\mu)$. In the proof of Proposition 4.3 we will use a refinement of this characterization, due to Conway and Olin [11, Theorem 4.11]. For the convenience of the reader, we will recall some definitions and facts from [11], and then state Conway and Olin's result. Let Ω be a bounded, simply connected region in $\mathbf{C}$, and let $K = \bar{\Omega}$ be the closure of Ω. Suppose that $R(K)$ (the closure in $C(K)$ of the set of rational functions with poles off K) is a Dirichlet algebra. Then the Dirichlet problem can be solved for Ω, and so for each $f \in C(\partial K)$ there is a unique function $\hat{f}$ in $C(K)$ such that $\hat{f}$ is harmonic in Ω and $\hat{f}|\partial K = f$. For each $z \in \Omega$, let m_z be the unique probability measure on ∂K such that

$$\hat{f}(z) = \int_{\partial K} f \, dm_z , \quad f \in C(\partial K).$$

The measures m_z are pairwise mutually absolutely continuous. Hence if we pick a fixed z in Ω and set $m = m_z$, then $L^\infty(\partial K) \equiv L^\infty(m)$ is independent of the choice of z. (The measure m is called *harmonic measure* for K.) Let $H^\infty(\partial K)$ denote the σ-weak closure of $R(K)$ in $L^\infty(\partial K)$, and let $H^\infty(\Omega)$ denote the algebra of bounded analytic functions on Ω. Then the map $f \to \hat{f}$: $H^\infty(\partial K) \to H^\infty(\Omega)$ defined by

$$\hat{f}(z) = \int_{\partial K} f \, dm_z , \quad z \in \Omega,$$

is an isometric isomorphism onto $H^\infty(\Omega)$. For $g \in H^\infty(\Omega)$, let f be the unique function in $H^\infty(\partial K)$ such that $\hat{f} = g$. Then we can define a function $\tilde{g} : K \to \mathbf{C}$ by setting $\tilde{g} = g$ on Ω and $\tilde{g} = f$ on ∂K. Now let μ be a measure supported in K and such that $\mu|\partial K \ll m$. Then every function in $L^\infty(\partial K)$ can be identified with a function in $L^\infty(\mu|\partial K)$. Hence $g \to \tilde{g}$ maps $H^\infty(\Omega)$ into $L^\infty(\mu)$. The image of $H^\infty(\Omega)$ under this map is denoted by $H^\infty(\Omega, \mu)$.

Conway and Olin's result [11, Theorem 4.11] is: if μ is a compactly supported measure on $\mathbf{C}$ then there are mutually singular measures $\mu_0, \mu_1, \mu_2, \ldots$ such that $\mu = \mu_0 \dotplus \mu_1 \dotplus \mu_2 \dotplus \ldots$, and there are bounded simply connected regions $\Omega_1, \Omega_2, \ldots$, such that

(a) $R(\bar{\Omega}_n)$ is a Dirichlet algebra;

(b) support $\mu_n \subset \bar{\Omega}_n$ and $\mu_n|\partial\Omega_n \ll$ harmonic measure for $\bar{\Omega}_n$;

(c) $\|f\|_{\mu_n} = \sup\{|f(z)|: z \in \Omega_n\}$ for every f in $H^\infty(\Omega_n)$;

(d) $P^\infty(\mu) = L^\infty(\mu_0) \oplus H^\infty(\Omega_1, \mu_1) \oplus \ldots$.

Proof of Proposition 4.3. Let $P^\infty(\mu) = L^\infty(\mu_0) \oplus H^\infty(\Omega_1, \mu_1) \oplus \ldots$ be the decomposition of $P^\infty(\mu)$ obtained in [11, Theorem 4.11]. To prove that $P^\infty(\mu)$ has Property S_σ, is suffices to show that each of the summands in the decomposition of $P^\infty(\mu)$ has Property S_σ [22, Proposition 1.11]. Since $L^\infty(\mu_0)$ is a type I von Neumann algebra, it has Property S_σ, and so it suffices to show that $H^\infty(\Omega_n, \mu_n)$ has Property S_σ for $n = 1, 2, \ldots$. Fix n, and set $\Omega = \Omega_n$ and $\nu = \mu_n$. Since Ω is bounded and simply connected, there is a bijection τ of the open unit disc D onto Ω. Define $\Phi: H^\infty(\Omega) \to H^\infty(D)$ by $\Phi(f) = f \circ \tau$. Then Φ is an isometric map onto $H^\infty(D)$. Define a map $\Psi: H^\infty(\Omega, \nu) \to H^\infty(\mathbf{T})$ as follows: if $f \in H^\infty(\Omega)$ and $\tilde{f}$ is the image of f in $H^\infty(\Omega, \nu)$, then $\Psi(\tilde{f})$ is the element of $H^\infty(\mathbf{T})$ obtained from $\Phi(f)$ by taking radial limits. Then Ψ maps $H^\infty(\Omega, \nu)$ onto $H^\infty(\mathbf{T})$, since Φ maps $H^\infty(\Omega)$ onto $H^\infty(D)$, and Ψ is isometric by [11, Lemma 4.2 and Theorem 4.11]. Since $H^\infty(\Omega, \nu)$ and $H^\infty(\mathbf{T})$ are both subspaces of abelian C^*-algebras, Ψ and Ψ^{-1} are completely bounded. We will next show that Ψ is σ-weakly bicontinuous. Since Ψ is an isometry and the predual of $H^\infty(\Omega, \nu)$ (with respect to the σ-weak topology) is a separable Banach space, it suffices to show that Ψ is σ-weakly sequentially continuous (see, e.g., [9, Theorems 2.3 and 2.7]). So suppose $\{\tilde{f}_n\}$ is a sequence in $H^\infty(\Omega, \nu)$ that converges σ-weakly to a function $\tilde{f}$ in $H^\infty(\Omega, \nu)$. Then $\sup_n \|\tilde{f}_n\|_\nu < \infty$, and so it follows from condition (c) of [11, Theorem 4.11] that $\{f_n\}$ is a uniformly bounded sequence in $H^\infty(\Omega)$. Moreover, for each z in Ω the functional on $H^\infty(\Omega, \nu)$ of evaluation at z is σ-weakly continuous [30, Remark 2, p. 11], and so $f_n(z) \to f(z)$ for all z in Ω. Hence the sequence $\{\Phi(f_n)\}$ is uniformly bounded in $H^\infty(D)$ and converges pointwise on D to $\Phi(f)$. It follows from this and definition of Ψ that $\Psi(\tilde{f}_n) \to \Psi(\tilde{f})$ σ-weakly (see, e.g., [11, Lemma 4.4]). Hence Ψ is σ-weakly sequentially continuous, and so it is σ-weakly bicontinuous. Since $H^\infty(\mathbf{T})$ has Property S_σ (Example 3.3), we can apply Proposition 4.2 to conclude that $H^\infty(\Omega, \nu) = H^\infty(\Omega_n, \mu_n)$ has Property S_σ. Hence $P^\infty(\mu)$ has Property S_σ. ▨

REMARK 4.4. Consider the following question: if $\mathscr{A}$ and $\mathscr{B}$ are maximal abelian subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively, is $\mathscr{A} \mathbin{\overline{\otimes}} \mathscr{B}$ a maximal abelian subalgebra of $B(\mathcal{H} \otimes \mathcal{K})$? If $\mathcal{A}$ and $\mathcal{B}$ are also self-adjoint, then it is well known (and follows from Tomita's theorem) that the answer is yes. The general question is open and is related to the question of whether every σ-weakly closed subspace has Property S_σ. For if $\mathcal{A}$ is a maximal abelian subalgebra of $B(\mathcal{H})$ with Property S_σ, and $\mathcal{B}$ is any maximal abelian subalgebra of $B(\mathcal{K})$, then $(\mathcal{A} \overline{\otimes} \mathcal{B})' = F(\mathcal{A}', \mathcal{B}') = F(\mathcal{A}, \mathcal{B}) = \mathcal{A} \overline{\otimes} \mathcal{B}$, so $\mathcal{A} \overline{\otimes} \mathcal{B}$ is maximal abelian. Hence, since abelian von Neumann algebras have Property S_σ, the tensor product of a m.a.s.a. with any maximal abelian algebra is maximal abelian. As another example, let S denote the unilateral shift, acting as multiplication by z on $\mathcal{H} = H^2(\mathbf{T})$. Then $\mathcal{A}(S)$ (which equals the algebra of analytic Toeplitz operators) is maximal abelian in $B(\mathcal{H})$ (see, e.g., [10, Corollary III.6.13]). Since S is subnormal, $\mathcal{A}(S)$ has Property S_σ, so $\mathcal{A}(S) \overline{\otimes} \mathcal{B}$ is maximal abelian in $B(\mathcal{H} \otimes \mathcal{K})$ whenever $\mathcal{B}$ is maximal abelian in $B(\mathcal{K})$.

If $T \in B(\mathcal{H})$, let $\mathcal{A}(T)_\perp$ denote the set $\{L \in \mathcal{C}_1(\mathcal{H}) : \mathrm{Tr}(AL) = 0$ for all $A \in \mathcal{A}(T)\}$ (where $\mathcal{C}_1(\mathcal{H})$ denotes the Banach space of trace-class operators on $\mathcal{H}$ with the trace norm) and let $Q(T)$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\mathcal{A}(T)_\perp$. Then $\mathcal{A}(T)$ is the dual space of $Q(T)$ under the pairing

$$\langle A, [L]\rangle = \mathrm{Tr}(AL), \quad A \in \mathcal{A}(T),\ [L] \in Q(T),$$

where $[L]$ is the image in $Q(T)$ of the operator L in $\mathcal{C}_1(\mathcal{H})$ (see, e.g., [9, Corollary 2.2]). If x and y are in $\mathcal{H}$, let $x \otimes y$ denote the rank-one operator $u \to (u, y)x$. Brown showed in [8] that if T belongs to a certain class of subnormal operators, then for every $[L]$ in $Q(T)$ there exist vectors x and y in $\mathcal{H}$ such that

$$[L] = [x \otimes y]. \tag{10}$$

He used this fact to show that every subnormal operator on $\mathcal{H}$ has a nontrivial invariant subspace. In [28] Olin and Thomson showed that every subnormal operator S on $\mathcal{H}$ has the property that the equation (10) can be solved for any $[L]$ in $Q(S)$, and used this result in proving that S is reflexive and $\mathcal{A}(S) = \mathcal{W}(S)$.

Brown's result was extended in another direction in [9], where it was shown that if T is a (BCP)-operator on $\mathcal{H}$, then the equation (10) can be solved for any $[L]$ in $Q(T)$, from which it follows that T has a nontrivial invariant subspace. (An operator T on $\mathcal{H}$ is said to be a (BCP)-operator if it is a completely nonunitary contraction (i.e., $\|T\| \leqslant 1$ and T has no nontrivial reducing subspace on which it acts as a unitary operator) and almost every point of $\mathbf{T}$ is a non-tangential limit point of the intersection of the essential spectrum of T with the open unit disc [4].) In [4] it was shown that (BCP)-operators are reflexive. This result was extended to a larger class of operators in [3]. A contraction is said to be absolutely continuous if its unitary part is absolutely continuous or acts on the space (0). The class $\mathrm{A}(\mathcal{H})$ [5] consists of all those absolutely continuous contractions T in $B(\mathcal{H})$ for which the

Sz.-Nagy—Foiaş functional calculus $\Phi_T\colon H^\infty(\mathbf{T}) \to \mathscr{A}(T)$ is an isometry. The class $\mathbf{A}_{\aleph_0}(\mathscr{H})$ [5] consists of all operators T in $\mathbf{A}(\mathscr{H})$ such that every system of simultaneous equations

$$[L_{ij}] = [x_i \otimes y_j], \quad 0 \leqslant i, j < \infty$$

(where the $[L_{ij}]$ are arbitrary elements in $Q(T)$) can be solved with vectors $\{x_i\}_{1 \leqslant i < \infty}$ and $\{y_j\}_{1 \leqslant j < \infty}$ from $\mathscr{H}$. It was shown in [6] that every (BCP)-operator on $\mathscr{H}$ is in $\mathbf{A}_{\aleph_0}(\mathscr{H})$, and it was shown in [3] that if $T \in \mathbf{A}_{\aleph_0}(\mathscr{H})$, then T is reflexive and $\mathscr{A}(T) = \mathscr{W}(T)$. This last result is of particular interest because, as demonstrated in [2], the class $\mathbf{A}_{\aleph_0}(\mathscr{H})$ is surprisingly large. (In particular, if T is either a unilateral or bilateral weighted shift in $B(\mathscr{H})$ that belongs to C_{00} and whose spectral radius and norm are both equal to 1, then $T \in \mathbf{A}_{\aleph_0}(\mathscr{H})$ [2, Theorem 3.6].) We refer the reader to [7] for a detailed survey (with proofs) of the results in [2, 3, 4, 5, 6, 9] and related papers.

We next show that the class of operators T in $B(\mathscr{H})$ for which $\mathscr{A}(T)$ has Property S_σ includes $\mathbf{A}(\mathscr{H})$ (and hence $\mathbf{A}_{\aleph_0}(\mathscr{H})$).

THEOREM 4.5. *Let $T \in B(\mathscr{H})$, $T \neq 0$, and suppose that $T/\|T\| \in \mathbf{A}(\mathscr{H})$. Then $\mathscr{A}(T)$ has Property S_σ.*

Proof. Since $\mathscr{A}(T) = \mathscr{A}(T/\|T\|)$, we can assume that $T \in \mathbf{A}(\mathscr{H})$. Let $\Phi = \Phi_T$ be the Sz.-Nagy—Foaiş functional calculus map for T. Since Φ is an isometry, it follows from [9, Theorem 3.2] that Φ is a σ-weakly bicontinuous map from $H^\infty(\mathbf{T})$ onto $\mathscr{A}(T)$. Moreover, Φ^{-1} is completely bounded, since $H^\infty(\mathbf{T})$ is contained in the abelian C^*-algebra $L^\infty(\mathbf{T})$. Hence, since $H^\infty(\mathbf{T})$ has Property S_σ, to complete the proof it suffices to show that Φ is completely bounded. Let U be the minimal unitary dilation of T [32, Theorem I.4.2], acting on a Hilbert space $\mathscr{K}$. Since T is an absolutely continuous contraction, it follows from [32, Theorem II.6.4] that U is absolutely continuous, and hence there is a Sz.-Nagy—Foiaş functional calculus $\Phi_U\colon H^\infty(\mathbf{T}) \to \mathscr{A}(U)$. Let P denote the projection from $\mathscr{K}$ onto $\mathscr{H}$. Then it follows from [32, Theorem III.2.1] that $\Phi(f) = P\Phi_U(f)|\mathscr{H}$ for every f in $H^\infty(\mathbf{T})$. Since $\mathscr{A}(U)$ is a subspace of the abelian C^*-algebra $W^*(U)$, Φ_U is completely bounded, and it is obvious that the map $A \to PA|\mathscr{H}$ from $B(\mathscr{K})$ to $B(\mathscr{H})$ is completely bounded, so Φ is completely bounded. Hence $\mathscr{A}(T)$ has Property S_σ. ▨

COROLLARY 4.6. *Let $T \in \mathbf{A}_{\aleph_0}(\mathscr{H})$, and let $\mathscr{L}_1 = \mathrm{lat}(T)$. Then for any subspace lattice $\mathscr{L}_2$ we have*

$$\mathrm{alg}\,\mathscr{L}_1 \mathbin{\overline{\otimes}} \mathrm{alg}\,\mathscr{L}_2 = \mathrm{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2).$$

Proof. Since $T \in \mathbf{A}(\mathscr{H})$, $\mathscr{A}(T)$ has Property S_σ. Since $T \in \mathbf{A}_{\aleph_0}(\mathscr{H})$, T is reflexive and $\mathscr{W}(T) = \mathscr{A}(T)$, so $\mathrm{alg}\,\mathscr{L}_1 = \mathscr{A}(T)$. The result now follows from Equation (3.3) on p. 372 of [22]. ▨

REMARK 4.7. If S is a subnormal operator on $\mathscr{H}$, then $\operatorname{alg}\operatorname{lat}(S) = \mathscr{A}(S)$ has Property S_σ by Theorem 4.1. Hence (2) is also valid when $\mathscr{L}_1 = \operatorname{lat}(S)$ and $\mathscr{L}_2$ is any subspace lattice. Thus the class of subspace lattices $\mathscr{L}_1$ for which (2) is valid for $\mathscr{L}_1$ and any subspace lattice $\mathscr{L}_2$ is greatly added to as a result of Theorem 4.1 and Corollary 4.6. It should be noted that the subspace lattices $\operatorname{lat}(T)$ (where T is subnormal or in $A_{\aleph_0}(\mathscr{H})$) can be quite complicated. For example, if $T \in A_{\aleph_0}(\mathscr{H})$, then $\operatorname{lat}(T)$ contains a lattice that is isomorphic to the lattice of all subspaces of $\mathscr{H}$ [7, Proposition 9.1]. For further information about $\operatorname{lat}(T)$ for T in $A_{\aleph_0}(\mathscr{H})$, we refer the reader to [7, Chapters IX and X].

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JON KRAUS
Department of Mathematics,
State University of New York at Buffalo,
Buffalo, New York 14214,
U.S.A.

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