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WEAK OPERATOR AND WEAK* TOPOLOGIES ON SINGLY GENERATED ALGEBRAS

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0. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. As is well known, $\mathscr{L}(\mathscr{H})$ can be identified as the dual of the trace class (τc), via the pairing

$$\langle A, L\rangle = \operatorname{tr}(AL), \quad A \in \mathscr{L}(\mathscr{H}),\ L \in (\tau c)$$

and thus carries a weak* topology.

For $T \in \mathscr{L}(\mathscr{H})$, let $\mathfrak{A}_T$ denote the smallest unital weak* closed subalgebra of $\mathscr{L}(\mathscr{H})$ containing T, and let $\mathscr{W}_T$ denote the smallest unital subalgebra of $\mathscr{L}(\mathscr{H})$ which is closed in the weak operator topology (WOT). Let $(\mathfrak{A}_T, \mathrm{w}^*)$ (respectively $(\mathfrak{A}_T, \mathrm{wot})$, $(\mathscr{W}_T, \mathrm{wot})$) denote the topological space $\mathfrak{A}_T$ (respectively $\mathfrak{A}_T$, $\mathscr{W}_T$) equipped with the weak* topology (respectively WOT, WOT).

In [3] it is shown that for a large class of operators $(\mathfrak{A}_T, \mathrm{w}^*) = (\mathscr{W}_T, \mathrm{wot})$. Furthermore, for certain operators T, agreement of the weak operator and weak* topologies on $\mathfrak{A}_T$ implies via [3, Theorem 6.13], among other things, that such T have large invariant subspace lattices and are reflexive. The questions then arise whether or not agreement of these topologies is a general phenomenon, and if there exist operators T for which $\mathfrak{A}_T \neq \mathscr{W}_T$. The answer to this second question is still unknown, even in the case that the weak* and relative weak operator topologies on $\mathfrak{A}_T$ are the same. However, below we show that there exist operators T for which $(\mathfrak{A}_T, \mathrm{w}^*) \neq (\mathfrak{A}_T, \mathrm{wot})$. In his survey article on shifts [9, Question 5], Allen Shields raises the question of what is the relationship between the weak* and weak operator topologies on $\mathscr{W}_T$ in the case T is an injective shift. (It is easy to show that $\mathfrak{A}_T = \mathscr{W}_T$ in this case.) The example given in Section 3 serves to show that even in this special case the two topologies can be different.

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1. PRELIMINARIES

Let ${}^{\perp}\mathfrak{A}_T$ denote the preannihilator of $\mathfrak{A}_T$ in (τc), and let Q_T denote the quotient space $(\tau c)/{}^{\perp}\mathfrak{A}_T$. It follows from general principles that $\mathfrak{A}_T$ may be regarded as the dual of Q_T via the pairing

$$\langle A, [L]\rangle = \mathrm{tr}(AL), \quad A \in \mathfrak{A}_T,\ [L] \in Q_T,$$

where $[L]$ denotes the equivalence class in Q_T which contains $L \in (\tau c)$. Also the weak* continuous linear functionals on $\mathfrak{A}_T$ are those that arise via the dual action of elements of Q_T. That is, φ is a weak* continuous linear functional on $\mathfrak{A}_T$ if, and only if, there exists an element $[L] \in Q_T$ such that

$$\varphi(A) = \langle A, [L]\rangle = \mathrm{tr}(AL), \quad A \in \mathfrak{A}_T.$$

The WOT continuous linear functionals on $\mathfrak{A}_T$ are those that arise via the dual action of finite rank operators on $\mathfrak{A}_T$. That is, see [3, Proposition 1.7], ψ is a WOT continuous linear functional on $\mathfrak{A}_T$, if, and only if, there exist finite sequences $\{x_i\}_{i=1}^n$, $\{y_i\}_{i=1}^n \subset \mathscr{H}$ such that

$$\psi(A) = \left\langle A, \sum_{i=1}^n [x_i \otimes y_i]\right\rangle = \mathrm{tr}\left(A \sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n (Ax_i, y_i), \quad A \in \mathfrak{A}_T,$$

where, as usual $x \otimes y$ denotes the rank-one operator given by $(x \otimes y)(u) = (u, x)y$. The following notion, introduced in [3], will prove useful. Here, and throughout, $\mathscr{H}$ is used to denote a possibly finite dimensional, separable, complex Hilbert space.

DEFINITION 1.1. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to prove property $(\mathbf{A}_{1/n})$ if every $[L] \in Q_T$ can be written in the form

$$[L] = \sum_{i=1}^n [x_i \otimes y_i] \tag{1.1}$$

for certain sequences $\{x_i\}_{i=1}^n$, $\{y_i\}_{i=1}^n$ of vectors from $\mathscr{H}$.

If T has property $(\mathbf{A}_{1/n})$, $r \geq 1$ and for every $s > r$ the sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ satisfying (1.1) can also be chosen to satisfy

$$\sum_{i=1}^n \|x_i\| \leq \sqrt{s}\,\|[L]\|, \quad \sum_{i=1}^n \|y_i\| \leq \sqrt{s}\,\|[L]\|,$$

then T is said to have property $(\mathbf{A}_{1/n}(r))$.

The following lemma can be proved in exactly the same way as [2, Corollary 1].

LEMMA 1.2. *If for some r and $n \geqslant 1$, $T \in \mathscr{L}(\mathscr{H})$ has property* $(\mathbf{A}_{1/n}(r))$, *then* $(\mathfrak{A}_T, \mathrm{w}^*) = (\mathscr{W}_T, \mathrm{wot})$.

DEFINITION 1.3. An operator $T \in \mathscr{L}(\mathscr{H})$ is called an *r-step unilateral weighted shift* if there exists a weight sequence $\{w_n\}_{n=1}^{\infty}$ and an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathscr{H}$ such that $Te_n = w_n e_{n+r}$, $n = 1, 2, \ldots$. For T acting on an m-dimensional space $\mathscr{K}$ we say T is an *r-step finite dimensional weighted shift* if there exists an orthonormal basis $\{e_n\}_{n=1}^{m}$ for $\mathscr{K}$ and a weight sequence $\{w_n\}_{n=1}^{m-r}$ such that

$$Te_n = w_n e_{n+r} \quad 1 \leqslant n \leqslant m - r,$$

$$Te_n = 0 \quad m - r + 1 \leqslant n \leqslant m,$$

and we denote this shift,

$$T = D_r(w_1, w_2, \ldots, w_{m-r}).$$

In the next section we show how, given n and $r \geqslant 1$, to construct a finite dimensional weighted shift which fails to have property $(\mathbf{A}_{1/n}(r))$. In the following section such operators are used to construct a shift T for which $(\mathfrak{A}_T, \mathrm{w}^*) \neq (\mathfrak{A}_T, \mathrm{wot})$. In the sequel the following notational convention will prove useful. If $x_i \in \mathbf{C}^n$, then x_i is written in component form $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$ and if $y_i \in \mathbf{C}^n$, then y_i is written in component form $y_i = (\overline{y}_{i,1}, \overline{y}_{i,2}, \ldots, \overline{y}_{i,n})$.

2. AN OPERATOR WITHOUT PROPERTY $(\mathbf{A}_{1/k}(k))$

Inspiration for this construction was drawn from an example of Hadwin and Nordgren [5], who showed that given $r > 1$, the operator on $\mathbf{C}^4$

$$\begin{bmatrix} 0 & n & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

fails to have property $(\mathbf{A}_1(r))$ if n is sufficiently large.

The operator constructed will not quite be the most straightforward possible since we wish to produce an operator which will be useful in later constructions. Namely, given positive integers k and v_k we require a 1-step shift T_k such that $T_k^{v_k}$ fails to have property $(\mathbf{A}_{1/k}(k))$.

The idea is to construct a v_k-step finite dimensional shifts $S_k = S_k(n)$, whose weights are powers of n, along with a functional $\varphi = \varphi(n)$ on $\mathfrak{A}_{S_k}$ such that the assumption

$$\varphi(A) = \sum_{i=1}^{k} (Ax_i, y_i), \quad A \in \mathfrak{A}_{S_k}, \ \|x_i\| \leqslant p, \ \|y_i\| \leqslant p \tag{2.1}$$

leads to conditions on the components of the x_i and y_i which cannot be simultaneously satisfied if n is sufficiently large. The next proposition indicates what these conditions are, and how they arise. It will be convenient to work with a condition weaker than that given in (2.1).

PROPOSITION 2.1. *Suppose* $U_n = D_r(n^{j_1}, n^{j_2}, \ldots, n^{j_{m-r}})$, *and that* $j_t > j_s$ *for* $s \neq t$. *(Note that this says the biggest weight of* U_n *is in the* $(t+r, t)^{\text{th}}$ *position of the matrix representation of* U_n.*) Suppose further that* φ *is a linear functional on* $\mathscr{L}(\mathbf{C}^m)$ *which satisfies*

$$\left| \varphi(U_n) - \sum_{i=1}^{k} (U_n x_i, y_i) \right| < 1, \quad \textit{where } \|x_i\| \leqslant p, \ \|y_i\| \leqslant p. \tag{2.2}$$

Then

$$\left| \sum_{i=1}^{k} x_{i,t} y_{i,t+r} - \frac{\varphi(U_n)}{n^{j_t}} \right| < \frac{1}{n}(1 + kp^2).$$

Proof.

$$\left\| \varphi(U_n) - \sum_{i=1}^{k} \sum_{l=1}^{m-r} n^{j_l} x_{i,l} y_{i,l+r} \right\| \leqslant 1 \qquad \Rightarrow$$

$$\Rightarrow \qquad \left| \frac{\varphi(U_n)}{n^{j_t}} - \sum_{i=1}^{k} x_{i,t} y_{i,t+r} \right| \leqslant \frac{1}{n^{j_t}} \left[1 + \sum_{i=1}^{k} \left| \sum_{\substack{l=1 \\ l \neq t}}^{m-r} n^{j_l} x_{i,l} y_{i,l+r} \right| \right] \leqslant$$

$$\leqslant \frac{1}{n} \left[1 + \sum_{i=1}^{k} \left| \sum_{l=1}^{m-r} x_{i,l} \cdot y_{i,l+r} \right| \right] \leqslant \frac{1}{n} [1 + kp^2].$$

Note that if $U_n = D_1(n^{j_1}, n^{j_2}, \ldots, n^{j_{m-1}})$, then $U_n^2 = D_2(n^{j_1+j_2}, n^{j_2+j_3}, \ldots$ $\ldots, n^{j_{m-2}+j_{m-1}})$ and so on. In case $\{j_i\}_{i=1}^{m-1}$ is the sequence $\alpha^{(1)}(k)$ given below, Lemma 2.2 shows how to locate the largest weights of U_n^r for $r = 1, 2, \ldots, m-1$.

For k a positive integer, order the first k^2 integers as follows.

$$k, 2k, \ldots, k^2, k-1, 2k-1, \ldots, k^2-1, \ldots, 1, k+1, \ldots, k^2-k+1. \tag{2.3}$$

Let $\alpha^{(1)} = \alpha^{(1)}(k) = \{\alpha_i^{(1)}(k)\}_{i=1}^{k^2}$ denote the sequence (2.3), and let $\alpha^{(r)} = \alpha^{(r)}(k) = \{\alpha_i^{(r)}(k)\}_{i=1}^{k^2-r+1}$ denote the sequence

$$\left\{\sum_{j=i}^{i+r-1} \alpha_j^{(1)}(k)\right\}_{i=1}^{k^2-r+1}.$$

Let $p_k(r)$ be the integer satisfying $1 \leqslant p_k(r) \leqslant k$ and $r + p_k(r) \equiv 1 \pmod k$.

LEMMA 2.2. *There is a unique largest element in the sequence $\alpha^{(r)}(k)$, which appears in the $p_k(r)^{\text{th}}$ position.*

Proof. Consider the sum

$$\sum_{j=i}^{i+r-1} \alpha_j^{(1)}(k). \tag{2.4}$$

The lemma says that for fixed r this sum is maximum when $0 \leqslant i \leqslant k$ and $i + r - 1 \equiv 0 \pmod k$. If $i > k$, then replacing (2.4) by

$$\sum_{j=i-k}^{i+r-1-k} \alpha_j^{(1)}(k)$$

produces a strictly larger sum. Hence for (2.4) to be maximum $0 \leqslant i \leqslant k$.

So suppose $0 \leqslant i \leqslant k$ and $i + r - 1 \equiv 0 \pmod k$. It is enough to show that varying i in the range $0 \leqslant i \leqslant k$ strictly decreases (2.4). First suppose $i > 1$ and is decreased by l, then terms dropped from (2.4) have sum at least

$$(k^2 - k + 1) + (k^2 - 2k + 1) + \ldots + (k^2 - lk + 1)$$

and terms added to (2.4) have sum

$$(i-1)k + (i-2)k + \ldots + (i-l)k.$$

Hence it is enough to show for $k \geqslant i > l$

$$\frac{(2k^2 - (l-1)k + 2)l}{2} > \frac{k(2i - l - 1)l}{2}$$

or, $2k^2 + 2 > 2ik$, which is clearly so.

Finally, suppose $i > k$ and is increased by l. Terms dropped from (2.4) have sum

$$ik + (i+1)k + \ldots + (i+l-1)k$$

and terms gained have sum at most

$$(k-1)+(2k-1)+\ldots+(lk-1).$$

Hence it is enough to show for $k \geqslant i+l$

$$\frac{kl(2i+l-1)}{2} > \frac{l(k+lk-2)}{2}$$

or, $2ki > k-1$, again which is clearly so.

We are now ready to construct S_k for $k = 1, 2, \ldots$. It is to be a v_k-step shift acting on a space $\mathscr{K}_k$ of dimension u_k. The following peculiar choice of u_k and v_k will be of benefit in the next section. If the only interest is to construct an operator without property $(\mathbf{A}_{1/k}(r))$, then it is enough to take $u_k = (k+1)^2+1$ and $v_k = 1$.

The sequence $\{u_k\}_{k=1}^{\infty}$ is determined from the following recurrence relation

$$u_1 = 4$$

$$u_k = ((k+1)^2+1)\sum_{i=1}^{k-1} u_i, \quad k = 2, 3, \ldots .$$

Let $v_k = \sum_{i=1}^{k-1} u_i$, then S_k is the block matrix

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \\ D_{1,1} & 0 & \cdot & \cdot & \cdot & \\ 0 & D_{1,2} & 0 & \cdot & \cdot & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & 0 \\ & & & 0 & & D_{1,(k+1)^2} \end{bmatrix},$$

where $D_{1,i}$ is a diagonal matrix acting on a v_k-dimensional space: $D_{1,i} = \operatorname{diag}\{n_k^{\alpha_i^{(1)}(k+1)}, 1, 1, \ldots, 1\}$, with $\alpha_i^{(1)}(k+1)$ being given by (2.3) and n_k being an integer to be specified later.

Note that S_k^r is the following matrix.

$$\begin{array}{l} \\ r \text{ blocks} \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right. \\ \\ \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccccc} 0 & 0 & \cdot & \cdot & & \\ 0 & 0 & \cdot & \cdot & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & 0 & & & & \\ D_{r,1} & 0 & 0 & \cdot & \cdot & \\ 0 & D_{r,2} & 0 & \cdot & & \\ 0 & 0 & \cdot & \cdot & & \\ & & & \ddots & & \\ & & & & D_{r,(k+1)^2+1-r} & \underbrace{0\ 0\ \cdot\ \cdot\ \cdot\ 0}_{r \text{ blocks}} \end{array}\right],$$

where $D_{r,i} = \operatorname{diag}\{n_k^{\alpha_i^{(r)}(k+1)}, 1, 1, \ldots, 1\}$. Hence, using Lemma 2.2 we assert that for $n_k > 1$ the shift S_k^r has a unique largest weight, which is the $((p_{k+1}(r) - 1)v_k + 1)^{\text{th}}$ element of its weight sequence.

We define a functional φ_k on $\mathscr{L}(\mathscr{H}_k)$ as follows: for $A = (a_{i,j}) \in \mathscr{L}(\mathscr{H}_k)$

$$\varphi_k(A) = (k+1)^{-3}(a_{(k+1)v_k+1,kv_k+1} + a_{2(k+1)v_k+1,(k-1)v_k+1} + \ldots$$
$$\ldots + a_{(k+1)^2 v_k+1,1}).$$

Note that $\|\varphi_k\| \leqslant (k+1)^{-2}$ and that the elements of $(a_{i,j})$ used in the definition of φ_k are all in positions occupied by the largest weights of S_k^r for $r = 1, (k+2)+1, 2(k+2)+1, \ldots, (k+1)^2$.

The next proposition, which has a corollary that given $m > 1$, for n_k sufficiently large S_k fails to have property $(\mathbf{A}_{1/k}(m))$, will be the key tool in the construction of the next section. The notation used in the proof should include evidence that $\{x_i\}$, $\{y_i\}$ and S_k depend on n_k, however we abuse notation to avoid a profusion of subscripts and superscripts.

PROPOSITION 2.3. *Given $m > 1$, there exists n_k such that if $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ satisfy*

$$\left| \varphi_k(S_k^r) - \sum_{i=1}^k (S_k^r x_i, y_i) \right| < 1, \quad r = 1, 2, \ldots, (k+1) \tag{2.5}$$

then $\max\{\|x_i\|, \|y_j\|\}_{i,j=1}^k \geqslant m$.

Proof. We suppose that for all n_k we can satisfy (2.5) with vectors of norm less than m and obtain a contradiction.

Let $\varepsilon > 0$ be given. Note that the largest weight of S_k is the $(kv_k + 1)^{\text{th}}$ and is equal to $n_k^{(k+1)^2}$. Hence since $\varphi_k(S_k) = (k+1)^{-3} n_k^{(k+1)^2}$ Proposition 2.1 implies that for n_k sufficiently large

$$\left| (k+1)^{-3} - \sum_{i=1}^{k} x_{i,kv_k+1} y_{i(k+1)v_k+1} \right| < \varepsilon. \tag{2.6.0.1}$$

Performing similar calculations for S_k^r for $r = 2, 3, \ldots, k+1$, and noting that $\varphi_k(S_k^r) = 0$ for such r gives the following inequalities if n_k is sufficiently large

$$\left| \sum_{i=1}^{k} x_{i,(k-1)v_k+1} y_{i,(k+1)v_k+1} \right| < \varepsilon \tag{2.6.1.1}$$

$$\left| \sum_{i=1}^{k} x_{i,(k-2)v_k+1} y_{i,(k+1)v_k+1} \right| < \varepsilon \tag{2.6.1.2}$$

$$\vdots$$

$$\left| \sum_{i=1}^{k} x_{i,1} y_{i,(k+1)v_k+1} \right| < \varepsilon. \tag{2.6.1.k}$$

Next, using the fact that

$$\varphi_k(S_k^r) = 0 \quad k+4 \leqslant r \leqslant 2(k+1)$$

and

$$\varphi_k(S_k^r) = n_k^{(k+1)^3} \quad r = k+3$$

and repeating the above calculations for these values of r shows that we can assume that n_k is so large that the following inequalities are also satisfied.

$$\left| (k+1)^{-3} - \sum_{i=1}^{k} x_{i,(k-1)v_k+1} y_{i,2(k+1)v_k+1} \right| < \varepsilon \tag{2.6.2.1}$$

$$\left| \sum_{i=1}^{k} x_{i,(k-2)v_k+1} y_{i,2(k+1)v_k+1} \right| < \varepsilon \tag{2.6.2.2}$$

$$\vdots$$

$$\left| \sum_{i=1}^{k} x_{i,1} y_{i,2(k+1)v_k+1} \right| < \varepsilon. \tag{2.6.2.k}$$

By examining $\varphi_k(S_k^r)$ for various values of r, we can assume n_k is so large that in all we get $k^2 + k + 1$ inequalities: (2.6.0.1) and $k + 1$ groups of k inequalities, the j^{th} of these groups being

(2.6.j.1) $$\left| \sum_{i=1}^{k} x_{i,(k-1)v_k+1} y_{i,j(k+1)v_k+1} \right| < \varepsilon$$

$$\vdots$$

(2.6.j.j) $$\left| (k+1)^{-3} - \sum_{i=1}^{k} x_{i,(j-1)v_k+1} y_{i,j(k+1)v_k+1} \right| < \varepsilon$$

$$\vdots$$

(2.6.j.k) $$\left| \sum_{i=1}^{k} x_{i,1} y_{i,j(k+1)v_k+1} \right| < \varepsilon$$

which is obtained by looking at $\varphi_k(S_k^r)$ for $j(k+1) - k + 1 \leqslant r \leqslant j(k+1)$.

Let $s_i \in \mathbf{C}^k$ be the vector

$$(x_{1,(k-i)v_k+1}, x_{2,(k-i)v_k+1}, \ldots, x_{k,(k-i)v_k+1}) \quad i = 0, 1, \ldots, k$$

and let $t_i \in \mathbf{C}^k$ be the vector

$$(\overline{y}_{1,i(k+1)v_k+1}, \overline{y}_{2,i(k+1)v_k+1}, \ldots, \overline{y}_{k,i(k+1)v_k+1}) \quad i = 1, 2, \ldots, k+1.$$

The system (2.6) can then be written

(2.7.0.1) $$|(k+1)^{-3} - (s_0, t_1)| < \varepsilon$$

(2.7.1.1) $$|(s_1, t_1)| < \varepsilon$$

(2.7.1.2) $$|(s_2, t_1)| < \varepsilon$$

$$\vdots$$

(2.7.1.k) $$|(s_k, t_1)| < \varepsilon$$

(2.7.2.1) $$|(k+1)^{-3} - (s_1, t_2)| < \varepsilon$$

(2.7.2.2) $$|(s_2, t_2)| < \varepsilon$$

$$\vdots$$

(2.7.2.k) $$|(s_k, t_2)| < \varepsilon$$

$$\vdots$$

(2.7.$k+1$.1) $$|(s_1, t_{k+1})| < \varepsilon$$

$$\vdots$$

(2.7.k+1.k−1) $$|(s_{k-1}, t_{k+1})| < \varepsilon$$

(2.7.k+1.k) $$|(k+1)^{-3} - (s_k, t_{k+1})| < \varepsilon.$$

The supposition is that we can solve the system (2.7) for all values of n_k. If this is so, letting ε tend to zero, noting that $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ belong to a compact set and that the left hand sides of the system (2.7) depend continuously on $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ we can obtain a solution to the system of equations obtained by setting the left hand sides of the system (2.7) all equal to zero. We label this system (2.8.0.1), (2.8.1.1),...,(2.8.k+1.k), but do not write it down.

It follows from (2.8.2.1)–(2.8.k+1.k) that $s_1, s_2, \ldots, s_k$ are linearly independent. Equation (2.8.0.1) shows that $t_1 \neq 0$, and it then follows that (2.8.1.1)–(2.8.1.k) cannot all be satisfied, and this contradiction completes the proof.

The following corollary is an obvious consequence of Proposition 2.3.

COROLLARY 2.4. *Suppose n_k is such that*

$$\left| \varphi_k(S_k^r) - \sum_{i=1}^{k} (s_k^r x_i, y_i) \right| < 1, \qquad r = 1, 2, \ldots, (k+1)^2$$

implies that $\max\{\|x_i\|, \|y_j\|\}_{i,j=1}^{k} \geqslant m$. *Then if $n \leqslant k$ and*

$$\left| \varphi_k(S_k^r) - \sum_{i=1}^{n} (S_k^r x_i', y_i') \right| < 1, \qquad r = 1, 2, \ldots, (k+1)^2,$$

we have $\max\{\|x_i'\|, \|y_j'\|\}_{i,j=1}^{n} \geqslant m$.

Finally we define the shift T_k mentioned at the beginning of this section. First, using Proposition 2.3 choose n_k, fixed once and for all, such that $n_k \geqslant n_{k-1}$ and

$$\left| \varphi_k(S_k^r) - \sum_{i=1}^{k} (S_k^r x_i, y_i) \right| < 1, \qquad r = 1, 2, \ldots, (k+1)^2$$

implies that $\max\{\|x_i\|, \|y_j\|\}_{i,j=1}^{k} \geqslant k$. Then define T_k to be a one step shift acting on $\mathscr{K}_k$ with weight sequence $\{\{w_{k,i}\}_{i=1}^{u_k-1}$, where we set $w_{k,1} = w_{k,2} = \ldots = w_{k,v_k-1} = 1$, and for $i = 1, 2, \ldots, u_k - v_k$ the value of w_{k,v_k+i-1} is found from the relation $w_{k,i} w_{k,i+1} \ldots w_{k,v_k+i-1} = z_i$, where z_i is the i^{th} weight of S_k. Note that $T_k^{v_k} = S_k$.

3. THE MAIN EXAMPLE

Let $\mathscr{K}_k$ and T_k be as given in Section 2, let $\mathscr{H} = \sum_{k=1}^{\infty} {}^{\oplus}\, \mathscr{K}_k$ and let $T \in \mathscr{L}(\mathscr{H})$ be the shift given by the following matrix.

$$T = \left[\begin{array}{c|c|c} c_1 T_1 & & \\ \hline \begin{matrix} \ldots 0 & 0 & \mu_1 \\ & 0 & 0 \\ & & 0 \end{matrix} & c_2 T_2 & \\ \hline & \begin{matrix} 0 & 0 & \mu_2 \\ & 0 & 0 \\ & & 0 \end{matrix} & c_3 T_3 \; \ddots \end{array} \right], \tag{3.1}$$

where $c_1 = 1$, and for $k > 1$

$$c_k = \frac{c_{k-1}}{2(k-1)^3 \|T_k\|} .$$

The choice of the sequence of scalars μ_k is more involved. For each k it is easy to see that μ_k is the $(v_{k+1})^{\text{th}}$ weight of T and, setting $b = \max\{1, v_{k+1} + 1 - r\}$, we have that for each r the b^{th} to $(v_{k+1})^{\text{th}}$ weights of T^r contain this weight as a factor. By induction choose μ_k to satisfy

(i) $\mu_k > 0$.

(ii) All weights of $(c_{k+1}^{-1}T)^r$ which contain the $(v_{k+1})^{\text{th}}$ weight of T as a factor are less than $(2(k+1))^{-3}$, for $1 \leqslant r \leqslant u_{k+1}$.

PROPOSITION 3.1. *If T is given by* (3.1), *then the weak operator and weak* topologies on $\mathfrak{A}_T$ are different.*

Proof. It will be enough to show that there exists a linear functional on $\mathfrak{A}_T$ which is weak* continuous, but not WOT continuous.

Let $\hat{\varphi}_k$ be the linear functional on $\mathscr{K}(\mathscr{H})$ which satisfies

$$\hat{\varphi}_k(e_i \otimes e_j) = 0 \quad \text{if } e_i \text{ or } e_j \leqslant v_k \text{ or, } e_i \text{ or } e_j > u_{k+1}$$

and

$$\hat{\varphi}_k(e_i \otimes e_j) = \varphi_k(e_{i-v_k}^{(k)} \otimes e_{j-v_k}^{(k)}) \quad \text{otherwise,}$$

where φ_k is the functional given in Section 2, $\{e_i^{(k)}\}_{i=1}^{u_k}$ is the orthonormal basis of $\mathscr{K}_k$ with respect to which T_k is written, and $\{e_i\}_{i=1}^{\infty}$ is the orthonormal basis of $\mathscr{H}$ with respect to which T is written.

It is easy to see that $\hat{\varphi}_k$ is a WOT continuous (therefore weak* continuous) linear functional on $\mathscr{L}(\mathscr{H})$ of norm at most $(k+1)^{-2}$. Hence, since a norm limit of weak* continuous linear functionals is weak* continuous, we get that $\hat{\varphi} = \sum_{k=1}^{\infty} \hat{\varphi}_k$ is a weak* continuous linear functional on $\mathscr{L}(\mathscr{H})$ of norm at most $\pi^2/6 - 1 < 1$.

Define φ to be $\hat{\varphi}$ restricted to $\mathfrak{A}_T$.

To complete the proof we assume that φ is WOT continuous and obtain a contradiction. So suppose that there exist $\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n \subset \mathscr{H}$ such that for $A \in \mathfrak{A}_T$

$$\varphi(A) = \sum_{i=1}^{n} (Ax_i, y_i).$$

Let $m = \max\{\|x_i\|, \|y_j\|\}_{i,j=1}^n$ and pick k so that $k \geqslant \max\{m, n\}$.

Define φ_1 to be the extension of φ to $\mathscr{L}(\mathscr{H})$ given by

$$\varphi_1(B) = \sum_{i=1}^{n} (Bx_i, y_i) \quad B \in \mathscr{L}(\mathscr{H}).$$

Note that if $A \in \mathfrak{A}_T$ is written $A = A_1 + A_2$, where A_1 and A_2 do not necessarily belong to $\mathfrak{A}_T$, then $\varphi(A) = \hat{\varphi}(A_1) + \hat{\varphi}(A_2) = \varphi_1(A_1) + \varphi_1(A_2)$. Let $S = (c_k^{-1}T)^{v_k}$ and look at $\varphi(S^r)$ for $r = 1, 2, \ldots, (k+1)^2$. Let

$$T_{1,r} = P_{\mathscr{H}_k} S^r|_{P_{\mathscr{H}_k}\mathscr{H}}, \quad T_{2,r} = T - T_{1,r}, \quad x_i' = P_{\mathscr{H}_k} x_i \text{ and } y_i' = P_{\mathscr{H}_k} y_i.$$

Then it is easy to see that

$$\varphi_1(T_{1,r}) = \sum_{i=1}^{n} (S_k^r x_i', y_i')$$

and

$$\hat{\varphi}(T_{1,r}) = \varphi_k(S_k^r),$$

where S_k is as given in Section 2. Note that

$$0 = \varphi(T_{1,r} + T_{2,r}) - \varphi(T_{1,r} + T_{2,r}) =$$

$$= \varphi_1(T_{1,r}) - \hat{\varphi}(T_{1,r}) + \varphi_1(T_{2,r}) - \hat{\varphi}(T_{2,r})$$

and thus

$$\left| \sum_{i=1}^{n} (S_k^r x_i', y_i') - \varphi_k(S_k^r) \right| \leqslant |\varphi_1(T_{2,r})| + |\hat{\varphi}(T_{2,r})|. \tag{3.2}$$

Also we have

$$|\hat{\varphi}(T_{2,r})| \leqslant \|T_{2,r}\| \tag{3.3}$$

and

$$|\varphi_1(T_{2,r})| \leqslant \|T_{2,r}\| \sum_{i=1}^{n} \|x_i\| \|y_i\| \leqslant nk^2 \|T_{2,r}\| \leqslant k^3 \|T_{2,r}\|. \tag{3.4}$$

Next we show that $\|T_{2,r}\| \leqslant 1/2k^3$. To do this it is enough to note that $T_{2,r}$ is an rv_k step shift and to show that every weight of $T_{2,r}$ is less than $1/2k^3$.

By the choice of $\{\mu_k\}$ and $\{c_k\}$ all weights of $c_k^{-1}T$ beyond and including the $(v_{k+1})^{\text{th}}$ are less than $1/2k^3$, thus all weights of $T_{2,r}$ beyond and including the $(v_{k+1})^{\text{th}}$ are less than $(2k^3)^{-rv_k} < 1/2k^3$.

The first v_k weights of $T_{2,r}$ contain the v_k^{th} weight of $c_k^{-1}T$ as a factor and are thus less than $1/2k^3$ by the choice of μ_{k-1}.

The $(v_k + 1)^{\text{th}}$ to $(v_{k+1} - rv_k)^{\text{th}}$ weights of $T_{2,r}$ are zero.

The remaining weights contain the $(v_{k+1})^{\text{th}}$ weight of $(c_k^{-1}T)^{rv_k}$ as a factor. By the choice of μ_k these weights must be less than $c_{k+1}^{rv_k}(2(k+1)^3 c_k^{rv_k})^{-1} < 1/2k^3$.

Finally, combining (3.2) to (3.4) and applying Corollary 2.4 we get

$$\max\{\|x_i'\|, \|y_j'\|\}_{i,j=1}^{n} \geqslant k.$$

This contradiction completes the proof.

4. FURTHER EXAMPLES

The operator T given by (3.1) can be used to construct operators S for which $(\mathfrak{A}_S, \text{w}^*) \neq (\mathcal{W}_S, \text{wot})$, but which do possess other interesting properties. Recall that a subset Λ of the open unit disc $\mathbf{D}$ is *dominating* if for every $h \in H^\infty(\mathbf{D})$ we have $\|h\|_\infty = \sup\{|h(\lambda)| : \lambda \in \Lambda\}$. A completely nonunitary contraction S is called a (BCP)-operator if the intersection of its essential spectrum, $\sigma_e(S)$, and $\mathbf{D}$ is dominating. The study of such operators was initiated in [4], where a more restrictive definition was in use. Subsequently Robel [7] showed that all (BCP)-operators have property $(\text{A}_1(1))$. It then follows from Lemma 2.1 that if S is a (BCP)-operator then $(\mathfrak{A}_S, \text{w}^*) = (\mathcal{W}_S, \text{wot})$. I am grateful to Hari Bercovici for suggesting the first example below.

PROPOSITION 4.1. *There exists an operator S such that S^{-1} is a* (BCP)*-operator, and hence* $(\mathfrak{A}_{S^{-1}}, \text{w}^*) = (\mathcal{W}_{S^{-1}}, \text{wot})$, *but* $(\mathfrak{A}_S, \text{w}^*) \neq (\mathcal{W}_S, \text{wot})$.

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be a dominating subset of $\mathbf{D}$ in which each point is isolated. (Such sets are shown to exist in [8]. See [1, Theorem 5] for a stronger statement.) Let

$$S = \sum_{n=1}^{\infty} {}^{\oplus} (\lambda_n^{-1} + r_n T)$$

where T is the operator given in (3.1) and $r_n = (3\|T\|)^{-1}\text{dist}(\lambda_n^{-1}, \{\lambda_j^{-1} : j \neq n\})$. Then it is easy to see that $\sigma_e(S)$ is the closure of $\{\lambda_n^{-1} : n = 1, 2, \ldots\}$ and thus S^{-1} is a (BCP)-operator. To complete the proof it is enough to show

$$\mathfrak{A}_S = \sum_{n=1}^{\infty} {}^{\oplus}\ \mathfrak{A}_{\lambda_n^{-1} + r_n T}.$$

For each $n = 1, 2, \ldots$ there clearly exists an analytic function which takes the value 1 in a neighborhood of λ_n^{-1} and 0 on an open set containing $\{\lambda_j^{-1} : j \neq n\}$. The result then follows from [6, Theorem 2.12].

Let $\{\lambda_n\}_{n=1}^{\infty}$ be as above and write $\lambda_n = r_n e^{i\theta(n)}$, where $0 \leqslant \theta(n) < 2\pi$, and let $\mu_n = r_n^{1/2} e^{i\theta(n)/2}$. Using an argument similar to the above, it is possible to construct an example of an operator S such that $\sigma_e(S) = \{\mu_n : h = 1, 2, \dots\}^-$ but $(\mathfrak{A}_S, \mathrm{w}^*) \neq (\mathcal{W}_S, \mathrm{wot})$. This leads to the following.

PROPOSITION 4.2. *There exists an operator S such that $(\mathfrak{A}_S, \mathrm{w}^*) \neq (\mathcal{W}_S, \mathrm{wot})$ but each of the following is a* (BCP)*-operator:*

(a) $S \oplus - S$,

(b) $S \oplus S^*$,

(c) S^2.

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