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OPERATORS WITH SPECTRAL SINGULARITIES

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1. INTRODUCTION

It is the basic idea in the theory of spectral operators that for a large number of linear operators (the closed spectral operators in a complex Banach space in the sense of Dunford and Bade, cf. [5]) there exists a strongly countably additive spectral measure, defined on the σ-algebra of all Borel subsets of the complex plane, which is closely connected with the given operator. J. T. Schwartz [17] noticed first that for some operators (even in Hilbert space) the construction of such a completely satisfying spectral measure is impeded by the fact that the natural candidate for the corresponding measure becomes unbounded in the neighborhood of and cannot be defined for some Borel subset of the spectrum. Similar phenomena have been pointed out by Ljance [11] in connection with a class of second-order non-selfadjoint ordinary differential operators, and by several authors (cf. Harvey [8], Langer [10]) studying definitizable operators in Kreĭn spaces.

Attempting to construct a general theory for such (bounded) operators, Bacalu [1], [2] and the present author [12] have studied the so called S-spectral operators, for which the spectral measure is defined only for a smaller σ-algebra of the Borel subsets which avoid or contain a bad exceptional closed subset S of the spectrum. However, such spectral measures must still be uniformly bounded, so cannot play the role of the measures mentioned in the preceding paragraph.

The main idea in this paper is that we consider such spectral measures (more exactly, resolutions of the identity) that are defined and strongly countably additive only on a Boolean algebra of the Borel subsets of the complex plane (which, in some specified sense, avoid or contain a "bad" exceptional closed subset S). Such measures can already be not uniformly bounded, moreover, it will be shown that for each closed operator there is a smallest one among these bad closed sets. It is then natural to call this smallest set the set of the spectral singularities (or the smallest critical set) of the operator, thereby giving the general definition of a concept, which has been up to now defined only in some particular cases (see, e.g., the papers mentioned above). Making use of the notion of a spectral operator of scalar type, similar, but stronger concepts will be defined, and the existence of the smallest exceptional set (in this sense) will be proved (in our notation the set $K_0(T)$ for the operator T).

In the final section we shall show that the set of the spectral singularities for differential operators in the class studied by Ljance [11] is equal to the set of the spectral singularities as defined by Ljance for this particular case.

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2. PRELIMINARIES AND NOTATIONS

Let X be a complex Banach space, let $L(X)$ and $C(X)$ denote the set of all bounded and closed linear operators in X, respectively, and let $\overline{\mathbf{C}}$ denote the compactification of the complex plane $\mathbf{C}$. For a set b in $\overline{\mathbf{C}}$ let b^c denote $\overline{\mathbf{C}}\setminus b$, and let $\overline{b}$, b^0 and $h(b)$ denote the closure, the interior and the boundary of b in the topology of $\overline{\mathbf{C}}$ (unless we say explicitly otherwise), respectively. For $T \in C(X)$ and a T-invariant closed subspace Y let $\sigma(T, Y)=\sigma(T \mid Y)$ denote the (extended) spectrum of the restriction $T|Y$ (the extended spectrum of T is its usual spectrum $s(T)$ if $T \in L(X)$, and is $s(T) \cup \{\infty\}$ otherwise).

Let A be a Boolean algebra (with respect to the usual set operations) of Borel sets in $\overline{\mathbf{C}}$. A homomorphism E of A into a Boolean algebra of projections in $L(X)$ is called an *A-spectral measure* if $E(u) = I$ (here u and I are the identities in A and $L(X)$, respectively), and E is countably additive on A in the strong operator topology. Such an E is called an *A-resolution of the identity* for $T \in C(X)$ if

$$E(a)T \subset TE(a), \quad \sigma(T, E(a)X) \subset \overline{a} \quad (a \in A).$$

We say that T is *A-spectral,* if T has an A-resolution of the identity E. The restriction $T \mid E(a)X$ will then occasionally be denoted by T_a.

Let S be a closed set in $\overline{\mathbf{C}}$, and let A be one of the following Boolean algebras of sets:

$$B_S = \{b \text{ Borel set in } \overline{\mathbf{C}} : b \cap S = \emptyset \text{ or } b \supset S\},$$

$$D_0(S) = \{b \text{ Borel set in } \overline{\mathbf{C}} : \overline{b} \cap S = \emptyset \text{ or } b^0 \supset S\}.$$

Then B_S is a σ-algebra, and instead of B_S-spectral, etc., we shall equivalently say S-spectral, etc.

Let $T \in C(X)$, S as above, and let $K = S \cap \sigma(T)$. If E is a B_S-resolution of the identity for T, then (cf. [12]) E extends to the following K-resolution of the identity F for T:

$$F(b) = \begin{cases} E(b \cup S) & \text{for } b \supset K \\ E(b \cap S^c) & \phantom{\text{for }} b \subset K^c \end{cases} \quad (b \in B_K).$$

Further, if E is a $D_0(S)$-resolution of the identity for T, and $D(K) = D(K, T) = \{b$ Borel subset of $\sigma(T) : \bar{b} \cap K = \emptyset$ or $b^{or} \supset K\}$ (where b^{or} denotes the interior of b in the relative topology of $\sigma(T)$), then (cf. [15]) with the definition $\rho(T) = \sigma(T)^c$ and

$$F(b) = \begin{cases} E(b) & \text{if } \bar{b} \cap K = \emptyset \\ E(b \cup \rho(T)) & \phantom{\text{if }} b^{or} \supset K \end{cases} \quad (b \in D(K))$$

F is a $D(K)$-resolution of the identity for T. So we can occasionally assume in both cases that $S \subset \sigma(T)$, and consider rather $D(S)$- than $D_0(S)$-resolutions of the identity.

Now we recall some definitions and notations from the theory of decomposable operators (cf., e.g., [14]). Let $T \in C(X)$ and $x \in X$. The open set $\delta_T(x)$ consists of those $z \in \bar{\mathbf{C}}$, in a neighborhood U of which there is a holomorphic X-valued function f such that $(u - T)f(u) = x$ $(u \in U \cap \mathbf{C})$. Such an f is called a *T-associated function* of x (in U). There is a largest one, N_T, among the open sets $N \subset \bar{\mathbf{C}}$, in every open subset of which $f \equiv 0$ is the unique T-associated function of $0 \in X$. Let

$$S_T = N_T^c, \quad \gamma_T(x) = \delta_T(x)^c, \quad \sigma_T(x) = \gamma_T(x) \cup S_T, \quad \rho_T(x) = \sigma_T(x)^c.$$

The set S_T is called the *analytic residuum* of T. If $S_T = \emptyset$, we say that T has the single-valued extension property. For any $e \subset \bar{\mathbf{C}}$ let

$$X_T(e) = \{x \in X : \sigma_T(x) \subset e\}.$$

For any closed set f in $\bar{\mathbf{C}}$ let $X_{T,f}$ or $X(T,f)$ denote the largest one (if exists) among the closed T-invariant subspaces Y such that $\sigma(T, Y) \subset f$. These subspaces $X_{T,f}$ are called *spectral maximal* for T.

Let $S = \bar{S} \subset \bar{\mathbf{C}}$. The operator $T \in C(X)$ is called *strongly S-decomposable* if for each open S-covering of $\sigma(T)$, i.e. for each pair of open sets (G_1, G_S) such that $G_1 \cup G_S \supset \sigma(T) \cup S$, $\bar{G}_1 \cap S = \emptyset$ there are T-invariant subspaces $X_1 \subset D(T)$ $X_S \subset X$ such that $\sigma(T, X_i) \subset G_i$ $(i = 1, S)$, and for every T-spectral maximal subspace Y we have

$$(*) \qquad Y = Y \cap X_1 + Y \cap X_S.$$

T is called *S-decomposable,* if we postulate (*) only for the case $Y = X$. The smallest one among the sets S for which T is (strongly) S-decomposable is called the (*strong*) *spectral residuum* of T ([14]), and will be denoted by $S^*(T)$ ($S^{**}(T)$).

Note that the point ∞ in the spectrum of an operator T in $C(X) \setminus L(X)$ could be handled in different ways. For example, a closed operator T such that $\sigma(T) = \{\infty\}$ is $\emptyset$-spectral in the sense given here. However, T is not spectral in the sense of [5; XVIII.2.1]. Clearly, T has this latter property if and only if T has a $\emptyset$-resolution of the identity E satisfying $E(\{\infty\}) = 0$. A similar problem causes an apparent slight inconsistency in the formulation of Theorem 3 below.

3. CLOSED S-SPECTRAL OPERATORS

Let $T \in C(X)$ and let S be a closed subset of $\sigma(T)$. With the aim of application in the next section we extend here the theory of bounded S-spectral operators ([12]) to this more general case. The development will closely parallel that of [12], therefore we omit several proofs and refer for them to [12]. However, since an immediate generalization of the fact "any spectral maximal subspace of a bounded operator is hyperinvariant" (cf. Foiaş [6]) is not available, there must be some changes and we shall present them.

LEMMA 1. *If E is an S-resolution of the identity for T, further $z \in S^c \cap \mathbf{C}$ and $(z - T)x = 0$, then $E(\{z\})x = x$.*

LEMMA 2. *If T is S-spectral, then $S_T \subset S$.*

LEMMA 3. *If E is an S-resolution of the identity for T, the set c is closed in $\overline{\mathbf{C}}$ and disjoint from S, $x \in E(c)X$ and $\sigma_T(x) \cap c = \emptyset$, then $x = 0$.*

LEMMA 4. *If E is an S-resolution of the identity for T, and the closed set $e \subset \overline{\mathbf{C}}$ contains S, then $E(e)X = X_T(e)$.*

COROLLARY. *If E is an S-resolution of the idenity for T, then $E(\sigma(T)) = I$.*

The proof of the following commutativity theorem makes use of the lemmas above, but differs from the corresponding proof for the bounded case in [12].

THEOREM 1. *If E is an S-resolution of the identity for T, and the operator $A \in L(X)$ satisfies $AT \subset TA$, then $AE(b) = E(b)A$ for every $b \in B_S$.*

Proof. By assumption, $(z - T)f(z) = x$ implies $(z - T)Af(z) = Ax$. Hence $\gamma_T(Ax) \subset \gamma_T(x)$ and $\sigma_T(Ax) \subset \sigma_T(x)$ for any x in X. This and Lemma 4 yield $E(c)AE(c) = AE(c)$ for any closed set c satisfying $S \subset c \subset \overline{\mathbf{C}}$.

Let the closed set d be a subset of S^c, and let $c = d \cup S$. For any closed T-invariant subspace W of X such that $\sigma(T | W) \subset d$, Lemmas 2 and 4 imply

$W \subset E(c)X$. Hence $\sigma(T_c | W) \subset d$. Let D be a Cauchy domain ([18; p. 288]) such that $d \subset D$, $\bar{D} \subset S^c$, and let $h(D)$ denote its positively oriented boundary. For every w in W then

$$w = k \int_{h(D)} (z - T_c | W)^{-1} w \, dz + jw = k \int_{h(D)} (z - T_c)^{-1} E(d \cup S) w \, dz + jw =$$

$$= k \int_{h(D)} (z - T_d)^{-1} E(d) w \, dz + jE(d) w + k \int_{h(D)} (z - T_S)^{-1} E(S) w \, dz + jE(S) w =$$

$$= E(d) w,$$

where $k = (2\pi i)^{-1}$, $j = 1$ if $\infty \in D$ and $j=0$ otherwise. Hence $E(d)X$ is the spectral maximal subspace $X_{T,d}$ for T.

Let $X(d)$ denote the range of the spectral projection $P(d)$ associated with the part in d of $\sigma(T_c)$. Then $\sigma(T|X(d)) = \sigma(T_c | X(d)) \subset d$ implies $X(d) \subset E(d)X$. On the other hand, $x = E(d)x$ implies

$$P(d)x = k \int_{h(D)} (z - T_c)^{-1} x \, dz + jx = k \int_{h(D)} (z - T_d)^{-1} x \, dz + jx = x.$$

Hence $X(d) = E(d)X$.

Let $x \in E(c)X \cap D(T)$. Since $AT \subset TA$, and $AE(c) = E(c)AE(c)$, we have $AT_c x = T_c Ax$, hence $A(z - T_c)^{-1} = (z - T_c)^{-1} A$ for every z in $\rho(T_c) \cap \mathbf{C}$. Thus for any y in $X(d)$

$$Ay = k \int_{h(D)} A(z - T_c)^{-1} y \, dz + j\, Ay = P(d)Ay.$$

Therefore $AE(d) = E(d)AE(d)$. Take an increasing sequence of closed sets $e_n \supset S$, converging to d^c, and let $x \in X$. Then

$$E(d)AE(d^c)x = \lim_n E(d)AE(e_n)x = \lim_n E(d)E(e_n)AE(e_n)x = 0.$$

Hence $E(d)A = E(d)A(E(d) + E(d^c)) = AE(d)$ for any closed set d in S^c. The countable additivity of E implies $E(b)A = AE(b)$ for every Borel set b in S^c. Taking complements of b's we end the proof.

COROLLARY 1. *If T is S-spectral, then its S-resolution of the identity is uniquely determined on B_S.*

COROLLARY 2. *If T is S_i-spectral with S_i-resolutions of the identity E_i with domains B_i and if $b_i \in B_i$ $(i = 1, 2)$, then $E_1(b_1)$ and $E_2(b_2)$ commute.*

THEOREM 2. *If T is S_i-spectral $(i = 1, 2)$, then T is $S_1 \cap S_2$-spectral.*

The result of Foiaş [6], cited at the beginning of this section and extended by Vasilescu [19; IV. 4. 6], can be generalized as follows.

LEMMA 5. *Let $T \in C(X)$ and $A \in L(X)$ satisfy $(z - A)^{-1}T \subset T(z - A)^{-1}$ for every z in $\rho(A) \cap \mathbf{C}$. If Y is a spectral maximal subspace for T, then $(z - A)^{-1}Y \subset Y$ $(z \in \rho(A) \cap \mathbf{C})$; hence $f(A)Y \subset Y$ for every f which is locally holomorphic on $\sigma(A)$.*

The proof will be omitted.

THEOREM 3. *Let T be S-spectral. If $T \in L(X)$, then T is strongly S-decomposable. If $T \in C(X) \setminus L(X)$, then T is strongly $S \cup \{\infty\}$-decomposable.*

Proof. Let E denote the S-resolution of the identity of T and let Y be a spectral maximal subspace for T. For any projection $E(b)$ $(b \in B_S)$ and for any $z \in \mathbf{C} \setminus \{0, 1\}$ we have $(z - E(b))^{-1} = z^{-1} + z^{-1}(z-1)^{-1}E(b)$. Thus for $z \in \rho(E(b)) \cap \mathbf{C}$ we have $(z - E(b))^{-1}T \subset T(z - E(b))^{-1}$. By Lemma 5, $E(b)Y \subset Y$.

Let (G_1, G_S) be an open S-covering (resp., $S \cup \{\infty\}$-covering) of $\sigma(T)$. The set $U = \bar{G}_1 \cup \bar{G}_S$ contains $\sigma(T)$, therefore $E(U) = I$. Hence for every y in Y

$$y = E(U)y = E(\bar{G}_1)y + E(\bar{G}_S \setminus \bar{G}_1)y.$$

By the preceding paragraph, the terms on the right hand side are in Y. By the proof of Theorem 1 and by Lemma 4, they are in the spectral maximal subspaces $X_{T,\bar{G}_1}$ and $X_T(\bar{G}_S)$, respectively, and $X_{T,\bar{G}_1} \subset D(T)$. Thus

$$Y = Y \cap X_{T,\bar{G}_1} + Y \cap X_T(\bar{G}_S),$$

which was to be proved.

LEMMA 6. *Let $T \in C(X)$, and let*

$$S(T) = \{S = \bar{S} \subset \bar{\mathbf{C}} : T \text{ is } S\text{-spectral}\}, \quad S_0(T) = \bigcap \{S : S \in S(T)\}.$$

Then each closed neighborhood of $S_0(T)$ belongs to the class $S(T)$. In particular, if $S_0(T) = \emptyset$, then T is $\emptyset$-spectral.

Proof. We may assume that $S_0(T) \neq \bar{\mathbf{C}}$. Let $N \neq \bar{\mathbf{C}}$ be an open neighborhood of $S_0(T)$. Then

$$\bigcap \{S : S \in S(T)\} \cap N^c = \emptyset.$$

Since $\overline{\mathbf{C}}$ is compact, there are sets $S_1, \ldots, S_k \in S(T)$ such that

$$\bigcap \{S_n : n = 1, \ldots, k\} \cap N^c = \emptyset .$$

By Theorem 2, the set $S_0 = \bigcap \{S_n : n = 1, \ldots, k\}$ belongs to $S(T)$. Since $N \supset S_0$, we have $\bar{N} \in S(T)$.

Remark. Example 1 in [12; p. 44] shows that it can happen that $S_0(T) \notin S(T)$, i.e. there is no smallest set in $S(T)$.

From Theorem 3 it is seen that for the spectral residues $S^*(T)$ and $S^{**}(T)$ we have

$$S^*(T) \subset S^{**}(T) \subset S_0(T) \cup \{\infty\},$$

and the last set can be omitted if $T \in L(X)$.

4. $A(K)$-SPECTRAL OPERATORS

Let T be a closed operator in X with spectrum $\sigma(T)$ and let K be a closed subset of $\sigma(T)$. Unless explicitly stated otherwise, all the occurring topological concepts in this section for subsets of $\sigma(T)$ will be understood in the induced topology of $\sigma(T)$ (as a subset of $\overline{\mathbf{C}}$). Thus $\bar{b}$ and b^0 will denote the closure and the interior of b in the induced topology of $\sigma(T)$, b^c is $\sigma(T) \setminus b$ and $h(b) = \bar{b} \cap \overline{b^c}$. Let

$$A(K) = \{b \text{ Borel subset of } \overline{\mathbf{C}} : h(b \cap \sigma(T)) \cap K = \emptyset\},$$

$$B(K) = \{b \text{ Borel subset of } \sigma(T) : h(b) \cap K = \emptyset\},$$

$$C(K) = \{b \in B(K) : \bar{b} \cap K = \emptyset\},$$

$$D(K) = \{b \in B(K) : \bar{b} \cap K = \emptyset \text{ or else } b^0 \supset K\}.$$

Since $\emptyset$, $\sigma(T) \in B(K)$, $h(b \cup b') \subset h(b) \cup h(b')$ and $h(b^c) = h(b)$, $B(K)$ is a Boolean algebra of subsets of $\sigma(T)$, containing the subalgebra $D(K)$ and the Boolean subring $C(K)$. Similarly, $A(K)$ is a Boolean algebra of subsets of $\overline{\mathbf{C}}$, containing the class $B(K)$.

Remark. Considering that particular case of an A-spectral measure (Section 2) when the Boolean algebra A is an algebra $D(K)$ (for an operator T), it is seen from [5; IV. 10.2] that each $D(K)$-spectral measure E is uniformly bounded on each subclass $a \cap D(K)$ if $\bar{a} \cap K = \emptyset$, $a \in D(K)$. However, E is not necessarily uniformly bounded on all of $D(K)$.

LEMMA 7. *Let $T \in C(X)$, and let K be a closed subset of $\sigma(T)$. T is $D(K)$-spectral if and only if T is S-spectral for every $S = \bar{S} \subset \sigma(T)$ such that $S^0 \supset K$.*

Proof. Let E be a $D(K)$-resolution of the identity for T, let S be as above, and let

$$E_S(b) = E(b \cap \sigma(T)) \quad (b \in B_S).$$

Then E_S is clearly the S-resolution of the identity for T.

Conversely, let S be as above, and let E_S be the S-resolution of the identity for T. We have

$$D(K) = (\bigcup\{B_S : S = \bar{S} \subset \sigma(T),\ S^0 \supset K\}) \cap \sigma(T).$$

So for every $b \in D(K)$ we have $b \in B_S$ for some S as above, and we can define

$$E(b) = E_S(b) \quad (b \in B_S \cap \sigma(T)).$$

By Corollary 1 to Theorem 1 and by Theorem 2, E is well-defined on all of $D(K)$, and it is easily seen that E is a $D(K)$-resolution of the identity for T.

COROLLARY. *An operator $T \in C(X)$ has at most one $D(K)$-resolution of the identity.*

The main statement of the following theorem is the extendability of the $D(K)$-resolution of the identity to the $B(K)$-resolution of the identity for T (in the case when the set K is not connected).

THEOREM 4. *Let $T \in C(X)$, and let K be a closed subset of $\sigma(T)$. The following statements are equivalent:*

1° *T is $A(K)$-spectral,*

2° *T is $B(K)$-spectral,*

3° *T is $D(K)$-spectral.*

If this is the case, the corresponding resolutions of the identity for T are uniquely determined, hence they are restrictions (extensions) of each other.

Proof. Let E be an $A(K)$-resolution of the identity for T, and let $S = \sigma(T)$. Then $E \mid B_S$ is the S-resolution of the identity for T. By the Corollary to Lemma 4, $E(\sigma(T)) = I$. The restriction $E \mid B(K)$ is a $B(K)$-resolution of the identity for T. For every $a \in A(K)$ we have $a \cap \sigma(T) \in B(K)$, and

$$E(a) = E(a)E(\sigma(T)) = E(a \cap \sigma(T)).$$

It follows that the $A(K)$-resolution of the identity for T is unique, if so is the $B(K)$-resolution of the identity.

If F is a $B(K)$-resolution of the identity for T, then

$$E(a) = F(a \cap \sigma(T)) \quad (a \in A(K))$$

is an $A(K)$-resolution of the identity for T. Hence 1° and 2° are equivalent. Further, $F \mid D(K)$ is the $D(K)$-resolution of the identity for T. Hence 2° implies 3°, and all $B(K)$-resolutions of the identity for T coincide on the algebra $D(K)$.

Now let F be the (unique) $D(K)$-resolution of the identity for T. We shall show that F extends to a $B(K)$-resolution of the identity in a unique way. This will prove the equivalence of 2° and 3° and the uniqueness statements.

If the set K is connected, then the class $B(K)\setminus D(K)$ is void, and there is nothing to be proved. If not, let $b \in B(K)\setminus D(K)$. There exist closed sets $c, f \subset \sigma(T)$ such that

$$\bar{b} \cap K \subset c^0, \quad c \subset b^0, \quad \overline{b^c} \cap K \subset f^0, \quad f \subset (b^c)^0. \tag{1}$$

Indeed, by assumption, $\bar{b} \cap \overline{b^c} \cap K = \emptyset$. Hence

$$\emptyset \neq \bar{b} \cap K \subset b^0, \quad \emptyset \neq \overline{b^c} \cap K \subset (b^c)^0.$$

So there exist closed neighborhoods c and f of $\bar{b} \cap K$ and $\overline{b^c} \cap K$, respectively, satisfying (1). Then $(c \cup f)^0 \supset c^0 \cup f^0 \supset K$, hence $F(c \cup f)$ is defined and

$$\sigma(T|F(c \cup f)X) \subset c \cup f.$$

Let $P(c,f)$ denote the spectral projection in $F(c \cup f)X$, associated with the part in c of the spectrum above, and let

$$Q(c,f) = P(c,f)F(c \cup f) \in L(X).$$

$Q(c,f)$ is a projection in X. We show that for fixed c it is independent of the choice of f. Let $f_1 \subset f_2$ be closed sets satisfying the conditions (1) above, and $x \in X$. Then, with obvious notations (where, e.g., $\alpha(c) = 1$ if ∞ lies inside the contour surrounding the set c and $= 0$ otherwise),

$$P(c, f_2)F(c \cup f_1)x =$$

$$= \alpha(c)F(c \cup f_1)x + (2\pi i)^{-1} \oint_c (z - T \mid F(c \cup f_1)X)^{-1} F(c \cup f_1)x \, dz =$$

$$= P(c, f_1)F(c \cup f_1)x.$$

Further, $F(c \cup f_2) = F(c \cup f_1) + F(f_2 \setminus f_1)$, and

$$P(c, f_2)F(f_2 \setminus f_1)x =$$

$$= \alpha(c)F(f_2 \setminus f_1)x + (2\pi i)^{-1} \oint_c (z - T|F(f_2 \setminus f_1)X)^{-1}F(f_2 \setminus f_1)x \, dz = 0.$$

Hence $Q(c, f_2) = Q(c, f_1)$. Now if f_1, f_2 are (possibly not comparable) closed sets satisfying the conditions above, then so is $f_1 \cap f_2 \subset f_i$. Thus $Q(c, f)$ is independent of f, and may be denoted by $Q(c)$.

Since, by (1), $\overline{b \setminus c} \cap K = \emptyset$, so $b \setminus c \in D(K)$, we may define

$$E(b, c) = Q(c) + F(b \setminus c).$$

We show that it is independent of c. If $c_1 \subset c_2$ and f satisfy (1), then the additivity of F on $D(K)$ implies

$$Q(c_2) = P(c_2, f)F(c_2 \cup f) =$$

$$= P(c_1, f)F(c_1 \cup f) + P(c_2, f)F(c_2 \setminus c_1) = Q(c_1) + F(c_2 \setminus c_1).$$

Hence $E(b, c_2) = E(b, c_1)$. The case of not comparable c_i's can be settled as above.

$E(b) = E(b, c)$ is a projection in X. Indeed, the equalities

$$Q(c)F(b \setminus c) = P(c, f)F(c \cup f)F(b \setminus c) = 0,$$

$$F(b \setminus c)Q(c) = F(b \setminus c)F(c \cup f)P(c, f)F(c \cup f) = 0$$

are evident in view of (1). For every $b \in B(K) \setminus D(K)$ we have $E(b)T \subset TE(b)$. Indeed, with c and f as above

$$Q(c, f)T = P(c, f)F(c \cup f)T \subset P(c, f)(T|F(c \cup f)X)F(c \cup f) \subset TQ(c, f),$$

by [5; VII. 9.8]. Further,

$$\sigma(T \mid E(b)X) = \sigma(T \mid (Q(c)X \oplus F(b \setminus c)X)) \subset c \cup \bar{b} = \bar{b}.$$

For $b \in D(K)$ let $E(b) = F(b)$. We show that the mapping E of $B(K)$ into $L(X)$ is a $B(K)$-spectral measure. We shall prove first that E is an algebra homomorphism. Let $b \in B(K) \setminus D(K)$, then $b^c \in B(K) \setminus D(K)$. For b pick c and f as above, then

$$E(b) + E(b^c) = Q(c, f) + F(b \setminus c) + Q(f, c) + F(b^c \setminus f) =$$

$$= (P(c, f) + P(f, c))F(c \cup f) + F((c \cup f)^c) = I.$$

Now we shall prove multiplicativity, and occasionally write b_1b_2 rather than $b_1 \cap b_2$. Note that the formula

$$E(b) = P(c,f)F(c \cup f) + F(b \setminus c)$$

is valid for $b \in D(K)$, too. Indeed, $b \in C(K)$ implies that we may pick $c = \emptyset$, hence $E(b) = F(b)$. If $b \in D(K) \setminus C(K)$, so $b^0 \supset K$, we have $c^0 \supset K$, therefore we may take $f = \emptyset$. Hence $E(b) = F(c) + F(b \setminus c) = F(b)$.

Let $b_1, b_2 \in B(K)$, and define

$$K_{12} = Kb_1b_2, \quad K_1 = Kb_1b_2^c, \quad K_2 = Kb_1^cb_2, \quad K_0 = Kb_1^cb_2^c.$$

These sets are pairwise disjoint with union K. By the definition of the algebra $B(K)$, we can choose closed sets k_{12}, k_1, k_2, k_0 in $\sigma(T)$ such that (cf. the consideration after (1))

$$K_{12} \subset k_{12}^0, \quad k_{12} \subset (b_1b_2)^0,$$

and similarly for the other ones. Let now

$$c_1 = k_{12} \cup k_1, \quad c_2 = k_{12} \cup k_2, \quad f_1 = k_0 \cup k_2, \quad f_2 = k_0 \cup k_1;$$

$$e = c_1 \cup f_1 = c_2 \cup f_2.$$

Then the sets b_i, c_i, f_i satisfy the relations (1) with the subscripts i added ($i = 1, 2$). Indeed, for example, we have

$$k_2^0 \subset k_2 \subset (b_1^cb_2)^0, \quad k_0^0 \subset k_0 \subset (b_1^cb_2^c)^0.$$

Therefore $k_2^0 \cup k_0^0 \subset b_1^{c0} \subset b_1^c$, thus $b_1 \subset k_2^{0c}k_0^{0c}$. Hence

$$\bar{b}_1 \cap K \subset k_2^{0c}k_0^{0c}K \subset K_{12} \cup K_1 \subset k_{12}^0 \cup k_1^0 \subset c_1^0.$$

Further,

$$c_1 = k_{12} \cup k_1 \subset (b_1b_2)^0 \cup (b_1b_2^c)^0 \subset b_1^0,$$

and we can verify the relations (1) similarly for the other cases, too. Therefore,

$$E(b_1)E(b_2) = (P(c_1,f_1)F(e) + F(b_1 \setminus c_1))(P(c_2,f_2)F(e) + F(b_2 \setminus c_2)).$$

Since $(b_i \setminus c_i)e = b_ic_i^c(c_i \cup f_i) \subset b_ic_i^cb_i^c = \emptyset$, we obtain $F(b_i \setminus c_i)F(e) = 0$ ($i = 1, 2$). Further,

$$P(c_1,f_1)P(c_2,f_2)F(e) = P(k_{12}, e \setminus k_{12})F(e).$$

Since $b_i \setminus c_i \in D(K)$, we have

$$F(b_1\setminus c_1)F(b_2\setminus c_2) = F(b_1b_2\setminus(c_1 \cup c_2)b_1b_2) = F(b_1b_2\setminus k_{12}).$$

Hence we obtain

$$E(b_1)E(b_2) = P(k_{12}, e\setminus k_{12})F(e) + F(b_1b_2\setminus k_{12}) = E(b_1 \cap b_2).$$

So E is an algebra homomorphism of $B(K)$ into a Boolean algebra of projections in $L(X)$. We shall show now that E on $B(K)$ is countably additive in the strong operator topology of $L(X)$.

Let the sequence $\{b_n\} \subset B(K)$ be such that the sets b_n are pairwise disjoint, and their union b belongs to $B(K)$. Let

$$H = \{b_n : b_n \cap K \neq \emptyset\}.$$

If the class H contains infinitely many sets, then we choose one element $k_n \in b_n \cap K$ from each set $b_n \in H$. Since K is compact, the sequence $\{k_n\}$ has an accumulation point k in K. Since $b \in B(K)$, so $h(b) \cap K = \emptyset$, we have $k \in b$. Hence for some positive integer r, $k \in b_r \in H$. Since $b_r \in B(K)$, we see that $k \in b_r^0$. This contradicts the construction of the sequence $\{k_n\}$ and of the point k.

So H contains a finite number of sets, say p. Let

$$b^p = \bigcup\{b_n : b_n \in H\}, \quad b' = b\setminus b^p.$$

Then $b^p \in B(K)$, so $b' \in B(K)$. By the definition of H, we obtain that $\overline{b'} \cap K = \emptyset$. Hence $b' \in D(K)$, and $\{b_n : b_n \notin H\} \subset D(K)$. Making use of the countable additivity of E on $D(K)$, we have for any x in X, in the norm topology of X,

$$E(b)x = E(b^p)x + E(b')x = \sum_{n=1}^{\infty} E(b_n)x,$$

where the convergence of the series is unconditional. So the $D(K)$-resolution of the identity F extends to a $B(K)$-resolution of the identity E for T, thus 2° and 3° are equivalent.

Now let G be an arbitrary $B(K)$-resolution of the identity for T. Since the $D(K)$-resolution of the identity is unique,

$$G(b) = F(b) \quad (b \in D(K)). \tag{2}$$

Let $b \in B(K)\setminus D(K)$, and let c, f be as in (1). By (2) and from $c \cup f \in D(K)$

$$\sigma(T, G(c)F(c \cup f)X) = \sigma(T, G(c)X) \subset c. \tag{3}$$

By [4; 1.3.10, p. 26] (which is valid for a closed operator, too), the spectral maximal space of the restriction $T \mid F(c \cup f)X$, determined by the set c, exists, and equals

$$X(T \mid F(c \cup f)X, c) = P(c,f)F(c \cup f)X. \tag{4}$$

From (3) and (4)

$$G(c)X = G(c)F(c \cup f)X \subset P(c,f)F(c \cup f)X. \tag{5}$$

By symmetry,

$$G(f)X \subset P(f,c)F(c \cup f)X. \tag{6}$$

Since $G(c)+G(f) = G(c \cup f)$, from (2) we obtain

$$G(c) + G(f) = F(c \cup f) = (P(c,f) + P(f,c))F(c \cup f).$$

Multiplying by $P(c,f)F(c \cup f)$ from the left, by (5) and (6) we have

$$G(c) = P(c,f)F(c \cup f).$$

Since $b \setminus c \in D(K)$,

$$G(b) = G(c) + G(b \setminus c) = P(c,f)F(c \cup f) + F(b \setminus c) = E(b).$$

So the $B(K)$-resolution of the identity for T is unique, if exists, and the proof is complete.

DEFINITION 1. Let $T \in C(X)$ and let

$$\mathrm{Kr}(T) = \{K = \overline{K} \subset \sigma(T) : T \text{ is } A(K)\text{-spectral}\}.$$

This class will be called *the class of the critical sets for T*, and the smallest element in it (if exists) *the set of the spectral singularities* (or *the smallest critical set*) *for T*.

THEOREM 5. *For each $T \in C(X)$ the set $S_0(T)$ (defined in Lemma 6) is the set of the spectral singularities for T.*

Proof. Let $K \in \mathrm{Kr}(T)$. Since the induced topology of $\sigma(T)$ (as asubset of $\overline{\mathbf{C}}$) is normal, K is the intersection of its closed neighborhoods S (in the induced topology). By Theorem 4 and Lemma 7, T is S-spectral for every such S. With the notation of Lemma 6, $S_0(T) \subset K$. Further, Lemmas 6, 7 and Theorem 4 together yield that $S_0(T) \in \mathrm{Kr}(T)$.

DEFINITION 2. Let $T \in C(X)$ and $K \in \mathrm{Kr}(T)$. T is called *$A(K)$-scalar* (or, equivalently, *$B(K)$- or $D(K)$-scalar*) if for every b in $C(K) \cap \mathbf{C}$ the operator $T|E(b)X$

(here E denotes the $A(K)$-resolution of the identity for T) is a spectral operator of scalar type in the sense of Bade (cf. [5; XVIII. 2.12]).

Note that the resolution of the identity for $T|E(b)X$ is then (the evident extension of) $E|E(b)X$.

THEOREM 6. *Let* $T \in C(X)$ *and let*

$$\mathrm{Kr}_0(T) = \{K \subset \sigma(T) : K \text{ is closed}, \ T \text{ is } A(K)\text{-scalar}\}.$$

There is a smallest set (which will be denoted by $K_0(T)$*) in the family* $\mathrm{Kr}_0(T)$.

Proof. $\sigma(T) \in \mathrm{Kr}_0(T)$, thus the family $\mathrm{Kr}_0(T)$ is nonvoid. Let $\{K_a : a \in A\}$ be a totally ordered (with respect to $\subset$) subfamily of $\mathrm{Kr}_0(T)$ and let $K' = \bigcap\{K_a : a \in A\}$. By Theorem 5, $K' \in \mathrm{Kr}(T)$. Let E denote the $A(K')$-resolution of the identity for T. Let $b \in C(K') \cap \mathbf{C}$. By the compactness of $\bar{\mathbf{C}}$, there is $a_0 \in A$ such that $\bar{b} \cap K_{a_0} = \emptyset$, i.e. b belongs to $C(K_{a_0}) \cap \mathbf{C}$. Hence $T|E(b)X$ is scalar, so $K' \in \mathrm{Kr}_0(T)$. Applying Zorn's lemma, it is sufficient to prove that $K_1, K_2 \in \mathrm{Kr}_0(T)$ implies $K = K_1 \cap K_2 \in \mathrm{Kr}_0(T)$. By Theorem 5, $K \in \mathrm{Kr}(T)$.

Let $b \in C(K) \cap \mathbf{C}$. There is a closed neighborhood $S \subset \sigma(T)$ of K such that $b \subset S^c$. Further, there are closed neighborhoods $S_i \subset \sigma(T)$ of K_i such that $S_1 \cap S_2 \subset S$. Then $b = bS^c = bS_1^c \cup bS_1S_2^c$, $bS_1^c \in C(K_1)$, $bS_1S_2^c \in C(K_2)$.

Let E denote now the $A(K)$-resolution of the identity for T. We shall show that $T \mid E(b)X$ is a scalar operator. Let

$$f(z) = z, \quad f_n(z) = \begin{cases} f(z) & \text{if } |f(z)| \leqslant n \\ 0 & \phantom{\text{if }} |f(z)| > n, \end{cases}$$

$$\bar{E}(f_n; h) = \int_h f_n(z) E(\mathrm{d}z) \qquad (h \in C(K)),$$

$$\bar{E}(f; h)x = \lim_n \bar{E}(f_n; h)x$$

with

$$D(\bar{E}(f; h)) = \{x \in E(h)X : \lim_n \bar{E}(f_n; h)x \text{ exists}\}.$$

If $x = E(b)x \in D(T)$, then $E(bS_1^c)x$, $E(bS_1S_2^c)x \in D(T)$, for T is $A(K)$-spectral. Since $K_1, K_2 \in \mathrm{Kr}_0(T)$, we have

$$TE(bS_1^c)x = \bar{E}(f; bS_1^c)x = \lim_n E(S_1^c)\bar{E}(f_n; b)x,$$

$$TE(bS_1S_2^c)x = \lim_n E(S_1S_2^c)\bar{E}(f_n; b)x.$$

Hence we obtain that $\bar{E}(f, b)x$ is defined, and

$$TE(b)x = \lim_n (E(S_1^c) + E(S_1 S_2^c))\bar{E}(f_n; b)x = \bar{E}(f; b)x.$$

On the other hand, if $\bar{E}(f; b)x$ exists, and $x = E(b)x$, then, by assumption,

$$E(S_1^c)\bar{E}(f; b)x = \bar{E}(f; bS_1^c)x = TE(bS_1^c)x,$$

$$E(S_1 S_2^c)\bar{E}(f; b)x = \bar{E}(f; bS_1 S_2^c)x = TE(bS_1 S_2^c)x.$$

Hence $\bar{E}(f; b)x = TE(b)x$ and, by [5; XVIII. 2.12], $T \mid E(b)X$ is a spectral operator of scalar type. Thus $K \in \mathrm{Kr}_0(T)$.

5. AN APPLICATION

The following ordinary differential operator, having at most a finite number of spectral singularities, has been considered, among others, by Naimark [16], Ljance [11] and Folland [7]. Let p be a Lebesgue measurable complex-valued function on $\mathbf{R}^+ = [0, \infty)$ which satisfies

$$\int_{\mathbf{R}^+} e^{\varepsilon x}|p(x)|\, dx < \infty,$$

where ε is a positive number. Let $\theta \in \mathbf{C}$, and let

$$\tau = -(d/dx)^2 + p(x), \quad Bf = f'(0) - \theta f(0).$$

Define the closed operator T in $H = L^2(\mathbf{R}^+)$ by $Tf = \tau f$ on the domain $D(T) = \{f \in H : f'$ exists and is locally absolutely continuous on $\mathbf{R}^+$, $\tau f \in H$, and $Bf = 0\}$. For preliminary results (from our point of view) on this operator, we refer to [7], [11] or [16]. We shall quote the most relevant results and show that the smallest critical set $S_0(T)$ in our terminology in the set S of all spectral singularities of T in the sense of Ljance [11; p. 528].

The spectrum of T, $\sigma(T) \cap \mathbf{C}$, consists of the continuous spectrum $\mathbf{R}^+$ plus a finite number of eigenvalues $z_1, \ldots, z_r$. Here $z_k = s_k^2$, $\operatorname{Im} s_k > 0$ and $A(s_k) = 0$, where the function A is holomorphic in the half-plane $\operatorname{Im} s > -\varepsilon/2$, has exactly the roots s_k in $\operatorname{Im} s > 0$, and exactly the roots $q_1, \ldots, q_n$ in $\mathbf{R} \setminus \{0\}$. The positive numbers $v_k = q_k^2$ are called the spectral singularities of the operator T, so we have for their set S

$$S = \{v_k = q_k^2 : k = 1, \ldots, n\}.$$

Note that the numbers v_k are distinct elements of the continuous spectrum of T.

For $z \in \mathbf{C}$ let $w(\cdot, z)$ denote the solution of the differential equation $\tau f = zf$ satisfying $w(0, z) = 1$, $w_x'(0, z) = 0$. If the order a_m of s_m is greater than 1, then we define the principal functions

$$w^{(j)}(x, z_m) = \frac{\partial^j w}{\partial z^j}(x, z_m) \quad (0 \leqslant j \leqslant a_m - 1).$$

Note that $w^{(j)}(\cdot, z_m) \in H$. The T-Fourier transform of $f \in H$ is the "function wf on the spectrum of T" defined by

$$(wf)(z) = \operatorname*{l.i.m.}_{t \to \infty} \int_0^t f(x) w(x, z) \mathrm{d}x \quad (z \in \mathbf{R}^+)$$

$$(wf)^{(j)}(z_m) = \int_0^\infty f(x) w^{(j)}(x, z_m) \mathrm{d}x \quad (0 \leqslant j \leqslant a_m - 1,\ 1 \leqslant m \leqslant r),$$

i.e. a function measurable on $\mathbf{R}^+$ and having an ordered a_m-tuple of complex numbers as its value at each point z_m.

With the help of the T-Fourier transforms two functional calculi have been defined and studied for the operator T, which will be of interest for us. One by Ljance in [11; Theorem 5.4], the other by Folland in [7; Theorem 3]. Though their domains of definition are different, we shall make use of the fact that on a wide class of functions they are identical. Let r_k denote the order of the root q_k of the function A $(k = 1, \ldots, n)$.

LEMMA 8. *Let F be a complex-valued function, bounded and measurable on $\mathbf{R}$, r_k times continuously differentiable in some $\mathbf{R}^+$-neighborhood of each $v_k \in S$ $(k = 1, \ldots, n)$, and locally holomorphic on the set $Z = \{z_1, \ldots, z_r\}$ of all eigenvalues. Let the operator corresponding to F be denoted by $F_1(T)$ in the Ljance and by $F_2(T)$ in the Folland calculus. Then*

$$F_1(T) = F_2(T) \in L(H).$$

(The operator $F_i(T)$ will be denoted in what follows by $F(T)$.)

Proof. Clearly, F belongs to both domains. By [7; Theorem 3], $F_2(T)$ is a bounded linear operator in H, and for every f in a dense set $X \subset H$ (see [7; pp. 222—223])

$$w(F_2(T)f) = F \cdot wf.$$

By [11; Theorem 5.4], this implies $f \in D(F_1(T))$, and $w(F_1(T)f) = w(F_2(T)f)$. By [11; Corollary 3.4], it follows that $F_1(T)f = F_2(T)f$ for all f in X. Since $F_1(T)$ is a closed operator ([11; Theorem 5.4]), $F_1(T) = F_2(T) \in L(H)$.

Ljance has proved in [11; Theorem 5.7] that the operator T is generalized spectral in the following sense: let

$$D = \{b \text{ Borel subset of } \sigma(T) : \bar{b} \cap S = \emptyset\} = C(S).$$

There is a map $P\colon D \to L(H)$, which is multiplicative and strongly coutably additive on D, satisfying $P(b)T \subset TP(b)$ $(b \in D)$, $P(\{\infty\}) = 0$, and $\sigma(T|P(b)H) \subset \bar{b}$ $(b \in D)$. We shall prove the stronger

THEOREM 7. $S_0(T) = S$.

Proof. First we prove that T is $D(S)$-spectral, which, by Theorem 4, will imply that $S_0(T) \subset S$. If $b \in D(S) \setminus C(S)$, then $b^c = \mathbf{C} \cap \sigma(T) \setminus b \in C(S)$. Defining $P(b) \doteq I - P(b^c)$, the extended map P is clearly a $D(S)$-spectral measure commuting with T.

We have still to show that

$$\sigma(T, P(b)H) \subset \bar{b} \quad (b \in D(S) \setminus C(S)).$$

Each b as above is the disjoint union of some $b_1 \in C(S)$ and some $b_2 \in D(S) \setminus C(S)$, where b_2 is contained in some bounded $\mathbf{R}^+$-neighborhood of S. If we prove that

$$\sigma(T, P(b_2)H) \subset \bar{b}_2 \quad (b_2 \text{ as above}),$$

then we shall have proved, by our preceding remark, that

$$\sigma(T, P(b)H) = \sigma(T, P(b_1)H) \cup \sigma(T, P(b_2)H) \subset \bar{b}_1 \cup \bar{b}_2 = \bar{b}.$$

So in what follows we may and will assume that each considered $b \in D(S) \setminus C(S)$ is contained in some bounded $\mathbf{R}^+$-neighborhood of S.

Let $k(b) = k(b, \cdot)$ denote the characteristic function of the set b. Assume that $b \in D(S) \setminus C(S)$ and that $z \in \mathbf{C}$, $z \notin \bar{b}$. Let $F(u) = (u - z)^{-1} k(b, u)$. Then F satisfies the conditions of Lemma 8, hence the operator $F(T)$ belongs to $L(H)$.

By [11; Theorem 4.3], for every f in H we have

$$wP(b^c)f = k(b^c)wf.$$

Hence $wP(b)f = k(b)wf$ and, by [11; Theorem 5.4], $P(b) = k(b, T)$. Assume now that $f \in D(T) \cap P(b)H$ and $u \in \sigma(T) \cap \mathbf{C}$. Then $wf = k(b)wf$ and, by [11; Lemma 2.18], $wTf(u) = uwf(u)$. Hence $wf(u) = (u-z)^{-1}k(b, u)(u-z)wf(u) = F(u)w(T-z)f(u)$. By [11; Theorem 5.4], $wF(T)(T-z)f = wf$, thus

$$F(T)(T|P(b)H - z)f = f.$$

Now let $f \in P(b)H$ and $u \in \sigma(T) \cap \mathbf{C}$. If $g = F(T)f$, then $wg = Fwf$. Indeed, the domain of $F(T)$ is all of H. On the other hand, formula (5.12) in [11; p. 558] shows that for every f in this domain $wF(T)f = Fwf$. So

$$wg(u) = wF(T)f(u) = F(u)wf(u) \quad (u \in \sigma(T) \cap \mathbf{C}).$$

Therefore

$$(u-z)wg(u) = (u-z)F(u)wf(u) = k(b, u)wf(u) = wf(u).$$

By [11; Theorem 5.4], then $g \in D(T-z) = D(T)$, and

$$w(T-z)F(T)f(u) = (u-z)wg(u) = wf(u).$$

Further,

$$k(b, u)wg(u) = k(b, u)F(u)wf(u) = wg(u)$$

implies that $g \in P(b)H$. Hence

$$(T|P(b)H - z)F(T)f = f,$$

so we have shown that $\sigma(T|P(b)H) \subset \bar{b} \cup \{\infty\}$.

We have assumed that the set $b \in D(S)\setminus C(S)$ is bounded. So the function $G(u) = uk(b, u)$ satisfies the conditions of Lemma 8, hence the operator $G(T)$ is in $L(H)$. Let $f \in D(T)$ and $u \in \sigma(T) \cap \mathbf{C}$. Then $P(b)f \in D(T)$, and

$$wTP(b)f(u) = uwP(b)f(u) = uk(b, u)wf(u) = G(u)wf(u).$$

By [11; Theorem 5.4], $TP(b)f = G(T)f$. Since the operator T is closed, $TP(b) = G(T) \in L(H)$. So T is $D(S)$-spectral with the $D(S)$-resolution of the identity P.

On the other hand, if $s \in S\setminus S_0(T)$ then, by Theorem 4 and Lemma 7, the $A(S_0(T))$-resolution on the identity E for T is uniformly bounded in some neighborhood of s. Further, for $b \in C(S)$ we have $E(b) = P(b)$, by Theorem 4 and the first part of this proof. Ljance [11, Theorem 4.3] has proved that $\lim |P(b)| = \infty$ as the distance of b to S tends to 0. Therefore $S_0(T) = S$.

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