J. OPERATOR THEORY
15(1986), 327–344

A DESCRIPTION OF INVARIANT SUBSPACES OF C_{11}-CONTRACTIONS

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INTRODUCTION

Sz.-Nagy and Foiaş have given a description of invariant subspaces of completely non-unitary (c.n.u.) contractions in terms of regular factorizations of their characteristic function (cf. [11, Chapter VII]). Since it is rather difficult to look over all regular factorizations of a contractive analytic function even in the simplest cases, Sickler [10] initiated to derive more explicit descriptions of invariant subspaces. His results concern C_{11}-contractions with scalar-valued characteristic function, and have been generalized by Wu [15], [16] to C_{11}-contractions with a finite matrix characteristic function. Their method is the following: construct a simple, canonical quasi-affinity, intertwining the functional model of the contraction with a unitary operator, and after that, using results about regular factorizations, show that this quasi-affinity implements the isomorphism of the invariant subspace lattices. The aim of the present work is to extend this method for arbitrary C_{11}-contractions.

Our paper is organized as follows. In Section I we examine when the isomorphism of different invariant subspace lattices of quasi-similar C_{11}-contractions can be implemented by intertwining quasi-affinities. It turns out that such an implementation is possible:

1) for the C_{11}-part of the hyperinvariant subspace lattices under quasi-similarity;

2) for the C_{11}-part of the biinvariant subspace lattices under weak similarity; and

3) for the invariant subspace lattices under a similarity relation which may be named analytic similarity.

In Section II first we show that the canonical, intertwining operator between the contraction and the corresponding unitary operator, introduced by Sickler and Wu, is a quasi-affinity for every c.n.u. C_{11}-contraction. Then we examine the isomorphism of which invariant subspace lattices can be implemented by this quasi--affinity. Applying results of Section I, we find that such an implementation can be realized:

1) between the C_{11}-parts of hyperinvariant subspace lattices, for every C_{11}-contraction;

2) between the C_{11}-parts of biinvariant subspace lattices, for contractions being weakly similar to unitaries; and

3) between the invariant subspace lattices, for C_{11}-contractions whose characteristic functions have scalar multiples.

As a consequence we obtain descriptions for the corresponding invariant subspace lattices of C_{11}-contractions belonging to the narrowing classes mentioned before.

We shall use the terminology of the monograph [11]. Here we only remind some basic notations.

If $\mathfrak{H}$ and $\mathfrak{K}$ are (complex, separable) Hilbert spaces, then $\mathscr{L}(\mathfrak{H}, \mathfrak{K})$ stands for the set of all linear, bounded operators mapping from $\mathfrak{H}$ into $\mathfrak{K}$. If $\mathfrak{H}$ and $\mathfrak{K}$ coincide then we shall write briefly $\mathscr{L}(\mathfrak{H})$ instead of $\mathscr{L}(\mathfrak{H}, \mathfrak{H})$. For any two operators $T \in \mathscr{L}(\mathfrak{H})$ and $S \in \mathscr{L}(\mathfrak{K})$, $\mathscr{I}(T, S)$ denotes the set of operators, intertwining T and S, i.e. $\mathscr{I}(T, S) = \{X \in \mathscr{L}(\mathfrak{H}, \mathfrak{K}) : XT = SX\}$. T can be injected into $S : T \overset{i}{\prec} S$, if $\mathscr{I}(T, S)$ contains an injection. T is a quasi-affine transform of $S : T \prec S$, if $\mathscr{I}(T, S)$ contains a quasi-affinity, i.e. an injection with dense range. T and S are quasi-similar: $T \sim S$, when they are quasi-affine transforms of each other.

For an operator $T \in \mathscr{L}(\mathfrak{H})$, $\{T\}'$ and $\{T\}''$ stand for the commutant and bicommutant of T, respectively, i.e. $\{T\}' = \mathscr{I}(T, T)$ and $\{T\}'' = \bigcap\{\{A\}' : A \in \{T\}'\}$. Moreover, $\operatorname{Lat} T$ denotes the lattice of invariant subspaces of T, while $\operatorname{Lat}'' T$ and $\operatorname{Hyplat} T$ are the biinvariant and hyperinvariant subspace lattices of T, i.e. $\operatorname{Lat}'' T = \bigcap\{\operatorname{Lat} A : A \in \{T\}''\}$ and $\operatorname{Hyplat} T = \bigcap\{\operatorname{Lat} A : A \in \{T\}'\}$.

A contraction $T \in \mathscr{L}(\mathfrak{H})$, i.e. an operator with norm $\|T\| \leq 1$, is of class C_{11} if for every non-zero vector $h \in \mathfrak{H}$ we have $\lim_{n\to\infty} \|T^n h\| \neq 0 \neq \lim_{n\to\infty} \|T^{*n} h\|$. The set $\operatorname{Lat}_1 T = \{\mathfrak{M} \in \operatorname{Lat} T : T|\mathfrak{M} \in C_{11}\}$ of C_{11}-invariant subspaces forms a complete lattice, whose sublattice is the C_{11}-part of the hyperinvariant subspace lattice: $\operatorname{Hyplat}_1 T = \operatorname{Hyplat} T \cap \operatorname{Lat}_1 T$. (For details we refer to [7].)

I. IMPLEMENTATION OF LATTICE ISOMORPHISMS BY INTERTWINING OPERATORS

1. HYPERINVARIANT SUBSPACES

Let T_1 and T_2 be quasi-similar C_{11}-contractions. It is known that their C_{11}-hyperinvariant subspace lattices are isomorphic

$$\operatorname{Hyplat}_1 T_1 \cong \operatorname{Hyplat}_1 T_2.$$

Moreover, there is only one isomorphism

$$q_{T_1, T_2} : \mathrm{Hyplat}_1 T_1 \to \mathrm{Hyplat}_1 T_2,$$

such that $T_1|\mathfrak{M}$ is quasi-similar to $T_2|q_{T_1, T_2}(\mathfrak{M})$, for every $\mathfrak{M} \in \mathrm{Hyplat}_1 T_1$. (Cf. [11], [13], [7].)

It is known also (cf. e.g. [7]) that if $\mathrm{Hyplat}_1 T_2 = \{(\mathrm{ran}\, B)^- : B \in \{T_2\}''\}$, then q_{T_1, T_2} can be implemented in the following way: for every quasi-affinity $X \in \mathcal{I}(T_1, T_2)$, q_{T_1, T_2} coincides with the mapping

$$q_X : \mathrm{Hyplat}_1 T_1 \to \mathrm{Hyplat}_1 T_2, \quad q_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-.$$

In the general case we can prove the following

PROPOSITION 1. *If T_1 and T_2 are quasi-similar C_{11}-contractions, then there exists a quasi-affinity $X \in \mathcal{I}(T_1, T_2)$ which implements the isomorphism q_{T_1, T_2}, i.e. $q_{T_1, T_2} = q_X$ for the mappings defined before.*

Proof. Let $U_+ \in \mathcal{L}(\mathfrak{K}_+)$ be the minimal isometric dilation of the contraction $T_2 \in \mathcal{L}(\mathfrak{H})$, and let us consider the unitary part $R = U_+ \mid \mathfrak{R}$ of U_+ in the Wold decomposition. It is known that T_1 and T_2 are quasi-similar to R (cf. [11, Propositions II.3.4 and II.3.5]).

Let $X_1 \in \mathcal{I}(T_1, R)$ be an arbitrary quasi-affinity. Since R being unitary, $\mathrm{Hyplat}_1 R = \mathrm{Hyplat}\, R = \{(\mathrm{ran}\, B)^- : B \in \{R\}''\}$ we infer that $q_{T_1, R} = q_{X_1}$.

On the other hand, let $X_2 \in \mathcal{I}(R, T_2)$ be the operator defined by $X_2 = P_{\mathfrak{H}}|\mathfrak{R}$ ($= (P_{\mathfrak{R}}|\mathfrak{H})^*$), where $P_{\mathfrak{H}}$ denotes the orthogonal projection onto $\mathfrak{H}$ in the space $\mathfrak{K}_+$. On account of [11, Proposition II.3.5] X_2 is a quasi-affinity. Moreover, an application of the Lifting theorem (cf. [11, Theorem II.2.3] and [12]) gives that for every operator $B \in \{T_2\}'$ we can find an operator $C \in \{R\}'$ such that $BX_2 = X_2C$. This relation shows that for any subspace $\mathfrak{M} \in \mathrm{Hyplat}_1 R$ the subspace $q_{X_2}(\mathfrak{M}) = (X_2\mathfrak{M})^-$ belongs to $\mathrm{Hyplat}_1 T_2$. Since $q_{R, T_2}(\mathfrak{M}) = \bigvee\{(AX_2\mathfrak{M})^- : AT_2 = T_2A\}$, for every $\mathfrak{M} \in \mathrm{Hyplat}_1 R$ (cf. e.g. [7, Theorem 5]), $(X_2\mathfrak{M})^- \in \mathrm{Hyplat}_1 T_2$ implies $q_{R, T_2}(\mathfrak{M}) = (X_2\mathfrak{M})^-$. Hence, we infer that $q_{X_2} = q_{R, T_2}$.

Consequently, for the quasi-affinity $X = X_2X_1 \in \mathcal{I}(T_1, T_2)$ we have

$$q_X = q_{X_2} \circ q_{X_1} = q_{R, T_2} \circ q_{T_1, R} = q_{T_1, T_2},$$

and the proof is finished.

In the sequel we shall examine how the subspace $\mathfrak{M} \in \text{Hyplat}_1 T_1$ can be recovered from the subspace $(X\mathfrak{M})^- \in \text{Hyplat}_1 T_2$. We shall need a lemma which expresses the maximality of C_{11}-hyperinvariant subspaces in a more explicit way than [11, Theorem VII.5.2].

LEMMA 2. *Let T be a C_{11}-contraction and $\mathfrak{M} \in \text{Hyplat}_1 T$, $\mathfrak{L} \in \text{Lat}_1 T$. If $T|\mathfrak{L}$ can be injected into $T|\mathfrak{M}$, or $(T|\mathfrak{L})^*$ can be injected into $(T|\mathfrak{M})^*$, then $\mathfrak{L}$ is contained in $\mathfrak{M}$.*

Proof. We may assume that T is a completely non-unitary contraction (cf. [6, Lemmas 1 and 2]). On account of [5, Corollary 1] and [3, Lemma 4.1] the relation $T|\mathfrak{L} \overset{i}{\prec} T|\mathfrak{M}$ implies that

$$\text{rank}\, \Delta_{T|\mathfrak{L}}(e^{it}) \leqslant \text{rank}\, \Delta_{T|\mathfrak{M}}(e^{it})$$

a.e. (with respect to the Lebesgue measure) on the unit circle C of the complex plane $\mathbf{C}$. (For a c.n.u. contraction S, Δ_S is defined by $\Delta_S(e^{it}) = [I - \Theta_S(e^{it})^* \Theta_S(e^{it})]^{1/2}$, where Θ_S is the characteristic function of S.) This inequality shows that $\Theta_{T|\mathfrak{L}}(e^{it})$ is isometric whenever $\Theta_{T|\mathfrak{M}}(e^{it})$ is so. Therefore, on account of [11, Theorem VII.5.2] we obtain that $\mathfrak{L} \subset \mathfrak{M}$.

The case $(T|\mathfrak{L})^* \overset{i}{\prec} (T|\mathfrak{M})^*$ follows from the previous one, taking into account that

$$\text{rank}\, \Delta_S(e^{it}) = \text{rank}\, \Delta_{S^*}(e^{-it})$$

holds a.e., for every c.n.u. C_{11}-contraction S. The last equality is a consequence of the relation $\Theta_S(e^{it})^* \Delta_{S^*}(e^{-it}) = \Delta_S(e^{it})\, \Theta_S(e^{it})$ (we remind that $\Theta_{S^*}(e^{it}) = \Theta_S(e^{-it})^*$) and the fact that, for the C_{11}-contraction S, Θ_S is outer from both sides, and so $\Theta_S(e^{it})$ is a quasi-affinity a.e. (cf. [11, Propositions VI.3.5 and V.2.4]).

PROPOSITION 3. *Let T_1 and T_2 be C_{11}-contractions, and let us assume that $X \in \mathscr{I}(T_1, T_2)$ is an injective operator. Then for every subspace $\mathfrak{M} \in \text{Hyplat}_1 T_1$ we have*

$$\mathfrak{M} = \{X^{-1}((X\mathfrak{M})^-)\}^{(1)}.$$

(We recall that for any C_{11}-contraction T, and for any invariant subspace $\mathfrak{L} \in \text{Lat}\, T$, $\mathfrak{L}^{(1)}$ denotes the C_{11}-part of $\mathfrak{L}$, i.e. the largest invariant subspace $\mathfrak{L}'$ included in $\mathfrak{L}$, such that $T|\mathfrak{L}' \in C_{11}$; cf. [7].)

Proof. It is clear that the subspace $\mathfrak{M}' = \{X^{-1}((X\mathfrak{M})^-)\}^{(1)} \in \text{Lat}_1 T_1$ contains $\mathfrak{M}$. On the other hand, both $T_1|\mathfrak{M}$ and $T_1|\mathfrak{M}'$ being the quasi-affine transforms of $T_2|(X\mathfrak{M})^- \in C_{11}$, it follows that $T_1|\mathfrak{M}$ is quasi-similar to $T_1|\mathfrak{M}'$. (We remind that if a C_{11}-contraction S_1 is a quasi-affine transform of a C_{11}-contraction S_2, then they are quasi-similar; cf. [11, Propositions II.3.4, II.3.5].) Since $\mathfrak{M}$ belongs to $\mathrm{Hyplat}_1 T_1$, Lemma 2 yields that $\mathfrak{M}$ includes $\mathfrak{M}'$. Therefore $\mathfrak{M} = \mathfrak{M}'$, and the proof is completed.

REMARK 4. We note that the operation of taking C_{11}-part can not be omitted in the preceding proposition. In fact, there exists a cyclic C_{11}-contraction T, quasi-similar to a reductive unitary operator U, such that $(\mathfrak{M}^{\perp_1})^{\perp} \supsetneqq \mathfrak{M}$, for a subspace $\mathfrak{M} \in \mathrm{Lat}_1 T$ (cf. [2, Remark 3.6]). (Here $\mathfrak{M}^{\perp_1}$ is defined by $\mathfrak{M}^{\perp_1} = (\mathfrak{M}^{\perp})^{(1)}$.) Now, on account of [6, Proposition 7] we get $(X((\mathfrak{M}^{\perp_1})^{\perp}))^{-} = (X\mathfrak{M})^{-}$, for any operator $X \in \mathscr{I}(T, U)$, while in virtue of [8, Theorem 15] we have $\mathrm{Lat}_1 T = \mathrm{Hyplat}_1 T$. Therefore $\mathfrak{M} \neq X^{-1}(X\mathfrak{M})^{-}$ for the previous subspace $\mathfrak{M} \in \mathrm{Hyplat}_1 T$.

2. BIINVARIANT SUBSPACES

In this point we examine the behaviour of $\mathrm{Lat}_1 T$ under quasi-similarity, for C_{11}-contractions. First of all we note that for any $T \in C_{11}$ we have

$$\mathrm{Lat}_1 T := \{\mathfrak{M} \in \mathrm{Lat}\, T : T|\mathfrak{M} \in C_{11}\} = \{\mathfrak{M} \in \mathrm{Lat}'' T : T \mid \mathfrak{M} \in C_{11}\}.$$

(Cf. [6, Lemma 5].)

Let $T_1 \in \mathscr{L}(\mathfrak{H}^{(1)})$ and $T_2 \in \mathscr{L}(\mathfrak{H}^{(2)})$ be weakly similar C_{11}-contractions. Weak similarity, which is an equivalence relation in C_{11}, stronger than quasi-similarity, has been introduced in [8] and [9], and means that there are basic systems $\{\mathfrak{H}_n^{(1)}\}_n$ and $\{\mathfrak{H}_n^{(2)}\}_n$ in $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$, respectively, such that $\mathfrak{H}_n^{(1)} \in \mathrm{Hyplat}_1 T_1$, $\mathfrak{H}_n^{(2)} \in \mathrm{Hyplat}_1 T_2$ and $T_1|\mathfrak{H}_n^{(1)}$ is similar to $T_2|\mathfrak{H}_n^{(2)}$, for every n. (We recall that $\{\mathfrak{H}_n\}_n$ is a basic system in $\mathfrak{H}$, if $\mathfrak{H} = \mathfrak{H}_n \dotplus (\bigvee_{k \neq n} \mathfrak{H}_k)$ for every n, and $\bigcap_{n \geqslant 0} (\bigvee_{k \geqslant n} \mathfrak{H}_k) = \{0\}$; cf. [1].)

It is easy to see that, for $i = 1, 2$, $\mathrm{Lat}_1 T_i$ can be decomposed into the direct sum,

$$\mathrm{Lat}_1 T_i = \dotplus_n \mathrm{Lat}_1 (T_i|\mathfrak{H}_n^{(i)}),$$

i.e. every subspace $\mathfrak{M} \in \mathrm{Lat}_1 T_i$ can be uniquely written in the form $\mathfrak{M} = \bigvee_n \mathfrak{M}_n$, where $\mathfrak{M}_n \in \mathrm{Lat}_1(T_i \mid \mathfrak{H}_n^{(i)})$, for every n.

Now, let us define the quasi-affinity $X \in \mathscr{I}(T_1, T_2)$ as follows:

$$Xh = \sum_n \alpha_n A_n P_n h \quad (h \in \mathfrak{H}^{(1)}),$$

where $A_n \in \mathscr{I}(T_1|\mathfrak{H}_n^{(1)}, T_2|\mathfrak{H}_n^{(2)})$ is an affinity, P_n denotes the projection onto $\mathfrak{H}_n^{(1)}$ with respect to the decomposition $\mathfrak{H}^{(1)} = \mathfrak{H}_n^{(1)} \dotplus (\bigvee_{k \neq n} \mathfrak{H}_k^{(1)})$, for every n, and the sequence $\{\alpha_n\}_n$ of positive numbers satisfies the inequality

$$\sum_n \alpha_n \|A_n\| \, \|P_n\| < \infty.$$

It is immediate that the mapping

$$\varphi_X : \mathrm{Lat}_1 T_1 \to \mathrm{Lat}_1 T_2, \quad \varphi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-,$$

corresponding to X, will be an isomorphism. Therefore, we obtain

PROPOSITION 5. *If T_1 and T_2 are weakly similar C_{11}-contractions, then there is a quasi-affinity $X \in \mathscr{I}(T_1, T_2)$ such that the mapping φ_X defined before is an isomorphism, moreover, $T_1 \mid \mathfrak{M}$ is weakly similar to $T_2 \mid \varphi_X(\mathfrak{M})$, for every $\mathfrak{M} \in \mathrm{Lat}_1 T_1$.*

Proof. We have only to verify the last statement. Let us consider the decomposition $\mathfrak{M} = \dot{+}_n \mathfrak{M}_n$ of $\mathfrak{M} \in \mathrm{Lat}_1 T_1$, where $\mathfrak{M}_n \subset \mathfrak{H}_n^{(1)}$, and $\mathfrak{M}_n \in \mathrm{Lat}_1(T_1 \mid \mathfrak{M})$, for every n. It is evident that $\{\mathfrak{M}_n\}_n$ forms a basic system in $\mathfrak{M}$. We shall show that $\mathfrak{M}_n \in \mathrm{Hyplat}(T_1 \mid \mathfrak{M})$, for every n. We may assume that T_1 is c.n.u. (cf. [6, Lemmas 1 and 2]). Then the relation $\mathfrak{H}_n^{(1)} \cap (\bigvee_{k \neq n} \mathfrak{H}_k^{(1)}) = \{0\}$ implies that

$$\mathrm{rank}\, \Delta_{T_1|\mathfrak{H}_n^{(1)}}(e^{it}) \cdot \mathrm{rank}\, \Delta_{T_1|\bigvee_{k\neq n} \mathfrak{H}_k^{(1)}}(e^{it}) = 0 \quad \text{a.e.}$$

(cf. [7]), and so in virtue of [11, Theorem VII.1.1 and Proposition VII.3.3.d] it follows that

$$\mathrm{rank}\, \Delta_{T_1|\mathfrak{M}_n}(e^{it}) \cdot \mathrm{rank}\, \Delta_{T_1|\bigvee_{k\neq n} \mathfrak{M}_k}(e^{it}) = 0 \quad \text{a.e.}.$$

Now, by [5, Corollary 1] and [3, Lemma 4.1] we infer that $\mathscr{I}(T_1|\mathfrak{M}_n, T_1| \bigvee_{k\neq n} \mathfrak{M}_k) = \{0\}$, hence $\mathfrak{M}_n \in \mathrm{Hyplat}(T_1 \mid \mathfrak{M})$.

An analogous argument yields that $\{(X\mathfrak{M}_n)^-\}_n$ forms a basic system in $(X\mathfrak{M})^-$, consisting of C_{11}-hyperinvariant subspaces of $T_2 \mid (X\mathfrak{M})^-$. Since $T_1 \mid \mathfrak{M}_n$ is similar to $T_2 \mid (X\mathfrak{M}_n)^-$ for every n, it follows that $T_1 \mid \mathfrak{M}$ and $T_2 \mid (X\mathfrak{M})^-$ are weakly similar, and the proof is completed.

We note that φ_X is not uniquely determined. For example, let T_1 and T_2 be the identity operator. Then, for every invertible operator X, φ_X will be an isomorphism. However, as the following proposition shows, the restriction of φ_X to C_{11}-hyperinvariant subspaces is unique.

PROPOSITION 6. *Let T_1 and T_2 be quasi-similar C_{11}-contractions, and let us assume that*

$$\varphi_X : \mathrm{Lat}_1 T_1 \to \mathrm{Lat}_1 T_2, \quad \varphi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$$

is an isomorphism, for a quasi-affinity $X \in \mathcal{I}(T_1, T_2)$. *Then for every subspace* $\mathfrak{M} \in \operatorname{Hyplat}_1 T_1$, $\varphi_X(\mathfrak{M})$ *belongs to* $\operatorname{Hyplat}_1 T_2$, *and the equation*

$$\varphi_X \mid \operatorname{Hyplat}_1 T_1 = q_{T_1, T_2}$$

is true.

Proof. Let $\mathfrak{M} \in \operatorname{Hyplat}_1 T_1$ be an arbitrary subspace. Taking into account that

$$T_2 \mid \varphi_X(\mathfrak{M}) \sim T_1 \mid \mathfrak{M} \sim T_2 \mid q_{T_1, T_2}(\mathfrak{M}),$$

we infer by Lemma 2 that $\varphi_X(\mathfrak{M}) \subset q_{T_1, T_2}(\mathfrak{M})$. Since φ_X is an isomorphism, it follows that $\mathfrak{M}' = \varphi_X^{-1}(q_{T_1, T_2}(\mathfrak{M})) \supset \mathfrak{M}$. On the other hand, the relations

$$T_1 \mid \mathfrak{M}' \sim T_2 \mid q_{T_1, T_2}(\mathfrak{M}) \sim T_1 \mid \mathfrak{M}$$

imply again by Lemma 2 that $\mathfrak{M}' \subset \mathfrak{M}$. Therefore, we obtain that $\mathfrak{M}' = \mathfrak{M}$, and so $\varphi_X(\mathfrak{M}) = q_{T_1, T_2}(\mathfrak{M})$.

Now we shall show that weak similarity can not be replaced by quasi-similarity in Proposition 5. This will immediately follow from the following two propositions. The first one can be contrasted with [8, Lemma 7].

PROPOSITION 7. *There exist a C_{11}-contraction T and subspaces* $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat}_1 T$ *such that*

$$\mathfrak{M}_1 \overset{(1)}{\bigcap} \mathfrak{M}_2 = \{0\} \neq \mathfrak{M}_1 \cap \mathfrak{M}_2,$$

and

$$\mathfrak{M}_1 = \{h \in \mathfrak{M}_1 : T^n h \in \mathfrak{M}_1 \cap \mathfrak{M}_2 \textit{ for some } n \geqslant 0\}^-.$$

(Here $\mathfrak{M}_1 \overset{(1)}{\bigcap} \mathfrak{M}_2$ is the greatest common lower bound of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in the lattice $\operatorname{Lat}_1 T$; cf. [7].)

Proof. Let $U \in \mathcal{L}(\mathfrak{K})$ be a bilateral shift of infinite multiplicity, and $\mathfrak{M}$ a wandering subspace such that $\mathfrak{K} = \bigoplus_{n=-\infty}^{\infty} U^n \mathfrak{M}$. Let $Q \in \mathcal{L}(\mathfrak{M})$ be a non-invertible C_{11}-contraction. By [2, Lemma 3.2] there exists a vector $f \in \mathfrak{M}$ such that the operator

$$\begin{bmatrix} 0 & 0 \\ f & Q \end{bmatrix} \in \mathcal{L}(\mathbf{C} \oplus \mathfrak{M})$$

is an injective contraction; in the matrix, f denotes the operator $f: \mathbf{C} \to \mathfrak{M}$ defined by $f: \lambda \mapsto \lambda f$.

Let $D \in \mathcal{L}(\mathfrak{K})$ be the operator which has the diagonal form $D = \bigoplus_{n=-\infty}^{\infty} Q_n$ in the decomposition $\mathfrak{K} = \bigoplus_{n=-\infty}^{\infty} U^n\mathfrak{M}$, with $Q_n = U^n Q U^{-n} | U^n\mathfrak{M} \in \mathcal{L}(U^n\mathfrak{M})$, and let us consider the operator $W = UD \in \mathcal{L}(\mathfrak{K})$. W is obviously a C_{11}-contraction.

Let $S \in \mathcal{L}(\mathfrak{K}_+)$ be a unilateral shift of multiplicity one, and let $e \in \operatorname{ran}(I - SS^*)$ be a unit vector. Moreover, let $f \otimes e \in \mathcal{L}(\mathfrak{K}_+, \mathfrak{K})$ denote the operator of rank one, which transfers the vector e into f, i.e. $(f \otimes e)k = \langle k, e\rangle f$, for every $k \in \mathfrak{K}_+$. Now we define the operator T on the space $\mathfrak{H} = \mathfrak{K}_+ \oplus \mathfrak{K}$ by the matrix

$$T = \begin{bmatrix} S^* & 0 \\ f \otimes e & W \end{bmatrix}.$$

Let $x \in \mathfrak{H}$ be an arbitrary non-zero vector and, for every integer n, let $P_n \in \mathcal{L}(\mathfrak{H})$ denote the orthogonal projection onto the subspace $U^n\mathfrak{M}$. Then we can find a positive integer n_0 such that $\left(\sum_{n=1}^{\infty} P_n\right) T^{n_0} x \neq 0$, hence there is an integer $n_1 \geqslant 1$ such that $y = P_{n_1} T^{n_0} x \neq 0$. Since, for every $n \geqslant n_0$, the vectors $T^{n-n_0} y$ and $T^{n-n_0}(T^{n_0}x - y)$ are orthogonal, it follows that $\|T^n x\| = \|T^{n-n_0} T^{n_0} x\| \geqslant \geqslant \|T^{n-n_0} y\|$, if $n \geqslant n_0$. Taking into account $Q \in C_{1\cdot}$, we infer that $\lim_{n\to\infty} \|T^n x\| \geqslant \geqslant \lim_{m\to\infty} \|T^m y\| = \lim_{m\to\infty} \|Q^m U^{-n_1} y\| > 0$. So we have obtained that $T \in C_{1\cdot}$. The relation $T \in C_{\cdot 1}$ can be proved similarly, therefore T is a C_{11}-contraction. Moreover, the subspaces $\mathfrak{M}_1 = \bigvee_{n \geqslant 0} T^n \mathfrak{K}_+$ and $\mathfrak{M}_2 = \mathfrak{K}$ clearly possess the required properties.

PROPOSITION 8. *Let T_1 be a C_{11}-contraction with properties in Proposition 7 and T_2 be a C_{11}-contraction, quasi-similar to T_1 such that* $\operatorname{Lat}_1 T_2$ *is a sublattice of* $\operatorname{Lat} T_2$ *(e.g. T_2 may be a unitary operator quasi-similar to T_1). Then for every operator $X \in \mathcal{I}(T_1, T_2)$ the mapping*

$$\varphi_X : \operatorname{Lat}_1 T_1 \to \operatorname{Lat}_1 T_2, \quad \varphi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$$

is not a lattice-isomorphism.

Proof. First of all we note that $(X\mathfrak{M})^- \in \operatorname{Lat}_1 T_2$, for every $\mathfrak{M} \in \operatorname{Lat}_1 T_1$ (cf. [6, Lemma 5]). Let $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat}_1 T_1$ be subspaces with properties described in Proposition 7, and let us assume that φ_X is an isomorphism for an operator $X \in \mathcal{I}(T_1, T_2)$. Then the relation $\mathfrak{M}_1 \overset{(1)}{\bigcap} \mathfrak{M}_2 = \{0\}$ implies that $(X\mathfrak{M}_1)^- \overset{(1)}{\bigcap} (X\mathfrak{M}_2)^- = \{0\}$.

Since $\mathrm{Lat}_1 T_2$ is a sublattice of $\mathrm{Lat}\, T_2$, it follows that $X(\mathfrak{M}_1 \cap \mathfrak{M}_2) \subset (X\mathfrak{M}_1)^- \cap \cap (X\mathfrak{M}_2)^- = (X\mathfrak{M}_1)^- \overset{(1)}{\bigcap} (X\mathfrak{M}_2)^- = \{0\}$. Now, on account of the relation $\mathfrak{M}_1 = = \{h \in \mathfrak{M}_1 : T_1^n h \in \mathfrak{M}_1 \cap \mathfrak{M}_2, n \geqslant 0\}^-$, and taking into consideration that T_2 is an injective operator, the intertwining relation $XT_1^n = T_2^n X$ $(n \geqslant 0)$ yields that $(X\mathfrak{M}_1)^- = \{0\}$. However $\mathfrak{M}_1 \in \mathrm{Lat}_1 T_1$ is a non-zero subspace, since $\mathfrak{M}_1 \supset \mathfrak{M}_1 \cap \cap \mathfrak{M}_2 \neq \{0\}$, hence φ_X is not an isomorphism, which is a contradiction.

3. INVARIANT SUBSPACES

Finally, for the sake of easy reference we mention the following fact, frequently used by Wu in connection with contractions of finite defect indices (cf. e.g. [17]).

PROPOSITION 9. *Let T_1 and T_2 be quasi-similar C_{11}-contractions with absolutely continuous unitary parts. Let us assume that there are quasi-affinities $X \in \mathscr{I}(T_1, T_2)$ and $Y \in \mathscr{I}(T_2, T_1)$ such that $YX = \delta_1(T_1)$ and $XY = \delta_2(T_2)$ for some outer functions $\delta_1, \delta_2 \in H^\infty$. Then the mapping*

$$\psi_X : \mathrm{Lat}\, T_1 \to \mathrm{Lat}\, T_2, \quad \psi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$$

will be an isomorphism such that $T_1 | \mathfrak{M}$ is quasi-similar to $T_2 | \psi_X(\mathfrak{M})$, for every $\mathfrak{M} \in \mathrm{Lat}\, T_1$.

Proof. We have only to note that for any contraction T with absolutely continuous unitary part, and for any outer function $\delta \in H^\infty$, $\delta(T)$ is a quasi-affinity (cf. [11, Proposition III. 3.1]). This fact immediately implies that ψ_X and ψ_Y are inverses of each other.

II. CHARACTERIZATION OF INVARIANT SUBSPACES

1. CANONICAL INTERTWINING QUASI-AFFINITIES

In this section, applying the results of Section I, we give descriptions of invariant subspaces of C_{11}-contractions by the aid of canonical quasi-affinities intertwining them with unitaries.

In virtue of [6, Lemmas 1 and 2] without loss of generality we may assume that $T \in C_{11}$ is a c.n.u. contraction, and so that $T = S(\Theta)$ is a model-operator (cf. [11, Chapter VI]). More precisely, let $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$ be a purely contractive, analytic function, outer from both sides (defined on the open unit disc D of $\mathbf{C}$, and with values in $\mathscr{L}(\mathfrak{E})$), and let $T = S(\Theta)$ be defined in the following way. $H^2(\mathfrak{E})$,

$L^2(\mathfrak{E})$ denote the usual Hilbert spaces of vector-valued functions, $\Delta(e^{it}) = [I - \Theta(e^{it})^* \Theta(e^{it})]^{1/2}$ is the defect function associated with Θ. Then

$$\mathfrak{K}_+ = H^2(\mathfrak{E}) \oplus (\Delta L^2(\mathfrak{E}))^-,$$

and $U_+ \in \mathcal{L}(\mathfrak{K}_+)$ stands for the isometry of multiplication by e^{it}. The subspace $\mathfrak{R} = (\Delta L^2(\mathfrak{E}))^-$ reduces U_+ to a unitary operator $R = U_+ | \mathfrak{R}$. Let us consider the isometry $V: H^2(\mathfrak{E}) \to \mathfrak{K}_+$, $Vu = \Theta u \oplus \Delta u \, (u \in H^2(\mathfrak{E}))$. Then

$$\mathfrak{H} = \mathfrak{K}_+ \ominus VH^2(\mathfrak{E})$$

will be a semi-invariant subspace of U_+, and $T = S(\Theta)$ is defined as the compression

$$T = P_{\mathfrak{H}} U_+ | \mathfrak{H}.$$

T is a c.n.u. C_{11}-contraction, and U_+ is its minimal isometric dilation.

It is known (cf. [11, Proposition II.3.5]) that T is quasi-similar to the unitary operator R, called the residual part of T, and that

(1) $$Y = P_{\mathfrak{H}} | \mathfrak{R} \in \mathcal{I}(R, T)$$

is an intertwining quasi-affinity.

Our aim is to provide another quasi-affinity, intertwining T with unitary, which has the advantage of being an operator of multiplication by an operator-valued function. Let us consider the function $\Delta_*(e^{it}) = [I - \Theta(e^{it})\Theta(e^{it})^*]^{1/2}$, and the space $\mathfrak{R}_* = (\Delta_* L^2(\mathfrak{E}))^-$. The unitary operator R_* of multiplication by e^{it} on $\mathfrak{R}_*$ is called the $*$-residual part of T. Our result is the following

THEOREM 10. *If $T = S(\Theta)$ is the C_{11}-contraction introduced before, then the mapping*

(2) $$X: \mathfrak{H} \to \mathfrak{R}_*, \quad X(u \oplus v) = -\Delta_* u + \Theta v \quad (u \oplus v \in \mathfrak{H})$$

is a (well-defined) quasi-affinity, belonging to $\mathcal{I}(T, R_*)$. Moreover, its product $Z = XY \in \mathcal{I}(R, R_*)$ with the operator Y defined in (1) *acts as a multiplication by Θ*, i.e.

$$(Zv)(e^{it}) = \Theta(e^{it}) v(e^{it})$$

holds a.e. *on C, for every $v \in \mathfrak{R}$.*

The operator X occurring in this theorem was introduced by Sickler [10] in the case when Θ is scalar-valued (i.e. $\dim \mathfrak{E} = 1$), and was studied by Wu [15], [16], when T has finite defect indices (i.e. $\dim \mathfrak{E} < \infty$).

To prove Theorem 10 we need a lemma, which is a slight modification of [14, Proposition 2]. (Contrast also with [11, Proposition III.1.1.])

LEMMA 11. *If* $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}_*\}$ *is a contractive, analytic, *-outer function, then*

$$\Theta L^2(\mathfrak{E}) \cap H^2(\mathfrak{E}_*) = \Theta H^2(\mathfrak{E}).$$

Proof. Let us introduce the unitary operators $V \in \mathscr{L}(L^2(\mathfrak{E}))$ and $V_* \in \mathscr{L}(L^2(\mathfrak{E}_*))$ by the definitions:

$$(Vf)(e^{it}) = e^{-it}f(e^{-it}) \quad (f \in L^2(\mathfrak{E})),$$

and

$$(V_*g)(e^{it}) = e^{-it}g(e^{-it}) \quad g \in L^2(\mathfrak{E}_*)).$$

Since Θ is *-outer, it follows that $\Theta^{\sim}(\lambda) = \Theta(\bar{\lambda})^*$ is outer, i.e.

$$(\Theta^{\sim} H^2(\mathfrak{E}_*))^- = H^2(\mathfrak{E}).$$

An easy computation shows that $\Theta^* = V\Theta^{\sim}V_*^{-1}$. Hence we infer that

$$(\Theta^*(L^2(\Theta_*) \ominus H^2(\mathfrak{E}_*)))^- = L^2(\mathfrak{E}) \ominus H^2(\mathfrak{E}). \tag{3}$$

Let us assume now that

$$\Theta\mathfrak{K} \subset H^2(\mathfrak{E}_*), \tag{4}$$

for a subspace $\mathfrak{K} \subset L^2(\mathfrak{E})$. Then (3) and (4) yeld that

$$L^2(\mathfrak{E}) \ominus H^2(\mathfrak{E}) = (\Theta^*(L^2(\mathfrak{E}_*) \ominus H^2(\mathfrak{E}_*)))^- \subset L^2(\mathfrak{E}) \ominus \mathfrak{K}.$$

Hence $\mathfrak{K} \subset H^2(\mathfrak{E})$, and the lemma is proved.

Proof of Theorem 10. For an arbitrary vector $u \oplus v \in \mathfrak{K}_+$ we define

$$\hat{X}(u \oplus v) = -\Delta_* u + \Theta v. \tag{5}$$

On account of the commuting relation

$$\Theta\Delta = \Delta_*\Theta, \tag{6}$$

it follows that $\hat{X}(u \oplus v) \in \mathfrak{R}_*$. Therefore, equation (5) defines an operator $\hat{X} \in \mathscr{L}(\mathfrak{K}_+, \mathfrak{R}_*)$. Being a multiplication operator by an operator-valued function, $\hat{X}$ clearly intetwines U_+ and R_*. On the other hand, for every $w \in H^2(\mathfrak{E})$ we have in virtue of (6) that

$$\hat{X}(\Theta w \oplus \Delta w) = -\Delta_*\Theta w + \Theta\Delta w = 0.$$

This immediately implies that the operator $X = \hat{X} \mid \mathfrak{H} \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}_*)$ intertwines T and R_*:

$$X \in \mathscr{I}(T, R_*).$$

For every vector $v \in L^2(\mathfrak{E})$ we have

$$Z(\Delta v) = XY(0 \oplus \Delta v) = X[(0 \oplus \Delta v) - (\Theta w \oplus \Delta w)] =$$

$$= X(-\Theta w \oplus (\Delta v - \Delta w)) = \Delta_* \Theta w + \Theta \Delta v - \Theta \Delta w = \Theta \Delta v,$$

where $w \in H^2(\mathfrak{E})$ is an appropriate vector, and we have used (6). Therefore, $Z \in \mathscr{L}(\mathfrak{R}, \mathfrak{R}_*)$ acts as multiplication by the operator-valued function Θ.

Now we prove that X has dense range. Since

$$\operatorname{ran} X \supset \operatorname{ran} Z \supset Z\Delta L^2(\mathfrak{E}) = \Theta \Delta L^2(\mathfrak{E}) = \Delta_* \Theta L^2(\mathfrak{E}),$$

it is enough to show that

$$(\Theta L^2(\mathfrak{E}))^- = L^2(\mathfrak{E}). \tag{7}$$

Let U denote the operator of multiplication by e^{it} on the space $L^2(\mathfrak{E})$. Taking into account that Θ is outer, we infer that for every integer n

$$(\Theta L^2(\mathfrak{E}))^- = (\Theta U^n L^2(\mathfrak{E}))^- = U^n (\Theta L^2(\mathfrak{E}))^- \supset U^n (\Theta H^2(\mathfrak{E}))^- = U^n H^2(\mathfrak{E}).$$

Hence $(\Theta L^2(\mathfrak{E}))^- \supset \bigvee_{n=-\infty}^{\infty} U^n H^2(\mathfrak{E}) = L^2(\mathfrak{E})$, which proves (7). Therefore, X has dense range.

Finally, we show that X is injective. Let us assume that

$$X(u \oplus v) = -\Delta_* u + \Theta v = 0, \tag{8}$$

for a vector $u \oplus v \in \mathfrak{H}$. Multiplying equation (8) by Δ_*, and applying (6), we obtain

$$0 = -\Delta_*^2 u + \Delta_* \Theta v = -u + \Theta \Theta^* u + \Theta \Delta v,$$

i.e.

$$H^2(\mathfrak{E}) \ni u = \Theta(\Theta^* u + \Delta v). \tag{9}$$

Since Θ is $*$-outer, an application of Lemma 11 results that

$$\Theta^* u + \Delta v \in H^2(\mathfrak{E}). \tag{10}$$

On the other hand, $u \oplus v$ being in $\mathfrak{H}$ is orthogonal to $VH^2(\mathfrak{E})$, i. e. for every vector

$w \in H^2(\mathfrak{E})$ we have

$$0 = \langle u \oplus v,\ \Theta w \oplus \Delta w\rangle_{\mathfrak{K}_+} = \langle \Theta^* u + \Delta v,\ w\rangle_{L^2(\mathfrak{E})}.$$

This yields that

$$\Theta^* u + \Delta v \in L^2(\mathfrak{E}) \ominus H^2(\mathfrak{E}). \tag{11}$$

Comparing relations (10) and (11) we conclude that

$$\Theta^* u + \Delta v = 0. \tag{12}$$

Now, we obtain by (12), (9) and (8) that $u = \Theta v = 0$, and since, Θ being $*$-outer, $\Theta(e^{it})$ is injective a. e. on C, this implies that $v = 0$. Consequently $u \oplus v = 0$, and the proof is completed.

2. HYPERINVARIANT SUBSPACES

As an application first we give a description of C_{11}-hyperinvariant subspaces.

Let $T = S(\Theta)$ be a C_{11}-contraction, and let $\mathscr{B}(C)$ denote the σ-algebra of Borel subsets of the unit circle. We call two sets $\alpha, \beta \in \mathscr{B}(C)$ to be T-equivalent, if $\operatorname{rank} \Delta(e^{it}) = 0$ a. e. on their symmetric difference. The inclusion relation in $\mathscr{B}(C)$ induces a partial ordering in the set $\mathscr{B}_T(C)$ of equivalence classes $\hat{\alpha}$ ($\alpha \in \mathscr{B}(C)$), making it a countable distributive, complete lattice.

THEOREM 12. *If $T = S(\Theta)$ is a C_{11}-contraction, then the mapping*

$$q : \mathscr{B}_T(C) \to \operatorname{Hyplat}_1 T,$$

$$q : \hat{\alpha} \mapsto \mathfrak{H}_\alpha = \{u \oplus v \in \mathfrak{H} : (-\Delta_* u + \Theta v)(e^{it}) = 0 \quad \text{a. e. } on \ C \setminus \alpha\}^{(1)}$$

is a (well-defined) lattice-isomorphism such that

$$\operatorname{rank} \Delta_{T|\mathfrak{H}_\alpha}(e^{it}) = \chi_\alpha(e^{it}) \operatorname{rank} \Delta(e^{it}) \quad \text{a. e.},$$

for every $\alpha \in \mathscr{B}(C)$, where χ_α stands for the characteristic function of α.

Proof. Let us consider the quasi-affinity $X \in \mathscr{I}(T, R_*)$ constructed in Theorem 10. Since R_* is unitary, the mapping

$$q_X : \operatorname{Hyplat}_1 T \to \operatorname{Hyplat}_1 R_* = \operatorname{Hyplat} R_*, \quad q_X(\mathfrak{M}) = (X\mathfrak{M})^-$$

($\mathfrak{M} \in \operatorname{Hyplat}_1 T$) will be an isomorphism such that $T \mid \mathfrak{M}$ is quasi-similar to $R_* \mid (X\mathfrak{M})^-$, for every $\mathfrak{M} \in \operatorname{Hyplat}_1 T$. In virtue of Proposition 3 every $\mathfrak{M} \in \operatorname{Hyplat}_1 T$ can be recovered from $q_X(\mathfrak{M})$ by the formula $\mathfrak{M} = \{X^{-1} q_X(\mathfrak{M})\}^{(1)}$.

On the other hand, the hyperinvariant subspaces of R_* have the form $\chi_\alpha(\Delta_* L^2(\mathfrak{E}))^-$, where $\alpha \in \mathscr{B}(C)$. Since $\Theta(e^{it})$ is a quasi-affinity a. e. and $\Theta\Delta = \Delta_*\Theta$, it follows that

$$\operatorname{rank} \Delta_*(e^{it}) = \operatorname{rank} \Delta(e^{it}) \quad \text{a. e.},$$

hence $\chi_\alpha(\Delta_* L^2(\mathfrak{E}))^- = \chi_\beta(\Delta_* L^2(\mathfrak{E}))^-$ if and only if α and β are T-equivalent.

So we have obtained that the mapping q defined in the theorem is a lattice-isomorphism. Finally, taking into account that $T \mid \mathfrak{H}_\alpha$ is quasi-similar to $R_* \mid \chi_\alpha(\Delta_* L^2(\mathfrak{E}))^-$ we infer by [5, Corollary 1] that

$$\operatorname{rank} \Delta_{T|\mathfrak{H}_\alpha}(e^{it}) = \chi_\alpha(e^{it}) \operatorname{rank} \Delta(e^{it}) \quad \text{a. e. .}$$

REMARK 13. On account of [7, Theorem 3] and [5, Corollary 1] there is only one mapping $q : \mathscr{B}_T(C) \to \mathrm{Hyplat}_1 T$ such that $\operatorname{rank} \Delta_{T|q(\hat{\alpha})}(e^{it}) = \chi_\alpha(e^{it}) \operatorname{rank} \Delta(e^{it})$ a.e. . Therefore, the isomorphism q occurring in Theorem 12 coincides with the mapping (from $\mathscr{B}_T(C)$ into $\mathrm{Hyplat}_1 T$) constructed by Sz.-Nagy and Foiaş via the regular factorization of Θ. (Cf. [11, Theorem VII.5.2], [13] and [7].) Actually, Theorem 12 provides a more explicit representation of this mapping. (This was the reason why Sickler has introduced it.)

3. BIINVARIANT SUBSPACES

Now we turn to the representation of $\mathrm{Lat}_1 T$, for $T = S(\Theta) \in C_{11}$. First of all we note that for the unitary operator R_* we have

$$\mathrm{Lat}_1 R_* = \mathrm{Lat}'' R_* = \{P(\Delta_* L^2(\mathfrak{E}))^- : P \in \{R_*\}',\ P^2 = P,\ P^* = P\},$$

where P is an operator of multiplication by a projection-valued function $P(e^{it})$ ($\in L(\mathfrak{E})$), with range in $(\Delta_* L^2(\mathfrak{E}))^-$ (cf. [11, proof of Lemma V.3.1]); let $\mathscr{P}_T$ denote the set of such functions.

THEOREM 14. *Let $T = S(\Theta)$ be a C_{11}-contraction, weakly similar to unitary. Then the C_{11}-invariant subspaces $\mathfrak{M} \in \mathrm{Lat}_1 T$ of T are exactly those of the form*

$$\mathfrak{M} = \{u \oplus v \in \mathfrak{H} : (-\Delta_* u + \Theta v)(e^{it}) \in \operatorname{ran} P(e^{it}) \quad \text{a. e.}\}^{(1)},$$

where P belongs to $\mathscr{P}_T$. Moreover, two subspaces $\mathfrak{M}$ and $\mathfrak{N} \in \mathrm{Lat}_1 T$, corresponding to P and $Q \in \mathscr{P}_T$, respectively, coincide if and only if $P(e^{it}) = Q(e^{it})$ a. e. .

Proof. Let us consider the quasi-affinity $X \in \mathscr{I}(T, R_*)$ defined in (2). It is sufficient to show that the mapping $\varphi_X : \mathrm{Lat}_1 T \to \mathrm{Lat}_1 R_*$, $\varphi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$ establishes an isomorphism. Then the relation $\mathfrak{M} = \{X^{-1}((X\mathfrak{M})^-)\}^{(1)}$, for every $\mathfrak{M} \in \mathrm{Lat}_1 T$, will immediately follow.

Since T is weakly similar to unitary, it follows by [8, Theorem 4] that $\Theta(e^{it})$ is (boundedly) invertible a. e. . For every natural number n, let α_n be the measurable set $\alpha_n = \{e^{it}: n \leqslant \|\Theta(e^{it})^{-1}\| < n+1\}$. Then the hyperinvariant subspaces $\mathfrak{R}_n = \chi_{\alpha_n}\mathfrak{R} \in \operatorname{Hyplat} R$ give an orthogonal decomposition of $\mathfrak{R}$: $\bigoplus_{n=1}^{\infty} \mathfrak{R}_n = \mathfrak{R}$.

Let us consider the quasi-affinities $Y \in \mathcal{I}(R, T)$ and $X \in \mathcal{I}(T, R_*)$ defined in (1) and (2). Taking into account the proof of Proposition 1 and the fact that $\operatorname{Hyplat}_1 R_* = \{(\operatorname{ran} A)^- : A \in \{R_*\}''\}$ we can see that

$$q_Y: \operatorname{Hyplat} R \to \operatorname{Hyplat}_1 T, \quad q_Y(\mathfrak{M}) = (Y\mathfrak{M})^- \quad (\mathfrak{M} \in \operatorname{Hyplat} R),$$

and

$$q_X: \operatorname{Hyplat}_1 T \to \operatorname{Hyplat} R_*, \quad q_X(\mathfrak{N}) = (X\mathfrak{N})^- \quad (\mathfrak{N} \in \operatorname{Hyplat}_1 T)$$

are isomorphisms. Hence $\mathfrak{H}_n \overset{(1)}{\bigcap} (\bigvee_{k\neq n} \mathfrak{H}_k) = \{0\}$ and $\mathfrak{H}_n \vee (\bigvee_{k\neq n} \mathfrak{H}_k) = \mathfrak{H}$ for every n and $\overset{(1)}{\bigcap_{n\geqslant 0}} (\bigvee_{k\geqslant n} \mathfrak{H}_k) = \{0\}$, for the subspaces $\mathfrak{H}_n = q_Y(\mathfrak{R}_n)$. An application of [8, Lemma 7] gives that $\mathfrak{H}_n \cap (\bigvee_{k\neq n} \mathfrak{H}_k) = \{0\}$ $(n \in \mathbf{N})$ and $\bigcap_{n\geqslant 0} (\bigvee_{k\geqslant n} \mathfrak{H}_k) = \{0\}$. Moreover, $T|\mathfrak{H}_n$ being quasi-similar to $R|\mathfrak{R}_n$ we infer that $\mathfrak{H}_n = q(\hat{\alpha}_n)$, where q is the mapping occurring in Theorem 12. On account of Remark 13 we get that the characteristic function $\Theta_{T|\mathfrak{H}_n}$, outer from both sides, satisfies the relation

$$\Theta_{T|\mathfrak{H}_n}(e^{it})^* \Theta_{T|\mathfrak{H}_n}(e^{it}) = \begin{cases} I & \text{a.e. on } C\setminus\alpha_n, \\ \Theta(e^{it})^*\Theta(e^{it}) & \text{a.e. on } \alpha_n. \end{cases}$$

This implies that $\|\Theta_{T|\mathfrak{H}_n}(e^{it})^{-1}\| \leqslant n+1$ a.e., and an argumentation applied in the proof of [8, Theorem 4] yields that $\mathfrak{H}_n \dotplus (\bigvee_{k\neq n} \mathfrak{H}_k) = \mathfrak{H}$. Therefore, $\{\mathfrak{H}_n\}_n \subset \operatorname{Hyplat}_1 T$ forms a basic system in $\mathfrak{H}$.

On the other hand, since q_X is also a lattice-isomorphism, we obtain that the subspaces $\mathfrak{R}_{*,n} = q_X(\mathfrak{H}_n) \in \operatorname{Hyplat} R_*$ give an orthogonal decomposition of $\mathfrak{R}_*$: $\bigoplus_n \mathfrak{R}_{*,n} = \mathfrak{R}_*$.

Taking into account that $Z = XY$ acts as the multiplication by Θ (see Theorem 10), and so $Z|\mathfrak{R}_n = (X|\mathfrak{H}_n)(Y|\mathfrak{R}_n)$ is bounded from below, we conclude that so is $X|\mathfrak{H}_n$ too, hence $X|\mathfrak{H}_n \in \mathcal{I}(T|\mathfrak{H}_n, R_*|\mathfrak{R}_{*,n})$ is an affinity.

Now, by the proof of Proposition 5 it follows that

$$\varphi_X: \operatorname{Lat}_1 T \to \operatorname{Lat}_1 R_*, \quad \varphi_X(\mathfrak{M}) = (X\mathfrak{M})^- \quad (\mathfrak{M} \in \operatorname{Lat}_1 T)$$

is an isomorphism, and the proof is finished.

4. INVARIANT SUBSPACES

Finally we give a description of all invariant subspaces under a more restrictive assumption on the C_{11}-contraction T.

PROPOSITION 15. *Let $T = S(\Theta)$ be a C_{11}-contraction and $X \in \mathscr{I}(T, R_*)$ the quasi-affinity defined in* (2). *If Θ has a scalar multiple, then there exists a quasi--affinity $\overline{Y} \in \mathscr{I}(R_*, T)$ such that*

$$X\overline{Y} = \delta(R_*) \quad \textit{and} \quad \overline{Y}X = \delta(T)$$

with an outer function $\delta \in H^\infty$.

Proof. By the assumption there are a contractive analytic function $\{\Omega(\lambda), \mathfrak{E}, \mathfrak{E}\}$ and a scalar-valued outer function $\delta \in H^\infty$ such that $\Theta\Omega = \Omega\Theta = \delta I$ (cf. [11, Theorem V.6.2]). Let $W \in \mathscr{I}(R_*, R)$ denote the quasi-affinity of multiplication by Ω and let $\overline{Y}$ be the product

$$\overline{Y} := YW \in \mathscr{I}(R_*, T),$$

where $Y \in \mathscr{I}(R, T)$ is the operator defined in (1). Then $\overline{Y}$ is also a quasi-affinity.

For any vector $v \in \mathfrak{R}_*$, we have on account of Theorem 10 that

$$(X\overline{Y})v = XYWv = ZWv = \Theta\Omega v = \delta v = \delta(R_*)v,$$

and so $X\overline{Y} = \delta(R_*)$.

On the other hand, for any vector $v \in \mathfrak{R}$ we can write

$$(\overline{Y}X)(Yv) = YWXYv = YWZv = Y\Omega\Theta v = Y\delta v = Y\delta(R)v = \delta(T)Yv.$$

Since ran Y is dense in $\mathfrak{H}$, we conclude that $\overline{Y}X = \delta(T)$.

In virtue of Propositions 15 and 9 we get

THEOREM 16. *Let $T = S(\Theta)$ be a C_{11}-contraction and $X \in \mathscr{I}(T, R_*)$ the quasi--affinity defined in* (2). *If Θ has a scalar multiple, then the mapping*

$$\psi_X : \operatorname{Lat} T \to \operatorname{Lat} R_*, \quad \psi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$$

will be a lattice-isomorphism such that $T|\mathfrak{M}$ is quasi-similar to $R_*|\psi_X(\mathfrak{M})$, for every $\mathfrak{M} \in \operatorname{Lat} T$.

SOME REMARKS. It is clear that the restriction $\varphi_X = \psi_X | \operatorname{Lat}_1 T : \operatorname{Lat}_1 T \to \to \operatorname{Lat}_1 R_*$ will be also an isomorphism. $T = S(\Theta)$ is in particular weakly similar to unitary under the assumption of Θ having a scalar multiple (cf. [8, Remark 5]).

Since the invariant subspaces of the unitary operator R_* are well-known (cf. e.g. [4]), a description of Lat T, analogous to Theorems 12 and 14 can be obtained. The rather complicated details are left to the reader.

Concluding our paper we prove that under the assumption of the last theorem the C_{11}-invariant subspace lattices coincide with the biinvariant subspace lattices This shows that our results are direct generalizations of the ones of Sickler and Wu (cf. [10], [15], [16]).

PROPOSITION 17. *If $T = S(\Theta)$ is a C_{11}-contraction such that Θ has a scalar multiple, then*

$$\mathrm{Lat}_1 T = \mathrm{Lat}'' T \quad \text{and} \quad \mathrm{Hyplat}_1 T = \mathrm{Hyplat}\, T.$$

Proof. It is enough to prove that every biinvariant subspace is C_{11}-invariant.

Let us consider the quasi-affinity $X \in \mathscr{I}(T, R_*)$ defined in (2). In virtue of Theorem 16 we know that the mapping

$$\psi_X : \mathrm{Lat}\, T \to \mathrm{Lat}\, R_*, \quad \psi_X : \mathfrak{M} \mapsto (X\mathfrak{M})^-$$

is an isomorphism, such that $T|\mathfrak{M}$ is quasi-similar to $R_*|\psi_X(\mathfrak{M})$, for every $\mathfrak{M} \in \mathrm{Lat}\, T$. Since R_* is unitary, we infer that $\mathrm{Lat}'' R_* = \mathrm{Lat}_1 R_*$. Hence, it is sufficient to verify the inclusion

$$\psi_X(\mathrm{Lat}'' T) \subset \mathrm{Lat}'' R_*.$$

So let $\mathfrak{M} \in \mathrm{Lat}'' T$ be an arbitrary subspace and $B \in \{R_*\}''$ an arbitrary operator. We have to check whether the relation

$$B(X\mathfrak{M})^- \subset (X\mathfrak{M})^-$$

holds.

On account of Proposition 15 there exists a quasi-affinity $\bar{Y} \in \mathscr{I}(R_*, T)$ such that

$$X\bar{Y} = \delta(R_*) \quad \text{and} \quad \bar{Y}X = \delta(T),$$

where $\delta \in H^\infty$ is an outer function. Let $A \in \{T\}'$ be arbitrary. Since $XA\bar{Y} \in \{R_*\}'$, B commutes with $XA\bar{Y}$: $BXA\bar{Y} = XA\bar{Y}B$. Multiplying this equality by $\bar{Y}$ and X from the left and from the right, respectively, and taking into consideration that $\bar{Y}X = \delta(T) \in \{T\}''$ is a quasi-affinity, we get

$$\delta(T)[(\bar{Y}BX)A] = \bar{Y}BXA\,\delta(T) = \bar{Y}(BXA\bar{Y})X = \bar{Y}(XA\bar{Y}B)X =$$

$$= \delta(T)[A(\bar{Y}BX)],$$

and so

$$(\bar{Y}BX)A = A(\bar{Y}BX).$$

Therefore $\bar{Y}BX$ belongs to $\{T\}''$, hence $\bar{Y}BX\mathfrak{M} \subset \mathfrak{M}$. Applying this relation and the fact that, δ being outer, the restriction of $\delta(R_*) = X\bar{Y}$ to any invariant subspace is a quasi-affinity, we conclude that

$$B(X\mathfrak{M})^- \subset (BX\mathfrak{M})^- = (\delta(R_*)\,BX\mathfrak{M})^- = (X\bar{Y}BX\mathfrak{M})^- \subset (X\mathfrak{M})^-.$$

Hence $(X\mathfrak{M})^-$ is invariant for B, and so the proof is finished.

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Received January 29, 1985; revised June 24, 1985.