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OPTIMIZATION OVER SPACES OF ANALYTIC FUNCTIONS AND THE CORONA PROBLEM

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1. INTRODUCTION

A classical pursuit in analysis is the problem of finding the distance of a function g on the circle $\mathbf{T}$ to H^{∞} and characterizing properties of the optima. The problem was solved when distance is in the L^{2} sense by Szegö while in the L^{∞} case much is known and there are over a hundred articles beginning with Carathéodory--Fejer and Pick in the early 1900's. In particular when g is continuous there is a direct characterization of the L^{∞} closest point f_0 to g provided f_0 is continuous. Namely,

(i) $|g - f_0|$ is constant,

(ii) the winding number of $(g - f_0)$ about zero is negative.

The article has two objectives. The first is to generalize this to nonlinear optimization problems. Indeed we find a strict generalization of this property. It applies to highly nonlinear optimization problems and gives a practical test to determine if a particular continuous function f_0 is or is not an optimum. We believe the result can be applied to many engineering situations, see for example [10] or [9].

The second objective of the article is to generalize the classical Corona theorem. Indeed this is forced upon us by our study of optimization.

The optimization problem this paper analyzes concerns a subset E of $\mathscr{C}_{\mathbf{T}}(\mathbf{C}^{N})$, the continuous $\mathbf{C}^{N}$-valued functions on the unit circle. We let $\Gamma(\mathrm{e}^{\mathrm{i}\theta}, w)$ be a function on $\mathbf{T}\times\mathbf{C}^{N}$, and study the optimization problem: Find

$$\text{(OPT)}\qquad \gamma_0 = \inf_{f\in E}\sup_{\mathrm{e}^{\mathrm{i}\theta}\in\mathbf{T}} \Gamma(\mathrm{e}^{\mathrm{i}\theta}, f(\mathrm{e}^{\mathrm{i}\theta})) = \|\Gamma(\cdot, f_0)\|_{\infty}.$$

Here $\|\ \ \|_{\infty}$ denotes the usual supremum norm. This article concerns qualitative properties of the optimum. Most of our attention focuses on $E = \mathfrak{A}$ the algebra of all functions in $\mathscr{C}_{\mathbf{T}}(\mathbf{C}^{N})$ with analytic continuation onto $\mathbf{D}$, the unit disk.

As we shall see two properties characterize solutions f_0 of (OPT) over $E = \mathfrak{A}$. The first property is that the function f_0 *flattens* Γ, that is, $\Gamma(\mathrm{e}^{\mathrm{i}\theta}, f_0(\mathrm{e}^{\mathrm{i}\theta}))$ is constant a.e. in θ. The complete characterization of the optimum for smooth Γ is.

THEOREM 1.1. ($N=1$). *A continuous function f_0 for which* $\frac{\partial \Gamma}{\partial z}(e^{i\theta}, f_0(e^{i\theta})) \stackrel{\Delta}{=} a(e^{i\theta})$ *never equals* 0 *is a strict local optimum for* (OPT) *over* $E = \mathfrak{A}$ *if and only if*

(1) f_0 *flattens* Γ.

(2) *The function a has positive winding number about* 0.

Note we do not guarantee existence of a continuous optimum f_0.

We also give conditions (1) and (2) when $N > 1$. While the main conditions (1) and (2) of Theorem 1.1 are stable under small changes in f_0, the most straightforward extension of (2) to $N > 1$ is not stable. Indeed we show that producing a stable version of (2) for $N > 1$ amounts to generalizing the Corona Theorem and computing the 'Corona constant' in this more general case.

There are two types of Corona Theorems, the classical one, cf. [7], and a newer one based on Toeplitz operators. Fortunately the Toeplitz version gives the exact 'Corona constant', is 'easy' to prove, and can be implemented on the computer. In this article we generalize the operator theoretic Corona Theorem. Surely this corresponds to some extension of the classical Corona Theorem and we hope that Corona specialists will find this an interesting open question. The section (§ 3) on the Corona Theorem is reasonably self contained.

To give the flavor of our Corona Theorem we state a weak corollary (of Theorem 4.2′). If $a \in L^\infty$ define T_a to be the Toeplitz operator $H^2 \to H^2$ for which $T_a g = P_{H^2} ag$ for $g \in H^2$; also H_a denotes the Hankel operator $H^2 \to H^{2\perp}$ given by $H_a g = P_{H^{2\perp}} ag$. Let $\bar{\mathbf{B}}L^\infty$ denote $\{f : \|f\|_\infty \leq 1\}$ the closed unit ball of L^∞. In fact $\mathbf{B}V$ where V is a vector space will always denote the open unit ball of V.

COROLLARY 1.2. *Suppose $a_j \in H^\infty$ for $j = 1, \ldots, N$, and define an operator* $\tau : H^2 \to H^2$ by

$$\tau \stackrel{\Delta}{=} \sum_{j=1}^{N} T_{a_j} T_{a_j}^* = T_{\sum_j |a_j|^2} - \sum_{j}^{N} H_{\bar{a}_j}^* H_{\bar{a}_j}.$$

Assume that $I - \tau$ has closed range. Then there exists a $\mathbf{C}^N$-valued function h in $\bar{\mathbf{B}}H^\infty(\mathbf{C}^N)$ for which $\sum_{j=1}^{N} a_j h_j$ is a Blaschke product of order L if and only if τ has at most L eigenvalues less than or equal to 1. *Furthermore, there is a simple linear fractional parametrization of all such h in $\bar{\mathbf{B}}H^\infty(\mathbf{C}^N)$, see Theorem* 4.3.

This corollary actually follows quickly from a simple construction and Theorem 4.2 of [4]. Our more general theorem extends this to functions a_j which have finitely many poles in the disk. One might think of this as a Corona theorem for H_L^∞ in the spirit of Takagi's extension of Pick interpolation from H^∞ to H_L^∞.

2. THE TEST FOR A LOCAL OPTIMA

We begin with a general result from which Theorem 1.1 immediately follows. It treats a set $E \subset \mathscr{C}(\mathbf{C}^N)$ and is stated in terms of the *tangent cone* $\boldsymbol{T}_{f_0}E$ to E at a point f_0. Recall that this is the set of all h in $\mathscr{C}(\mathbf{C}^N)$ for which there is a sup norm differentiable curve $f_t \subset E$ for t in $[0, 1]$ with $f_0 = f_0$ and $\frac{\partial f_t}{\partial t}(0) = h$. Let wno a denote *the winding number of the function* a about 0.

THEOREM 2.1. *Suppose $\Gamma(e^{i\theta}, w)$ is continuous in θ and thrice continuously diffeferentiable in w. If f_0 in E is continuous and the tangent cone $\boldsymbol{T}_{f_0}\Gamma E$ to $\Gamma(e^{i\theta}, E)$ at f_0 satisfies*

$$\psi\mathfrak{A} \subset \boldsymbol{T}_{f_0}E \subset \varphi\mathfrak{A}$$

for some continuous functions φ and ψ which never vanish on $\mathbf{T}$, *then f_0 is a strict local optimum for* (OPT) *if*

(i) *f_0 flattens Γ;* (ii) wno φ *is positive,*

and only if

(i) *f_0 flattens Γ;* (ii) wno ψ *is positive.*

Proof. If $f_t \in E$ for $t \geqslant 0$ is (sup norm) differentiable at $t = 0$, then Taylor's Theorem says

$$\Gamma(e^{i\theta}, f_t(e^{i\theta})) = \Gamma(e^{i\theta}, f_0(e^{i\theta})) + t\, 2\,\mathrm{Re} \sum_{j=1}^{N} a_j(e^{i\theta}) \frac{df_0^j}{dt}(e^{i\theta}) + O(t^2) \tag{2.1}$$

where f_t^j is the j^{th} co-ordinate function of the $\mathbf{C}^N$-valued function $f_t = (f_t^1, f_t^2, \ldots$ $\ldots, f_t^N)$ and $a_j(e^{i\theta}) = \frac{\partial \Gamma}{\partial w_j}(e^{i\theta}, f_0(e^{i\theta}))$. That is

$$\Gamma(e^{i\theta}, f_t(e^{i\theta})) = g_0(e^{i\theta}) + t\, 2\,\mathrm{Re}\, v(e^{i\theta}) + O(t^2) \tag{2.2}$$

for some v in $\boldsymbol{T}_{f_0}\Gamma E$. Conversely, the definition of $\boldsymbol{T}_{f_0}\Gamma E$ insures that for any v in $\boldsymbol{T}_{f_0}\Gamma E$ there is a f_t in E satisfying (2.2).

The fact that $\boldsymbol{T}_{f_0}\Gamma E \supset \psi$ and the proof of Theorem 3 in [10] imply that if f_0 is an optimum it flattens Γ. However, for the moment assume that f_0 flattens Γ, so g_0 is a constant. Then (2.1) implies that f_0 is an optimum if and only if there is no v in $\boldsymbol{T}_{f_0}\Gamma E$ with $\mathrm{Re}\, v \leqslant -\delta < 0$. If every v in $\boldsymbol{T}_{f_0}\Gamma E$ has the form φh with $h \in \mathfrak{A}$, then wno v = wno φ + wno h which by hypothesis is greater than zero; thus every v has wno $v > 0$. Now wno $v > 0$ implies $\mathrm{Re}\, v$ has mixed sign, so the wno $\varphi > 0$ side of the theorem is proved.

Conversely, suppose wno $\psi \leqslant 0$. Recall

THEOREM 2.3. [11] (or for another proof see [6]). *If* wno $f \geqslant 0$ *and* $\varepsilon > 0$, *then there is polynomial* p *which is never* 0 *on* $\mathbf{T}$ *for which*

$$|\arg p - \arg f| < \varepsilon.$$

Apply this to obtain a p with $\arg p$ as close as we like to $\pi - \arg\psi$. By hypothesis $\arg\psi p$ is very close to π. Thus $\operatorname{Re}\psi p < 0$. This concludes the proof.

We sketch another proof (for $g_0 > 0$) since different proofs will ultimately generalize in different ways. This is a direct reduction to the linear case. Let h denote $\frac{df_0}{dt}$ of (2.1). A basic lemma (cf. Wulbert [14]) in approximation theory says that f_0 is a strict local optimum if and only if

$$\|g_0 + 2\operatorname{Re} ah\|_{L^\infty} > \|g_0\|_{L^\infty}$$

for small enough $h \neq 0$ in $\boldsymbol{T}_{f_0}E$. If g_0 is not identically zero and if for each θ, $Q(e^{i\theta}, w)$ is a positive definite quadratic form in $w \in \mathbb{C}$, then (since $g_0 \geqslant 0$) for small enough h we have

$$\|\mu(h)\|_{L^\infty} \geqslant \|g_0 + 2\operatorname{Re} ah\|_{L^\infty} \geqslant \|g_0\|_{L^\infty}.$$

Here $\mu(h) = g_0 + 2\operatorname{Re} ah + Q(\ ,h)$. Thus $\inf_{h \in \boldsymbol{T}_{f_0}E} \|\mu(h)\|_{L^\infty}$ occurs at $h = 0$.

We now choose Q to make μ a well understood quadratic form. For example, choose $Q(e^{i\theta}, w) = \frac{|a|^2}{g_0} |w|^2$ to obtain $\mu(h) = \left|\sqrt{g_0} + \frac{a}{\sqrt{g_0}} j\right|^2$. Then we have f_0 is a strict local optimum for (OPT) if and only if 0 is the strict local optimum for

$$\inf_{h \in \boldsymbol{T}_{f_0}E} \left\| \sqrt{g_0} + \frac{a}{\sqrt{g_0}} h \right\|_{L^\infty}. \tag{2.3}$$

Note that to this point we have not used that g_0 or a are continuous. Now we aply the qualitative theory for this standard problem to conclude properties of g_0 and a. We get the conclusion of Theorem 2.1. In addition we obtain strong motivation for

CONJECTURE. *For generic* Γ *the problem* (OPT) *over* $\mathfrak{A}$ *with continuous strict local optimum* f_0 *yields* wno $\frac{\partial}{\partial w} \Gamma(e^{i\theta}, f_0(e^{i\theta})) = +1$.

The rationale is that (2.3) converts to saying

$$\inf_{h \in \mathfrak{A}} \|k_0 - h\|_{L^\infty} \quad \text{occurs at } h = 0 \tag{2.4}$$

for a function k_0 with wno $k_0 = -$wno a. Now most non-vanishing continuous k_0 for which (2.4) occurs have wno $k_0 = -1$. This can be seen from the dual extremal method, since the dual extremal measure $\lambda_0 d\theta$ satisfies $e^{i\theta}k_0\lambda_0 = \|k_0\|_{L^\infty}$. Now λ_0 can be chosen the be an extreme point of $\overline{\mathbf{B}}H^1$. Extreme points are outer functions and for generic k_0 they do not vanish. Thus generally wno $k_0 = -1$.

Theorem 1.1 is an immediate consequence of Theorem 2.1 since the hypothesis of Theorem 1.1 implies that $\varphi = \psi = a$. Also Theorem 2.1 yields an elegant theoretical solution for $N \geqslant 1$:

THEOREM 2.4. *If f_0 is continuous, each $a_j(e^{i\theta}) = \frac{\partial \Gamma}{\partial z_j}(e^{i\theta}, f_0(e^{i\theta}))$ is rational and not all a_j vanish at the same 0, then f_0 is a* **strict local optimum** *for* (OPT) *over $E =$ the $\mathbf{C}^N$-valued functions continuous and analytic on the disk if and only if*

(1) *f_0 flattens Γ.*

(2) *Write $a_j = \frac{p_j}{q_j}$ with p_j and q_j coprime polynomials. The integer i defined by*

$i(f_0) =$ number of zeroes which the greatest common divisor of

$p_1, \ldots, p_N$ has inside the unit disk minus the number of zeroes

of the least common multiple of $q_1, \ldots, q_N$ inside the disk

is strictly greater than 0.

Roughly speaking i is the number of common zeroes minus the total number of poles (inside the disk) of the a_j.

Proof. Equation (2.1) implies that

$$\boldsymbol{T}_{f_0}\Gamma = \left\{\sum_{j=1}^{N} a_j h_j : h_j \in \mathfrak{A}\right\}.$$

One can write

$$\sum_{j=1}^{N} a_j h_j = \frac{\delta}{\mu}(\rho_1 h_1 + \ldots + \rho_N h_N) \tag{2.5}$$

where δ is the greatest common divisor of $p_1, p_2, \ldots, p_N$ and μ is the least common multiple of the q's. Also the polynomials ρ_j have no common factor and as a consequence it is possible to select $\varkappa_j$'s in $\mathfrak{A}$ for which $\sum \rho_j \varkappa_j = 1$. Since we may select $h_j = \varkappa_j h$ for any h in $\mathfrak{A}$, we see that $\boldsymbol{T}_{f_0}\Gamma = \frac{\delta}{\mu}\mathfrak{A}$. Thus the key to f_0 being optimum is wno $\frac{\delta}{\mu} > 0$, which is what (2) of Theorem 2.4 says.

3. STABILITY OF THE TEST

The integer $i(f)$ can change wildly with small changes in f, since functions a_j can have many common zeroes but most small perturbations of the a_j will have no common divisor. Thus the test for N dimensional optimality in Theorem 2.4 is not practical (while the test for one dimensional optimality in Theorem 2.1 is extremely practical). The rest of the article is devoted to developing a numerically plausible test and proving properties of it.

To obtain a version of test (2) in Theorem 2.4 which depends continuously on f_0 we introduce a 'condition number' for test (2). Intuitively the reason test (2) is unstable is that it makes no attempt to measure the size of the perturbation h of f_0 required to make a modest improvement in $\|\Gamma(\cdot, f+h)\|_{L^\infty}$. Whether it takes a modest, or a very large h to produce a modest improvement in Γ completely escapes our test. The (crude) measure of this phenomenon appropriate to test (2) is $\varkappa(A)$ defined for each N-tuple A of functions $A = \{a_1, \ldots, a_N\}$ by

$$\varkappa(A) = \sup\{\inf_\theta \left| \sum_{j=1}^{\infty} a_j(e^{i\theta})h_j(e^{i\theta}) \right| : h \in \mathbf{B}\mathfrak{A}, \text{ wno} \sum_{j=1}^{N} a_j h_j < 1\} \tag{3.1}$$

or by $\varkappa(A) = 0$ if any $\sum a_j h_j$ which never vanishes has wno$\geqslant 1$.

The number $\varkappa(A)$ depends continuously on A and can be regarded as a measure of how close the winding number critical to test (2) is to changing from being $\leqslant 0$ to being > 0. To wit if $\varkappa(A)$ is large (resp. small) clearly no (resp. some) small change in A produces an A' for which test (2) holds. Thus a sensible numerically stable generalization of Theorem 1.1 to $N \geqslant 1$ is

(1) f_0 nearly flattens Γ.

(2) $\varkappa(A)$ is small for $A = \left\{ \frac{\partial \Gamma}{\partial w_j} (e^{i\theta}, f_0(e^{i\theta})) \right\}$.

Henceforth, our major objective will be to compute $\varkappa(A)$. Section 4 is devoted to this.

Note that while $\varkappa(A)$ is a 'condition number' appropriate for test (2) of Theorem 2.3 it may not be a reasonable 'condition number' for (OPT). Section 5 discusses this and gives estimates to quantify the term $\varkappa(A)$ is small.

4. A CORONA THEOREM

This section concerns a set $A = \{a_j\}_1^N$ of continuous functions and computation of $\varkappa(A)$ defined (recall (3.1)) by

$$\varkappa(A) = \sup\left\{\inf_\theta \left| \sum_{j=1}^{N} a_j h_j \right| : h \in \mathbf{B}\mathfrak{A}, \text{ wno}(\textstyle\sum a_j h_j) \leqslant 0\right\} \tag{4.0}$$

and

$$\varkappa(A) = 0 \quad \text{if } \{\quad\} \text{ is empty.}$$

We begin with an example.

EXAMPLE. Assume each a_j is in H^∞. The classical Corona Theorem gives conditions on A so that there is a number $C(A)$ called a 'Corona constant' with the property that there is an $h^0 \in H^\infty(\mathbf{C}^N)$ for which $\sum_1^N a_j h_j^0 = 1$ and $\|h^0\| < C(A)$. Let $\varkappa_\infty(A)$ denote the value obtained in (4.0) the definition of $\varkappa$ by maximizing over $h \in \mathbf{B}H^\infty(\mathbf{C}^N)$ rather than $\mathbf{B}\mathfrak{A}(\mathbf{C}^N)$. Then $\varkappa_\infty(A) \geqslant 1/\|h^0\| \geqslant 1/C(A)$. Conversely, if $\varkappa_\infty(A) > 1/C$, then $\exists\, h^0$ such that $\sum a_j h_j^0 = q$ and $\inf_\theta |q(e^{i\theta})|/\|h\| > 1/C$. Thus $h^1 = h^0/q$ solves the 'Corona equation' $\sum a_j h_j^1 = 1$ and $\|h^1\| \leqslant \|h^0\|/\inf_\theta |q(e^{i\theta})| < C$. Therefore *the 'best' Corona constant* equals $1/\varkappa_\infty(A)$.

One classical estimate at the moment for the Corona constant goes like this: If

$$\sum_{j=1}^N |a_j(z)|^2 \geqslant \delta \tag{4.1}$$

then $C(A) = 65 \left\| \sum_j^\infty |a_j|^2 \right\|_{L^\infty}^{3/2} \delta^{-4}$ is a Corona constant, (see [12]). We shall ultimately see that for rational a_j we have $\varkappa_\infty(A) = \varkappa(A)$.

The example makes it clear that finding a formula such as (4.1) for estimating (A) amounts to generalizing the Corona theorem in a certain way. This leads us o the operator theoretic Corona theorem mentioned in the introduction. The operator theoretic theorem has the advantage that it produces the *best* Corona constant — not just an approximate one. We now introduce the necessary terminology.

Let $P_{[m,n]}$ denote the orthogonal projection of L^2 onto the trig polynomials $\sum_{j=m}^n a_j e^{ij\theta}$; here m can be negative to include $-\infty$. If $A = (a_1, \ldots, a_N)$ is an N-tuple of L^∞ functions define an operator ${}_AH: H^2 + \ldots + H^2 \to (H^2)^\perp$ by

$${}_AH(h_1, h_2, \ldots, h_N) = \sum_{j=1}^N H_j h_j.$$

σ_A by $\sigma_A = H_{a_1}(H_{a_1})^* h_1 + \ldots + H_{a_N}(H_{a_N})^* h_N$, and $\#$ by

$${}^\# A = \dim \operatorname{Rg} {}_AH = \dim \operatorname{Rg} \sigma_A. \tag{4.3}$$

Let H_l^∞ denote the functions in L^∞ which are boundary values of functions bounded and analytic in $\mathbf{D}$ except for possibly l poles inside $\mathbf{D}$. If each a_j is in H_l^∞, then ${}^\# A$ is finite.

Our first observation is

LEMMA 4.1. *If $A = \{a_j\}$ are rational functions with no poles on the unit disk, then $^{\#}A$ equals the number of zeroes in the unit disk of the least common multiple μ of the denominators q_j of a_j. Roughly $^{\#}A$ is the total number of poles of a_j discounting overlaps.*

Proof. The key is (2.5). It and the nearby discussion implies that

$$\operatorname{Rg}_A H = P_{[-\infty,-1]} \frac{\delta}{\mu} H^2$$

where δ is a trig polynomial in H^∞ whose zeroes never intersect the zeroes of μ. It is easy to check that the range of such a Hankel matrix equals the span of the functions

$$\frac{1}{z - z_i}, \quad \frac{1}{(z - z_i)^2}, \quad \frac{1}{(z - z_i)^{r_i}}$$

where z_i is a zero of μ and r_i is its multiplicity. Since these functions are linearly independent and there are $^{\#}A$ of them the formula in the lemma is proved.

Our main result on $\varkappa(A)$ is

THEOREM 4.2. *Suppose $A = \{a_j\}_1^N$ is a set of $H_{L_1}^\infty$ functions for some L_1. Let $\mathscr{W}$ denote the space $\operatorname{Rg}_A H \oplus H^2$ and define an operator τ on $\mathscr{W}$ by*

$$\tau g \triangleq \sum_{j=1}^{N} P_{\mathscr{W}} a_j P_{H^2} \bar{a}_j g.$$

Assume that $\varkappa^2 I_{\mathscr{W}} - \tau$ has closed range for all $\varkappa$ slightly less than $\varkappa(A)$. If each a_j is rational, then

(4.4) $$\varkappa(A)^2 = (^{\#}A + 1)^{\text{th}} \textit{ smallest eigenvalue of } \tau.$$

Here we count any continuous spectrum as an infinite number of eigenvalues.

We now take a brief aside and present a generalization of the Corona theorem. In this generalization we are given a number $\varkappa > 0$ and an integer L. For given a_j in $H_{L_1}^\infty$ we must satisfy

4.5) $$\sum_{j=1}^{N} a_j h_j = \varkappa \psi$$

for $h = (h_1, h_2, \ldots, h_N) \in \overline{\mathbf{B}}H^\infty(\mathbf{C}^N)$ with ψ an L^∞ function whose modulus is $\geqslant 1$ on $\mathbf{T}$ a.e. and whose winding number about 0 is $\leqslant L$. Here we take $\operatorname{wno}\psi = \infty$ if ψ is not continuous. If a solution to (4.5) exists then a solution h to

(4.5′) Add to (4.5) the restriction that $|\psi(e^{i\theta})| \equiv 1$ a.e.

exists, since we can multiply equation (4.5) by the Wiener-Hopf factorization q of $|\psi|^{-2}$. We find that as $\varkappa$ gets larger we must take a larger L in order for a solution to exist.

The following theorem tells exactly what the tradeoff is.

THEOREM 4.2.′ *Suppose $\varkappa^2 I_{\mathscr{W}} - \tau$ has closed range. If $\varkappa^2$ is between the ν^{th} and $\nu+1)^{th}$ smallest eingevalue of τ, then* (4.5) *has a solution h for any $L \geqslant -{}^{\#}A + \nu$ and for no $L < -{}^{\#}A + \nu$. Here any continuous spectrum of τ counts as an infinite number of eigenvalues.*

Further detail is possible, namely, we can parameterize all functions satisfying (4.5). This is done in terms of a linear fractional map

$$G_{\Xi}(s) = (\alpha s + \beta)(\eta s + \gamma)^{-1}$$

with matrix function coefficients $\Xi = \begin{pmatrix} \alpha & \beta \\ \eta & \gamma \end{pmatrix}$ acting on s in $\mathbf{B}H^{\infty}(\mathbf{C}^N)$. Here $\alpha \in L^{\infty}(M_{N,N-1}), \beta \in L^{\infty}(M_{N,1}), \eta \in L^{\infty}(M_{1,N-1})$ and $\gamma \in L^{\infty}(M_{1,1})$ with $M_{m,n}$ denoting the $m \times n$ matrices. Let $[u, v]_{\mathbf{C}^{N+1}}$ denote the signed sesquilinear form

$$- u_1\bar{v}_1 \dots - u_N\bar{v}_N + u_{N+1}\bar{v}_{N+1}$$

an $\mathbf{C}^{N+1}$ and let $[\ ,\ ]_{L^2(\mathbf{C}^{N+1})}$ denote the one it naturally induces on $L^2(\mathbf{C}^{N+1})$. Frequently we abbreviate this to $[\ ,\ ]$. A $M_{N+1,K}$ valued function Ξ is called a phase function provided that $[\Xi(e^{i\theta})u, \Xi(e^{i\theta})v]_{\mathbf{C}^{N+1}} = [u, v]_{\mathbf{C}^K}$. Under the hypothesis of theorem 4.2′ we have

THEOREM 4.3. *Assume that A and $\varkappa$ satisfy the hypothesis of Theorem* 4.2′. *If $\varkappa^2$ is the ν^{th} smallest eigenvalue of τ, then a solution to* (4.5) *with $L = -{}^{\#}A + \nu$ isunique. If $\varkappa^2$ is between the ν^{th} and $(\nu+1)^{th}$ eigenvalue there is a $[\ ,\ ]_N$ to $[\ ,\ ]_{N+1}$ phase function Ξ in $H^{\infty}(M_{N+1,N})$ so that the set $G_{\Xi}(\mathbf{B}H^{\infty}(\mathbf{C}^{N-1}))$ equals all $h\psi^{-1}$ for which h is a solution to* (4.5) *with $L = -{}^{\#}A + \nu$. If the a_j are all rational, then Ξ is rational; also a unique solution will be rational.*

A similar parameterization for the solutions L of the traditional Corona theorem (${}^{\#}A = 0$, $\psi = 1$, $\nu = 0$) was discovered by C. Foiaş and mentioned in his Toeplitz lectures. While a description of his parameterization has not been published the basic method is presented in [1].

Proof of Theorems 4.2, 4.2′, and 4.3. Note that Theorem 4.2 is an immediate consequence of Theorem 4.2′. So we turn to 4.2′ and 4.3. The first step is a trivial reduction. Given a_j and $\varkappa > 0$ set $a_j = a_j/\varkappa$, then the relationship between $\varkappa^2$ and the spectrum of τ is exactly the same as the relationship between 1 and the spectrum of the operator τ' built from the a'_j. Thus without loss of generality we henceforth take $\varkappa = 1$.

The proof of the theorem requires the generalized commutant lifting theorem in [4, Theorem 4.2]. In fact the theorem fits very naturally into the [4] setting and so this is the approach we take. Familiarity with the basic [4] construction is very helpful, but not absolutely essential to understanding the forthcoming proof.

Define a subspace $\mathscr{M} \subset L^2(\mathbf{C}^{N+1})$ by

$$\mathscr{M} = \left\{ \left(f_1, f_2, \ldots, f_N, \sum_{j=1}^{N} a_j f_j \right)^{\mathrm{T}} : f_j \in H^2 \right\}. \tag{4.6}$$

Let $\mathscr{S} \subset L^2(\mathbf{C}^{N+1})$ be the subspace

$$\mathscr{S} = \{(h_1 f, \ldots, h_N f, f)^{\mathrm{T}} : f \in \psi H^2\} \tag{4.7}$$

where ψ is a rational function of modulus one on $\mathbf{T}$. Then $\mathscr{S}$ is clearly invariant under the operator χ on $L^2(\mathbf{C}^{N+1})$ which *multiplies each function* by e^{i0}. Moreover $h = (h_1, \ldots, h_N)^{\mathrm{T}}$ is in $\overline{\mathbf{B}}L^\infty(\mathbf{C})^N$ if and only if $\mathscr{S}$ is a *positive subspace* with respect to $[\ ,\ ]$, that is, $[v, v] \geq 0$ for all $v \in \mathscr{S}$. Conversely, if $\mathscr{S} \subset \mathscr{M}$ is χ invariant and maximal positive in $\mathscr{M}$ (or of finite codimension in a maximal positive subspace of $\mathscr{M}$), then we shall soon see that $\mathscr{S}$ has the form (4.7).

The key observation is that $\mathscr{S}$ of the form (4.7) is contained in $\mathscr{M}$ means

$$(h_1 f, \ldots, h_N f, f)^{\mathrm{T}} = \left(f_1, \ldots, f_N, \sum_{j=1}^{N} a_j f_j \right)^{\mathrm{T}},$$

that is, $h_j f = f_j \in H^2$ and $f = \sum_{j=1}^{N} a_j f_j$. Consequently $\mathscr{S} \subset \mathscr{M}$ if and only if $h_j \psi \in H^\infty$ and

$$ah = 1. \tag{4.8}$$

Thus to study equation (4.8) with $h_j \in \psi^{-1}\overline{\mathbf{B}}H^\infty(\mathbf{C}^N)$ or equivalently to study (4.5′) we study invariant (nearly) maximal positive subspaces of $\mathscr{M}$. As we shall see the main theorem of [4] parameterizes all of these spaces.

The next item is to represent positive invariant subspaces of $\mathscr{M}$ as (4.7). Define a new space $\mathscr{R}$ by $\mathscr{R} = \mathscr{M} + \begin{pmatrix} H^2(\mathbf{C}^N) \\ 0 \end{pmatrix}$. Note that $\mathscr{R}$ equals $H^2(\mathbf{C}^{N+1}) + \begin{pmatrix} 0 \\ 0 \\ \mathrm{Rg}_A H \end{pmatrix}$.

Let β be the reciprocal of a Blaschke product which represents $\mathrm{Rg}_A H \oplus H^2(\mathbf{C}^N)$ as βH^2.

If $\mathscr{S}$ is a positive subspace of $\mathscr{R}$ extend it to a maximal positive subspace $\mathscr{S}_1$ of $\mathscr{R}$ and define the positive cosignature of $\mathscr{S}$ (in $\mathscr{R}$) to be the co-dimension of $\mathscr{S}$ in $\mathscr{S}_1$. The definition is independent of how we extend $\mathscr{S}$ to $\mathscr{S}_1$, see [4, § 1]. A trivial lemma (1.1 of [4]) says that:

The positive cosignature of any maximal positive subspace of $\mathcal{M}$ equals the dimension of the largest positive subspace of $\mathcal{R} - \mathcal{M}$, called the positive signature of $\mathcal{R} - \mathcal{M}$.

Also Lemma 1.1 of [4] implies that if $\mathcal{S} \subset \mathcal{R}$ is positive with cosignature λ in $\mathcal{R}$, then $\mathcal{S}$ has the form (4.7) with $\lambda = \dim [\beta H^2 - \psi H^2] = {}^{\#}A -$ num. poles ψ inside the disk + num. zeroes ψ inside the disk $= {}^{\#}A + \text{wno}\,\psi$. Combine these formulas to get

LEMMA 4.4. *Every positive χ invariant subspace $\mathcal{S}$ of $\mathcal{M}$, has a representation* (4.7) *which satisfies*

$$\text{wno}\,\psi = -\ {}^{\#}A + \textit{the positive cosignature of } \mathcal{S} \textit{ in } \mathcal{M} + $$
$$+ \textit{ the positive signature of } \mathcal{R} - \mathcal{M}.$$

Concentrate on invariant maximal positive subspaces of $\mathcal{M}$. We have shown that each such space corresponds to a solution to (4.8) with a ψ having winding number which we shall soon compute directly from the definition of $\mathcal{M}$ and $\text{Rg}_A H$. The main theorem in [4] says that if $\mathcal{M}$ is 'regular' and contains no isotropic vector (one which is [,] orthogonal to all of $\mathcal{M}$), then there exists an integer $K > 0$ and a $(N + 1) \times K$ matrix valued L^∞ function Ξ which is $[\ ,\ ]_{\mathbf{C}^K}$ to $[\ ,\ ]_{\mathbf{C}^{N+1}}$ phase and which represents $\mathcal{M}$ as

$$\mathcal{M} = \Xi H^2(\mathbf{C}^K).$$

That $\mathcal{M}$ is regular is guaranteed by hypothesis that $I_{\mathscr{W}} - \tau$ has closed range. Now Ξ maps the invariant maximal positive subspaces of $H^2(\mathbf{C}^K)$ onto the invariant maximal positive subspaces of $\mathcal{M}$. This gives the parameterization in the second part of Theorem 4.3 once we rule out isotropic vectors and compute K. Since at fixed z inside the disk, the vector space $\{F(z) : F \in \mathcal{M}\}$ has dimension N, we see that $K = N$. If $\mathcal{M}$ contains a positive space then $[\ ,\ ]_{\mathbf{C}^K}$ has positive signature $\geqslant 1$. Since $H^2(\mathbf{C}^{N+1}) \supset \supset \mathcal{M}$ the positive signature of $[\ ,\ ]_{\mathbf{C}^K}$ is $\leqslant 1$.

It is so easy to see that the representation $\mathcal{M} = \Xi H^2(\mathbf{C}^N)$ gives the G_Ξ parameterization that we sketch this here. An invariant maximal positive $\mathcal{S}_0$ in $H^2(\mathbf{C}^N)$ has the form $\{(h_0 x, x)^{\mathrm{T}} : x \in H^2(\mathbf{C}^1)\}$ with $h_0 \in \overline{\mathbf{B}}H^\infty(M_{N-1,1})$. Thus $\Xi\mathcal{S}_0 = \mathcal{S}$ is gotten by

$$\begin{pmatrix} \alpha & \beta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} h_0 x \\ x \end{pmatrix} = \begin{pmatrix} [\alpha h_0 + \beta] x \\ [\eta h_0 + \gamma] x \end{pmatrix} \triangleq \begin{pmatrix} ax \\ bx \end{pmatrix} = \begin{pmatrix} hy \\ y \end{pmatrix}$$

where $h = ab^{-1}$ and $y = bx$. Note that $b(z)$ is a 1×1 matrix so $b(z)^{-1}$ is well defined at almost all points of the disk.

Isotropic vectors and all issues of signature are determined by the following lemma:

LEMMA 4.5. *The positive signature of $\mathcal{R} - \mathcal{M}$ equals the number of nonnegative eigenvalues of*

$$I_{\mathscr{W}} - \tau.$$

Also $\mathcal{M}$ contains an isotropic vector if and only if this operator has a null vector.

Proof. A vector x in $\mathcal{R}$ is [,] orthogonal to $\mathcal{M}$ if and only if $(x_1, f_1) + \ldots$ $\ldots + (x_N, f_N) - \left(x_{N+1}, \sum_{j+1}^{N} a_j f_j\right) = 0$ for all $f_j \in H^2$. So $(x_j - \overline{a}_j x_{N+1}, f_j) = 0$, from which we conclude $x_j = P_{H^2}\overline{a}_j x_{N+1}$. Thus

$$[x, x] = -\sum_{j=1}^{N} \| P_{H^2}\overline{a}_j x_{N+1}\|^2 + \|x_{N+1}\|^2 =$$

$$= -\sum_{j=1}^{N} \|P_{H^2}\overline{a}_j \beta g\|^2 + \|g\|^2 = -\sum_{j=1}^{N} \|P_{H^2}\overline{a}_j \beta g\|^2 + \|\beta g\|^2$$

for a g in H^2. Consequently the positive signature of $\mathcal{R} - \mathcal{M}$ equals the number of nonnegative eigenvalues of $I_{\mathcal{W}} - \tau$. We interpret this to be ∞ if $I_{\mathcal{W}} - \tau$ has positive continuous spectrum. Also x is isotropic in $\mathcal{R} - \mathcal{M}$ if and only if g is a null vector of this operator. Now $\mathcal{M}$ has an isotropic vector if and only if its [,] orthogonal complement does. Q.E.D.

It remains to show that if 1 equals the ν^{th} eigenvalue of τ, then (4.5) has a unique solution with wno equal to $-{}^{\#}A + \nu$. From Lemma 4.5 we see that $\mathcal{M}$ contains an isotropic subspace N_0. If S is a positive subspace of $\mathcal{M}$, then $\mathcal{S} + \mathcal{N}_0$ is positive since $\mathcal{N}_0$ is [,] orthogonal to $\mathcal{S}$. Thus a maximal positive $\mathcal{S}$ contains q. Define $\mathcal{N}$ to be the norm closure of $\bigvee_j e^{ij\theta}\mathcal{N}_0$; since $[e^{ij\theta}\mathcal{N}_0, e^{im\theta}\mathcal{N}_0] = 0$ for $m \neq j$. Then we have that any invariant maximal positive subspace $\mathcal{S}$ of $\mathcal{M}$ contains $\mathcal{N}$. Now we must invoke some substantial machinery from [4] and the correction, which implies that $\mathcal{N}$ is maximal positive in $\mathcal{M}$. Since we have seen that each invariant maximal positive $\mathcal{S}$ in $\mathcal{M}$ contains $\mathcal{N}$, each such $\mathcal{S}$ equals $\mathcal{N}$. That is (4.5) has a unique solution.

We remark that the assumption that the a_j are rational is likely much too strong. Probably a_j once differentiable would do. The main point is to show that *continuous* solutions h to (4.8) exist. To prove this one would need to go through the proof of Theorem 4.2 in [4] and establish that a_j smooth implies Ξ is continuous.

(*Added in proof.* Joe Ball checked this and said it is true. Also D. Marshall said the result is known classically.)

REMARKS. (1) We present a way to find the number of positive eigenvalues of $\varkappa^2 I - \tau$ on a computer. It would be accurate provided the functions a_j have Fourier coefficients which go quickly to zero and $\sum_j \dim \operatorname{Rg} H_{a_j}$ is 'numerically' small. The method is based on operator identities which convert the non-compact operator τ to an operator whose complicated part is compact. For motivation recall the old identity $T_a T_a^* = T_{|a|^2} - H_{\overline{a}}^* H_{\overline{a}}$; the Hankel term is, of course, compact. We begin by deriving this for τ. Suppose P is an orthogonal projection on L^2 and that a is an L^∞ function. Then

$$P_{\mathscr{W}} a P \bar{a} P_{\mathscr{W}} = P_{\mathscr{W}} a \bar{a} P_{\mathscr{W}} - P_{\mathscr{W}} a P^{\perp} \bar{a} P_{\mathscr{W}}. \tag{4.9}$$

Now suppose that a is invertible outer and that $P = P_{H^2}$. Since $\mathscr{W}$ is invariant under multiplication by a,

$$P_{\mathscr{W}} b a P_{\mathscr{W}} = P_{\mathscr{W}} b P_{\mathscr{W}} a P_{\mathscr{W}}$$

for b in L^∞. Consequently $P_{\mathscr{W}} a P_{\mathscr{W}}$ is invertible on $\mathscr{W}$ and its inverse is $P_{\mathscr{W}} a^{-1} P_{\mathscr{W}}$.

To compute the number of positive eigenvalues of $\varkappa^2 - \tau$ note that this equals the number of positive eigenvalues of

$$P_{\mathscr{W}} \bar{\beta} P_{\mathscr{W}} [\varkappa^2 - \tau] P_{\mathscr{W}} \beta P_{\mathscr{W}}. \tag{4.10}$$

Take β to be the outer Wiener-Hopf factor of $\left[\sum_1^N |a_j|^2 - \varkappa\right]^{-1} \stackrel{\Delta}{=} [\rho - \varkappa^2]_{-1}$. Then 4.10) is

$$P_{\mathscr{W}} \bar{\beta} P_{\mathscr{W}} \left\{ \sum_{j=1} P_{\mathscr{W}}\, a_j P^{\perp} \bar{a}_j P_{\mathscr{W}} - P_{\mathscr{W}} \rho^{-1} P_{\mathscr{W}} \right\} P_{\mathscr{W}} \beta P_{\mathscr{W}} =$$

$$= \sum_{j=1}^{N} P_{\mathscr{W}} \bar{\beta} a_j P^{\perp} \bar{a}_j \beta - I_{\mathscr{W}} \tag{4.11}$$

acting on $\mathscr{W}$.

In summary we read off the following from our computations:

PROPOSITION 4.6. *If $\mathscr{W}$ is finite dimensional and if the a_j are continuous, then the continuous spectrum of τ equals the interval* $[\inf_\theta \rho(e^{i\theta}), \sup_\theta \rho(e^{i\theta})]$. *Moreover for $\varkappa^2$ beneath this interval $\varkappa^2 - \tau$ and* (4.11) *have the same number of positive eingenvalues.*

Computing the number of positive eigenvalues for (4.11) is a reliable and fairly routine procedure.

(2) One natural basis to use in studying τ is $\dfrac{1}{1 - \bar{z}_j e^{ij\theta}} \stackrel{\Delta}{=} e_j(\theta)$. Certain choices of z_j yield a dense set. We restrict to the case where the $a_j \in H^\infty$ since the following computation has already been done there. Sarason [14] showed that $\left([\varkappa^2 - \tau] \sum_1^\infty \alpha_n e_n, \sum_1^\infty {}_n e \alpha_n\right) = (A_k \vec{\alpha}, \vec{\alpha})$ where $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots)$ and

$$A_k = \left\{ \frac{\varkappa^2 - \sum_1^N a_j(z_m)\overline{a_j(z_n)}}{1 - z_m \bar{z}_m} \right\}_{m,\,n=1}^{\infty}.$$

More general classes of canonical functions than e_n are found in [3], [13], and [8]. Possibly one of these would efficate computer use.

Since a positive definite matrix has positive diagonal, $\Lambda_k \leqslant 0$ for all z_m implies $\sum_1^N |a_j(z_m)|^2 \geqslant \varkappa^2$ all z_m. The classical Corona theorem implies that for some $\varkappa$ this implies $\Lambda_k \leqslant 0$. However a simple direct proof of this fact is not known.

(3) A question of possible interest to Corona theorists is what is the classical analog of Corollary 1.2. What is the generalization of the classical condition $\inf_z \sum_j |a_j(z)|^2 > \delta$ which guarantees that $\sum a_j h_j = \varkappa\varphi$ has a solution $h \in \overline{\mathbf{B}}H^\infty$ with wno $\varphi = L \leqslant 0$ with δ and $\max_j \| a_j \|_\infty$ controlling the size of $\varkappa$?

Obvious possibilities are for fixed $L \geqslant 0$.

(a) Λ_k has its $(L+1)^{\text{st}}$ eigenvalue uniformly bigger than δ for all choices of $z_1, z_2, \ldots, z_{L+1}$.

(b)

$$\sup_{\substack{\text{wno}\,\varphi \leqslant L \\ \varphi = \text{Blaschke}}} \inf_z \sum_j \left| \frac{a_j(z)}{\varphi(z)} \right|^2 \geqslant \delta > 0.$$

This problem has not yet received attention.

5. A CONDITION NUMBER FOR OPT

We have devoted much attention to $\varkappa(A)$ our condition number for test (2). There also is a condition number $S(A)$ appropriate to OPT. Had we been able to compute much about $S(A)$ it would have been the main subject of this paper rather than $\varkappa(A)$ or possibly even test (2). What little we have derived about $S(A)$ is presented in this section.

The key equation (2.1) prompts us to define

$$S(A) = \max \left\{ \left| \sup_\theta \text{ the negative part of Re} \sum_{j=1}^N a_j h_j(e^{i\theta}) \right| : h \in \overline{\mathbf{B}}\mathfrak{A}(\mathbf{C}^N) \right\}$$

for $A = \{a_j\}_1^N$. For $f \in \mathfrak{A}(\mathbf{C}^N)$ define S_f to be $S(A)$ for $a_j(e^{i\theta}) = \frac{\partial \Gamma}{\partial z_j}(e^{i\theta}, f(e^{i\theta}))$. Clearly S_f is connected with the first order term of (2.1). The zero$^{\text{th}}$ order term is judged by its non-constancy, the simplest measure of which is

$$S_f^0 = \sup_\theta \left| \|\Gamma(e^{i\theta}, f(e^{i\theta}))\|_{L^\infty} - \Gamma(e^{i\theta}, f(e^{i\theta})) \right|.$$

Clearly if f is a solution of OPT, then S_f and S_f^0 equal zero ; conversely if the a_j's do not all vanish at the same θ, then S_f and $S_f^0 = 0$ imply that f is a local optimum. Indeed a (non-computable) alternative to Theorem 1.1 is

THEOREM 5.1. *Suppose f_0 satisfies the hypothesis of Theorem* 1.1. *For h in $\mathfrak{A}$ define $\gamma(h) = \|\Gamma(e^{i\theta}, f_0 + h)\|_{L^\infty} - \|\Gamma(e^{i\theta}, f(e^{i\theta})\|_{L^\infty}$ and suppose that it is negative. Then*

$$\gamma(h) \leqslant S^0_{f_0} + S_{f_0}\|h\| + O(\|h\|^2)$$

and

$$\sup_{\substack{\|h\|=\mu \\ \gamma(h)\text{negative}}} |\gamma(h)| \geqslant S_{f_0}\mu + O(\mu^2).$$

Since $\gamma(h)$ measures how much improvement $f_0 + h$ makes over f_0, this implies that locally for $S^0_{f_0}$ small, S_{f_0} determines local approximate optimality.

Proof. Define

$G(h)(e^{i\theta}) =$ the negative part of $\Gamma^{i\theta}(e^{i\theta}, f(e^{i\theta}) + h(e^{i\theta})) - \|\Gamma(e^{i\theta}, f(e^{i\theta}))\|_{L^\infty}$.

Note that if $\gamma(h) < 0$, then $\sup_\theta |G(h)(e^{i\theta})| = \gamma(h)$. Define $\Delta_f(h) = \operatorname{Re}\sum a_j h_j$. Equation (2.2) implies

$$G(h) = (\Gamma(\ ,f) - \|\Gamma(\ ,f)\|_{L^\infty}) + \text{neg. part } \Delta_f(h) +$$

$$+ \text{ pos. part } \Delta_f(h) + O(\|h\|^2).$$

Drop the positive part of Δ to get the first inequality on G. The second inequality comes from dropping the (negative) term in parenthesis and noting that when h is producing $\sup_h$ we have pos. part $\Delta_f(h) = 0$.

There is an obvious relationship between $\varkappa$ and S given by

LEMMA 5.2.

$$\varkappa(A) \geqslant S(A).$$

Consequently if $\varkappa_{f_0}$ is small for f_0, then $\Gamma(e^{i\theta}, f_0(e^{i\theta}))$ is hard to improve by a small perturbation of f_0.

Proof. Given $\varepsilon > 0$ there is an $h \in \mathbf{B}\mathfrak{A}(\mathbf{C}^N)$ such that $\sum a_j h_j$ equals a function q with $S(A) - \varepsilon \leqslant \inf_\theta |\operatorname{Re} q(e^{i\theta})|$. Thus

$$S(A) - \varepsilon \leqslant \inf_\theta |q(e^{i\theta})| \leqslant \sup_{h\in\mathbf{B}\mathfrak{A}} (\inf_\theta |\sum a_j h_j|) = \varkappa(A).$$

Now that we have an upper bound on $S(A)$ we compute a lower bound on $S(A)$ when A is just a single function a. If a has negative winding number ρ we may choose $h = z^{-p}h_1$ to convert to $\dfrac{\operatorname{Re} a_1 h_1}{\|h_1\|}$ with $a_1 = az^{-p}$. Without loss of generality assume a and h have winding number 0. Thus $\arg a$ and $\arg h \triangleq b$ are continuous. Write

$$\frac{\operatorname{Re} ah}{\|h\|} = \frac{|a|\, e^{\tilde{b}} \operatorname{Re} e^{i[\arg a + b]}}{\|e^{\tilde{b}}\|} = |a| \exp[\tilde{b} - \sup_\theta \tilde{b}] \cos(\arg a + b)$$

where $\tilde{b}$ is the harmonic conjugate of b. Set $\arg a = \alpha$. Thus

$$\inf_\theta \frac{\operatorname{Re} ah}{\|h\|} \geqslant (\inf_\theta |a|) \exp(\inf_\theta \tilde{b} - \sup_\theta \tilde{b}) \cos\|\alpha + b\|.$$

A standard estimate (see Theorem 1.3 of [7]) on the size of $\tilde{b}$ implies

$$\omega_\pi(\tilde{b}) \leqslant \frac{4\pi + 3}{\pi^2} \int_0^\pi \frac{1}{t}\, \omega_t(b)\, dt$$

where $\omega_t(f)$ is the modulus of continuity

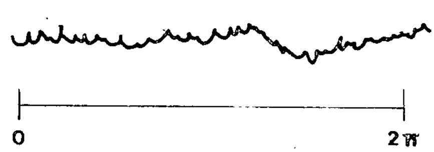
$$\omega_t(f) \triangleq \sup_{|\theta - \psi| < t} |f(e^{i\theta}) - f(e^{i\psi})|.$$

Combine the preceding to get

$$S(A) \geqslant (\inf_\theta |a|)[\exp - 2 \int_0^\pi \frac{1}{t}\, \omega_t(\mu - \alpha)\, dt] \cos\|\mu\| \tag{5.1}$$

where μ is any continuous function. In particular $\mu = 0$ gives a simple estimate on $S(A)$ in terms of a and the derivative of $\arg a$.

Intuitively, the freedom in μ can be used to remove rapid small oscillations in α For example, an α like



would have a large modulus of continuity, but a μ with $\|\mu\|$ small could be chosen which would make $\alpha - \mu$ equal

which has a very small modulus of continuity. Thus small oscillations in arg a do not force $S(A)$ to increase much.

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BIBLIOGRAPHY

1. ARSENE, GR.; CEAUŞESCU, Z.; FOIAŞ, C., On interwining dilations. VIII, *J. Operatory Theory* **4**(1980), 55–91.
2. ARVESON, W., Interpolation problems in nest algebras, *J. Functional Analysis*, **20**(1975), 208–235.
3. BALL, J. A., Interpolation problems of Pick-Nevanlinna and Löewner types for meromorphic matrix functions, *Integral Equations Operator Theory*, 6(1983), 801–810.
4. BALL, J. A.; HELTON, J. W., A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory, *J. Operator Theory*, 9(1983), 107–142.
5. BALL, J. A.; HELTON, J. W., Beurling-Lax representations using classical Lie groups with many applications. IV: $\mathrm{GL}(n, \mathbf{R})$, $U^*(2n)$, $\mathrm{SL}(n, \mathbf{C})$ and a solvable group, in preparation.
6. GAMLIN, T.; GARNET, J.; RUBEL, L.; SHIELDS, A., On badly approximable functions, *J. Approx. Theory*, 17(1976), 280–296.
7. GARNET, J., *Bounded analytic functions*, Academic Press, New York, 1981.
8. HELTON, J. W., Orbit structure of the Möbius transformation semigroup acting on H^∞ (broadband matching), in *Topics in Functional Analysis*, Advances in Mathematics Supplementary Studies, vol. 3(1978), 129–157.
9. HELTON, J. W., Worst case analysis in the frequency domain: the H^∞ approach to control, preprint.
10. HELTON, J. W.; HOWE, R. E., A bang-bang theorem for optimization over spaces of analytic functions, preprint.
11. POREDA, S. J., A characterization of badly approximable functions, *Trans. Amer. Math. Soc.*, **169**(1972), 249–256.
12. ROSENBLUM, M., A Corona theorem for countably many functions, *Integral Equations Operator Theory*, 3(1980), 127–137.
13. ROSENBLUM, M.; ROVNAYK, J., private communication.
14. SARASON, D., Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127**(1967), 179–203.
15. WULBERT, D., Uniqueness and differential characterization of approximations from manifolds of functions, *Amer. J. Math.*, **93**(107), 350–366.

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