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# PRODUCT STATES OF CERTAIN GROUP-INVARIANT AF-ALGEBRAS

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## INTRODUCTION

In this paper we study the restriction of product states of $n^\infty$ UHF-algebras $\mathfrak{A} = \bigotimes_{k=1}^{\infty} \mathfrak{B}_k$ (where $\mathfrak{B}_k$ is an $(n \times n)$-matrix algebra) to group invariant subalgebras $\mathfrak{A}^G$. The groups $G$ considered are the unitary groups of any $*$-subalgebra of the $(n \times n)$-matrices. For example, we could have $G = \mathrm{U}(1) \times \mathrm{U}(3)$ in the case $n = 6$, or $G = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(4) \times \mathrm{U}(7)$ in the case $n = 14$, or $G = \mathrm{U}(n)$ for arbitrary $n$. The group action is of the obvious (adjoint) product type (see Section II) and invariant algebras $\mathfrak{A}^G$ are the AF-subalgebras of the fixed points of $\mathfrak{A}$ under the group action. We give computable necessary and sufficient conditions that the restriction of a product state of $\mathfrak{A}$ to $\mathfrak{A}^G$ yield a factor state (Theorem 4.13). We determine when such restrictions are pure (Theorem 4.16) and when such restrictions give type I or type $\mathrm{II}_1$ representations (Theorem 4.17). We find necessary and sufficient conditions that the restrictions of two product states of $\mathfrak{A}$ to $\mathfrak{A}^G$ yield quasi-equivalent representations of $\mathfrak{A}^G$ (Theorem 4.14).

In this paper we generalize the results of [4] from the case $n = 2$ to arbitrary positive $n$. Although this paper is self contained, seeing how our arguments in our earlier paper go for SU(2) and U(1) may make our present analysis of much more general groups appear less mysterious.

We consider the two important ideas of this paper to be the following. The first was introduced in our earlier paper and extended here. If $\pi$ is a representation of $\mathfrak{A}$ we define $H^G(\pi) = \{g \in G \,;\, \pi \underset{q}{\sim} \pi \circ \alpha_g\}$. In Section III we show that for $*$-representations coming from product states of $\mathfrak{A}$, $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)''$ with $H = H^G(\pi)$.

The second idea is the use of a probability result of [1] to determine the factoriality of the restriction of product states for the case when $G$ is an $n$-torus (see Lemma 4.12). Moreover the proof of Lemma 4.12 relies on techniques developed in the Aldous and Pitman paper.

Although most of this paper is concerned with product states many of the techniques have more general applicability. For example, in Theorem 4.18 we give a sufficient condition for the restriction of a factor state of $\mathfrak{A}$ to $\mathfrak{A}^G$ to yield a factor state and in Theorem 4.19 we give conditions for the quasi-equivalence of the restrictions of factor states.

## 2. NOTATION AND DEFINITIONS

Let $\mathfrak{B}_0$ be an $(r \times r)$-matrix algebra. For each $k = 1, 2, \ldots$ let $\gamma_k$ be a $*$-isomorphism of $\mathfrak{B}_0$ with $\mathfrak{B}_k$ and let $\mathfrak{A}_n = \bigotimes_{k=1}^{n} \mathfrak{B}_k$ be the tensor product of the first $n$ algebras. Note $\mathfrak{A}_n$ is an $(r^n \times r^n)$-matrix algebra. We denote by $\mathfrak{A}$ the $C^*$-algebra obtained from the norm closure of the inductive limit of the $\mathfrak{A}_n$, i.e., $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$. This $C^*$-algebra is a UHF-algebra of type $r^\infty$ (see [8]).

For each unitary $U \in \mathfrak{B}_0$ we define a product automorphism $\alpha_U$ of $\mathfrak{A}$ by requiring $\alpha_U(\gamma_k(A)) = \gamma_k(UAU^{-1})$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Note that if $A = \gamma_1(A_1)\gamma_2(A_2)\ldots \gamma_n(A_n)$ then $\alpha_U(A) = \gamma_1(UA_1U^{-1})\gamma_2(UA_2U^{-1})\ldots \gamma_n(UA_nU^{-1})$ so $\alpha_U$ is completely determined on $\mathfrak{A}_n$. Since the $\mathfrak{A}_n$ are dense in $\mathfrak{A}$, $\alpha_U$ is then uniquely determined on $\mathfrak{A}$.

Suppose $G$ is a group of unitary elements $U \in \mathfrak{B}_0$. We denote by $\mathfrak{A}^G$ the $C^*$-subalgebra of $\mathfrak{A}$ consisting of $\alpha_U$ invariant elements of $\mathfrak{A}$ with $U \in G$, i.e.

$$\mathfrak{A}^G = \{A \in \mathfrak{A} ; \alpha_U(A) = A \text{ for all } U \in G\}.$$

Note that if $H$ is a subgroup of $G$ $(H \subset G)$ then $\mathfrak{A}^H \supset \mathfrak{A}^G$. Note that $\mathfrak{A}^G$ is the norm closure of the union of the $\mathfrak{A}_n^G$ where

$$\mathfrak{A}_n^G = \{A \in \mathfrak{A}_n ; \alpha_U(A) = A \text{ for all } U \in G\}.$$

Since the $\mathfrak{A}_n^G$ are finite dimensional algebras $\mathfrak{A}^G$ is an AF-algebra (see [5]).

In this paper we will study the restriction of product states to $\mathfrak{A}^G$. The groups $G$ which we will consider are not arbitrary subgroups of the unitary group of $\mathfrak{B}_0$ but groups which fix a $*$-subalgebra $R_0$ of $\mathfrak{B}_0$. Suppose $R_0$ is a $*$-subalgebra of $\mathfrak{B}_0$ which contains the unit $I$ of $\mathfrak{B}_0$. We will restrict our attention to groups $G$ of the form,

$$G = G(R_0) = \{U \in R_0' ; U \text{ is unitary}\},$$

where $R_0'$ denotes the elements of $\mathfrak{B}_0$ which commute with $R_0$.

The invariant algebra $\mathfrak{A}^G$ is closely related to the symmetric group or permutation group $S_\infty$ of all finite permutations of the positive integers, i.e., $\sigma \in S_\infty$ if and only if $i \to \sigma(i)$ is a one-to-one mapping so that $i \neq \sigma(i)$ for only finitely many

$i = 1, 2, \ldots$ . If $\sigma \in S_\infty$ there is a unique unitary element $U_\sigma$ of $\mathfrak{A}$ so that

$$U_\sigma \gamma_k(A) U_\sigma^{-1} = \gamma_{\sigma(k)}(A)$$

for all $A \in \mathfrak{B}_0$, $k = 1, 2, \ldots$ and $\tau(U_\sigma) > 0$ where $\tau$ is the normalized trace on $\mathfrak{A}$. For a permutation $\sigma$ which is a cycle of length $n$, so $\sigma(i_k) = i_{k+1}$ for $k = 1, 2, \ldots$ $\ldots, n-1$ and $\sigma(i_n) = i_1$, Price has calculated (see [15]) that $U_\sigma$ is given by

$$U_\sigma = \sum_{j_1, j_2, \ldots, j_n = 1}^{r} \gamma_{i_1}(e_{j_1 j_2}) \gamma_{i_2}(e_{j_2 j_3}) \ldots \gamma_{i_n}(e_{j_n j_1})$$

where the $\{e_{ij} ; i,j = 1, \ldots, r\}$ are any set of matrix units for $\mathfrak{B}_0$.

It follows from Weyl's classical work on group invariants (see [16] or [15]) that if $G = \{\lambda I\}' = U(r)$ then $\mathfrak{A}_n^G$ is generated by the $U_\sigma$ with $\sigma \in S_n$ ($\sigma \in S_n$ if $\sigma(i) = i$ for $i > n$). A similar result we will use many times in this paper is that if $G = G(R_0)$ then $\mathfrak{A}_n^G$ is generated by $\gamma_1(R_0)$ and the $U_\sigma$ with $\sigma \in S_n$. Here is a proof of this fact.

THEOREM 2.1. *$\mathfrak{A}_n^G$ is generated by $\gamma_1(R_0)$ and the $U_\sigma$ with $\sigma \in S_n$.*

*Proof.* Let $\mathfrak{B}_n^G$ be the algebra generated by $\gamma_1(R_0)$ and the $U_\sigma$ with $\sigma \in S_n$. Clearly, $\mathfrak{B}_n^G \subset \mathfrak{A}_n^G$. If $S$ is a subset of $\mathfrak{A}_n$ we denote by $S^c$ the set of $A \in \mathfrak{A}_n$ which commute with each element of $S$ (i.e., $S^c$ is the commutant of $S$ in $\mathfrak{A}_n$ and $\mathfrak{A}_n$ is isomorphic to the set of all bounded operators on a Hilbert space of dimension $n^r$). We will show $(\mathfrak{B}_n^G)^c \subset (\mathfrak{A}_n^G)^c$. Suppose $A \in (\mathfrak{B}_n^G)^c$. Since $\mathfrak{B}_n^G$ contains the permutation unitaries $U_\sigma$ it follows that $A$ commutes with the $U_\sigma$ for $\sigma \in S_n$. Then it follows from Weyl's theorem that

$$A = \sum_{i=1}^{m} \alpha_i \gamma_1(U_i) \gamma_2(U_i) \ldots \gamma_n(U_i)$$

where the $\alpha_i$ are complex numbers and the $U_i$ are unitary elements of $\mathfrak{B}_0$. Since $\gamma_1(U) \in \mathfrak{B}_n^G$ for all unitary $U \in R_0$ we have $A = \gamma_1(U) A \gamma_1(U^{-1})$ for all unitary $U \in R_0$. Averaging over the unitary group of $R_0$ we find,

$$A = \sum_{i=1}^{m} \alpha_i \gamma_1(\varphi(U_i)) \gamma_2(U_i) \ldots \gamma_n(U_i)$$

where

$$\varphi(B) = \int UBU^{-1} \mathrm{d}\nu(U)$$

for $B \in \mathfrak{B}_0$ and $\nu$ is Haar measure on the unitary group of $R_0$. Since $\mathfrak{B}_n^G$ contains $U_{\sigma(1,k)}$ for $k = 2, \ldots, n$ ($\sigma(1, k)$ is the permutation which transposes 1 and $k$)

we have $\gamma_k(B) = U_{\sigma(1,k)}\gamma_1(B)U_{\sigma(1,k)}^{-1} \in \mathfrak{B}_n^G$ for all $B \in R_0$ and $k = 2,\ldots,n$. Hence, we have $\gamma_2(U)A\gamma_2(U^{-1}) = A$ for all unitary $U \in R_0$. Then, averaging $\gamma_2(U)A\gamma_2(U^{-1})$ over the unitary group of $R_0$ we find,

$$A = \sum_{i=1}^{m} \alpha_i\gamma_1(\varphi(U_i))\gamma_2(\varphi(U_i))\gamma_3(U_i)\ldots\gamma_n(U_i).$$

Continuing this averaging process applied to $\gamma_k(U)A\gamma_k(U^{-1})$ for $k = 3,\ldots,n$ we find,

$$A = \sum_{i=1}^{m} \alpha_i\gamma_1(\varphi(U_i))\gamma_2(\varphi(U_i))\ldots\gamma_n(\varphi(U_i)).$$

Let $\Gamma(B) = \gamma_1(B)\gamma_2(B)\ldots\gamma_n(B)$ for $B \in \mathfrak{B}_0$. Then we have

$$A = \sum_{i=1}^{m} \alpha_i\Gamma(\varphi(U_i)). \tag{2.1}$$

We will show $\Gamma(\varphi(B)) \in (\mathfrak{A}_n^G)^c$ for any $B \in \mathfrak{B}_0$. Suppose $B \in \mathfrak{B}_0$. Since $\varphi(B) \in R_0'$ we have by the polar decomposition that $\varphi(B) = V_0T$ with $V_0 \in R_0'$ a partial isometry and $T \in R_0'$ is positive. Since $R_0'$ is finite dimensional $V_0$ can be extended to a unitary $V \in R_0'$. Then we have $\Gamma(\varphi(B)) = \Gamma(VT) = \Gamma(V)\Gamma(T)$. Since $V \in G$ we have $\Gamma(V) \in$ $\in (\mathfrak{A}_n^G)^c$. Then to show $\Gamma(\varphi(B)) \in (\mathfrak{A}_n^G)^c$ it suffices to show that $\Gamma(T) \in (\mathfrak{A}_n^G)^c$. We will show this.

By the spectral theorem we have $T = \sum_{i=1}^{s} \lambda_iE_i$ where $\lambda_i \geqslant 0$, $E_i = E_i^* \in R_0'$, $E_iE_j = \delta_{ij}E_i$ and $\sum_{i=1}^{s} E_i = I$. Let

$$U(t_1, t_2, \ldots, t_s) = e^{it_1}E_1 + e^{it_2}E_2 + \ldots + e^{it_s}E_s$$

and

$$F(t_1, t_2, \ldots, t_s) = \sum_{\substack{k_1+k_2+\cdots+k_s=n \\ k_1,k_2,\ldots,k_s\geqslant 0}} \lambda_1^{k_1}e^{-ik_1t}\lambda_2^{k_2}e^{-ik_2t}\ldots\lambda_s^{k_s}e^{-ik_st}.$$

Then a straightforward computation shows that

$$(2\pi)^{-s}\int_{-\pi}^{\pi}\ldots\int_{-\pi}^{\pi} dt_1\ldots dt_sF(t_1,\ldots,t_s)\Gamma(U(t_1,\ldots,t_s)) = \Gamma(T).$$

Since $U(t_1, t_2, \ldots, t_s) \in G$ we have $\Gamma(U(t_1, t_2, \ldots, t_s)) \in (\mathfrak{A}_n^G)^c$. Hence, $\Gamma(T) \in (\mathfrak{A}_n^G)^c$. Hence, $\Gamma(\varphi(U_i)) \in (\mathfrak{A}_n^G)^c$ and from equation (2.1) it follows that $A \in (\mathfrak{A}_n^G)^c$. Hence, we have shown that $(\mathfrak{B}_n^G)^c \subset (\mathfrak{A}_n^G)^c$. By the double commutant theorem we have $\mathfrak{B}_n^G \supset \mathfrak{A}_n^G$. Hence, $\mathfrak{B}_n^G = \mathfrak{A}_n^G$. ▨

It follows immediately that $\mathfrak{A}^G$ is generated by $\gamma_1(R_0)$ and the $U_\sigma$ with $\sigma \in S_\infty$.

In this paper we will be primarily concerned with product states. Given states $\omega_k$ of $\mathfrak{B}_0$ for $k = 1, 2, \dots$ the product state $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ is defined by the requirement that $\omega(\gamma_1(A_1)\gamma_2(A_2) \dots \gamma_n(A_n)) = \omega_1(A_1)\omega_2(A_2) \dots \omega_n(A_n)$ for $A_i \in \mathfrak{B}_0$ and $i = 1, \dots, n$. This uniquely determines $\omega$ on $\mathfrak{A}_n$ and since the $\mathfrak{A}_n$ are dense in $\mathfrak{A}$ this uniquely determines $\omega$ on $\mathfrak{A}$. It is well known that product states are factor states, i.e., they induce cyclic $*$-representations $\pi$ so that $\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \{\lambda I\}$.

We will need some estimates on the norm differences of two product states of $\mathfrak{A}$. Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ and $\omega' = \bigotimes_{k=1}^{\infty} \omega'_k$ are product states of $\mathfrak{A}$. For each $k = 1, 2, \dots$ let $\Omega_k$ and $\Omega'_k$ be the positive trace one elements in $\mathfrak{B}_0$ so that $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ and $\omega'_k(A) = \mathrm{tr}(A\Omega'_k)$ for all $A \in \mathfrak{B}_0$, where tr is the trace, normalized (as will be our convention throughout this paper) so that $\mathrm{tr}(E) = 1$ for a rank one projection $E$. It follows from [13] that

$$2(1 - s) \leqslant \|\omega - \omega'\| \leqslant 2\sqrt{1 - s^2}$$

where

$$s = \prod_{k=1}^{\infty} \mathrm{tr}(\Omega_k^{1/2}\Omega_k'^{1/2}).$$

It follows (see [13] for details) that $\omega$ and $\omega'$ are quasi-equivalent, denoted $\omega \underset{q}{\sim} \omega'$ (i.e., they induce quasi-equivalent $*$-representations) if and only if

$$\sum_{k=1}^{\infty} \|\Omega_k^{1/2} - \Omega_k'^{1/2}\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} 2 - 2\,\mathrm{tr}(\Omega_k^{1/2}\Omega_k'^{1/2}) < \infty$$

where $\|A\|_{\mathrm{HS}} = \mathrm{tr}(A^*A)^{1/2}$ is the Hilbert-Schmidt norm of $A$. We note (as we will need this in the next sections) that if the above sum is finite and $\|\Omega_k^{1/2} - \Omega_k'^{1/2}\|_{\mathrm{HS}}^2 < 2$ for all $k$ then $\|\omega - \omega'\| < 2$.

We denote by $\omega^G$ the restriction of the state $\omega$ of $\mathfrak{A}$ to the algebra $\mathfrak{A}^G$. In this paper we will be concerned with determining when $\omega^G$ is a factor state and when $\omega_1^G$ and $\omega_2^G$ are quasi-equivalent (i.e., when they induce cyclic $*$-representations which are quasi-equivalent) where $\omega_1$ and $\omega_2$ are product states of $\mathfrak{A}$. We will often make use of the fact (see [7]) that two $*$-representations $\pi_1$ and $\pi_2$ are quasi-equivalent if and only if the mapping $\varphi$ defined by $\varphi(\pi_1(A)) = \pi_2(A)$ for all $A$ in the algebra is $\sigma$-strongly bicontinuous (or $\sigma$-weakly bicontinuous). The extension of $\varphi$ to the weak closures is a $*$-isomorphism of $\pi_1(\mathfrak{A})''$ with $\pi_2(\mathfrak{A})''$.

For the case of factor states the induced representations are either quasi-equivalent or disjoint. It follows that if $\omega_1$ and $\omega_2$ are factor states and $\|\omega_1 - \omega_2\| < 2$ then $\omega_1 \underset{q}{\sim} \omega_2$.

3. RELATION BETWEEN $\pi(\mathfrak{A}^G)''$ AND $\pi(\mathfrak{A})''$

Let $R_0$ be a $*$-subalgebra of $\mathfrak{B}_0$ and let $G = G(R_0)$ be the group of unitaries in $\mathfrak{B}_0$ which commute with $R_0$. Throughout this section $R_0$ and $G$ will be related as just described. In this section we will show that if $\pi$ is a representation of $\mathfrak{A}$ induced by a product state then $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)''$ where $H$ is a subgroup of $G$ which can be calculated from the product state $\omega$.

DEFINITION 3.1. Suppose $\pi$ is a $*$-representation of $\mathfrak{A}$. We define

$$R^G(\pi) = \{A \in \mathfrak{B}_0 \,;\, \pi(\gamma_1(A)) \in \pi(\mathfrak{A}^G)''\}.$$

Note $R^G(\pi)$ is a $*$-subalgebra of $\mathfrak{B}_0$ containing $R_0$. If $\omega$ is a state of $\mathfrak{A}$ we denote by $R^G(\omega)$ the algebra $R^G(\pi)$ where $\pi$ is the cyclic $*$-representation induced by $\omega$.

DEFINITION 3.2. Suppose $\pi$ is a $*$-representation of $\mathfrak{A}$. We define

$$H^G(\pi) = \{U \in G; \pi \circ \alpha_U \underset{q}{\sim} \pi\}.$$

Note $H^G(\pi)$ is a subgroup of $G$. Again if $\omega$ is a state of $\mathfrak{A}$ we denote by $H^G(\omega)$ the group $H^G(\pi)$ where $\pi$ is the $*$-representation induced by $\omega$.

LEMMA 3.3. *$R^G(\pi)$ and $H^G(\pi)$ only depend on the quasi-equivalence class of $\pi$, i.e., if $\pi_1 \underset{q}{\sim} \pi_2$ then $R^G(\pi_1) = R^G(\pi_2)$ and $H^G(\pi_1) = H^G(\pi_2)$.*

*Proof.* Suppose $\pi_1$ and $\pi_2$ are quasi-equivalent $*$-representations of $\mathfrak{A}$. Then there is a $\sigma$-strongly bicontinuous $*$-isomorphism $\varphi$ of $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$ so that $\varphi(\pi_1(A)) = \pi_2(A)$ for all $A \in \mathfrak{A}$. Clearly, we have $\varphi(\pi_1(\mathfrak{A}^G)'') = \pi_2(\mathfrak{A}^G)''$ so $R^G(\pi_1) = R^G(\pi_2)$. Since $\pi_1 \underset{q}{\sim} \pi_2$ and $\pi_1 \circ \alpha_U \underset{q}{\sim} \pi_2 \circ \alpha_U$ for any unitary $U \in \mathfrak{B}_0$ we have $\pi_1 \circ \alpha_U \underset{q}{\sim} \pi_1$ if and only if $\pi_2 \circ \alpha_U \underset{q}{\sim} \pi_2$. Hence, $H^G(\pi_1) = H^G(\pi_2)$.

LEMMA 3.4. *Suppose $\omega$ is a factor state of $\mathfrak{A}$ and $\Omega_k \in \mathfrak{B}_0$ is the unique positive trace one element so that $\operatorname{tr}(A\Omega_k) = \omega(\gamma_k(A))$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$. Suppose $\Omega$ is a limit point of the sequence $\{\Omega_k\}$. Then $\Omega \in R^G(\omega)$.*

*Proof.* Suppose the hypothesis and notation of the lemma are satisfied. Since $\Omega$ is a limit point there is a subsequence $k \to q(k)$ so that $\Omega_{q(k)} \to \Omega$ as $k \to \infty$. Let $U_k = U_\sigma$ where $\sigma$ is the permutation which transposes 1 and $q(k)$. We have

$$U_k = \sum_{i,j=1}^{r} \gamma_1(e_{ij})\gamma_{q(k)}(e_{ji})$$

where $\{e_{ij}\,;\, i,j = 1, \ldots, r\}$ are a set of matrix units for $\mathfrak{B}_0$. Let $(\pi, \mathscr{H}, f_0)$ be a cyclic

$*$-representation induced by $\omega$. We show that $\pi(U_k)$ converges weakly to $\pi(\gamma_1(\Omega))$ as $k \to \infty$. Suppose $A, B \in \mathfrak{A}_n$. Then for $k$ sufficiently large so $q(k) > n$ we have

$$(\pi(A)f_0, \pi(U_k)\,\pi(B)f_0) = \omega(A^* U_k B) =$$

$$= \sum_{i,j=1}^{r} \omega(A^*\gamma_1(e_{ij})\gamma_{q(k)}(e_{ji})B) =$$

$$= \sum_{i,j=1}^{r} \omega(A^*\gamma_1(e_{ij})B\gamma_{q(k)}(e_{ji})).$$

Since $\omega$ is a factor state it follows from Theorem 2.5 of [12] that $\omega$ has the cluster property so that $\omega(C\gamma_k(A)) - \omega(C)\omega(\gamma_k(A)) \to 0$ as $k \to \infty$. Hence, we have

$$(\pi(A)f_0, \pi(U_k)\,\pi(B)f_0) \to \sum_{i,j=1}^{r} \omega(A^*\gamma_1(e_{ij})B)\omega(\gamma_{q(k)}(e_{ji})) =$$

$$= \omega(A^*\gamma_1(\Omega_{q(k)})B) \to \omega(A^*\gamma_1(\Omega)B) = (\pi(A)f_0, \pi(\gamma_1(\Omega))\pi(B)f_0)$$

as $k \to \infty$. Since the $\pi(U_k)$ are uniformly bounded and the vectors $\pi(A)f_0$ and $\pi(B)f_0$ with $A, B \in \mathfrak{A}_n$ for $n < \infty$ are dense in $\mathscr{H}$ we have $\pi(U_k)$ converges weakly to $\pi(\gamma_1(\Omega))$. Hence, $\pi(\gamma_1(\Omega)) \in \pi(\mathfrak{A}^G)''$ (since $U_k \in \mathfrak{A}^G$) and $\Omega \in R^G(\omega)$. ▨

LEMMA 3.5. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$ *and* $\rho$ *is a limit point of the sequence* $\omega_k$. *Then if* $\rho(A) = \operatorname{tr}(A\Omega)$ *for* $A \in \mathfrak{B}_0$ *we have* $\Omega \in R^G(\omega)$.

The proof follows immediately from the previous lemma.

LEMMA 3.6. *Suppose* $\pi$ *is a* $*$*-representation of* $\mathfrak{A}$. *Then* $H^G(\pi) \subset G \cap R^G(\pi)'$.

*Proof.* Suppose $U \in H^G(\pi)$ and $A \in R^G(\pi)$. Then there is a sequence $A_n \in \mathfrak{A}^G$ so that $\pi(A_n) \to \pi(\gamma_1(A))$ $\sigma$-strongly as $n \to \infty$. Since $\pi \underset{q}{\sim} \pi \circ \alpha_U$ the mapping $\varphi(\pi(B)) = \pi(\alpha_U(B))$ for $B \in \mathfrak{A}$ is $\sigma$-strongly bicontinuous. Hence, $\pi(\alpha_U(A_n)) \to \pi(\gamma_1(UAU^{-1}))$ $\sigma$-strongly as $n \to \infty$. But $\alpha_U(A_n) = A_n$ and $\pi$ is faithful so $UAU^{-1} = A$. Since $A \in R^G(\pi)$ is arbitrary we have $U \in R^G(\pi)'$. ▨

LEMMA 3.7. *Suppose* $\pi$ *is a* $*$*-representation of* $\mathfrak{A}$ *and* $H = G \cap R^G(\pi)'$. *Then* $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)''$.

*Proof.* Since $\mathfrak{A}^H \supset \mathfrak{A}^G$ we have $\pi(\mathfrak{A}^H)'' \supset \pi(\mathfrak{A}^G)''$. By the definition of $R^G(\pi)$ we have $\pi(\gamma_1(R^G(\pi))) \subset \pi(\mathfrak{A}^G)''$. As was pointed out in the last section, $\gamma_1(R^G(\pi))$ and $\mathfrak{A}^G$ generate $\mathfrak{A}^H$. Hence, $\pi(\mathfrak{A}^G)'' \subset \pi(\mathfrak{A}^H)''$. ▨

THEOREM 3.8. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$. *Then*

$$H^G(\omega) = G \cap R^G(\omega)'.$$

*Furthermore, let $\Phi$ be the conditional expectation of $\mathfrak{B}_0$ onto $R^G(\omega)$ which preserves the trace. Let $\bar{\omega}_k(A) = \omega_k(\Phi(A))$ for $A \in \mathfrak{B}_0$ and let $\bar{\omega} = \bigotimes_{k=1}^{\infty} \bar{\omega}_k$. Then $\omega \underset{q}{\sim} \bar{\omega}$.*

*Proof.* Suppose $\omega$, $\Phi$ and $\bar{\omega}$ are as given in the statement of the theorem. We will begin by proving $\omega \underset{q}{\sim} \bar{\omega}$.

Suppose $\omega$ and $\bar{\omega}$ are not quasi-equivalent. Let $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$. Then $\bar{\omega}_k(A) = \mathrm{tr}(\Phi(A)\Omega_k) = \mathrm{tr}(A\Phi(\Omega_k))$. Since $\omega$ and $\bar{\omega}$ are not quasi-equivalent we have

$$\sum_{k=1}^{\infty} \|\Omega_k^{1/2} - \Phi(\Omega_k)^{1/2}\|_{\mathrm{HS}}^2 = \infty.$$

Let $S_k = \Omega_k^{1/2} - \Phi(\Omega_k)^{1/2}$, $s_k = \|S_k\|_{\mathrm{HS}}$ and $T_k = \Omega_k^{1/2} + \Phi(\Omega_k)^{1/2}$. Let

$$A_k = \int_{-\infty}^{0} \exp\left(\frac{1}{2}\, tT_k\right) S_k \exp\left(\frac{1}{2}\, tT_k\right) \mathrm{d}t.$$

The hermitian element $A_k \in \mathfrak{B}_0$ is the unique solution to the equation $(1/2)(A_kT_k + T_kA_k) = S_k$ subject to the requirement that the null space of $A_k$ contain the null space of $T_k$ (see [11] for a general discussion of this equation). One property of the $A_k$ that we shall need is that $\|A_k\| \leqslant 1$. To see this suppose $\lambda$ is an eigenvalue for $A_k$ and $f$ is an associated eigenvector. Then $(f, S_kf) = (1/2)(f, (A_kT_k + T_kA_k)f) = \lambda(f, T_kf)$. Since $T_k \geqslant S_k$ and $T_k \geqslant -S_k$ we have $|\lambda| \leqslant 1$. Hence, $\|A_k\| \leqslant 1$.

Since the operators, $s_k^{-1}S_k$, $A_k$ and $\Omega_k$ are uniformly bounded in a finite dimensional space, there is by a routine compactness argument (see the proof of Lemma 3.5 of [4] for details) a subsequence $k \to q(k)$ so that $q(1) \geqslant 2$,

$$\sum_{k=1}^{\infty} \|S_{q(k)}\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} s_{q(k)}^2 = \infty,$$

$$s_{q(k)}^{-1} S_{q(k)} \to X, \quad \Omega_{q(k)} \to \Omega \quad \text{and} \quad A_{q(k)} \to Y$$

as $k \to \infty$. Let $Q(n)$ be the first $n$ numbers of the subsequence $k \to q(k)$, i.e., $Q(n) = \{q(1), q(2), \ldots, q(n)\}$. Let $Q$ be the union of the $Q(n)$ for $n = 1, 2, \ldots$. To avoid the cumbersome subscript $q(k)$ we will write $A_k \to Y$ as $k \to \infty$ in $Q$. With this notation we have

$$\sum_{k\in Q} s_k^2 = \infty, \quad s_k^{-1}S_k \to X, \quad \Omega_k \to \Omega \quad \text{and} \quad A_k \to Y \tag{3.0}$$

as $k \to \infty$ in $Q$.

Let $H = G \cap R^G(\omega)'$ and let $\Gamma$ be the conditional expectations of $\mathfrak{A}$ onto $\mathfrak{A}^H$ given by

$$\Gamma(A) = \int_H \alpha_U(A) \mathrm{d}\nu(U)$$

where $\nu$ is Haar measure on $H$. Let $W = W^* \in \mathfrak{B}_0$. We will specify $W$ more precisely later on. Let

$$C_n = Z_n^{-1} \sum_{k \in Q(n)} \Gamma(\gamma_1(W)(\gamma_k(A_k) - \bar{\omega}_k(A_k)))$$

with

$$Z_n = \sum_{k \in Q(n)} s_k^2$$

set

$$B = \int_H U W U^{-1} f(U) \mathrm{d}\nu(U)$$

with

$$f(U) = \mathrm{tr}((X - (1/2)XY - (1/2)YX + YU^{-1}XU)U^{-1}XU).$$

Note $|f(U)| \leqslant 3$. We will show that

$$\pi(U_n(t)) = \pi(\exp(itC_n)) \to \pi(\exp(it\gamma_1(B))) = \pi(V(t))$$

strongly as $n \to \infty$, where $(\pi, \mathscr{H}, f_0)$ is a cyclic $*$-representation of $\mathfrak{A}$ induced by $\omega$ and $U_n(t)$, and $V(t)$ are defined as implied above.

Consider vectors $f \in \mathscr{H}$ of the form $f = \pi(\gamma_1(X_1)\gamma_2(X_2) \dots \gamma_m(X_m))f_0$ with $\|f\| = 1$ with $X_i \in \mathfrak{B}_0$. Since the linear span of such vectors is dense in $\mathscr{H}$ and the $U_n(t)$ and $V(t)$ are uniformly bounded, it is sufficient to show that $\|\pi(U_n(t) - V(t))f\|^2 = \omega_f((U_n(t) - V(t))^*(U_n(t) - V(t))) \to 0$ as $n \to \infty$, where $\omega_f(A) = (f, \pi(A)f)$. Note that $\omega_f = \bigotimes_{k=1}^{\infty} \omega_k'$ is a product state with $\omega_k' = \omega_k$ for $k$ sufficiently large ($k > m$). Now we have

$$\|\pi(U_n(t) - V(t))f\|^2 = 2 - 2\,\mathrm{Re}(\omega_f(U_n(t)^* V(t))).$$

Differentiating $U_n(t)^* V(t)$ and integrating the resulting differential equation we find

$$U_n(t)^* V(t) = I - \mathrm{i}\int_0^t U_n(s)^*(C_n - \gamma_1(B))V(s)\mathrm{d}s.$$

Inserting this in the above expression we find

$$\|\pi(U_n(t) - V(t))f\|^2 = 2\,\mathrm{Re}\left(i\int_0^t \omega_f(U_n(s)^*(C_n - \gamma_1(B))V(s))\mathrm{d}s\right) \leqslant$$

(3.1)

$$\leqslant 2\int_0^t \|\pi((C_n - \gamma_1(B))V(s))f\|\,\mathrm{d}s.$$

If $g = \pi(V(s))f$ then $\omega_f$ and $\omega_g$ are product states which differ only on $\mathfrak{B}_1$. Since $\pi(V(s))f$ is of the same general form as $f$ and since the estimates we will obtain are independent of the first few terms ($\omega_k'$) of the product state $\omega_f = \bigotimes_{k=1}^{\infty} \omega_k'$ we will simply replace $\pi(V(s))f$ by $f$ in estimating the right hand side of inequality (3.1). Now we have

$$C_n - \gamma_1(B) = Z_n^{-1} \sum_{k\in Q(n)} \int_H \gamma_1(UWU^{-1})L_k(U)\mathrm{d}\nu(U)$$

with

$$L_k(U) = \gamma_k(UA_kU^{-1}) - \bar{\omega}_k(A_k) - s_k^2 f(U).$$

Using the fact that $\omega_f = \bigotimes_{k=1}^{\infty} \omega_k'$ is a product state we have

$$\|\pi(C_n - \gamma_1(B))f\|^2 = \omega_f((C_n - \gamma_1(B))^2) =$$

$$= Z_n^{-2} \sum_{k,j\in Q(n)} \int \omega_1'(U_1WU_1^{-1}U_2WU_2^{-1})\omega_f(L_k(U_1)L_j(U_2))\mathrm{d}\nu(U_1)\mathrm{d}\nu(U_2).$$

Now for $k \neq j$ we have $\omega_f(L_k(U_1)L_j(U_2)) = \omega_f(L_k(U_1))\omega_f(L_j(U_2))$. Hence, we have

$$\|\pi(C_n - \gamma_1(B))f\|^2 =$$

(3.2)

$$= \int \omega_1'(U_1WU_1^{-1}U_2WU_2^{-1})(F_n(U_1)F_n(U_2) - G_n(U_1, U_2))\mathrm{d}\nu(U_1)\mathrm{d}\nu(U_2)$$

where

$$F_n(U) = Z_n^{-1} \sum_{k\in Q(n)} \omega_f(L_k(U))$$

and

$$G_n(U_1, U_2) = Z_n^{-2} \sum_{k\in Q(n)} \omega_f(L_k(U_1)L_k(U_2)) - \omega_f(L_k(U_1))\omega_f(L_k(U_2)).$$

We will show that $\|\pi(C_n - \gamma_1(B))f\| \to 0$ as $n \to \infty$ by showing the $F_n$ and $G_n$ are uniformly bounded and tend to zero pointwise (in fact, uniformly) as $n \to \infty$. We begin with $F_n(U)$. We have

$$\omega_f(L_k(U)) = \omega_k'(UA_kU^{-1}) - \bar{\omega}_k(A_k) - s_k^2 f(U). \tag{3.3}$$

For $k$ sufficiently large we have $\omega_k' = \omega_k$. We have

$$\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k) = \operatorname{tr}(\Omega_k UA_kU^{-1} - \Omega_k\Phi(A_k)) =$$

$$= \operatorname{tr}((U^{-1}\Omega_k U - \Phi(\Omega_k))A_k).$$

Since $\Omega_k^{1/2} = (1/2)(T_k + S_k)$ and $\Phi(\Omega_k)^{1/2} = (1/2)(T_k - S_k)$ we have

$$\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k) = (1/4)\operatorname{tr}(U^{-1}(T_k - S_k + 2S_k)^2UA_k - (T_k - S_k)^2A_k).$$

Since $T_k - S_k = 2\Phi(\Omega_k)^{1/2}$ commutes with $U \in R^G(\omega)'$ we have

$$\begin{aligned}\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k) &= \operatorname{tr}(U^{-1}S_kU(1/2)(A_kT_k + T_kA_k)) - \\ &- (1/2)\operatorname{tr}((S_kU^{-1}S_kU + U^{-1}S_kUS_k)A_k) + \operatorname{tr}(U^{-1}S_k^2UA_k).\end{aligned} \tag{3.4}$$

Since $S_k = (1/2)(A_kT_k + T_kA_k)$ we have

$$\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k) =$$

$$= \operatorname{tr}((S_k - (1/2)S_kA_k - (1/2)A_kS_k + A_kU^{-1}S_kU)U^{-1}S_kU).$$

If $J, K \in \mathfrak{B}_0$ with $\|J\| \leqslant 1$ and $\|K\| \leqslant 1$ then $|\operatorname{tr}(S_kJS_kK)| \leqslant \|JS_k\|_{HS}\|S_kK\|_{HS} \leqslant \|S_k\|_{HS}^2 = s_k^2$. Since $\|U\| = \|U^{-1}\| = 1$ and $\|A_k\| \leqslant 1$ we have

$$|\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k)| \leqslant 3s_k^2.$$

For $k$ sufficiently large we have $\omega_k' = \omega_k$ so from equation (3.3) and the fact that $|f(U)| \leqslant 3$ we have

$$\omega_f(L_k(U)) \leqslant 6s_k^2$$

for $k$ sufficiently large ($k > m$) independent of $U \in H$. Since $Z_n = \sum_{k \in Q(n)} s_k^2$ we have that $F_n(U)$ is uniformly bounded. In fact, we have $\limsup|F_n(U)| \leqslant 6$. As $k \to \infty$ in $Q$ we have from (3.0) that

$$\begin{aligned}&s_k^{-2}(\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k)) \to \\ &\to \operatorname{tr}((X - (1/2)XY - (1/2)YX + YU^{-1}XU)U^{-1}XU) = f(U).\end{aligned} \tag{3.5}$$

Hence, $s_k^{-2}\omega_f(L_k(U)) \to 0$ as $k \to \infty$ in $Q$. Hence, $|F_n(U)| \to 0$ as $n \to \infty$ for $U \in H$. Since the $F_n(U)$ are uniformly bounded and converge pointwise to zero the terms coming from $F_n(U_1)F_n(U_2)$ in equation (3.2) go to zero as $n \to \infty$.

Now we consider the expression for $G_n(U_1, U_2)$ in equation (3.2). For $k$ sufficiently large ($k > m$) we have $\omega_k' = \omega_k$. A small computation shows that for $k > m$ we have

$$\omega_f(L_k(U_1)L_k(U_2)) - \omega_f(L_k(U_1))\,\omega_f(L_k(U_2)) =$$

$$= \operatorname{tr}(\Omega_k U_1 A_k U_1^{-1} U_2 A_k U_2^{-1}) - \operatorname{tr}(\Omega_k U_1 A_k U_1^{-1}) \operatorname{tr}(\Omega_k U_2 A_k U_2^{-1}).$$

Computations of the kind made in equation (3.4) show

$$\operatorname{tr}(\Omega_k U_1 A_k U_1^{-1}) \leqslant \|T_k\|_{\mathrm{HS}}\|S_k\|_{\mathrm{HS}} + \text{terms of order } \|S_k\|_{\mathrm{HS}}^2.$$

Hence, $\operatorname{tr}(\Omega_k U_1 A_k U_1^{-1}) \leqslant (\text{constant})\, s_k$. We have

$$|\operatorname{tr}(\Omega_k U_1 A_k U_1^{-1} U_2 A_k U_2^{-1})|^2 = |\operatorname{tr}(\Omega_k^{1/2} U_1 A_k U_1^{-1} U_2 A_k U_2^{-1} \Omega_k^{1/2})|^2 \leqslant$$

$$\leqslant \operatorname{tr}(\Omega_k U_1 A_k^2 U_1^{-1}) \operatorname{tr}(\Omega_k U_2 A_k^2 U_2^{-1}).$$

Calculations like those done in equation (3.4) and repeated use of the inequality $\operatorname{tr}(T_k^2 A_k^2) \leqslant 2 s_k^2$ (derived by noting that $s_k^2 = \operatorname{tr}(S_k^2) = \operatorname{tr}((1/4)(A_k T_k + T_k A_k)^2) = (1/2)\operatorname{tr}(T_k^2 A_k^2) + (1/2)\operatorname{tr}(T_k A_k T_k A_k) \geqslant (1/2)\operatorname{tr}(T_k^2 A_k^2)$) shows that

$$\operatorname{tr}(\Omega_k U_1 A_k^2 U_1^{-1}) \leqslant (\text{constant})\, s_k^2.$$

Hence, $\operatorname{tr}(\Omega_k U_1 A_k U_1^{-1} U_2 A_k U_2^{-1}) \leqslant (\text{constant})\, s_k^2$. Hence, we have (for $k > m$)

$$|\omega_f(L_k(U_1)L_k(U_2)) - \omega_f(L_k(U_1))\omega_f(L_k(U_2))| \leqslant (\text{constant})\, s_k^2.$$

Since the expression for $G_n(U_1, U_2)$ has a $Z_n^{-2}$ term multiplying the sum which is bounded by a constant times $Z_n$ we see that $G_n(U_1, U_2) \to 0$ uniformly as $n \to \infty$. Hence, we have from equation (3.2) that $\|\pi(C_n - \gamma_1(B))f\| \to 0$ as $n \to \infty$. Then it follows from inequality (3.1) that $\|\pi(U_n(t) - V(t))f\| \to 0$ as $n \to \infty$. Hence, we have shown that $\pi(U_n(t)) \to \pi(V(t))$ strongly as $n \to \infty$.

From Lemma 3.7 we have $\pi(\mathfrak{A}^H)'' = \pi(\mathfrak{A}^G)''$. Since $C_n \in \mathfrak{A}^G$ and $U_n(t) = \exp(itC_n)$ we have $\pi(V(t)) \in \pi(\mathfrak{A}^G)''$. Since $V(t) = \exp(it\gamma_1(B))$ we have $\pi(\gamma_1(B)) \in \pi(\mathfrak{A}^G)''$. Hence, $B \in R^G(\omega)$.

We recall that

$$B = \int_H U W U^{-1} f(U) \mathrm{d}\nu(U).$$

Since $\Phi$ can be expressed in the form

$$\Phi(A) = \int_H UAU^{-1} \mathrm{d}\nu(U) \tag{3.6}$$

and $\nu$ is invariant under left and right translation we have $\Phi(UWU^{-1}) = \Phi(W)$, for $U \in H$. Hence,

$$\Phi(B) = \int_H \Phi(W) f(U) \mathrm{d}\nu(U). \tag{3.7}$$

From relation (3.5) we see that $s_k^{-2}(\omega_k(UA_kU^{-1}) - \bar{\omega}_k(A_k)) = f_k(U) \to f(U)$ as $k \to \infty$ in $Q$. Since $\bar{\omega}_k(A) = \omega_k(\Phi(A))$ it follows from (3.6) that

$$\int_H f_k(U) \mathrm{d}\nu(U) = 0 \quad \text{and hence,} \quad \int_H f(U) \mathrm{d}\nu(U) = 0.$$

Hence, from (3.7) we have that $\Phi(B) = 0$.

Next we show that for some choice of $W = W^* \in \mathfrak{B}_0$ we have $B \neq 0$. To this end suppose $B = 0$ for all choices $W = W^* \in \mathfrak{B}_0$. Then for all hermitian $A, W \in \mathfrak{B}_0$ we have

$$\mathrm{tr}(AB) = \int \mathrm{tr}(AUWU^{-1}) f(U) \mathrm{d}\nu(U) = 0.$$

We recall that $f(U) = \mathrm{tr}(XU(X - (1/2)XY - (1/2)YX)U^{-1})) + \mathrm{tr}(X^2UYU^{-1})$. Since the above expression is zero for all choices of hermitian $A$ and $W$ and $f(U)$ is a linear combination of two terms of the form $\mathrm{tr}(AUWU^{-1})$ with $A$ and $W$ chosen appropriately, it follows

$$\int f(U)^2 \mathrm{d}\nu(U) = \int |f(U)|^2 \mathrm{d}\nu(U) = 0.$$

But this is a contradiction since $f(I) = \mathrm{tr}(X(X - (1/2)XY - (1/2)YX + XY)) = \mathrm{tr}(X^2) = 1$ and $f$ is continuous. Hence, for some hermitian $W \in \mathfrak{B}_0$ we have $B \neq 0$.

We have finally reached the desired contradiction. By assuming $\omega$ and $\bar{\omega}$ are not quasi-equivalent we have constructed a $B \in R^G(\omega)$ with $B \neq 0$. Since $\Phi$ is the conditional expectation of $\mathfrak{B}_0$ onto $R^G(\omega)$ preserving the trace we have $\Phi(B) = B$. But we have shown that $\Phi(B) = 0$ and $B \neq 0$. Hence, we have shown that $\omega \underset{q}{\sim} \bar{\omega}$.

The proof of the first statement of the theorem is now easy. From Lemma 3.6 we have $H^G(\omega) \subset G \cap R^G(\omega)'$. Since $\bar{\omega}$ is invariant under $\alpha_U$ with $U \in H = G \cap \cap R^G(\omega)'$ we have $H^G(\bar{\omega}) \supset G \cap R^G(\omega)'$. Since $\omega \underset{q}{\sim} \bar{\omega}$ we have from Lemma 3.3 that $H^G(\bar{\omega}) = H^G(\omega)$. Hence, $H^G(\omega) = G \cap R^G(\omega)'$, by Lemma 3.6. ▨

## 4. FACTOR STATES OF $\mathfrak{A}^G$

Let $R_0$ be a $*$-subalgebra of $\mathfrak{B}_0$ and let $G = G(R_0)$ be the group of unitaries in $\mathfrak{B}_0$ which commute with $R_0$. Throughout this section $R_0$ and $G$ will be related as just described. The results we obtain are valid for any choice of $R_0$. Throughout this section the word projection will mean hermitian projection.

In this section we analyze when the restriction of a product state $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ to $\mathfrak{A}^G$ is a factor state (i.e., it induces a factor representation of $\mathfrak{A}^G$).

We determine when they are pure states and we obtain necessary and sufficient conditions that two such factor states be quasi-equivalent.

We apologize for the complexity of the following definition. Its purpose is to eliminate certain problems associated with finite-dimensional representations which would otherwise obscure the statements of our main results.

DEFINITION 4.1. Suppose $\omega$ is a state of $\mathfrak{A}$. We say $E \in \mathfrak{B}_0$ is the *$G_\gamma$-support projection of* $\omega$ if $E$ is the smallest projection in $R_0'$ so that $\omega(\gamma_k(E)) = 1$ for all $k = 1, 2, \ldots$ .

We say a state $\omega$ has *minimal $G_\gamma$-support* if for every projection $e \in R_0'$ so that $\omega(\gamma_k(e)) = 1$ exactly once, say for $k = k_0$, and $\omega(\gamma_k(e)) = 0$ for $k \neq k_0$, then there is a projection $f \in R_0'$ with $f \leqslant e$ so that $\omega(\gamma_{k_0}(f)) = 1$ and $fe_n$ is either zero or a minimal projection in $R_0'$ for each minimal central projection $e_n \in R_0 \cap R_0'$.

We show that if $\omega$ is a state of $\mathfrak{A}$ which is not of minimal $G_\gamma$-support then $\omega$ can be modified to produce a quasi-equivalent state $\omega'$ which has minimal $G_\gamma$-support and $\omega$ and $\omega'$ coincide on $\mathfrak{A}^G$ (i.e., $\omega^G = \omega'^G$).

Suppose $\omega$ is a state of $\mathfrak{A}$ and $e \in R_0'$ is a projection so that $\omega(\gamma_1(e)) = 1$ and $\omega(\gamma_k(e)) = 0$ for $k \geqslant 2$. Suppose $e_1, \ldots, e_m$ are the minimal central projections in $R_0 \cap R_0'$ so that $ee_i \neq 0$ for $i = 1, \ldots, m$. Let $\{e_{ij}'^{(k)}; i, j = 1, \ldots, s_k\}$ be matrix units for $e_k R_0' = R_0' e_k$ chosen so that $e_{11}'^{(k)} \leqslant e$ for $k = 1, \ldots, m$. Let

$$\omega'(A) = \sum_{k=1}^{m} \sum_{i=1}^{s_k} \omega(\gamma_1(e_{i1}'^{(k)}) A \gamma_1(e_{1i}'^{(k)}))$$

for $A \in \mathfrak{A}$. Now $\mathfrak{A}_n^G$ is spanned by elements of the form $A_0 = U_\sigma \gamma_1(A_1) \gamma_2(A_2) \ldots \ldots \gamma_n(A_n)$ with $A_i \in R_0$ and $\sigma$ a permutation so that $\sigma(i) = i$ for $i > n$. Because

$\omega(A) = \omega(\gamma_1(e)A\gamma_2(I-e)\gamma_3(I-e)\ldots\gamma_n(I-e))$ for all $A \in \mathfrak{A}_n^G$ it follows that $\omega(A_0) = 0$ unless the permutation $\sigma$ fixes one (i.e., $\sigma(1) = 1$). If $\sigma(1) = 1$ it follows that $\omega'(A_0) = \omega(A_0)$. Hence, $\omega' | \mathfrak{A}_n^G = \omega | \mathfrak{A}_n^G$. Since $n$ is arbitrary we have $\omega'^G = \omega^G$. Let $f = \sum_{k=1}^{m} e_{11}'^{(k)}$. We have that $f \leqslant e$ and $\omega'(\gamma_1(f)) = 1$ and $fe_k$ is either zero or a minimal projection in $R_0'$ for each minimal central projection $e_k \in R_0 \cap R_0'$.

Hence, we see that if $\omega$ is a state of $\mathfrak{A}$ we can, modifying $\omega$ in the manner indicated for a finite number of $k$ (in fact, the number of modifications must be less than $r/2$ where $\mathfrak{B}_0$ is an $(r \times r)$-matrix algebra), produce a state $\omega'$ which is quasi-equivalent with $\omega$ such that $\omega'$ has minimal $G_\gamma$-support and $\omega'^G = \omega^G$. If $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ is a product state then the modified state $\omega'$ is also a product state $\omega' = \bigotimes_{k=1}^{\infty} \omega_k'$ with $\omega_k' = \omega_k$ for all but a finite number of $k$.

DEFINITION 4.2. Suppose $\omega$ is a state of $\mathfrak{A}$. We define $M^G(\omega)$ to be the set of projections $e \in R_0'$ so that

$$\sum_{k=1}^{\infty} \omega(\gamma_k(e))(1 - \omega(\gamma_k(e))) < \infty.$$

We define $N^G(\omega)$ to be the set of projections $e \in R_0'$ so that

$$\sum_{k=1}^{\infty} \omega(\gamma_k(e)) < \infty.$$

Note $N^G(\omega) \subset M^G(\omega)$. The next few lemmas lead up to Theorem 4.7 in which we determine $M^G(\omega)$ for product states and its relation to $R^G(\omega)$.

LEMMA 4.3. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product of* $\mathfrak{A}$ *and* $e \in M^G(\omega)$. *Then* $e \in R^G(\omega)'$.

*Proof.* Suppose the hypothesis and notation of the lemma are valid. Suppose for each $k = 1, 2, \ldots$ $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for all $A \in \mathfrak{B}_0$. Suppose $U(t) = \exp(ite)$. We show $U(t) \in H^G(\omega)$ for all real $t$. We have

$$U(t) \in H^G(\omega) \Leftrightarrow \omega \underset{q}{\sim} \omega \circ \alpha_{U(t)} \Leftrightarrow \sum_{k=1}^{\infty} \|\Omega_k^{1/2} - U(t)^{-1}\Omega_k^{1/2}U(t)\|_{\mathrm{HS}}^2 < \infty.$$

We have

$$\|\Omega_k^{1/2} - U(t)^{-1}\Omega_k^{1/2}U(t)\|_{\mathrm{HS}}^2 = 4(1 - \cos(t))s_k$$

with

$$s_k = \mathrm{tr}(\Omega_k^{1/2}e\Omega_k^{1/2}(I - e)).$$

Since $s_k \leqslant \mathrm{tr}(e\Omega_k) = \omega_k(e)$ and $s_k \leqslant \mathrm{tr}((I-e)\Omega_k) = \omega_k(I-e)$ we have $s_k \leqslant \leqslant 2\omega_k(e)(1-\omega_k(e))$. Since $e \in M^G(\omega)$ we have

$$\sum_{k=1}^{\infty} \|\Omega_k^{1/2} - U(t)^{-1}\Omega_k^{1/2}U(t)\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} 4(1-\cos(t))s_k \leqslant$$

$$\leqslant 16 \sum_{k=1}^{\infty} \omega_k(e)(1-\omega_k(e)) < \infty.$$

Hence, $U(t) \in H^G(\omega)$. By Lemma 3.6 we have $H^G(\omega) \subset G \cap R^G(\omega)'$. Hence, $e \in R^G(\omega)'$. ▨

We remark that one can prove Lemma 4.3 holds for any factor state $\omega$ of $\mathfrak{A}$.

LEMMA 4.4. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *and* $\omega' = \bigotimes_{k=1}^{\infty} \omega_k'$ *and* $\omega \underset{q}{\sim} \omega'$. *Then* $M^G(\omega) = M^G(\omega')$ *and* $N^G(\omega) = N^G(\omega')$.

*Proof.* Suppose the hypothesis and notation of the lemma are valid. Suppose $e \in M^G(\omega)$ so $\sum_{k=1}^{\infty} \omega_k(e)(1-\omega_k(e)) < \infty$. For each $k = 1, 2, \ldots$ and $A \in \mathfrak{B}_0$ let $\rho_k(A) = \omega_k(e)^{-1}\omega_k(eAe)$ if $\omega_k(e) \geqslant 1/2$ and $\rho_k(A) = \omega_k(I-e)^{-1}\omega_k((I-e)A(I-e))$ if $\omega_k(e) < 1/2$. Let $\rho = \bigotimes_{k=1}^{\infty} \rho_k$. We will show $\omega \underset{q}{\sim} \rho$.

Let $x_k = \omega_k(e)^{-1/2}e$ if $\omega_k(e) \geqslant 1/2$ and $x_k = \omega_k(I-e)^{-1/2}(I-e)$ if $\omega_k(e) < 1/2$. Let $C_n = x_1x_2\ldots x_n$ and $\tilde{\rho}_n(A) = \omega(C_nAC_n)$ for $A \in \mathfrak{A}$. Note $\tilde{\rho}_n | \mathfrak{A}_n = \rho | \mathfrak{A}_n$. Now a routine estimate shows $\|\omega - \tilde{\rho}_n\| \leqslant 2(1-|\omega(C_n)|^2)^{1/2}$ (see [13], Lemma 2.4). We have $\|\omega - \rho\| = \lim_{n\to\infty} \|(\omega-\rho)| \mathfrak{A}_n\| \leqslant \lim_{n\to\infty} 2(1-|\omega(C_n)|^2)^{1/2}$. We have

$$|\omega(C_n)|^2 = s_1 s_2 \ldots s_n \quad \text{with } s_k = \omega_k(e) \text{ if } \omega_k(e) \geqslant 1/2$$
$$\text{and } s_k = \omega_k(I-e) \text{ if } \omega_k(e) < 1/2.$$

Since $1 - s_k \leqslant 2\omega_k(e)(1-\omega_k(e))$ we have $\sum_{k=1}^{\infty} 1 - s_k < \infty$. We have

$$s = \lim_{n\to\infty} |\omega(C_n)|^2 \geqslant s_1 s_2 \ldots s_{m-1}\left(1 - \sum_{k=m}^{\infty}(1-s_k)\right).$$

Since $s_k \geqslant 1/2$ for all $k$ and $\sum_{k=1}^{\infty} 1 - s_k < \infty$ we see from the above inequality that $s > 0$. Hence $\|\omega - \rho\| \leqslant 2(1-s)^{1/2} < 2$. Since $\omega$ and $\rho$ are factor states we have $\omega \underset{q}{\sim} \rho$.

Since $\omega \underset{q}{\sim} \omega'$ we have $\rho \underset{q}{\sim} \omega'$. Therefore, $\omega'$ and $\rho$ are asymptotically equal (see [12], Theorem 2.7) so there is an integer $n$ so that $\|(\omega' - \rho)| \mathfrak{A}_n^c\| < 1$, where $\mathfrak{A}_n^c = \bigotimes_{k=n+1}^{\infty} \mathfrak{B}_k$ is the relative commutant of $\mathfrak{A}_n$ in $\mathfrak{A}$. Let $e_k = \rho_k(e)e + \rho_k(I - e)(I - e)$ and $E_m = e_{n+1}e_{n+2} \ldots e_m$. We have $2E_m - I \in \mathfrak{A}_n^c$ and $\|2E_m - I\| = 1$. Hence,

$$|\omega'(2E_m - I) - \rho(2E_m - I)| = 2\left(1 - \prod_{k=n+1}^{m} \omega_k'(e_k)\right) < 1,$$

for all $m$. It follows that the infinite product $\prod_{k=n+1}^{\infty} \omega_k'(e_k)$ converges to a non-zero limit. Hence, we have $\sum_{k=1}^{\infty} 1 - \omega_k'(e_k) < \infty$. Since $e_k$ is either $e$ or $(I - e)$ we have $1 - \omega_k'(e_k) \geqslant \omega_k'(e)(1 - \omega_k'(e))$ for each $k = 1, 2, \ldots$ . Hence, $\sum_{k=1}^{\infty} \omega_k'(e)(1 - \omega_k'(e)) < \infty$. Hence, we have shown that $e \in M^G(\omega)$ implies that $e \in M^G(\omega')$. The same argument gives the reverse implication so we have $M^G(\omega) = M^G(\omega')$.

Now suppose $e \in N^G(\omega)$ so $\sum_{k=1}^{\infty} \omega_k(e) < \infty$. Then $e \in M^G(\omega) = M^G(\omega')$ so $\sum_{k=1}^{\infty} \omega_k'(e)(1 - \omega_k'(e)) < \infty$. Since $\omega \underset{q}{\sim} \omega'$ we have $\omega$ and $\omega'$ are asymptotically equal so $|\omega_k(e) - \omega_k'(e)| \to 0$ as $k \to \infty$. Since $\omega_k(e) \to 0$ as $k \to \infty$ we have $\omega_k'(e) \to 0$ as $k \to \infty$. Hence, $\sum_{k=1}^{\infty} \omega_k'(e) < \infty$. Hence, $N^G(\omega) \subset N^G(\omega')$. The same argument gives the reverse inclusion so $N^G(\omega) = N^G(\omega')$. ▨

LEMMA 4.5. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$. *Suppose* $e_1, e_2 \in M^G(\omega)$ *and* $e_1e_2 = e_2e_1$. *Then* $e_1e_2 \in M^G(\omega)$ *and* $e_1 + e_2 - e_1e_2 \in M^G(\omega)$.

*Proof.* Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ and $e_1$ and $e_2$ are commuting projections in $M^G(\omega)$. From the proof of the previous lemma there is a product state $\rho = \bigotimes_{k=1}^{\infty} \rho_k$ so that $\rho_k(e_1) = 0$ or $1$ for all $k = 1, 2, \ldots$ and $\rho \underset{q}{\sim} \omega$. Since by the previous lemma $M^G(\omega) = M^G(\rho)$ we have $\sum_{k=1}^{\infty} \rho_k(e_2)(1 - \rho_k(e_2)) < \infty$. Let $Q = \{k ; \rho_k(e_1) = 1\}$. Then

$$\sum_{k=1}^{\infty} \rho_k(e_1e_2)(1 - \rho_k(e_1e_2)) = \sum_{k \in Q} \rho_k(e_2)(1 - \rho_k(e_2)) \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} \rho_k(e_2)(1 - \rho_k(e_2)) < \infty.$$

Hence, $e_1e_2 \in M^G(\rho) = M^G(\omega)$.

To show $e_1 + e_2 - e_1e_2 \in M^G(\omega)$ we proceed as follows. Since $e_1, e_2 \in M^G(\omega)$ we have $I - e_1$, $I - e_2 \in M^G(\omega)$. Hence, $(I - e_1)(I - e_2) \in M^G(\omega)$. Hence, $e_1 + e_2 - e_1e_2 = I - (I - e_1)(I - e_2) \in M^G(\omega)$. ▨

LEMMA 4.6. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$. *Then* $N^G(\omega)$ *contains a unique maximal projection* $e_0$ *so that if* $e \in R_0'$ *is a projection then* $e \in N^G(\omega)$ *if and only if* $e \leqslant e_0$. *Furthermore,* $e_0 \in R^G(\omega) \cap R^G(\omega)'$.

*Proof.* Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$. Suppose $e_1, e_2 \in N^G(\omega)$. Let $e = e_1 \vee e_2$ (i.e., $e$ is the smallest projection in $\mathfrak{B}_0$ with $e \geqslant e_1$ and $e \geqslant e_2$). Note $e \in R_0'$. Since $e_1$ and $e_2$ are in a finite dimensional algebra there is a constant $\lambda$ so that $\lambda(e_1 + e_2) \geqslant e$. Then

$$\sum_{k=1}^{\infty} \omega_k(e) \leqslant \lambda \sum_{k=1}^{\infty} \omega_k(e_1) + \omega_k(e_2) < \infty.$$

Hence, $e \in N^G(\omega)$. Let $e_0$ be the smallest projection in $\mathfrak{B}_0$ so that $e_0 \geqslant e$ for all $e \in N^G(\omega)$. Since $\mathfrak{B}_0$ is finite dimensional $e_0 = e_1 \vee e_2 \vee \ldots \vee e_s$ with $e_i \in N^G(\omega)$ (i.e., $e_0$ is the sup of a finite number of projections in $N^G(\omega)$). Hence, $e_0 \in N^G(\omega)$. Clearly if $e \in R_0'$ is a projection then $e \in N^G(\omega)$ if and only if $e \leqslant e_0$.

Since $e_0 \in N^G(\omega) \subset M^G(\omega)$, it follows from Lemma 4.3 that $e_0 \in R^G(\omega)'$. We show $e_0 \in R^G(\omega)$. Suppose $U \in H^G(\omega)$. Then $\omega \underset{q}{\sim} \omega \circ \alpha_U$ and by Lemma 4.4 we have $N^G(\omega) = N^G(\omega \circ \alpha_U)$. Hence, $Ue_0U^{-1} \in N^G(\omega)$. Since $e_0$ is maximal in $N^G(\omega)$ we have $e_0 \geqslant Ue_0U^{-1}$. Since $e_0$ is finite dimensional we have $e_0 = Ue_0U^{-1}$. Hence, $e_0$ commutes with all $U \in H^G(\omega)$. By Theorem 3.8 $H^G(\omega) = G \cap R^G(\omega)'$ so $e_0 \in R^G(\omega)'' = R^G(\omega)$. Hence, $e_0 \in R^G(\omega) \cap R^G(\omega)'$. ▨

REMARK. We note that if $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ is a product state with $G_\gamma$-support projection $E$, then $E \in M^G(\omega)$, and, hence, $E \in R^G(\omega)'$.

THEOREM 4.7. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$. *Then there are orthogonal projections* $e_0, e_1, \ldots, e_n \in M^G(\omega) \cap R^G(\omega) \cap R^G(\omega)'$ *so that* $I = e_0 + e_1 + \ldots + e_n$,

$$\sum_{k=1}^{\infty} \omega_k(e_0) < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} \omega_k(e_i) = \infty \quad \textit{for } i = 1, \ldots, n$$

*and every projection* $e \in M^G(\omega)$ *is of the form* $e = f_0 + s_1e_1 + s_2e_2 + \ldots + s_ne_n$ *where* $s_i = 0$ *or* $1$ *for* $i = 1, \ldots, n$ *and* $f_0 \in R_0'$ *and* $f_0 \leqslant e_0$. *(Note* $e_0$ *may be zero.)*

*Proof*. Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$. Suppose $e \in M^G(\omega)$. Let $\{E_i ; i = 1, \dots s\}$ be the minimal central projections of $R^G(\omega) \cap R^G(\omega)'$. Let $e_i = E_i e$ for $i = 1, \dots, s$. Clearly $e = e_1 + \dots + e_s$. Suppose $0 \neq e_j \neq E_j$. We will show that $e_j \in N^G(\omega)$. Let $\{f_i ; i = 1, \dots, s_j\}$ be minimal orthonormal projections in $E_j R^G(\omega)' = R^G(\omega)' E_j$ (note that $E_j R^G(\omega)'$ is an $(s_j \times s_j)$-matrix algebra) chosen so that $e_j = f_1 + f_2 + \dots \dots + f_m$ with $0 < m < s_j$. Let $W$ be a unitary in $E_j R^G(\omega)'$ (consider $E_j$ the unit for $E_j R^G(\omega)'$) so that $W f_1 W^{-1} = f_{m+1}$, $W f_{m+1} W^{-1} = f_1$ and $W f_i W^{-1} = f_i$ for $i \neq 1$ or $i \neq m + 1$. Let $V = I - E_j + W$ (so $V$ is unitary in $\mathfrak{B}_0$). Since $V \in G \cap R^G(\omega)' = H^G(\omega)$ we have $\omega \underset{q}{\sim} \omega \circ \alpha_V$. Then by Lemma 4.4 we have $M^G(\omega) = M^G(\omega \circ \alpha_V)$. Since $e \in M^G(\omega)$ we have $VeV^{-1}$ and $V(I - e)V^{-1}$ are in $M^G(\omega)$. Since $e$, $V(I - e)V^{-1} \in M^G(\omega)$ are commuting projections, we have $eV(I - e)V^{-1} = f_1 \in M^G(\omega)$ by Lemma 4.5.

We will show that $f_1 \in N^G(\omega)$. To this end suppose $f_1 \notin N^G(\omega)$. Then $\sum_{k=1}^{\infty} \omega_k(f_1) = \infty$. Since $f_1 \in M^G(\omega)$ we have $\sum_{k=1}^{\infty} \omega_k(f_1)(1 - \omega_k(f_1)) < \infty$. Hence, there is a subsequence $k \to q(k)$ so that $\omega_{q(k)}(f_1) \to 1$ as $k \to \infty$. Since $V \in H^G(\omega)$ we have $\omega \underset{q}{\sim} \omega \circ \alpha_V$. Since $\omega \underset{q}{\sim} \omega \circ \alpha_V$ these states must be asymptotically equal (see [12], Theorem 2.7) so $\omega(\gamma_k(f_1)) - \omega(\alpha_V(\gamma_k(f_1))) \to 0$ as $k \to \infty$. But this is impossible because $\omega_{q(k)}(f_1) \to 1$ as $k \to \infty$ and $\omega_{q(k)}(V f_1 V^{-1}) = \omega_{q(k)}(f_{m+1}) \to 0$ as $k \to \infty$. Hence, $f_1 \in N^G(\omega)$.

Let $e_0$ be the maximal projection in $N^G(\omega)$. By the previous lemma $e_0 \in R^G(\omega) \cap R^G(\omega)'$. Since $f_1 \in N^G(\omega)$ we have $f_1 \leqslant e_0$. Since $E_j$ is the smallest central projection with $E_j \geqslant f_1$ we have $E_j \leqslant e_0$. Since $e_j \leqslant E_j$ we have $e_j \in N^G(\omega)$. Hence, we have shown that if $0 \neq e_j \neq E_j$ then $e_j \in N^G(\omega)$. Hence, each $e \in M^G(\omega)$ can be written in the form $e = f_0 + h$ with $f_0 \in R_0'$ and $f_0 \leqslant e_0$ and $h$ is a central projection of $R^G(\omega) \cap R^G(\omega)'$ which is orthogonal to $e_0$. Furthermore, since $f_0$, $e \in M^G(\omega)$ are commuting projections we have from Lemma 4.5 that $(I - f_0)e = h \in M^G(\omega)$.

In the decomposition $e = f_0 + h$ consider the $h$'s that occur. Suppose $h_1$ and $h_2$ are two such $h$'s. Since they lie in the center of $R^G(\omega)$ they must commute. Then by Lemma 4.5 $h_1 h_2 \in M^G(\omega)$ and $h_1 + h_2 - h_1 h_2 \in M^G(\omega)$. Hence, the set of such $h$'s form a complete Boolean lattice. Let $e_1, e_2, \dots, e_n$ be minimal projections in $M^G(\omega)$ with $e_0 e_i = 0$. Then each $h \in M^G(\omega)$ with $h e_0 = 0$ can be written in the form $h = s_1 e_1 + s_2 e_2 + \dots + s_n e_n$ with $s_i = 0$ or $1$. Hence, each $e \in M^G(\omega)$ can be written in the form $e = f_0 + h$ which is the form given in the statement of the theorem. ▧

LEMMA 4.8. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *and e is a central projection in* $R^G(\omega) \cap R^G(\omega)'$. *Suppose*

$$(\#) \qquad 0 < \sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) < \infty.$$

*Then* $\omega^G$ *is not a factor state.*

*Proof.* Assume the hypothesis and notation of the lemma are valid. Let $(\pi, \mathscr{H}, f_0)$ be a cyclic $*$-representation of $\mathfrak{A}$ induced by $\omega$. Let $\mathscr{H}^G$ be the closure of $\{\pi(\mathfrak{A}^G)f_0\}$ and let $(\pi^G, \mathscr{H}^G, f_0)$ be the restriction of $\pi$ to $\mathfrak{A}^G$ and $\mathscr{H}^G$ (i.e., $\pi^G(A)f = \pi(A)f$ for all $A \in \mathfrak{A}^G$ and $f \in \mathscr{H}^G$). Note $(\pi^G, \mathscr{H}^G, f_0)$ is a cyclic $*$-representation of $\mathfrak{A}^G$ induced by $\omega^G$.

Let $H = H^G(\omega)$. Let $\mathfrak{A}^H$ be the closure of $\{\pi(\mathfrak{A}^H)f_0\}$ and let $(\pi^H, \mathscr{H}^H, f_0)$ be the restriction of $\pi$ to $\mathfrak{A}^H$ and $\mathscr{H}^H$ (i.e., $\pi^H(A)f = \pi(A)f$ for all $A \in \mathfrak{A}^H$ and $f \in \mathscr{H}^H$). Note $(\pi^H, \mathscr{H}^H, f_0)$ is a cyclic $*$-representation of $\mathfrak{A}^H$ induced by $\omega^H$. From Lemma 3.7 and Theorem 3.8 we have $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)''$. Hence, $\mathscr{H}^G = \mathscr{H}^H$ and $\pi^G(\mathfrak{A}^G)'' = \pi^H(\mathfrak{A}^H)''$.

Since $\gamma_k(e) \in \mathfrak{A}^H$ we have $\pi^H(\gamma_k(e)) \in \pi(\mathfrak{A}^G)''$ for $k = 1, 2, \ldots$ . Let

$$J_n = \sum_{k=1}^{n} \gamma_k(e) - \omega_k(e)I \quad \text{and} \quad U_n(t) = \exp(itJ_n).$$

We will show that $\pi^H(U_n(t))$ converges strongly to a one-parameter unitary group in the center of $\pi^G(\mathfrak{A}^G)''$. Suppose $A \in \mathfrak{A}_r^G$. Then using the fact that $\omega$ is a product state we have for $n > m > r$,

$$\|\pi^H(U_n(t) - U_m(t))\pi^G(A)f_0\|^2 = 2\,\omega(A^*A) - 2\,\mathrm{Re}\,\omega(A^*U_m(t)^*U_n(t)A) =$$

$$= 2\,\omega(A^*A)(1 - \mathrm{Re}\,\omega(U_m(t)^*U_n(t))) =$$

$$= 2\,\omega(A^*A)(1 - \mathrm{Re}\,\omega(\exp(it(J_n - J_m)))) \leqslant \omega(A^*A)t^2\omega((J_n - J_m)^2) =$$

$$= \omega(A^*A)t^2 \sum_{k=m+1}^{n} \omega_k(e)(1 - \omega_k(e)).$$

Since the sum $(\#)$ is finite we have $\|\pi^H(U_n(t) - U_m(t))\,\pi^G(A)f_0\| \to 0$ as $n, m \to \infty$ and the convergence is uniform for $t$ in a compact set. Since the vectors $\pi^G(A)f_0$ with $A \in \mathfrak{A}_r^G$ for some $r = 1, 2, \ldots$ are dense in $\mathscr{H}^G$ and the $\pi^H(U_n(t))$ are uniformly bounded we have $\pi^H(U_n(t)) \to V(t)$ strongly as $n \to \infty$ where $V(t)$ is a strongly continuous unitary group. Since $\pi^H(U_n(t)) \in \pi^G(\mathfrak{A}^G)''$ for each $n$ we have $V(t) \in \pi^G(\mathfrak{A}^G)''$. For $A \in \mathfrak{A}_m^G$ we have $U_n(t)A = AU_n(t)$ for $n > m$. Hence, $V(t)\,\pi^G(A) =$

$= \pi^G(A)V(t)$ for $A \in \mathfrak{A}_m^G$. Since the $\mathfrak{A}_m^G$ are dense in $\mathfrak{A}^G$ we have $V(t) \in \pi^G(\mathfrak{A}^G)'$. Hence, $V(t)$ is in the center of $\pi^G(\mathfrak{A}^G)''$. A straightforward computation using the fact that the sum (#) is neither zero nor infinite shows that $|(f_0, V(t)f_0)| = \lim_{n\to\infty} |\omega(U_n(t))| < 1$ for some $t \neq 0$. Hence, $V(t)$ is not a multiple of the identity for some $t \neq 0$. Hence, $\pi^G(\mathfrak{A}^G)''$ is not a factor and, thus, $\omega^G$ is not a factor state. ▨

THEOREM 4.9. *Suppose $\{Q_1, Q_2, \ldots, Q_s\}$ is a partition of the positive integers into s disjoint subsets. Suppose $\{e_1, e_2, \ldots, e_s\}$ are orthogonal central projections in $R_0 \cap R_0'$. Suppose $\mathfrak{A}(Q_i) = \bigotimes_{k\in Q_i} \mathfrak{B}_k$ and $\mathfrak{A}^G(Q_i)$ are the $\alpha_U$ invariant elements of $\mathfrak{A}(Q_i)$ with $U \in G$. Suppose $\omega_i$ are states of $\mathfrak{A}(Q_i)$ so that $\omega_i(\gamma_k(e_i)) = 1$ for $k \in Q_i$. Let $\omega = \bigotimes_{i=1}^{s} \omega_i$ on $\mathfrak{A} = \bigotimes_{i=1}^{s} \mathfrak{A}(Q_i)$. Then $\omega^G$ is a factor state if and only if $\omega_i^G$ is a factor state of $\mathfrak{A}^G(Q_i)$ for each $i = 1, \ldots, s$.*

*Proof.* Suppose the hypothesis and notation of the theorem are valid. Let $(\pi, \mathscr{H}, f_0)$ be a cyclic $*$-representation of $\mathfrak{A}^G$ induced by $\omega^G$. Let $i \to r(i)$ be the function defined on the positive integers by the requirement that $r(k) = i$ if and only if $k \in Q_i$. Let $E_0 = \lim_{n\to\infty} \pi(\gamma_1(e_{r(1)})\gamma_2(e_{r(2)}) \ldots \gamma_n(e_{r(n)}))$. Clearly $E_0$ is the limit of a decreasing sequence of projections in $\pi(\mathfrak{A}^G)''$ so $E_0$ is a projection in $\pi(\mathfrak{A}^G)''$. We have $E_0 \neq 0$ since $E_0 f_0 = f_0$.

Suppose $\sigma$ is a finite permutation. Then we have $\pi(U_\sigma)E_0\pi(U_\sigma^{-1}) = E_\sigma$ is a projection and $E_\sigma = E_0$ if $\sigma(Q_i) = Q_i$ for each $i = 1, \ldots, s$ and $E_0 E_\sigma = 0$ if $\sigma(Q_i) \neq Q_i$ for some $i = 1, \ldots, s$. Let $\Theta$ be the set of equivalence classes $\theta$ of finite permutations where $\sigma_1 \sim \sigma_2$ if and only if $\sigma_1(Q_i) = \sigma_2(Q_i)$ for $i = 1, \ldots, s$. Then for each $\theta \in \Theta$ we define the projection $E_\theta = E_\sigma$ for $\sigma$ any permutation in the equivalence class $\theta$. One easily sees that the $E_\theta$ form an orthonormal set of projections and $I = \sum_{\theta\in\Theta} E_\theta$.

One can show that $E_0 \in \pi(\mathfrak{A}^G(Q_i))'$ for $i = 1, \ldots, s$. Now $\mathfrak{A}_n^G$ is the linear span of elements of the form $A_0 = U_\sigma\gamma_1(A_1)\,\gamma_2(A_2) \ldots \gamma_n(A_n)$ with $\sigma(i) = i$ for $i > n$ and $A_i \in R_0$ for $i = 1, \ldots, n$. For such an element one finds $R_0 E_0 \pi(A_0)E_0 = 0$ unless $\sigma(Q_i) = Q_i$ for $i = 1, \ldots, s$ or equivalently unless $\sigma = \sigma_1\sigma_2 \ldots \sigma_s$ where $\sigma_i(k) = k$ if $k > n$ or $k \notin Q_i$. Then if $A_0$ is an element so that $E_0\pi(A_0)E_0 \neq 0$, by regrouping the terms in $A_0$ one can write $A_0$ in the form $A_0 = U_{\sigma_1}D_1U_{\sigma_2}D_2 \ldots U_{\sigma_s}D_s = B_1B_2 \ldots \ldots B_s$ where the $D_i$ are products of the $\gamma_k(A_k)$ with $k \in Q_i$ and hence, the $B_i$ are elements of $\mathfrak{A}^G(Q_i)$. Hence, it follows that if $A \in \mathfrak{A}_n^G$ then $E_0\pi(A)E_0$ is a linear combination of terms of the form $E_0\pi(B_1B_2 \ldots B_s)E_0$ with $B_i \in \mathfrak{A}^G(Q_i)$. Hence we have

$$\text{strong closure of } E_0\pi(\mathfrak{A}^G)E_0 =$$

$$= \text{strong closure of } E_0\pi(\mathfrak{A}^G(Q_1) \otimes \mathfrak{A}^G(Q_2) \otimes \ldots \otimes \mathfrak{A}^G(Q_s))E_0.$$

Now suppose $\omega_i^G$ is a factor state for each $i = 1, \ldots, s$. Suppose $C \in \pi(\mathfrak{A}^G)'' \cap \pi(\mathfrak{A}^G)'$. Since $E_0 \in \pi(\mathfrak{A}^G)''$, $CE_0 = E_0C$ and $E_0CE_0$ is in the center of $E_0\pi(\mathfrak{A}^G)E_0$ (acting on $E_0\mathscr{H}$). But we have the center of $E_0\pi(\mathfrak{A}^G(Q_1) \otimes \mathfrak{A}^G(Q_2) \otimes \ldots \otimes \mathfrak{A}^G(Q_s))''E_0$ is the center of $E_0\pi(\mathfrak{A}^G)''E_0$. Since $\omega = \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_s$ so $\omega(A_1A_2 \ldots A_s) = \omega_1(A_1)\omega_2(A_2) \ldots \omega_s(A_s)$ for $A_i \in \mathfrak{A}^G(Q_i)$ and since $\omega_i^G$ is a factor state we have $E_0\pi(\mathfrak{A}^G(Q_1) \otimes \mathfrak{A}^G(Q_2) \otimes \ldots \otimes \mathfrak{A}^G(Q_s))''E_0$ is the tensor product of factors $\pi_i(\mathfrak{A}^G(Q_i))''$ where $\pi_i$ is a $*$-representation of $\mathfrak{A}^G(Q_i)$ induced by $\omega_i^G$. Since the tensor product of factors is a factor we have $E_0CE_0 = \lambda E_0$. Hence, $Cf_0 = CE_0f_0 = E_0CE_0f_0 = \lambda f_0$. Hence, $C\pi(A)f_0 = \pi(A)Cf_0 = \lambda\pi(A)f_0$ for all $A \in \mathfrak{A}^G$. Hence $C = \lambda I$. Hence, the center of $\pi(\mathfrak{A}^G)''$ is trivial so $\omega^G$ is a factor state.

Next suppose one of the states $\omega_i^G$ is not a factor state. Then $E_0\pi(\mathfrak{A}^G(Q_1) \otimes \mathfrak{A}^G(Q_2) \otimes \ldots \otimes \mathfrak{A}^G(Q_s))''E_0$ is not a factor and there is a $C_0 \neq \lambda E_0$ in its center. Let

$$C = \sum_{\theta \in \Theta} \pi(U_{\sigma(\theta)})C_0\pi(U_{\sigma(\theta)}^{-1})$$

where $\sigma(\theta)$ is a permutation in the equivalence class $\theta$. A straightforward computation shows that $\pi(\gamma_k(A))C = C\pi(\gamma_k(A))$ for $A \in R_0$ and $k = 1, 2, \ldots$ and $\pi(U_\sigma)C = C\pi(U_\sigma)$ for any finite permutation $\sigma$. Since the $U_\sigma$ and the $\gamma_k(A)$ with $A \in R_0$ generate $\mathfrak{A}^G$ we have $C \in \pi(\mathfrak{A}^G)'$. Since $C \in \pi(\mathfrak{A}^G)''$ and $E_0CE_0 \neq \lambda E_0$ we have $\pi(\mathfrak{A}^G)''$ is not a factor. ▨

LEMMA 4.10. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *and* $\omega = \omega \circ \alpha_U$ *for all* $U \in G$. *Suppose* $e$ *is a minimal central projection in* $R_0 \cap R_0'$ *and* $\omega_k(e) = 1$ *for all* $k = 1, 2, \ldots$. *Then* $\omega^G$ *is a factor state.*

*Proof.* Suppose $\omega$ satisfies the hypothesis and notation of the lemma. Let $(\pi, \mathscr{H}, f_0)$ be a cyclic $*$-representation of $\mathfrak{A}^G$ induced by $\omega$. Let $\{e_{ij} ; i, j = 1, \ldots, r\}$ be matrix units for $M_0 = eR_0 = R_0e$ and $\{e'_{ij} ; i, j = 1, \ldots, s\}$ be matrix units for $N_0 = eR_0' = R_0'e$. For $\sigma$ a cyclic permutation of $(i_1, i_2, \ldots, i_n)$ with $\sigma(i_k) = i_{k+1}$ for $k = 1, \ldots, n-1$ and $\sigma(i_n) = i_1$ we define

$$V_\sigma = \sum_{j_1, \ldots, j_n = 1}^{r, \ldots, r} \gamma_{i_1}(e_{j_1j_2})\gamma_{i_2}(e_{j_2j_3}) \ldots \gamma_{i_n}(e_{j_nj_1})$$

$$W_\sigma = \sum_{j_1, \ldots, j_n = 1}^{s, \ldots, s} \gamma_{i_1}(e'_{j_1j_2})\gamma_{i_2}(e'_{j_2j_3}) \ldots \gamma_{i_n}(e'_{j_nj_1}).$$

If $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ is a product of cycles we define $V_\sigma = V_{\sigma_1}V_{\sigma_2} \ldots V_{\sigma_m}$ and $W_\sigma = W_{\sigma_1}W_{\sigma_2} \ldots W_{\sigma_m}$.

The combined matrix units $\{e_{ij}e'_{kl}\,;\, i,j = 1,\ldots,r$ and $k,l = 1,\ldots,s\}$ form a complete set of matrix units for the matrix algebra $e\mathfrak{B}_0 e$. Then one can choose a set of matrix units $\{E_{ij}\}$ for $\mathfrak{B}_0$ so that $E_{pq} = e_{i(p)j(q)}e'_{k(p)l(q)}$ for $p, q = 1, 2,\ldots, rs$. If one constructs the permutation operators $U_\sigma$ in terms of the $\gamma_k(E_{ij})$ (see Section 2) one finds

$$V_p W_p = U_p P(Q) \quad \text{with } P(Q) = \prod_{i\in Q} \gamma_i(e)$$

where $Q$ is the set of integers where $\sigma(i) \neq i$. Note that for the representation $\pi$ one can show $\pi(\gamma_k(e)) = I$ for all $k = 1, 2,\ldots$ so $\pi(P(Q)) = I$ for all finite sets $Q$. Hence, $\pi(V_\sigma)\pi(W_\sigma) = \pi(U_\sigma)$ and the $\pi(U_\sigma)$ and $\pi(W_\sigma)$ are unitary.

We show $\pi(\mathfrak{A}^G)$ is generated by the $\pi(\gamma_k(e_{ij}))$ and the $\pi(W_\sigma)$. First we note $\pi(V_\sigma) \in \pi(\mathfrak{A}^G)$ and $\pi(U_\sigma) \in \pi(\mathfrak{A}^G)$ so $\pi(W_\sigma) = \pi(V_{\sigma^{-1}})\pi(U_\sigma) \in \pi(\mathfrak{A}^G)$. Next we note that since $\pi(V_\sigma)$ lies in the algebra generated by the $\pi(\gamma_k(e_{ij}))$ we have $\pi(U_\sigma) = \pi(V_\sigma)\pi(W_\sigma)$ lies in the algebra generated by the $\pi(\gamma_k(e_{ij}))$ and the $\pi(W_\sigma)$. Since $\pi(\mathfrak{A}^G)$ is generated by the $\pi(\gamma_k(e_{ij}))$ and the $\pi(U_\sigma)$ we have $\pi(\mathfrak{A}^G)$ is generated by the $\pi(\gamma_k(e_{ij}))$ and the $\pi(W_\sigma)$. Let $M$ be the algebra generated by the $\pi(\gamma_k(e_{ij}))$ for $i,j = 1,\ldots,r$ and $k = 1, 2,\ldots$ and let $N$ be the algebra generated by the $\pi(W_\sigma)$ with $\sigma \in S_\infty$. We have that $M$ and $N$ commute and together they generate $\pi(\mathfrak{A}^G)$ as a $C^*$-algebra.

Next we show that the state $\omega_0(A) = (f_0, Af_0)$ (so $\omega(A) = \omega_0(\pi(A))$ for $A \in \mathfrak{A}^G$) factorizes on $M$ and $N$ (i.e., $\omega_0(AB) = \omega_0(A)\omega_0(B)$ for $A \in M$ and $B \in N$). Since $\omega = \omega \circ \alpha_U$ we have $\omega_k(A) = \omega_k(UAU^{-1})$ for all $A \in \mathfrak{B}_0$ and all unitary $U \in R_0'$. Hence, $\omega_k(e_{ij}e'_{kl}) = \omega_k(e_{ij})s^{-1}\delta_{kl}$. Hence, for $A_i \in M_0$ and $B_i \in N_0$ and $i = 1,\ldots,n$ we have $\omega(\gamma_1(A_1)\,\gamma_2(A_2)\ldots\gamma_n(A_n)\,\gamma_1(B_1)\,\gamma_2(B_2)\ldots\gamma_n(B_n)) = \omega_1(A_1)\,\omega_2(A_2)\ldots\omega_n(A_n)s^{-n}\operatorname{tr}(B_1)\operatorname{tr}(B_2)\ldots\operatorname{tr}(B_n)$. Since the $W_\sigma$ can be written in terms of the $\gamma_k(e'_{ij})$ we have for $A, B \in \mathfrak{A}_n$ and $\pi(A) \in M$ and $\pi(B) \in N$ that $\omega(AB) = \omega(A)\omega(B)$. Since such $\pi(A)$ and $\pi(B)$ are dense in $M$ and $N$ (respectively) we have $\omega_0(AB) = \omega_0(A)\omega_0(B)$ for $A \in M$ and $B \in N$. Furthermore, $\omega_0$ restricted to $M$ is just a product state (a product of the $\omega_k|R_0$) and $\omega_0$ restricted to $N$ is just the trace state on the tensor product of $(s \times s)$-matrix algebras restricted to the permutation algebra. Since $\omega_0|M$ is a product state $\omega_0|M$ is a factor state. It is known that the restriction of the trace to the permutation algebra is a factor state (see [14]) so $\omega_0|N$ is a factor state. Hence, $\omega_0$ on $\pi(\mathfrak{A}^G)$ is the tensor product of two factor states, so it must be a factor state. ▨

LEMMA 4.11. *Suppose* $\omega = \bigotimes_{k=1}^{n} \omega_k$ *is a state of* $\mathfrak{A}_n$. *Suppose e is a minimal central projection in* $R_0 \cap R_0'$ *and* $\omega_k(e) = 1$ *for* $k = 1,\ldots,n$. *Then* $\omega^G$ *(the restriction of* $\omega$ *to* $\mathfrak{A}_n^G$*) is a factor state if and only if* $n = 1$ *or there is a projection* $e_1 \in R_0'$ *which is minimal in* $R_0'$ *so that* $\omega_k(e_1) = 1$ *for* $k = 1,\ldots,n$.

*Proof.* Suppose the hypothesis and notation of the lemma are valid. When $n = 1$ the lemma is trivial so assume $n > 1$. Let $\{e_{ij}\,;\, i,j = 1,\ldots,r\}$ be matrix units for $M_0 = eR_0 = R_0e$ and $\{e'_{ij}\,;\, i,j = 1,\ldots,s\}$ be matrix units for $N_0 = eR'_0 = R'_0e$. Let $V_\sigma$ and $W_\sigma$ be as defined in the previous lemma for permutation $\sigma \in S_n$ where $S_n$ is the group of permutations of $(1, 2, \ldots, n)$. Let $E = (1/n!) \sum_{\sigma \in S_n} W_\sigma \gamma_1(e)\gamma_2(e) \ldots$ $\ldots \gamma_n(e)$. As we saw in the previous lemma we have $\pi(W_\sigma) \in \pi(\mathfrak{A}^G_n)$ where $(\pi, \mathscr{H}, f_0)$ is a cyclic $*$-representation of $\mathfrak{A}^G_n$ induced by $\omega^G$. Hence, $\pi(E) \in \pi(\mathfrak{A}^G_n)$. Since $W_\sigma$ is defined in terms of the $\gamma_k(e'_{ij})$ it follows that $E$ commutes with the $\gamma_k(A)$ with $A \in R_0$. From the form of $E$ it follows that $E$ commutes with the permutation elements $U_\sigma$. Hence $\pi(E)$ is the in center of $\pi(\mathfrak{A}^G_n)$. A straightforward computation shows that $\pi(E) = \pi(E)^2$. Then if $\omega^G$ is a factor state we must have $\omega(E) = 0$ or $1$. Let $\Omega'_k(i,j) = \omega_k(e'_{ji})$ for $i,j = 1,\ldots,s$ and $k = 1,\ldots,n$. The number $\omega(E)$ is directly computable from the matrices $\Omega'_k$. One can show using the fact that $\omega$ is a product state that $\omega(E) \geqslant 1/n!$. Unfortunately, we do not know a quick proof of this fact, but since it only involves somewhat long but routine computations in a finite dimensional algebra we omit the proof. Then if $\omega^G$ is a factor state we must have $\omega(E) = 1$. Then $\omega(W_\sigma) = 1$ for all $\sigma \in S_n$. In particular if $\sigma$ is a transposition interchanging $i$ and $j$ one computes $\omega(W_\sigma) = \mathrm{tr}(\Omega'_i\Omega'_j)$. Since the $\Omega'_k$ are positive trace one matrices it follows that $\mathrm{tr}(\Omega'_i\Omega'_j) = 1$ if and only if $\Omega'_i = \Omega'_j$ and $\Omega'_i$ is of rank one. Hence, if $\omega^G$ is a factor state we have $\Omega'_k = \Omega_0$ for $k = 1,\ldots,n$ where $\Omega_0$ is a rank one projection. Let $e_1 = \sum_{i,j=1}^{s} \Omega_0(i,j)e'_{ij}$. Then $e_1$ is a minimal projection in $R'_0$ and $\omega_k(e_1) = 1$ for $k = 1,\ldots,n$.

Conversely, if there is a rank one projection $e_1 \in R'_0$ so that $\omega_k(e_1) = 1$ for $k = 1,\ldots,n$ then $\Omega'_k = \Omega_0$ for $k = 1,\ldots,n$ and $\Omega_0$ is a rank one projection corresponding to $e_1$. Then $\omega(W_\sigma) = 1$ for $\sigma \in S_n$ and $\omega(E) = 1$. We have $EW_\sigma E = E$ for all $\sigma \in S_n$ and from the proof of the previous lemma $\pi(E\mathfrak{A}^G_nE)$ is the tensor product of an $(r^n \times r^n)$-matrix algebra and the algebra generated by the $\pi(EW_\sigma E) = I$. Hence, $\pi(\mathfrak{A}^G)$ is an $(r^n \times r^n)$-matrix algebra. Hence, $\omega^G$ is a factor state. ▨

LEMMA 4.12. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *and* $\omega = \omega \circ \alpha_U$ *for all* $U \in G$. *Suppose* $e_0$ *is the maximal projection in* $N^G(\omega)$ *and* $\omega_k(e_0) = 0$ *for all* $k = 1, 2, \ldots$ . *Suppose* $\sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) = \infty$ *for all non zero projections* $e \in R'_0$ *with* $e \leqslant I - e_0$ *and* $e \neq I - e_0$. *Then* $\omega^G$ *is a factor state.*

*Proof.* Suppose the hypothesis and notation of the lemma are valid. Let $(\pi, \mathscr{H}, f_0)$ be a cyclic $*$-representation of $\mathfrak{A}^G$ induced by $\omega^G$. Suppose $0 \leqslant C \leqslant I$ and $C \in \pi(\mathfrak{A}^G)'' \cap \pi(\mathfrak{A}^G)'$. Let $\rho(A) = (f_0, \pi(A)Cf_0)$. We will show that $\rho = \lambda\omega$.

Let $\{e_i \,;\, i = 1, \ldots, q\}$ be the minimal central projections of $R_0 \cap R_0'$. Let $\mathfrak{D}_n$ be the $C^*$-algebra generated by the $\gamma_k(e_i)$ for $i = 1, \ldots, q$ and $k = 1, \ldots, n$. Let $S_n$ be the permutations $\sigma \in S_\infty$ so that $\sigma(i) = i$ for $i > n$. Let $\mathfrak{D}_n^s = \{A \in \mathfrak{D}_n \,;\, U_\sigma A U_\sigma^{-1} = A$ for all $\sigma \in S_n\}$. Note that $\mathfrak{D}_n^s$ is contained in the center of $\mathfrak{A}_n^G$. We will show that for every integer $m > 0$ and $\varepsilon > 0$ there are an integer $n > m$ and a $C_n \in \mathfrak{D}_n^s$ with $0 \leqslant C_n \leqslant I$ so that $|\rho(A) - \omega(AC_n)| \leqslant \varepsilon\|A\|$ for all $A \in \mathfrak{A}_m^G$. Then using a probability result of Aldous and Pitman we will show that for such $C_n$ the only limit points are multiples of the identity. Since any central $C$ is a strong limit of such $C_n$ one concludes $C = \lambda I$.

We note the following general fact. Suppose $A_i, B_i \in \mathfrak{A}^G$ for $i = 1, \ldots, m$. Then

$$(1!) \qquad \begin{matrix} \sum_{i=1}^{m} \omega(A_i A B_i) = 0 \\ \text{for all } A \in \mathfrak{A}^G \end{matrix} \quad \Rightarrow \quad \begin{matrix} \sum_{i=1}^{m} \rho(A_i A B_i) = 0 \\ \text{for all } A \in \mathfrak{A}^G. \end{matrix}$$

This may be seen as follows. Suppose the left hand side of (1!) holds. Then $\sum_{i=1}^{m} (\pi(A_i^*)f_0, \pi(A)\pi(B_i)f_0) = 0$ for all $A \in \mathfrak{A}^G$. Clearly, this equation extends to the weak closure of $\pi(\mathfrak{A}^G)$ so $\sum_{i=1}^{m} (\pi(A_i^*)f_0, A\pi(B_i)f_0) = 0$ for all $A \in \pi(\mathfrak{A}^G)''$. Since $C \in \pi(\mathfrak{A}^G)'' \cap \pi(\mathfrak{A}^G)'$ we have for $A \in \mathfrak{A}^G$ that $\pi(A)C \in \pi(\mathfrak{A}^G)''$. Hence, we have

$$\sum_{i=1}^{m} \rho(A_i A B_i) = \sum_{i=1}^{m} (f_0, \pi(A_i A B_i)Cf_0) =$$

$$= \sum_{i=1}^{m} (\pi(A_i^*)f_0, \pi(A)C\pi(B_i)f_0) = 0$$

for all $A \in \mathfrak{A}^G$. This establishes the implication (1!).

Let $X_n$ be the cartesian product of $(1, 2, \ldots, q)$ with itself $n$ times. Each $x = (x(1), x(2), \ldots, x(n)) \in X_n$ corresponds to a minimal projection in $\mathfrak{D}_n$ given by

$$e_x = \gamma_1(e_{x(1)})\, \gamma_2(e_{x(2)}) \ldots \gamma_n(e_{x(n)}).$$

We define $x, y \in X_n$ to be equivalent (denoted $x \sim y$) if and only if there is a permutation $\sigma \in S_n$ so that $x(i) = y(\sigma(i))$ for $i = 1, \ldots, n$. Clearly, $x \sim y$ if and only if $t(i, x) = t(i, y)$ for $i = 1, \ldots, q$ where $t(i, x)$ is the number of times $x(k)$ assumes the value $i$ for $k = 1, \ldots, n$. If $\theta$ is an equivalence class of $X_n$ we define

$$e_\theta = \sum_{x \in \theta} e_x .$$

One sees that the $e_\theta$ are the minimal projections in $\mathfrak{D}_n^s$. Let $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Since $\omega = \omega \circ \alpha_U$ we have $U\Omega_k U^{-1} = \Omega_k$ for all $U \in G$.

Hence, $\Omega_k \in R_0$. Since $e_i \in R_0 \cap R_0'$ the $\Omega_k$ and the $e_i$ commute. A small computation shows that the $\gamma_k(e_i)$ are in the centralizer of $\omega$ (i.e., $\omega(A\gamma_k(e_i)) = \omega(\gamma_k(e_i)A)$ for all $A \in \mathfrak{A}$, $i = 1, \ldots, q$ and $k = 1, 2, \ldots$). One first shows this for $A \in \mathfrak{A}_n$ and since the $\mathfrak{A}_n$ are dense in $A$ one has this for all $A \in \mathfrak{A}$. Then products of the $\gamma_k(e_i)$ are in the centralizer of $\omega$ so $e_x$ is in the centralizer of $\omega$ for all $x \in X_n$.

We define a probability measure $\mu$ on $X_n$ by defining $\mu_x = \omega(e_x)$ where $\mu_x$ is the measure of the point $x$. Since $\omega$ is a product state we have $\mu_x = \omega_1(e_{x(1)})\omega_2(e_{x(2)}) \ldots \ldots \omega_n(e_{x(n)})$ so $\mu$ is a product measure. If $\mu_x \neq 0$ we define $\omega_x(A) = \mu_x^{-1}\omega(Ae_x)$. Note that if $\mu_x \neq 0$, $\omega_x$ is a state since $\omega(Ae_x) = \omega(e_xAe_x)$. If $\mu_x = 0$ we define $\omega_x = 0$. Then we have $\omega = \sum_{x \in X_n} \mu_x \omega_x$.

Let $\{e_{ij}^{(k)}\,;\, i, j = 1, \ldots, r_k\}$ be a set of matrix units for $M_k = e_k R_0 = R_0 e_k$ and $\{e_{ij}'^{(k)};\, i, j = 1, \ldots, s_k\}$ be a set of matrix units for $N_k = e_k R_0' = R_0' e_k$, for $k = 1, \ldots, q$. As we saw in the proof of Lemma 4.10 it follows from the $\alpha_U$ invariance of $\omega$ for $U \in G$ that $\omega_k(e_{ij}^{(m)} e_{kl}'^{(m)}) = \omega_k(e_{ij}^{(m)})\, s_m^{-1}\delta_{kl}$ or equivalently for $A \in M_m$ and $B \in N_m$ we have $\omega_k(AB) = \omega_k(A)s_m^{-1}\mathrm{tr}(B)$.

Let $M_x = \bigotimes_{k=1}^{n} \gamma_k(M_{x(k)})$ and $N_x = \bigotimes_{k=1}^{n} \gamma_k(N_{x(k)})$. Note $e_x$ is the unit of $M_x$ and $N_x$. From the above remarks one sees that if $\mu_x \neq 0$ then $\omega_x$ is a product state of $M_x \otimes N_x$ so that $\omega_x(AB) = \omega_x(A)\omega_x(B)$ for $A \in M_x$ and $B \in N_x$. Furthermore $\omega_x | N_x$ is the normalized trace on $N_x$.

Let $\{E_{ij}^{(x)}\,;\, i, j = 1, \ldots, m(x)\}$ be a set of matrix units for $M_x$ chosen so that if $\mu_x \neq 0$ the $E_{ij}^{(x)}$ diagonalize $\omega_x$ in the following sense. We require $\omega_x(E_{ij}^{(x)}) = \lambda_i^{(x)}\delta_{ij}$ with $\lambda_1^{(x)} \geqslant \lambda_2^{(x)} \geqslant \ldots \geqslant \lambda_{m(x)}^{(x)}$. It follows that the $E_{ii}^{(x)}$ are in the centralizer of $\omega_x$ and $\omega$. The fact that the matrix units can be so chosen follows from the fact that a hermitian matrix can be diagonalized. One checks that for $\mu_x \neq 0$ we have

$$\omega_x(A) = \sum_{i=1}^{m(x)} (\lambda_i^{(x)}/\lambda_1^{(x)})\, \omega_x(E_{1i}^{(x)} A E_{i1}^{(x)})$$

for all $A \in \mathfrak{A}$.

Now suppose $\mu_x \neq 0$ and $\mu_y \neq 0$ and $x \sim y$ so $x(k) = y(\sigma(k))$ for $\sigma \in S_n$. Then $M_y = U_\sigma M_x U_\sigma^{-1}$. Since $U_\sigma^{-1}E_{11}^{(y)}U_\sigma$ and $E_{11}^{(x)}$ are minimal projections in $M_x$ there is a partial isometry $V \in M_x$ so that $V^*V = E_{11}^{(x)}$ and $VV^* = U_\sigma^{-1}E_{11}^{(y)}U_\sigma$. Direct computation then shows that

$$\omega_y(A) = \sum_{i=1}^{m(y)} (\lambda_i^{(y)}/\lambda_1^{(x)})\, \omega_x(V^* U_\sigma^{-1} E_{1i}^{(y)} A E_{i1}^{(y)} U_\sigma V)$$

for all $A \in \mathfrak{A}$.

Suppose $\theta$ is an equivalence class of points in $X_n$. Note that if $x, y \in \theta$ (so $x \sim y$) then $m(x) = m(y)$ since $M_x$ and $M_y$ are unitarily equivalent. Let $m(\theta) = m(x)$ for any $x \in \theta$. We define elements $B(\theta, x, i)$ for $x \in \theta$ and $i = 1, \ldots, m(\theta)$

as follows. If $\mu_x = 0$ for all $x \in \theta$ let $B(\theta, x, i) = 0$ for all $x \in \theta$ and $i = 1, \ldots, m(\theta)$. If $\mu_x \neq 0$ for some $x \in \theta$ pick an element $x(\theta) \in \theta$ so that $\mu_{x(\theta)}$ equals the maximum value of $\mu_x$ for $x \in \theta$. For $i = 1, \ldots, m(\theta)$ and $y \in \theta$ with $\mu_y \neq 0$ let

$$B(\theta, y, i) = (\lambda_i^{(y)}/\lambda_1^{(x)})^{1/2} E_{i1}^{(y)} U_\sigma V$$

where $V$ is a partial isometry in $M_{x(\theta)}'$ so that $V^*V = E_{11}^{(x)}$ and $VV^* = U_\sigma^{-1} E_{11}^{(y)} U_\sigma$ and $\sigma$ is a permutation so that $x(i) = y(\sigma(i))$ for $i = 1, \ldots, n$. If $\mu_y = 0$ let $B(\theta, y, i) = 0$. Then, provided $u_{x(\theta)} \neq 0$ we have

$$\omega_y(A) = \sum_{i=1}^{m(\theta)} \mu_{x(\theta)}^{-1} \omega(B(\theta, y, i)^* A B(\theta, y, i))$$

for all $A \in \mathfrak{A}$ and $y \in \theta$.

If $\Theta$ is the set of equivalence classes $\theta$ so that $\mu_{x(\theta)} \neq 0$ we have

$$\omega(A) = \sum_{\theta \in \Theta} \sum_{x \in \theta} \mu_x \sum_{i=1}^{m(\theta)} \mu_{x(\theta)}^{-1} \omega(B(\theta, x, i)^* A B(\theta, x, i)) \tag{2!}$$

for all $A \in \mathfrak{A}$. Since $B(\theta, x, i) \in \mathfrak{A}_n^G$ we have from (1!) that

$$\rho(A) = \sum_{\theta \in \Theta} \sum_{x \in \theta} \mu_x \sum_{i=1}^{m(\theta)} \mu_{x(\theta)}^{-1} \rho(B(\theta, x, i)^* A B(\theta, x, i)) \tag{3!}$$

for all $A \in \mathfrak{A}^G$.

For each $\theta \in \Theta$ let $F_\theta = E_{11}^{(x(\theta))}$. As mentioned earlier $F_\theta$ is in the centralizer of $\omega$ so $\omega(AF_\theta) = \omega(F_\theta A)$ for all $A \in \mathfrak{A}$. Since $F_\theta \in \mathfrak{A}_n^G$ we have by (1!) that $\rho(AF_\theta) = \rho(F_\theta A)$ for all $A \in \mathfrak{A}^G$. For $\theta \in \Theta$ we have $\mu_{x(\theta)} \neq 0$ so $\omega(F_\theta) \neq 0$. For $\theta \in \Theta$ let $\omega_\theta(A) = \omega(F_\theta)^{-1}\omega(AF_\theta)$ and $\rho_\theta(A) = \omega(F_\theta)^{-1}\rho(AF_\theta)$ for all $A \in \mathfrak{A}$. Let $x = x(\theta)$. As we saw earlier $\omega_x|M_x \otimes N_x = \omega_x|M_x \otimes \omega_x|N_x$ and $\omega_x|N_x$ is the normalized trace on $N_x$. Since $F_\theta$ is a minimal projection in $M_x$ we have $\omega_\theta$ is the normalized trace on $F_\theta \mathfrak{A}_n F_\theta$, so $\omega_\theta(AB) = \omega_\theta(BA)$ for all $A, B \in F_\theta \mathfrak{A}_n F_\theta$. Since $\omega_\theta(A) = \omega_\theta(F_\theta A F_\theta)$ for all $A \in \mathfrak{A}$ it follows that $\omega_\theta(AB) = \omega_\theta(BA)$ for all $A \in F_\theta \mathfrak{A}_n F_\theta$ and $B \in \mathfrak{A}_n$. Since $\omega$ is a product state $\omega_\theta$ is a product state with respect to $\mathfrak{A}_n$ and $\mathfrak{A}_n^c$. Hence, we have for $A \in F_\theta \mathfrak{A}_n F_\theta$ and $B = B_1 B_2$ with $B_1 \in \mathfrak{A}_n$ and $B_2 \in \mathfrak{A}_n^c$ that $\omega_\theta(AB) = \omega_\theta(AB_1)\omega_\theta(B_2) = \omega_\theta(B_1 A)\omega_\theta(B_2) = \omega_\theta(BA)$. Since every element $B$ of $\mathfrak{A}$ can be expressed as a linear combination of terms of form $B_1 B_2$ with $B_1 \in \mathfrak{A}_n$ and $B_2 \in \mathfrak{A}_n^c$ we have $\omega_\theta(AB) = \omega_\theta(BA)$ for all $A \in F_\theta \mathfrak{A}_n F_\theta$ and $B \in \mathfrak{A}$. Hence, from (1!) we have that $\rho_\theta(AB) = \rho_\theta(BA)$ for all $A \in F_\theta \mathfrak{A}_n^G F_\theta$ and $B \in \mathfrak{A}^G$.

We now examine the algebra $F_\theta \mathfrak{A}_n^G F_\theta$. Let $x = x(\theta)$ and let $Q_i = \{k\,;\, x(k) = i$ for $k = 1, \ldots, n\}$ for $i = 1, \ldots, q$. Let $t(i, x)$ be the number of elements in $Q_i$ $\left(\text{note } \sum_{i=1}^{q} t(i, x) = n\right)$. We have $\mathfrak{A}_n^G$ is the linear span of elements of the form

$U_\sigma\gamma_1(A_1)\,\gamma_2(A_2)\ldots\gamma_n(A_n)$ with $A_i \in R_0$ and $\sigma \in S_n$. We have $e_x = \gamma_1(e_{x(1)})\gamma_2(e_{x(2)})\ldots$ $\ldots\gamma_n(e_{x(n)})$ commutes with the $\gamma_k(A_k)$ with $A_k \in R_0$ and $e_x U_\sigma e_x = 0$ unless $\sigma(Q_i) = Q_i$ for $i = 1,\ldots, q$. Since $e_x F_\theta = F_\theta e_x = F_\theta$ it follows that $F_\theta U_\sigma \gamma_1(A_1)\,\gamma_2(A_2)\ldots\gamma_n(A_n)F_\theta = 0$ unless $\sigma(Q_i) = Q_i$ for $i = 1,\ldots, q$. Recall that the $\{e_{ij}^{(k)}\}$ and the $\{e_{ij}'^{(k)}\}$ are sets of matrix units for $M_k = e_k R_0$ and $N_k = e_k R_0'$, respectively. Let $V_\sigma^{(k)}$ and $W_\sigma^{(k)}$ be defined in terms of the $e_{ij}^{(k)}$ and the $e_{ij}'^{(k)}$ as was done in Lemma 4.10. As we saw in Lemma 4.10 we have

$$V_\sigma^{(k)} W_\sigma^{(k)} = U_\sigma P_k(Q) \quad \text{with } P_k(Q) = \prod_{i \in Q} \gamma_i(e_k)$$

where $Q$ is the set of integers where $\sigma(i) \neq i$. Since $U_\sigma$, $P_k(Q)$, $V_{\sigma^{-1}}^{(k)} \in \mathfrak{A}^G$ we have $W_\sigma^{(k)} = V_{\sigma^{-1}}^{(k)} U_\sigma P_k(Q) \in \mathfrak{A}^G$. Suppose $\sigma \in S_n$ and $\sigma(Q_i) = Q_i$ for $i = 1,\ldots, q$. Then we can write $\sigma$ as a product $\sigma = \sigma_1\sigma_2\ldots\sigma_q$ where $\sigma_i$ acts on $Q_i$ (i.e., $\sigma_i(k) = k$ if $k \notin Q_i$). A straightforward computation using the fact that $F_\theta$ is a minimal projection in $M_x$ shows that for such a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_q$ and $A_i \in R_0$ we have

$$F_\theta U_\sigma \gamma_1(A_1)\,\gamma_2(A_2)\ldots\gamma_n(A_n)F_\theta = \lambda F_\theta W_{\sigma_1}^{(1)} W_{\sigma_2}^{(2)}\ldots W_{\sigma_q}^{(q)}$$

where $\lambda$ is a complex number. Hence, $F_\theta \mathfrak{A}_n^G F_\theta = W_1 \otimes W_2 \otimes \ldots \otimes W_q$ where $W_i$ is the $C^*$-algebra generated by the $W_\sigma^{(i)}$ with $\sigma$ a permutation of $Q_i$. Furthermore, one finds that $\omega_\theta$ is a product state on $F_\theta \mathfrak{A}_n^G F_\theta$ in that if $A_i \in W_i$ for $i = 1,\ldots, q$ then $\omega_\theta(A_1A_2\ldots A_q) = \omega_\theta(A_1)\,\omega_\theta(A_2)\ldots\omega_\theta(A_q)$.

To further describe $\omega_\theta$ we introduce some notation. For $i = 1,\ldots, q$ let $\tilde{\mathfrak{B}}_{i0}$ be an $(s_i \times s_i)$-matrix algebra. Let $\gamma_k^{(i)}$ be a $*$-isomorphism of $\tilde{\mathfrak{B}}_{i0}$ with $\tilde{\mathfrak{B}}_{ik}$ and let $\tilde{\mathfrak{A}}_{im} = \bigotimes_{k=1}^{m} \tilde{\mathfrak{B}}_{ik}$. Let $\tilde{\mathfrak{A}}_i$ be the norm closure of $\bigcup_{k=1}^{\infty} \tilde{\mathfrak{B}}_{ik}$. Let $\tilde{U}_\sigma^{(i)}$ be the corresponding permutation elements of $\tilde{\mathfrak{A}}_i$ constructed as the $U_\sigma$ were in Section 2. Identify $Q_i$ with the integers $(1, 2, \ldots, t(i, x(\theta)))$ and thereby identify each permutation $\sigma_i$ of $Q_i$ with a permutation $\sigma_i$ of $(1, 2, \ldots, t(i, x(\theta)))$. We define a $*$-isomorphism $\varphi$ of $F_\theta \mathfrak{A}_n^G F_\theta$ with $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}_1 \otimes \tilde{\mathfrak{A}}_2 \otimes \ldots \otimes \tilde{\mathfrak{A}}_q$ by requiring that if $\sigma_i$ is a permutation of $Q_i$ for $i = 1,\ldots, q$ we have

$$\varphi(W_{\sigma_1}^{(1)} W_{\sigma_2}^{(2)}\ldots W_{\sigma_q}^{(q)}) = \tilde{U}_{\sigma_1}^{(1)}\tilde{U}_{\sigma_2}^{(2)}\ldots\tilde{U}_{\sigma_q}^{(q)}.$$

Let $\tau$ be the unique tracial state of $\tilde{\mathfrak{A}}$ (note $\tau$ is a product state). Then we have

$$\omega_\theta(A) = \tau(\varphi(A)) \quad \text{for all } A \in F_\theta \mathfrak{A}_n^G F_\theta.$$

Let $\tilde{W}_i$ be the $C^*$-algebra generated by the $\tilde{U}_\sigma^{(i)}$ for all $\sigma \in S_\infty$. Note $\tilde{W}_i = \tilde{\mathfrak{A}}_i^{G_i}$ where $G_i$ is the full unitary group of $\tilde{\mathfrak{B}}_{i0}$. Then by Lemma 4.10, $\tau \mid \tilde{W}_i$ is a factor state (in fact, an extremal trace). Let $\tilde{W} = \tilde{W}_1 \otimes \tilde{W}_2 \otimes \ldots \otimes \tilde{W}_q$. Since $\tau|\tilde{W} = \tau|\tilde{W}_1 \otimes \tau|\tilde{W}_2 \otimes \ldots \otimes \tau|\tilde{W}_q$ we have $\tau|\tilde{W}$ is the tensor product of factor states so $\tau^W = \tau|\tilde{W}$ is a factor state (in fact, an extremal trace). Very roughly speaking, this means $\omega_\theta$ is an extremal trace if $t(i, x(\theta)) = \infty$ for $i = 1, \ldots, q$ and since $\rho_\theta$ is a tracial functional dominated by $\omega_\theta$ we must have $\rho_\theta = \lambda\omega_\theta$. In the following paragraphs we will make this idea more precise.

Since $0 \leqslant \rho \leqslant \omega$ we have $0 \leqslant \rho_\theta \leqslant \omega_\theta$ and $0 \leqslant \rho_\theta|F_\theta\mathfrak{A}_n^G F_\theta \leqslant \omega_\theta|F_\theta\mathfrak{A}_n^G F_\theta$. We have seen that $\omega_\theta$ and $\rho_\theta$ are tracial functionals on $F_\theta F\mathfrak{A}_{n\theta}^G$ (e.g., $\rho_\theta(AB) = \rho_\theta(BA)$ for $A, B \in F_\theta\mathfrak{A}_n^G F_\theta$). Since tracial functionals are uniquely determined by their values on the center of $F_\theta\mathfrak{A}_n^G F_\theta$ there is a $C'$ in the center of $F_\theta\mathfrak{A}_n^G F_\theta$ with $0 \leqslant C' \leqslant I$ so that $\rho_\theta(A) = \omega_\theta(AC')$ for all $A \in F_\theta\mathfrak{A}_{n\theta}^G F$.

We will show that for each $\varepsilon > 0$ and positive integer $m$ there is an integer $N = N(\varepsilon, m)$ depending only on $\varepsilon$ and $m$ so that $|\rho_\theta(A) - \rho_\theta(I)\omega_\theta(A)| \leqslant \varepsilon\|A\|$ for all $A \in F_\theta\mathfrak{A}_m^G F_\theta$ provided $t(i, x(\theta)) \geqslant N$ for $i = 1, \ldots, q$. Suppose for some $\varepsilon > 0$ and positive integer $m$ no such $N$ existed. Let $(\pi_1, \mathscr{H}_1, f_1)$ be a cyclic $*$-representation of $\tilde{W}$ induced by $\tau^W$. Then for each integer $n > 0$ we can find functionals $\rho_\theta$ and $\omega_\theta$ having the properties we have shown such functionals have and an $A_n \in F_\theta\mathfrak{A}_m^G F_\theta$ with $\|A_n\| = 1$ so that $|\rho_\theta(A_n) - \rho_\theta(I)\omega_\theta(A_n)| \geqslant \varepsilon$ and $t(i, x(\theta)) \geqslant n$ for $i = 1, \ldots, q$. Then there is a $C_n'$ in the center of $F_\theta\mathfrak{A}_m^G F_\theta$ so that $0 \leqslant C_n' \leqslant I$ and $\omega_\theta(A_n C_n') - \omega_\theta(C_n')\omega_\theta(A_n)| \geqslant \varepsilon$. Then we have $|\tau^W(\varphi(A_n)\varphi(C_n')) - \tau^W(\varphi(A_n))\tau^w(\varphi(C_n'))| \geqslant \varepsilon$ and $\varphi(C_n')$ is in the center of $\varphi(F_\theta\mathfrak{A}_m^G F_\theta)$. Since $0 \leqslant \pi_1(\varphi(C_n')) \leqslant I$ we may assume by passing to a subsequence that $\pi_1(\varphi(C_n')) \to C_0$ weakly as $n \to \infty$. We have $|(f_1, \pi_1(\varphi(A_n)\varphi(C_n'))f_1) - (f_1, \pi_1(\varphi(A_n))f_1)(f_1, \pi_1(\varphi(C_n'))f_1)| = |\tau^W(\varphi(A_n C_n')) - \tau^W(\varphi(A_n))\tau^W(\varphi(C_n'))| \geqslant \varepsilon$. Since $\|A_n\| = 1$ and $A_n \in F_\theta\mathfrak{A}_m^G F_\theta$ the set $\{\varphi(A_n)\}$ is contained in a finite dimensional subalgebra of $\tilde{W}$ and, thus, this set is compact. Hence, there is an $A \in \tilde{W}$ so that $|(f_1, \pi_1(A)C_0 f_1) - (f_1, \pi_1(A)f_1)(f_1, C_0 f_1)| \geqslant \varepsilon$. Hence, $C_0 \neq \lambda I$. Clearly, we have $C_0 \in \pi_1(\tilde{W})''$. Each $C_n'$ is contained in the center of $F_\theta\mathfrak{A}_m^G F_\theta$. Since $t(i, x(\theta)) \geqslant n$ the sequence $\varphi(F_\theta\mathfrak{A}_m^G F_\theta)$ contains a subsequence which is increasing and whose union is dense in $\tilde{W}$. Hence, $C_0 \in \pi_1(\tilde{W})'$. Hence, $C_0$ is contained in the center of $\pi_1(\tilde{W})''$. But this is a contradiction because $\pi_1(\tilde{W})''$ is a factor and $C_0 \neq \lambda I$. Hence, for each $\varepsilon > 0$ and positive integer $m$ an $N = N(\varepsilon, m)$ exists.

Now suppose $\varepsilon > 0$ and $m$ is a positive integer. Let $T_{in} = \{x \in X_n \,;\, t(i, x) < N(\varepsilon/2, m)\}$ and $T_n = \bigcup_{i=1}^{q} T_{in}$. Let $s_{in}$ be the $\mu$-measure of $T_{in}$ and $s_n$ the $\mu$-measure of $T_n$. Let $\sigma_i^{(k)} = \omega_k(e_i)$. One finds

$$s_{in} = \mu(T_{in}) = \sum_{x \in T_{in}} \mu_x = \sum_{x \in T_{in}} \prod_{k=1}^{n} \sigma_{x(k)}^{(k)}.$$

The numbers $s_{in}$ have the following probabilistic interpretation. Suppose a Markov coin has a probability $\sigma_i^{(k)}$ of producing a "heads" on the $k^{\text{th}}$ toss. The number $s_{in}$ is the probability of getting $N = N(\varepsilon/2, m)$ "heads" or less in the first $n$ tosses. Since $\sum_{k=1}^{\infty} \sigma_i^{(k)} = \infty$ (since $e_i \leqslant I - e_0$ and $0 \neq e_i \neq I - e_0$) we have from the Borel-Cantelli lemma that the coin will almost surely produce an infinite number of "heads" (see [10]). Hence, $s_{in} \to 0$ as $n \to \infty$. Hence, there is an integer $n$ so that $s_{in} < \varepsilon/2q$ for $i = 1, \ldots, q$. Then, we have

$$s_n = \mu(T_n) \leqslant \sum_{i=1}^{q} \mu(T_{in}) = \sum_{i=1}^{q} s_{in} < \varepsilon/2.$$

We now define $C_n \in \mathfrak{D}_n^s$ as follows. Let

$$C_n = \sum_{\theta \in \Theta} \rho_\theta(I) e_\theta \quad \text{where } e_\theta = \sum_{x \in \theta} e_x .$$

Since $0 \leqslant C \leqslant I$ we have $0 \leqslant \rho_\theta(I) \leqslant 1$, so $0 \leqslant C_n \leqslant I$. We show that for $A \in \mathfrak{A}_m^G$ we have $|\rho(A) - \omega(AC_n)| \leqslant \varepsilon \|A\|$. Suppose $A \in \mathfrak{A}_m^G$. Then we have from (2!) and (3!)

$$\rho(A) - \omega(AC_n) = \sum_{\theta \in \Theta} \sum_{x \in \theta} \mu_x \sum_{i=1}^{m(\theta)} \mu_{x(\theta)}^{-1} (\rho(B(\theta, x, i)^* AB(\theta, x, i)) -$$

$$- \omega(B(\theta, x, i)^* AC_n B(\theta, x, i))). \tag{4!}$$

From the definition of $B(\theta, x, i)$ and $C_n$ we have that these elements commute and $C_n B(\theta, x, i) = \rho_\theta(I) B(\theta, x, i)$. Since $D(\theta, x, i) = B(\theta, x, i)^* AB(\theta, x, i) \in F_\theta \mathfrak{A}_n^G F_\theta$ we have

$$|\rho_\theta(D(\theta, x, i)) - \rho_\theta(I)\omega_\theta(D(\theta, x, i))| \leqslant \frac{1}{2} \varepsilon \|D(\theta, x, i)\| \leqslant$$

$$\leqslant \frac{1}{2} \varepsilon (\lambda_i^{(x)}/\lambda_1^{(x(\theta))}) \|A\| ,$$

for $x(\theta) \notin T_n$. Since $\rho_\theta(A) = \rho(AF_\theta)/\omega(F_\theta)$ and $\omega_\theta(A) = \omega(AF_\theta)/\omega(F_\theta)$ for $A \in \mathfrak{A}$ and $\omega(F_\theta) = \mu_{x(\theta)} \lambda_1^{(x(\theta))}$ we have

$$\mu_{x(\theta)}^{-1} |\rho(B(\theta, x, i)^* AB(\theta, x, i)) - \omega(B(\theta, x, i) AC_n B(\theta, x, i))| \leqslant \frac{1}{2} \varepsilon \lambda_i^{(x)} \|A\|$$

for $x(\theta) \notin T_n$. For $x(\theta) \in T_n$ the above expression is bounded by $\lambda_i^{(x)} \|A\|$ since $\|\omega_\theta - \rho_\theta\| \leqslant 1$. Hence, from (4!) we have

$$|\rho(A) - \omega(AC_n)| \leqslant \sum_{\substack{\theta \in \Theta \\ x(\theta) \notin T_n}} \sum_{x \in \theta} \mu_x \sum_{i=1}^{m(\theta)} \frac{1}{2} \varepsilon \lambda_i^{(x)} \|A\| +$$

$$+ \sum_{\substack{\theta \in \Theta \\ x(\theta) \notin T_n}} \sum_{x \in \theta} \mu_x \sum_{i=1}^{m(\theta)} \lambda_i^{(x)} \|A\| .$$

Since $\sum_{i=1}^{m(\theta)} \lambda_i^{(x)} = 1$ and since for $x \in \theta$ we have $x \in T_n$ if and only if $x(\theta) \in T_n$ we have from the above estimate

$$|\rho(A) - \omega(AC_n)| \leqslant \sum_{\substack{x \in X_n \\ x \notin T_n}} \mu_x \frac{1}{2} \varepsilon \|A\| + \sum_{\substack{x \in X_n \\ x \notin T_n}} \mu_x \|A\| \leqslant$$

$$\leqslant \frac{1}{2} \varepsilon \|A\| \sum_{x \in X_n} \mu_x + \|A\| \mu(T_n) \leqslant \frac{1}{2} \varepsilon \|A\| + \frac{1}{2} \varepsilon \|A\| = \varepsilon \|A\| .$$

Let $\mathfrak{D}$ be the norm closure of the union of the $\mathfrak{D}_n$. We claim $C \in \pi(\mathfrak{D})''$. To see this suppose $\varepsilon > 0$ and $m$ is a positive integer. Suppose $A_i, B_i \in \mathfrak{A}_m^G$ for $i = 1, \ldots, k$. Then there are an integer $n > m$ and a $C_n \in \mathfrak{D}_n^s \subset \mathfrak{D}_n \subset \mathfrak{D}$ with $0 \leqslant C_n \leqslant I$ so that

$$|(\pi(A_i)f_0, (C - \pi(C_n))\pi(B_i)f_0)| = |(f_0, \pi(A_i^* B_i)(C - \pi(C_n))f_0)| =$$

$$= |\rho(A_i^* B_i) - \omega(A_i^* B_i C_n)| \leqslant \varepsilon \|A_i^* B_i\|$$

for $i = 1, \ldots, k$. Since $\varepsilon > 0$ is arbitrary, the $\pi(C_n)$ are uniformly bounded and the vector $\pi(A_i)f_0$ and $\pi(B_i)f_0$ with $A_i, B_i \in \mathfrak{A}_m^G$ for some $m = 1, 2, \ldots$ are dense in $\mathscr{H}$, we have $C$ is in the weak closure of $\pi(\mathfrak{D})$. Hence, $C \in \pi(\mathfrak{D})''$.

We define $X_\infty$ with its infinite product measure (which we also denote by $\mu$) by taking the limit as $n \to \infty$ of the spaces $X_n$. We define a $*$-homomorphism $\Psi$ of $\mathfrak{D}$ into the algebra of bounded functions on $X_\infty$. For $x = (x(1), x(2), \ldots) \in X_\infty$ we define

$$\Psi(\gamma_k(e_i))(x) = \begin{cases} 1 & \text{if } x(k) = i \\ 0 & \text{if } x(k) \neq i \end{cases}$$

for $i = 1, \ldots, q$ and $k = 1, 2, \ldots$ . Since the $\gamma_k(e_i)$ generate $\mathfrak{D}$ as a $C^*$-algebra there is a unique $*$-homomorphism $\Psi$ satisfying the above requirements. From the construction of $\Psi$ we have $\mu(\Psi(A)) = (f_0, \pi(A)f_0)$ for all $A \in \mathfrak{D}$. Hence, the mapping $\Gamma$ given by $\Gamma(\pi(A)) = \Psi(A)$ for $A \in \mathfrak{D}$ extends to a $*$-homomorphism of $\pi(\mathfrak{D})''$ into the bounded $\mu$-measurable functions on $X_\infty$. Consider the $\mu$-measurable function $\Gamma(C)$. Since $C$ is in the center of $\pi(\mathfrak{A}^G)$, $C$ commutes with the permutation elements $\pi(U_\sigma)$. It follows that $\Gamma(C)$ is a permutation invariant function on $X_\infty$. It is shown in [1] that the only permutation invariant functions on $X_\infty$ are trivial (almost everywhere equal to a constant function) if and only if for each set $S \subset (1, 2, \ldots, q)$

$$\sum_{k=1}^{\infty} \sigma^{(k)}(S)(1 - \sigma^{(k)}(S)) = 0 \quad \text{or } \infty,$$

where $\sigma^{(k)}(S) = \sum_{i\in S} \sigma_i^{(k)}$. Since $\sigma_i^{(k)} = \omega_k(e_i)$ it follows from the assumption of the lemma that the condition of Aldous and Pitman is satisfied. Hence, $\Gamma(C)$ is a constant. Hence, we have $(f_0, C^2 f_0) = \mu(\Gamma(C^2)) = \mu(\Gamma(C)\Gamma(C)) = \mu(\Gamma(C))^2 = (f_0, Cf_0)^2$. Hence, $Cf_0 = (f_0, Cf_0)f_0 = \lambda f_0$. Since for $A \in \mathfrak{A}^G$ we have $C\pi(A)f_0 = \pi(A)Cf_0 = \lambda\pi(A)f_0$ we have $C = \lambda I$. Hence, the center of $\pi(\mathfrak{A}^G)''$ consists of multiples of the identity. Hence, $\omega^G$ is a factor state.

THEOREM 4.13. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of minimal* $G_\gamma$-*support. Suppose* $E$ *is the* $G_\gamma$-*support projection of* $\omega$. *Then,* $\omega^G$ *is a factor state if and only if*

$$(*) \qquad \sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) = 0 \quad \text{or } \infty$$

*for all* $e \in R_0'$ *with* $e \leqslant E$.

*Proof.* Suppose the hypothesis and notation of the theorem are satisfied Let $H = H^G(\omega)$ and let $(\pi, \mathscr{H}, f_0)$, $(\pi^G, \mathscr{H}^G, f_0)$ and $(\pi^H, \mathscr{H}^H, f_0)$ with $\mathscr{H}^H = \mathscr{H}^G$ be cyclic $*$-representations of $\mathfrak{A}$, $\mathfrak{A}^G$ and $\mathfrak{A}^H$ induced by $\omega$, $\omega^G$ and $\omega^H$, respectively, as constructed in Lemma 4.8. Recall that $\pi^G(\mathfrak{A}^G)'' = \pi^H(\mathfrak{A}^H)''$. Hence, $\omega^G$ is a factor state if and only if $\omega^H$ is a factor state.

First we prove the implication ($\Rightarrow$). Assume $\omega^G$ is a factor state. Suppose $e \in R_0'$, $e \leqslant E$ and the sum $(*)$ is finite. We will show that the sum $(*)$ is zero. We have $e \in M^G(\omega)$. Then, by Theorem 4.7 we have $e = e_0' + e'$ where $e' \in M^G(\omega)$ is a central projection in $R^G(\omega) \cap R^G(\omega)'$ and $e_0' \in N^G(\omega)$. Since $\omega^G$ is a factor state we have from Lemma 4.8 that $\omega_k(e') = 0$ or $1$ for all $k = 1, 2, \ldots$ . Hence,

$$\sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) \leqslant \sum_{k=1}^{\infty} \omega_k(e_0')(1 - \omega_k(e_0')) < \infty.$$

We will show the right hand sum is zero. Let $\{e_1, \ldots, e_m\}$ be the minimal central projections in $R_0 \cap R_0'$ so that $e_i e_0' \neq 0$ for $i = 1, \ldots, m$. Let $e_0$ be the maximal projection in $N^G(\omega)$ as described in Lemma 4.6. Since $e_i \in R_0 \cap R_0'$ and $e_0 \in R^G(\omega) \cap R^G(\omega)'$ we have $e_i e_0 = e_0 e_i \in R^G(\omega)$. And since $e_i e_0 \leqslant e_0$ we have $e_i e_0 \in N^G(\omega) \subset M^G(\omega)$ and by Lemma 4.3 we have $e_i e_0 \in R^G(\omega)'$. Hence, $e_i e_0 \in R^G(\omega) \cap R^G(\omega)'$. Since $\omega^H$ is a factor state we have from Lemma 4.8 that $\omega_k(e_i e_0) = 0$ or $1$ for all $i = 1, \ldots, m$ and $k = 1, 2, \ldots$ . Let $Q_i = \{k; \omega_k(e_i e_0) = 1\}$ for $i = 1, \ldots, m$. Note each $Q_i$ is a finite set since $e_i e_0 \in N^G(\omega)$. Since $\omega^H$ is a factor state it follows from Theorem 4.9 that $\omega|\mathfrak{A}^H(Q_i)$ is a factor state for $i = 1, \ldots, m$. Then it follows from Lemma 4.11 that for each $i = 1, \ldots, m$ either $Q_i$ has only one element or there is a projection $e_i' \in R^G(\omega)'$ with $e_i' \leqslant e_i e_0$ which is minimal in $R^G(\omega)'$ so that $\omega_k(e_i') = 1$

for all $k \in Q_i$. Since $e_i' \in R^G(\omega)'$ we have $e_i' \in R_0'$. We claim $e_i'$ is minimal in $R_0'$. If $e_i'$ is not minimal in $R_0'$ there is a unitary $V \in R_0'$ which commutes with $e_i'$ and $Ve_i' \neq \lambda e_i'$. Let $U = I - e_i' + Ve_i'$. Since $\omega_k(e_i') = 0$ except for a finite number of $k$ we have $\omega_k(UAU^{-1}) = \omega_k(A)$ for all $A \in \mathfrak{B}_0$ except for a finite number of $k$. Hence, $\omega \underset{q}{\sim} \omega \circ \alpha_U$ and $U \in H^G(\omega)$. Then, by Lemma 3.5 we have $U \in R^G(\omega)'$. Since $e_i'$ is minimal in $R^G(\omega)'$ we must have $Ue_i' = Ve_i' = \lambda e_i'$. But this contradicts the fact that $Ve_i' \neq \lambda e_i'$. Hence, $e_i'$ must be minimal in $R_0'$. If $Q_i$ contains only one element then it follows from the fact that $\omega$ is of minimal $G_\gamma$-support that there is a projection $e_i' \in R_0'$ which is minimal in $R_0'$ so that $\omega_k(e_i') = 1$ for $k \in Q_i$. Hence, for $i = 1, \ldots, m$ we have found projections $e_i' \in R_0'$ which are minimal in $R_0'$ so that $\omega_k(e_i') = 1$ for $k \in Q_i$. From the definition of the $G_\gamma$-support projection $E$ it follows that $Ee_ie_0 = e_i'$ for $i = 1, \ldots, m$. Since $e_0' \leqslant E$, $e_0' \leqslant e_0$ and $e_ie_0' = e_0'e_i \neq 0$ it follows that $0 \neq Ee_ie_0' \leqslant e_i'$ for $i = 1, \ldots, m$. Since $e_i'$ is minimal in $R_0'$ we have $Ee_ie_0' = e_i'$ for $i = 1, \ldots, m$. Hence, we have $\omega_k(e_ie_0') = \omega_k(Ee_ie_0') = \omega_k(e_i') = \omega_k(Ee_ie_0) = \omega_k(e_ie_0)$ and as we noted earlier $\omega_k(e_ie_0) = 0$ or $1$ for all $i = 1, \ldots, m$ and $k = 1, 2, \ldots$ . Since $\omega_k(e_0') = \sum_{i=1}^{m} \omega_k(e_ie_0')$ it follows that $\omega_k(e_0') = 0$ or $1$ for all $k = 1, 2, \ldots$ . Hence, $\sum_{k=1}^{\infty} \omega_k(e_0')(1 - \omega_k(e_0')) = 0$ and as we have seen this implies $\sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) = 0$. Hence, we have shown the sum condition $(*)$ is satisfied for all $e \in R_0'$ with $e \leqslant E$.

We now prove the implication $(\Leftarrow)$. We assume the sum condition $(*)$ is satisfied for all $e \in R_0'$ with $e \leqslant E$. We will show that $\omega^H$ and, therefore, $\omega^G$ is a factor state. Let $\Phi$ be the conditional expectation of $\mathfrak{B}_0$ onto $R^G(\omega)$ relative to the trace. We define a new product state $\bar{\omega} = \bigotimes_{k=1}^{\infty} \bar{\omega}_k$ as follows. Let $e_0$ be the maximal projection in $N^G(\omega)$. Since $e_0 \in R_0'$, $I - e_0 \leqslant E$ and $\omega$ is assumed to satisfy $(*)$ we have $\omega_k(e_0) = 0$ or $1$ for all $k = 1, 2, \ldots$ and $\omega_k(e_0) = 1$ for only a finite number of $k$. We define $\bar{\omega}_k = \omega_k(\Phi(A))$ if $\omega_k(e_0) = 0$ and $\bar{\omega}_k(A) = \omega_k(A)$ if $\omega_k(e_0) = 1$. From Theorem 3.8 it follows that $\bar{\omega} \underset{q}{\sim} \omega$ (note the $\bar{\omega}_k$ of Theorem 3.8 and the $\bar{\omega}_k$ just defined differ for only a finite number of $k$ and so the $\bar{\omega}$ of Theorem 3.8 and the $\bar{\omega}$ just defined are quasi-equivalent). Then, from Lemma 3.3 we have $R^G(\omega) = R^G(\bar{\omega})$ and $H^G(\omega) = H^G(\bar{\omega})$ and from Lemma 4.4 we have $M^G(\omega) = M^G(\bar{\omega})$ and $N^G(\omega) = N^G(\bar{\omega})$. First we will show that $\bar{\omega}^H$ is a factor state and then we will show that $\omega^H \underset{q}{\sim} \bar{\omega}^H$.

From Theorem 4.7 we have that every projection $e \in M^G(\omega)$ is of the form $e = e_0' + s_1e_1 + s_2e_2 + \ldots + s_ne_n$ where $e_0' \in R_0'$ with $e_0' \leqslant e_0$, $s_i = 0$ or $1$ and $e_i \in M^G(\omega) \cap R^G(\omega) \cap R^G(\omega)'$ for $i = 1, \ldots, n$. Since $\omega$ satisfies the sum condition $(*)$

we have $\omega_k(e_i) = 0$ or 1 for $i = 1, \ldots, n$ and $k = 1, 2, \ldots$ . Since $\Phi(e_i) = e_i$ for $i = 0, 1, \ldots, n$ we have $\bar{\omega}_k(e_i) = 0$ or 1 for $i = 0, 1, \ldots, n$ and $k = 1, 2, \ldots$ . Let $Q_i = \{k \,;\, \omega_k(e_i) = 1\}$ for $i = 0, 1, \ldots, n$. Note $Q_0$ is finite and $Q_i$ is infinite for $i = 1, \ldots, n$. Let $\mathfrak{A}(Q_i) = \bigotimes_{k \in Q_i} \mathfrak{B}_k$ and $\rho_i = \bigotimes_{k \in Q_i} \bar{\omega}_k$ be defined on $\mathfrak{A}(Q_i)$ for $i = 0, 1, \ldots, n$. We see that for $i = 1, \ldots, n$, $\rho_i$ satisfies the hypothesis of Lemma 4.12 with $G = H$ and $R_0 = R^G(\omega)$. Hence, $\rho_i^H = \rho_i | \mathfrak{A}^H(Q_i)$ is a factor state for $i = 1, \ldots, n$. Then applying Theorem 4.9 with the partitions $\{Q_0, Q_1, \ldots, Q_n\}$ and $\{e_0, e_1, \ldots, e_n\}$ and with $R_0 = R^G(\omega)$ and $G = H$ we find that $\omega^H$ is a factor state if and only if $\rho_0$ is a factor state of $\mathfrak{A}^H(Q_0)$. Note that $\rho_0 = \omega | \mathfrak{A}(Q_0)$ since $\bar{\omega}_k = \omega_k$ for $k \in Q_0$.

By the remark after Lemma 4.6 we have that $e_0$ and $E$ commute and $Ee_0 \in R^G(\omega)'$. Let $\{e_1', \ldots, e_m'\}$ be minimal central projections of $R^G(\omega) \cap R^G(\omega)'$ so that $Ee_0e_i' \neq 0$ for $i = 1, \ldots, m$. Since by Lemma 4.6 we have $e_0 \in R^G(\omega) \cap R^G(\omega)'$ we have $e_0e_i' = e_i'$ for $i = 1, \ldots, m$. Since $E \geqslant Ee_i \in R_0'$ we have from the sum condition $(*)$ that $\omega_k(e_i') = \omega_k(Ee_i') = 0$ or 1 for all $i = 1, \ldots, m$ and $k = 1, 2, \ldots$ . Now we further partition $Q_0$ as follows. Let $Q_{0i} = \{k \,;\, \omega_k(e_i') = 1\}$. Note $Q_0 = \bigcup_{i=1}^{m} Q_{0i}$. Let $\rho_{0i} = \rho_0 | \mathfrak{A}(Q_{0i})$. Again it follows from Theorem 4.9 that $\rho_0^H$ is a factor state if and only if $\rho_{0i}^H$ is a factor state for each $i = 1, \ldots, m$.

Note $Ee_i' \in R_0'$ and $Ee_i' \leqslant e_0$ so $Ee_i' \in N^G(\omega) \subset M^G(\omega)$. Then by Lemma 4.3 we have $Ee_i' \in R^G(\omega)'$. We claim $Ee_i'$ is minimal in $R^G(\omega)'$ for each $i = 1, \ldots, m$. Suppose $Ee_r'$ is not minimal in $R^G(\omega)'$. Let $\{e_{ij}' \,;\, i, j = 1, \ldots, s\}$ be a set of matrix units for $e_r'R^G(\omega)'e_r'$ so that $Ee_r' = e_{11}' + e_{22}' + \ldots + e_{nn}'$. Since $e_{11}' \in R_0'$ and $e_{11}' \leqslant E$ we have from the sum condition $(*)$ that $\omega_k(e_{11}') = 0$ or 1 for all $k = 1, 2, \ldots$ . Furthermore, $\omega_k(e_{11}') \neq 0$ for some $k$, for if $\omega_k(e_{11}') = 0$ for all $k$ then $\omega_k(E - e_{11}') = 1$ for all $k$ and by definition $E$ is the smallest projection in $R_0'$ with $\omega_k(E) = 1$ for all $k$. Hence, there is an integer $k$ so that $\omega_k(e_{11}') = 1$. Let $f = (1/2)(e_{11}' + e_{12}' + e_{21}' + e_{22}')$. We have $f \in R_0'$ and $f \leqslant E$ so $\omega_k(f) = 0$ or 1, but $\omega_k(f) = (1/2)\omega_k(e_{11}') = 1/2$. We have a contradiction so $Ee_i'$ is minimal in $R^G(\omega)'$ for $i = 1, \ldots, m$.

Since $e_i'$ is a minimal central projection in $R^G(\omega) \cap R^G(\omega)'$ and $Ee_i'$ is a minimal projection in $R^G(\omega)'$ so that $\omega_k(Ee_i') = 1$ for all $k \in Q_{0i}$ it follows from Lemma 4.11 (with $G = H$ and $R_0 = R^G(\omega)$) that $\rho_{0i}^H = \omega | \mathfrak{A}^H(Q_{0i})$ is a factor state. Hence, $\rho_{0i}^H$ is a factor state for $i = 1, \ldots, m$. Hence, from Theorem 4.9 we have $\rho_0^H$ is a factor state. As we have seen, this implies $\bar{\omega}^G$ is a factor state.

We now show $\omega^G$ is a factor state by showing $\omega^G \underset{q}{\sim} \bar{\omega}^G$. Let $\Omega_k \in \mathfrak{B}_0$ be defined by $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for all $A \in \mathfrak{B}_0$. For $k \notin Q_0$ we have $\bar{\omega}_k(A) = \mathrm{tr}(A\Phi(\Omega_k))$ for $A \in \mathfrak{B}_0$. If $\Phi(\Omega_k) = \sum_{i=1}^{s} \lambda_i E_i$ with $\lambda_i > 0$ is the spectral decomposition of $\Phi(\Omega_k)$, we define $\Phi(\Omega_k)^{-1} = \sum_{i=1}^{s} \lambda_i^{-1} E_i$ so $\Phi(\Omega_k)\Phi(\Omega_k)^{-1} = \Phi(\Omega_k)^{-1}\Phi(\Omega_k) = E_0$ where $E_0$

is the support projection of $\Phi(\Omega_k)$. For $k \in Q_0$ let $B_k = I$ and for $k \notin Q_0$ let $B_k = \Omega_k \Phi(\Omega_k)^{-1}$. Since the support projection of $\Phi(\Omega_k)$ is greater than the support projection of $\Omega_k$ we have $B_k\Phi(\Omega_k) = \Omega_k$. Then we have $\bar{\omega}_k(AB_k) = \omega_k(A)$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Let $\rho_n = \bigotimes_{k=1}^{n} \omega_k \bigotimes_{k=n+1}^{\infty} \bar{\omega}_k$ and let $C_n = \gamma_1(B_1)\gamma_2(B_2)\ldots \ldots \gamma_n(B_n)$. We have $\rho_n(A) = \bar{\omega}(AC_n)$ for all $A \in \mathfrak{A}$. Since $\omega \underset{q}{\sim} \bar{\omega}$ we have (see [12], Theorem 2.7) $\|(\omega - \bar{\omega})| \mathfrak{A}_n^c\| \to 0$ as $n \to \infty$. Since $\|\rho_n - \omega\| = \|(\omega - \bar{\omega})|\mathfrak{A}_n^c\|$ we have $\|\rho_n - \omega\| \to 0$ as $n \to \infty$. Let

$$D_n = \Gamma(C_n) = \int_G \alpha_U(C_n)\mathrm{d}\nu(U),$$

where $\nu$ is Haar measure on $G$. We have $D_n \in \mathfrak{A}_n^G$ and for $A \in \mathfrak{A}^G$

$$\bar{\omega}(AD_n) = \bar{\omega}(A\Gamma(C_n)) = \bar{\omega}(\Gamma(AC_n)) = \bar{\omega}(AC_n) = \rho_n(A).$$

Hence, $\bar{\omega}^G(AD_n) = \rho_n^G(A)$ for all $A \in \mathfrak{A}^G$. Since $\rho_n^G$ is a density matrix state of the representation of $\mathfrak{A}^G$ induced by $\bar{\omega}^G$ we have $\rho_n^G$ is a factor state. Since $\|\rho_n - \omega\| \to 0$ as $n \to \infty$ we have $\|\rho_n^G - \omega^G\| \to 0$ as $n \to \infty$. Since the factor states of a $C^*$-algebra form a norm closed set (see [6]) it follows that $\omega^G$ is a factor state. Since the states $\rho_n^G$ are quasi-equivalent to $\bar{\omega}^G$ it follows that $\bar{\omega}^G \underset{q}{\sim} \omega^G$. Note that we have also shown $\bar{\omega}^H \underset{q}{\sim} \omega^H$. ▨

THEOREM 4.14. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *and* $\omega' = \bigotimes_{k=1}^{\infty} \omega_k'$ *are product states of* $\mathfrak{A}$ *of minimal* $G_\gamma$*-support and* $E$ *and* $E'$ *are the* $G_\gamma$*-support projections of* $\omega$ *and* $\omega'$*, respectively. Suppose* $\omega^G$ *and* $\omega'^G$ *are factor states of* $\mathfrak{A}^G$*. Then* $\omega^G \underset{q}{\sim} \omega'^G$ *if and only if there is a unitary* $U \in G$ *satisfying the following conditions:*

i) $\omega \underset{q}{\sim} \omega' \circ \alpha_U$;

ii) $E' = UEU^{-1}$;

iii) *If* $e \in R_0'$ *and* $e \leqslant E$ *then* $\omega_k(e) = 0$ *or* $1$ *for all* $k$ *if and only if* $\omega_k'(UeU^{-1}) = 0$ *or* $1$ *for all* $k$ *and, furthermore, if* $\omega_k(e) = 0$ *or* $1$ *for all* $k$ *then*

$$\sum_{k=1}^{\infty} |\omega_k(e) - \omega_k'(UeU^{-1})| < \infty$$

*and*

$$\sum_{k=1}^{\infty} \omega_k(e) - \omega_k'(UeU^{-1}) = 0.$$

*Proof.* Suppose $\omega, \omega'$, $E$ and $E'$ satisfy the hypothesis and notation of the theorem. Suppose there is a $U \in G$ satisfying conditions i), ii) and iii). We will show $\omega^G \underset{q}{\sim} \omega'^G$.

Recalling the proof of the previous theorem we see that the projections $e \in R'_0$ with $e \leqslant E$ for which $\omega_k(e) = 0$ or $1$ for all $k$ are of the form $e = \sum_{i=1}^{n} s_i e_i$ where $s_i = 0$ or $1$, $e_i$ for $i = 1, \ldots, m$ are minimal projections in $R'_0$ (and they are minimal projections in $R^G(\omega)'$) with $e_i \leqslant e_0$ where $e_0$ is the maximal projection in $N^G(\omega)$ and $e_i$ for $i = m+1, \ldots, n$ are central projections in $R^G(\omega) \cap R^G(\omega)'$ with $e_i e_0 = 0$. Furthermore, for any choice of zero's and one's for the $s_i$ for $i = 1, \ldots, n$ one obtains a projection $e$ so that $\omega_k(e) = 0$ or $1$ for all $k = 1, 2, \ldots$ . Also, note that $E = \sum_{i=1}^{n} e_i$. Let $Q$ be the set of $k = 1, 2, \ldots$ so that $\omega_k(e_i) \neq \neq \omega'_k(Ue_iU^{-1})$ for some $i = 1, \ldots, n$. One sees from condition iii) that $Q$ is finite. Furthermore, it follows from condition iii) that

$$\sum_{k \in Q} \omega_k(e_i) - \omega'_k(Ue_iU^{-1}) = 0$$

for $i = 1, \ldots, n$. Hence, there is a permutation $\sigma$ with $\sigma(j) = j$ for $j \notin Q$ so that $\omega_k(e_i) = \omega'_{\sigma(k)}(Ue_iU^{-1})$ for $i = 1, \ldots, n$ and $k = 1, 2, \ldots$ .

Let $\rho(A) = \omega'(\alpha_U(U_\sigma A U_\sigma^{-1}))$ for all $A \in \mathfrak{A}$. We have $\rho = \bigotimes_{k=1}^{\infty} \rho_k$ where $\rho_k(A) = \omega'_{\sigma(k)}(UAU^{-1})$ for all $A \in \mathfrak{B}_0$. Since $\rho(A) = \omega'(U_\sigma A U_\sigma^{-1})$ for all $A \in \mathfrak{A}^G$ we have $\rho^G \underset{q}{\sim} \omega'^G$. Since $\rho \underset{q}{\sim} \omega' \circ \alpha_U$ and $\omega' \circ \alpha_U \underset{q}{\sim} \omega$ we have $\rho \underset{q}{\sim} \omega$.

Summarizing the situation at this point we have $\omega \underset{q}{\sim} \rho$, $E$ is the $G_\gamma$-support projection for both $\omega$ and $\rho$, from Lemma 3.3 we have $R^G(\omega) = R^G(\rho)$, from Lemma 4.4 we have $M^G(\omega) = M^G(\rho)$ and $N^G(\omega) = N^G(\rho)$. Let $\omega_k(A) = \operatorname{tr}(A\Omega_k)$ and $\rho_k(A) = \operatorname{tr}(A\Omega'_k)$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Since $\omega \underset{q}{\sim} \rho$ we have

$$\sum_{k=1}^{\infty} \|\Omega_k^{1/2} - \Omega_k'^{1/2}\|_{\mathrm{HS}}^2 < \infty.$$

Let $Q_1 = \{k \, ; \, \|\Omega_k^{1/2} - \Omega_k'^{1/2}\|_{\mathrm{HS}}^2 = 2\}$. We have $Q_i$ is a finite set. We define a new state $\bar{\rho} = \bigotimes_{k=1}^{\infty} \bar{\rho}_k$ as follows. Let $k \in Q_1$. Since $e_i \in M^G(\omega)$ then one and only one of the numbers $\omega_k(e_i) = \rho_k(e_i)$ is equal to one for $i = 1, \ldots, n$. If $e_i \leqslant e_0$ then $e_i$ is minimal in $R'_0$. Hence, there is a unitary $U_k \in R_0$ so that $|\omega_k(A) - \rho_k(U_k A U_k^{-1})| < 2$ and $|\rho_k(A) - \rho_k(U_k A U_k^{-1})| < 2$ for all $A \in \mathfrak{B}_0$ with $\|A\| \leqslant 1$. (If such a $U_k$ did not exist then $\omega_k$ and $\rho_k$ would have disjoint support in $R'_0$ which is impossible.) Then we define $\bar{\rho}_k(A) = \rho_k(U_k A U_k^{-1})$ for $A \in \mathfrak{B}_0$. If $e_i e_0 = 0$ we define $\bar{\rho}_k = (1/2)\rho_k + + (1/2)\omega_k$. One can easily check that $\bar{\rho}$ satisfies the sum condition $(*)$ of the previous theorem so $\bar{\rho}^G$ is a factor state. Since $\rho_k = \bar{\rho}_k$ for all but a finite number of $k$ we have $\rho \underset{q}{\sim} \bar{\rho}$. If $\bar{\rho}_k(A) = \mathrm{tr}(A\bar{\Omega}'_k)$ for all $A \in \mathfrak{B}_0$ we have

$$\sum_{k=1}^{\infty} \|\Omega_k'^{1/2} - \bar{\Omega}_k'^{1/2}\|_{\mathrm{HS}}^2 < \infty.$$

By construction we have $\|\Omega_k^{1/2} - \bar{\Omega}_k'^{1/2}\|_{\mathrm{HS}}^2 < 2$ and $\|\Omega_k'^{1/2} - \bar{\Omega}_k'^{1/2}\|_{\mathrm{HS}}^2 < 2$ for all $k = 1, 2, \ldots$ . Hence $\|\omega - \bar{\rho}\| < 2$ and $\|\bar{\rho} - \rho\| < 2$. Thus, $\|\omega^G - \bar{\rho}^G\| < 2$ and $\|\bar{\rho}^G - \rho^G\| < 2$. Hence, $\omega^G \underset{q}{\sim} \bar{\rho}^G$ and $\bar{\rho}^G \underset{q}{\sim} \rho^G$. Hence, $\omega^G \underset{q}{\sim} \rho^G$. Since $\rho^G \underset{q}{\sim} \omega'^G$ we have $\omega^G \underset{q}{\sim} \omega'^G$.

Now suppose $\omega^G \underset{q}{\sim} \omega'^G$. We will show there exists a $U \in G$ satisfying conditions i), ii) and iii). Let $(\pi_i, \mathscr{H}_i, f_i)$ be cyclic $*$-representations of $\mathfrak{A}$ induced by $\omega$ for $i = 1$ and $\omega'$ for $i = 2$. Let $P_i$ be the orthogonal projection of $\mathscr{H}_i$ onto $\mathscr{H}_i^G =$ closure of $\{\pi_i(\mathfrak{A}^G)f_i\}$. Let $(\pi_i^G, \mathscr{H}_i^G, f_i)$ be the restriction of $\pi_i$ to $\mathfrak{A}^G$ and $\mathscr{H}_i^G$ (i.e., $\pi_i^G(A)f = \pi_i(A)f$ for $A \in \mathfrak{A}^G$ and $f \in \mathscr{H}_i^G$). Note $\pi_1^G$ and $\pi_2^G$ are cyclic $*$-representations of $\mathfrak{A}^G$ induced by $\omega^G$ and $\omega'^G$, respectively.

Let $H_1 = H^G(\omega)$ and $H_2 = H^G(\omega')$. From Lemma 3.6 we have $\pi_i(\mathfrak{A}^G)'' = \pi_i(\mathfrak{A}^{H_i})''$ and, therefore, $\mathscr{H}_i^G = \mathscr{H}_i^H =$ closure of $\{\pi_i(\mathfrak{A}^{H_i})f_i\}$. Let $(\pi_i^H, \mathscr{H}_i^H, f_i)$ be the restriction of $\pi_i$ to $\mathfrak{A}^{H_i}$ and $\mathscr{H}_i^H$ (i.e., $\pi_i^H(A)f = \pi_i(A)f$ for all $A \in \mathfrak{A}^{H_i}$ and $f \in \mathscr{H}_i^H$). Since $\pi_i(\mathfrak{A}^G)'' = \pi_i(\mathfrak{A}^{H_i})''$ we have $\pi_i^G(\mathfrak{A}^G)'' = \pi_i^H(\mathfrak{A}^{H_i})''$.

Since $\omega^G \underset{q}{\sim} \omega'^G$ there is a $\sigma$-strongly ($\sigma$-weakly) bicontinuous $*$-isomorphism $\Phi$ of $\pi_1^G(\mathfrak{A}^G)''$ onto $\pi_2^G(\mathfrak{A}^G)''$ so that $\Phi(\pi_1^G(A)) = \pi_2^G(A)$ for all $A \in \mathfrak{A}^G$. Suppose $A \in R^G(\omega)$. Then, $\pi_1^H(\gamma_1(A)) \in \pi_1^H(\mathfrak{A}^{H_1})'' = \pi_1^G(\mathfrak{A}^G)''$. We will show that there is a $B \in R^G(\omega')$ so that $\Phi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(B))$.

Let $\overline{\mathfrak{A}}_n$ be the $C^*$-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_1 = \gamma_1(\mathfrak{B}_0)$ and $\mathfrak{B}_k = \gamma_k(\mathfrak{B}_0)$ for $k > n$, i.e., $\overline{\mathfrak{A}}_n = \mathfrak{B}_1 \otimes \bigotimes_{k=n+1}^{\infty} \mathfrak{B}_k$. Let $\overline{\mathfrak{A}}_n^G$ be the $\alpha_U$-invariant elements of $\overline{\mathfrak{A}}_n$ with $U \in G$. If one examines the proofs of Lemma 3.3 and Theorem 3.7 one sees that these proofs only depend on the asymptotic properties of the state $\omega$. It follows that for the product state $\omega$, if $A \in R^G(\omega)$ there is a sequence $A_k \in \overline{\mathfrak{A}}_{n_j}^G$ so that $\pi_1(A_k)$ converges strongly to $\pi_1(\gamma_1(A))$ as $k \to \infty$. It then follows that if $A \in R^G(\omega)$ there is a sequence $A_n \in \overline{\mathfrak{A}}_n^G$ with $\|A_n\| \leqslant \|A\|$ so that $\pi_1(A_n)$ converges strongly to $\pi_1(\gamma_1(A))$ as $n \to \infty$.

Since $A_n \subset \overline{\mathfrak{A}}_n^G \subset \overline{\mathfrak{A}}_n$, $A_n$ can be expressed in the form

$$A_n = \sum_{i,j=1}^{r} \gamma_1(e_{ij})A_n(i,j)$$

where the $\{e_{ij}\,;\, i,j = 1, \ldots, r\}$ are matrix units for $\mathfrak{B}_0$ and the $A_n(i,j) \in \mathfrak{A}_n^c$. Explicitly, the $A_n(i,j)$ are given by the formula,

$$A_n(i,j) = \sum_{k=1}^{r} \gamma_1(e_{ki}) A_n \gamma_1(e_{jk}).$$

Now for each $n$ the $(r \times r)$-matrix of operators $\pi_2(A_n(i,j))$ is bounded in norm by $\|A\|$. Since the unit ball of $\mathfrak{B}(\mathscr{H}_2)$ is weakly compact there is a cluster point $C = C(i,j)$ of the sequence $\pi_2(A_n(i,j))$ in the weak topology. Since the $C(i,j)$ are weak limit points of the $\pi_2(A_n(i,j)) \in \pi_2(\mathfrak{A})$ we have $C(i,j) \in \pi_2(\mathfrak{A})''$. And since $A_n(i,j) \in \mathfrak{A}_n^c$ we have $C(i,j) \in \pi_2(\mathfrak{A}_n)'$ for all $n = 1, 2, \ldots$ . Hence, $C(i,j) \in \pi_2(\mathfrak{A})'' \cap \cap \pi_2(\mathfrak{A})'$. Since $\omega'$ is a factor state of $\mathfrak{A}$ the $C(i,j)$ must be multiples of the identity (i.e. $C(i,j) = c_{ij}I$). Let $B = \sum_{i,j=1}^{r} c_{ij} e_{ij}$. It follows that $\pi_2(\gamma_1(B))$ is a cluster point of the sequence $\pi_2(A_n)$. Since $A_n \in \mathfrak{A}^G$ we have that $B \in R^G(\omega')$. One sees that $\varphi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(B))$ as follows.

Since $\pi_1(A_n) \to \pi_1(\gamma_1(A))$ strongly we have $\pi_1^H(A_n) \to \pi_1^H(\gamma_1(A))$ strongly as $n \to \infty$. Since $\Phi$ is strongly bicontinuous on bounded sets we have $\pi_2^H(A_n) = = \Phi(\pi_1^H(A_n)) \to \Phi(\pi_1^H(\gamma_1(A)))$ strongly as $n \to \infty$. Since $\pi_2(\gamma_1(B))$ is a weak cluster point of the $\pi_2(A_n)$ we have $\pi_2^H(\gamma_1(B))$ is a weak cluster point of the $\pi_2^H(A_n)$. Since $\pi_2^H(A_n) \to \Phi(\pi_1^H(\gamma_1(A)))$ strongly as $n \to \infty$ we have $\Phi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(B))$.

Hence, for each $A \in R^G(\omega)$ there is a $B \in R^G(\omega')$ so that $\Phi(\pi_1^H(\gamma_1(A))) = = \pi_2^H(\gamma_1(B))$. We could have just as well started with a $B \in R^G(\omega')$ and then found a corresponding $A \in R^G(\omega)$. It is very tempting to conclude at this point that we have a $*$-isomorphism of $R^G(\omega)$ with $R^G(\omega')$. However, all we can safely conclude is that $\Phi$ provides a $*$-isomorphism of $\pi_1^H(\gamma_1(R^G(\omega)))$ with $\pi_2^H(\gamma_1(R^G(\omega')))$.

One easily checks that for $A \in R^G(\omega)$ we have $\pi_1^H(\gamma_1(A)) = 0$ if and only if $AE = 0$ and for $A \in R^G(\omega')$, $\pi_2^H(\gamma_1(A)) = 0$ if and only if $AE' = 0$, where $E$ and $E'$ are the $G_\gamma$-support projections for $\omega$ and $\omega'$. Hence, $\Phi$ provides a $*$-isomorphism of $R^G(\omega)E$ with $R^G(\omega')E'$.

Let $e_0$ and $e_0'$ be the maximal projections in $N^G(\omega)$ and $N^G(\omega')$. Let $N = = \sum_{k=1}^{\infty} \omega_k(e_0)$. Since $\omega^G$ is a factor state it follows from the previous theorem that $\omega_k(e_0) = 0$ or $1$ so $N$ is an integer. We show $\Phi(\pi_1^H(\gamma_1(e_0))) = \pi_2^H(\gamma_1(e_0'))$ and $\sum_{k=1}^{\infty} \omega_k'(e_0') = N$. Let $e$ be the largest projection in $R^G(\omega')$ so that $\Phi(\pi_1^H(\gamma_1(e_0))) = = \pi_2^H(\gamma_1(e))$. Let $D_n = \sum_{k=1}^{n} \pi_1^H(\gamma_k(e_0))$. Since the $D_n$ converge to a central element and $\omega(D_n) \to N$, we have $D_n \to N \cdot I$ strongly as $n \to \infty$. Since $\Phi$ is bicontinuous we have $\sum_{k=1}^{\infty} \omega_k'(e) = N$. Hence, $e \leqslant e_0'$. If $e \neq e_0'$ then let $f$ be the largest projection in $R^G(\omega)$ so that $\Phi(\pi_1^H(\gamma_1(f))) = \pi_2^H(\gamma_1(e_0'))$. Clearly, $f \neq e_0$, $f \geqslant e_0$ and $\sum_{k=1}^{\infty} \omega_k(f) = \sum_{k=1}^{\infty} \omega_k'(e_0') < \infty$. But this contradicts the fact that $e_0$ is maximal in $N^G(\omega)$. Hence, $e = e_0'$, $\Phi(\pi_1^H(\gamma_1(e_0))) = \pi_2^H(\gamma_1(e_0'))$ and $\sum_{k=1}^{\infty} \omega_k'(e_0') = N$.

Since $I - e_0 \leqslant E$ and $I - e_0' \leqslant E'$ the mappings $A \to \pi_1^H(\gamma_1(A))$ and $A' \to \pi_2^H(\gamma_1(A'))$ are faithful for $A \in R^G(\omega)(I - e_0)$ and $A' \in R^G(\omega')(I - e_0')$, respectively. Hence, there is a $*$-isomorphism $\alpha$ of $R^G(\omega)(I - e_0)$ onto $R^G(\omega')(I - e_0')$ so that $\Phi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(\alpha(A)))$ for all $A \in R^G(\omega)(I - e_0)$. We will show that there is a unitary $U_1 \in \mathfrak{B}_0$ so that $\alpha(A) = U_1 A U_1^{-1}$ for all $A \in R^G(\omega)(I - e_0)$.

It is well known that a $*$-isomorphism $\alpha$ between two finite dimensional algebras acting on a finite dimensional Hilbert space is unitarily implementable if and only of rank$(e)$ = rank$(\alpha(e))$ for all minimal projections $e$. Suppose that $e$ is a minimal projection in $R^G(\omega)(I - e_0)$ and $r$ = rank$(e)$. Let $Q$ be a finite set of positive integers. Suppose $f_1 \in R^G(\omega)$ and $f_2 \in R^G(\omega')$ and $f_1$ and $f_2$ are projections. For $i = 1, 2$ we define

$$\Gamma_i(Q, f_i) = \pi_i^H\Big( \sum_{\sigma \in S(Q)} \delta(\sigma) U_\sigma \prod_{k \in Q} \gamma_k(f_i)\Big)$$

where $S(Q)$ is the set of permutations $\sigma$ so that $\sigma(i) = i$ for $i \notin Q$, $\delta(\sigma) = 1$ for even permutations and $\delta(\sigma) = -1$ for odd permutations. One finds that $\Gamma_i(Q, f_i) = 0$ if the number of elements in $Q$ exceeds the rank of $f_i$. (This corresponds to the fact that the $n$ fold antisymmetric tensor product of a Hilbert space is zero if $n$ exceeds the dimension of the Hilbert space.) We show that if $Q$ has $r$ elements then $\Gamma_1(Q, e) \neq 0$.

Suppose $Q$ has $r$ elements and $\Gamma_1(Q, e) = 0$. Then we have $\pi_1^H(U_\sigma)\Gamma_1(Q, e) \pi_1^H(U_\sigma^{-1}) = 0$ for all finite permutations $\sigma$. Hence, $\Gamma_1(Q, e) = 0$ for all sets $Q$ containing $r$ elements. Now let $\bar{\omega} = \bigotimes_{k=1}^{\infty} \bar{\omega}_k$ be the state constructed from $\omega$ as was done in the proof of the previous theorem. Let $(\bar{\pi}_1, \bar{\mathscr{H}}_1, \bar{f}_1)$, $(\bar{\pi}_1^G, \bar{\mathscr{H}}_1^G, \bar{f}_1)$ and $(\bar{\pi}_1^H, \bar{\mathscr{H}}_1^H, \bar{f}_1)$ be the associated representations constructed from $\bar{\omega}$. Let $\bar{\Gamma}_1(Q, e)$ be defined as $\Gamma_1(Q, e)$, using the representation $\bar{\pi}_1^H$. We saw in the proof of the previous theorem that $\omega^G \underset{q}{\sim} \bar{\omega}^G$ and, therefore, $\omega^{H_1} \underset{q}{\sim} \bar{\omega}^{H_1}$. Hence, if $\Gamma_1(Q, e) = 0$ we must have $\bar{\Gamma}_1(Q, e) = 0$ (since quasi-equivalent representations have the same kernel). A somewhat lengthy computation shows

$$(\bar{f}_1, \bar{\Gamma}_1(Q, e)^2 \bar{f}_1) \geqslant \prod_{k \in Q} \bar{\omega}_k(e) = \prod_{k \in Q} \omega_k(e).$$

Since $\sum_{k=1}^{\infty} \omega_k(e) = \infty$ we can certainly find a set $Q$ of $r$ elements so that $\prod_{k \in Q} \omega_k(e) \neq 0$.

Hence, $\bar{\Gamma}_1(Q, e) \neq 0$. This is the desired contradiction, so we have shown that $\Gamma_1(Q, e) \neq 0$ for $Q$ having $r$ elements. Since $\Phi$ is a $*$-isomorphism we have $\Phi(\Gamma_1(Q, e)) = \Gamma_2(Q, \alpha(e)) \neq 0$. Hence, the rank of $\alpha(e)$ must equal or exceed $r$. Hence, we have shown that $\operatorname{rank}(e) \leqslant \operatorname{rank}(\alpha(e))$ for all minimal projections in $R^G(\omega)(I - e_0)$.

Repeating this argument with $\pi_2^H$ to $\pi_1^H$ reversed we find $\operatorname{rank}(e) \geqslant \operatorname{rank}(\alpha(e))$ for minimal projections $e \in R^G(\omega)(I - e_0)$. Hence, $\operatorname{rank}(e) = \operatorname{rank}(\alpha(e))$ and there is a unitary $U_1 \in \mathfrak{B}_0$ so that $\alpha(A) = U_1AU_1^{-1}$ for all $A \in R^G(\omega)(I - e_0)$.

Let $V = U_1(I - e_0)$. Note $V$ is a partial isometry so that $\alpha(A) = VAV^*$ for all $A \in R^G(\omega)(I - e_0)$. We will extend $V$ to a unitary $U$ so that $\Phi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(UAU^{-1}))$ for all $A \in R^G(\omega)$.

As we have seen $e_0$ and $E$ are commuting projections in $R_0'$ so $Ee_0$ is a projection in $R_0'$. Let $\{e_1, \ldots, e_m\}$ be minimal central projections in $R_0 \cap R_0'$ so that $Ee_0e_i \neq 0$ for $i = 1, \ldots, m$. It follows from the first part of the proof of the previous theorem that $e_0e_i$ is a minimal central projection in $R^G(\omega) \cap R^G(\omega)'$ and $Ee_0e_i$ is a minimal projection in $R_0'$. Let $N_i = \sum_{k=1}^{\infty} \omega_k(e_0e_i)$. Note $N = \sum_{i=1}^{m} N_i$.

Since $\gamma_1(e_i) \in \mathfrak{A}^G$ we have $\Phi(\pi_1^H(\gamma_1(e_i))) = \pi_2^H(\gamma_1(e_i))$. Since $\Phi$ is an isomorphism and $\Phi(\pi_1^H(\gamma_1(e_0))) = \pi_2^H(\gamma_1(e_0'))$ we have $\Phi(\pi_1^H(\gamma_1(e_0e_i))) = \pi_2^H(\gamma_1(e_0'e_i))$ for $i = 1, \ldots, m$. Again it follows from the first part of the proof of the previous theorem that $e_0'e_i$ is a minimal central projection in $R^G(\omega') \cap R^G(\omega')'$ and $E'e_0'e_i$ is a minimal projection in $R_0'$. Furthermore, $\sum_{k=1}^{\infty} \omega_k'(e_0'e_i) = N_i$ since $D_{in} = \sum_{k=1}^{n} \pi_1^H(\gamma_k(e_0e_i)) \to N_iI$ as $n \to \infty$ so we have $(f_2, \varphi(D_{in})f_2) = \sum_{k=1}^{n} \omega_k'(e_0'e_i) \to N_i$ as $n \to \infty$.

Since $e_i$ is a minimal projection in $R_0 \cap R_0'$ we have $e_iR_0' = R_0'e_i$ is an $(s_i \times s_i)$-matrix algebra. Since $Ee_0e_i$ and $E'e_0'e_i$ are minimal projections in $e_iR_0'$ there is a partial isometry $V_{1i} \in e_iR_0'$ so that $V_{1i}^*V_{1i} = Ee_0e_i$ and $V_{1i}V_{1i}^* = E'e_0'e_i$. Since $\gamma_1(e_i) \in \mathfrak{A}^G$, $\alpha((I - e_0)e_i) = U_1(I - e_0)e_iU_1^{-1} = (I - e_0')e_i$ and we have $\operatorname{rank}((I - e_0)e_i) = \operatorname{rank}((I - e_0')e_i)$. Hence, $\operatorname{rank}(e_0e_i) = \operatorname{rank}(e_0'e_i)$ and, thus, $\operatorname{rank}(e_0e_i - Ee_0e_i) = \operatorname{rank}(e_0'e_i - E'e_0'e_i)$. Since two projections of the same rank in an $(s_i \times s_i)$-matrix algebra are related by a partial isometry there is a partial isometry $V_{2i} \in e_iR_0'$ so that $V_{2i}^*V_{2i} = e_0e_i - Ee_0e_i$ and $V_{2i}V_{2i}^* = e_0'e_i - E'e_0'e_i$. Let $V_i = V_{1i} + V_{2i}$. Then $V_i^*V_i = e_0e_i$, $V_iV_i^* = e_0'e_i$ and $V_iEe_0e_iV_i^* = E'e_0'e_i$.

Finally let $e_{m+1}$ be the largest central projection in $R_0 \cap R_0'$ so that $Ee_{m+1} = 0$. Since $Ee_{m+1} = 0$ we have $\pi_1^H(\gamma_1(e_{m+1})) = 0$ and, thus, $\Phi(\pi_1^H(\gamma_1(e_{m+1}))) = \pi_2^H(\gamma_1(e_{m+1})) = 0$. Hence, $E'e_{m+1} = 0$. Let $V_{m+1} = e_{m+1}$ and let $U = V + \sum_{i=1}^{m+1} V_i$. A little checking shows $U$ is unitary, $UAU^{-1} = A$ for all $A \in R_0$, so $U \in R_0'$, and $\Phi(\pi_1^H(\gamma_1(A))) = \pi_2^H(\gamma_1(UAU^{-1}))$ for all $A \in R^G(\omega)$ and, finally, $E' = UEU^{-1}$.

We will show that $\omega \underset{q}{\sim} \omega' \circ \alpha_U$. Let $\rho = \omega' \circ \alpha_U$. Note $\omega'^G = \rho^G$, $R^G(\rho) = U^{-1}R^G(\omega')U = R^G(\omega)$ and, hence, by Theorem 3.8, $H^G(\rho) = H^G(\omega) = H_1$. Let $\pi_3(A) = \pi_2(\alpha_U(A))$ for $A \in \mathfrak{A}$, $\pi_3^G(A) = \pi_2^G(\alpha_U(A)) = \pi_2^G(A)$ for $A \in \mathfrak{A}^G$ and $\pi_3^H(A) = \pi_2^H(\alpha_U(A))$ for $A \in \mathfrak{A}^{H_1}$ (note that $\alpha_U(\mathfrak{A}^{H_1}) = \mathfrak{A}^{H_2}$). Note $\pi_3$, $\pi_3^G$ and $\pi_3^H$ are cyclic $*$-representations induced by $\rho, \rho^G$ and $\rho^{H_1}$, respectively. Note $\pi_3(\mathfrak{A}^G)'' = \pi_3(\mathfrak{A}^{H_1})''$. Since $\omega'^G = \rho^G$ we have $\pi_2^G \underset{q}{\sim} \pi_3^G$. Let $\Phi_1$ be the unique $*$-isomorphism of $\pi_2^H(\mathfrak{A}^{H_2})'' = \pi_2^G(\mathfrak{A}^G)''$ onto $\pi_3(\mathfrak{A}^{H_1})'' = \pi_3(\mathfrak{A}^G)''$ so that $\Phi_1(\pi_2^G(A)) = \pi_3^G(A)$ for all $A \in \mathfrak{A}^G$. Clearly, we have $\Phi_1(\pi_2^H(\gamma_1(A))) = \pi_3^H(\gamma_1(U^{-1}AU))$ for $A \in R^G(\omega')$. Let $\Psi = \Phi_1 \circ \Phi$. We have $\Psi$ is a $*$-isomorphism of $\pi_1^H(\mathfrak{A}^{H_1})'' = \pi_1(\mathfrak{A}^G)''$ onto $\pi_3^H(\mathfrak{A}^{H_1})'' = \pi_3(\mathfrak{A}^G)''$ so that $\Psi(\pi_1^G(A)) = \pi_3^G(A)$ for $A \in \mathfrak{A}^G$ and $\Psi(\pi_1^H(\gamma_1(A))) = \pi_3^H(\gamma_1(A))$ for $A \in R^G(\omega)$. Since $\mathfrak{A}^{H_1}$ is generated by $\gamma_1(R^G(\omega))$ and $\mathfrak{A}^G$, and $\Psi$ is a $*$-isomorphism, we have $\Psi(\pi_1^H(A)) = \pi_3^H(A)$ for all $A \in \mathfrak{A}^{H_1}$. Hence, $\omega^{H_1} \underset{q}{\sim} \rho^{H_1}$.

Since $\omega^{H_1} \underset{q}{\sim} \rho^{H_1}$ it follows from ([5], Theorem 4.5) that for every $\varepsilon > 0$ (and in particular for $\varepsilon = 1/2$) there is an integer $n_1$ so that

$$\|(\omega^{H_1} - \rho^{H_1})|(\mathfrak{A}_{n_1}^{H_1})^c \cap \mathfrak{A}^{H_1}\| < \frac{1}{2}.$$

Since $\mathfrak{A}_{n_1}^c \cap \mathfrak{A}^{H_1} \subset (\mathfrak{A}_{n_1}^{H_1})^c \cap \mathfrak{A}^{H_1}$ we have

$$\|(\omega^{H_1} - \rho^{H_1})|\mathfrak{A}_{n_1}^c \cap \mathfrak{A}^{H_1}\| < \frac{1}{2}.$$

Let $\bar{\omega}$ and $\bar{\rho}$ be the $H_1$-invariant states constructed from $\omega$ and $\rho$ by the procedure described in the statement of Theorem 3.8. By Theorem 3.8 we have $\omega \underset{q}{\sim} \bar{\omega}$ and $\rho \underset{q}{\sim} \bar{\rho}$. Hence, there are integers $n_2$ and $n_3$ so that

$$\|(\omega - \bar{\omega})|\mathfrak{A}_{n_2}^c\| < \frac{1}{2} \quad \text{and} \quad \|(\rho - \bar{\rho})\,|\mathfrak{A}_{n_3}^c\| < \frac{1}{2}.$$

Combining these inequalities with the above inequality we find

$$\|(\bar{\omega} - \bar{\rho})|\mathfrak{A}_n^c \cap \mathfrak{A}^{H_1}\| < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3/2,$$

where $n = \max(n_1, n_2, n_3)$. Suppose $\Gamma$ is the conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}^{H_1}$ given by

$$\Gamma(A) = \int_{U \in H_1} \alpha_U(A)\, d\nu(U)$$

where $\nu$ is Haar measure on $H_1$. Suppose $A \in \mathfrak{A}_n^c$. Since $\bar{\omega}$ and $\bar{\rho}$ are $H_1$-invariant we have $\bar{\omega}(A) = \bar{\omega}(\Gamma(A))$ and $\bar{\rho}(A) = \bar{\rho}(\Gamma(A))$. Since $\Gamma(A) \in \mathfrak{A}_n^c \cap \mathfrak{A}^{H_1}$ we have

$$|\bar{\omega}(A) - \bar{\rho}(A)| = |\bar{\omega}(\Gamma(A)) - \bar{\rho}(\Gamma(A))| \leqslant (3/2)\|\Gamma(A)\| \leqslant (3/2)\|A\|.$$

Hence, $\|(\bar{\omega} - \bar{\rho})|\mathfrak{A}_n^c\| \leqslant 3/2 < 2$ and, thus, $\bar{\omega} \underset{q}{\sim} \bar{\rho}$. Since $\omega \underset{q}{\sim} \bar{\omega}$ and $\rho \underset{q}{\sim} \bar{\rho}$ we have $\omega \underset{q}{\sim} \rho = \omega' \circ \alpha_U$.

Hence, $U$ satisfies conditions i) and ii) of the theorem. We show $U$ satisfies condition iii). Suppose $e \in R_0'$ and $e \leqslant E$ and $\omega_k(e) = 0$ or 1 for all $k$. Then $e \in M^G(\omega)$. Since $\omega \underset{q}{\sim} \rho$ we have from Lemma 4.4 that $M^G(\omega) = M^G(\rho)$. Since $\rho^G$ is a factor state we have from the previous theorem that $\rho_k(e) = \omega_k'(UeU^{-1}) = 0$ or 1 for all $k = 1, 2, \ldots$ . Similarly if $\rho_k(e) = 0$ or 1 for all $k$ we find $\omega_k(e) = 0$ or 1 for all $k$. Hence, if $e \in R_0'$ and $e \leqslant E$ we have $\omega_k(e) = 0$ or 1 for all $k$ if and only if $\omega_k'(UeU^{-1}) = 0$ or 1 for all $k$.

If $e \in R_0'$, $e \leqslant E$ and $\omega_k(e) = 0$ or 1 for all $k$ then by Theorem 4.7 we have $e = f_0 + \sum_{i=1}^{r} s_i f_i$ where $s_i = 0$ or 1, $f_0 \leqslant e_0$ and the $f_i$ are central projections in $R^G(\omega) \cap R^G(\omega)'$ with $f_i e_0 = 0$ for $i = 1, \ldots, r$. To show $U$ satisfies the sum condition in iii) it is sufficient to show the sum conditions are satisfied for the projections $f_i$ for $i = 0, 1, \ldots, r$ individually. Recall $e_1, \ldots, e_m$ are minimal projections in $R_0 \cap R_0'$ such that $Ee_0e_i \neq 0$. Since $Ee_0e_i$ is minimal in $R_0'$ we have $f_0$ is a sum of these projections, i.e., $f_0 = Ee_0e_{i_1} + \ldots + Ee_0e_{i_s}$. Then $Uf_0U^{-1} = E'e_0'e_{i_1} + \ldots + E'e_0'e_{i_s}$. Since $\sum_{k=1}^{\infty} \omega_k(e_0e_i) = N_i = \sum_{k=1}^{\infty} \omega_k'(e_0'e_i)$ we have

$$\sum_{k=1}^{\infty} \omega_k(f_0) = N_{i_1} + \ldots + N_{i_s} = \sum_{k=1}^{\infty} \omega_k'(Uf_0U^{-1}).$$

Hence, the sum condition iii) is satisfied for $f_0$.

Now we consider the $f_i$ for $i = 1, \ldots, r$. We have $\omega_k(f_i) = 0$ or 1 and $\rho_k(f_i) = 0$ or 1 for all $k = 1, 2, \ldots$ . If $\omega_k(f_i) \neq \rho_k(f_i)$ we have $\|\omega_k - \rho_k\| = 2$, since they have disjoint support. Since $\omega \underset{q}{\sim} \rho$ we have that $\|\omega_k - \rho_k\| = 2$ for only a finite number of $k$ (or else $\|(\omega - \rho)|\mathfrak{A}_n^c\| = 2$ for all $n = 1, 2, \ldots$). Hence, we have $\sum_{n=1}^{\infty} |\omega_k(f_i) - \omega_k'(Uf_iU^{-1})| < \infty$. For $i$ satisfying $1 \leqslant i \leqslant r$ let

$$J_n = \sum_{k=1}^{n} \gamma_k(f_i) - \omega_k(f_i)I \quad \text{and} \quad U_n(t) = \exp(\mathrm{i}tJ_n).$$

Recalling the proof of Lemma 4.8 we see that $\pi_1^H(U_n(t)) \to V(t) \in \pi_1^H(\mathfrak{A}^{H_1})'' \cap$

$\cap \pi_1^H(\mathfrak{A}^{H_1})' = \{\lambda I\}$. Since $\omega(J_n) = 0$ we see that $\pi_1^H(U_n(t)) \to I$ strongly as $n \to \infty$. Since $\Psi = \Phi_1 \circ \Phi$ is strongly continuous on bounded sets we have $\Psi(\pi_1^H(U_n(t))) \to I$ strongly as $n \to \infty$. Now,

$$(f_2, \Psi(\pi_1^H(U_n(t)))f_2) = \rho(U_n(t)) =$$

$$= \prod_{k=1}^{n} (1 + (e^{it} - 1)\rho_k(f_i))\exp(-\,it\omega_k(f_i)) = \prod_{k=1}^{n} S_k(t) \to 1$$

as $n \to \infty$. Since $S_k(t) = 1$ for all $t$ (except for a finite number of $k$) we have $\prod_{k=1}^{n} S_k(t) = 1$ for $n$ sufficiently large. Then for $n$ sufficiently large we have $(\mathrm{d}/\mathrm{d}t) \prod_{k=1}^{n} S_k(t)|_{t=0} = \mathrm{i} \sum_{k=1}^{n} \rho_k(f_i) - \omega_k(f_i) = 0$. Hence, we have $\sum_{k=1}^{\infty} \omega_k(f_i) -$ $- \omega_k'(Uf_iU^{-1}) = 0$ for each $i = 1, \ldots, r$. Hence, we have shown condition iii) is satisfied for each $f_i$ for $i = 0, 1, \ldots, r$. Hence, $U$ satisfies condition iii). ▨

Lemma 4.15. *Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ is a product state of $\mathfrak{A}$ of minimal $G_\gamma$-support. Suppose $\omega^G$ is pure. Then $\omega_k$ is pure for each $k = 1, 2, \ldots$ .*

*Proof.* Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$, $\omega$ is of minimal $G_\gamma$-support and $\omega^G$ is pure. Suppose $\omega_i$ is not pure. Let $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for all $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Suppose $\sigma = \sigma(i, j)$ is the permutation which transposes $i$ and $j$ and suppose $V \in R_0$ is unitary and $\{e_{kl}\,; k, l = 1, \ldots, r\}$ are matrix units for $\mathfrak{B}_0$. Then we have

$$\omega(\gamma_i(V)U_{\sigma(i,j)}\gamma_i(V^{-1})) = \sum_{k,l=1}^{r} \omega(\gamma_i(V)\gamma_i(e_{kl})\gamma_j(e_{lk})\gamma_i(V^{-1})) =$$

$$= \sum_{k,l=1}^{r} \mathrm{tr}(Ve_{kl}V^{-1}\Omega_i)\mathrm{tr}(e_{lk}\Omega_j) = \mathrm{tr}(V\Omega_jV^{-1}\Omega_i).$$

Let $E_i$ and $E_j$ be the support projections for $\Omega_i$ and $\Omega_j$. Let $e_i$ and $e_j$ be rank one projections in $\mathfrak{B}_0$ with $e_i \leqslant E_i$ and $e_j \leqslant E_j$. Let $\omega_i'(A) = \mathrm{tr}(Ae_i)$, $\omega_j'(A) =$ $= \mathrm{tr}(Ae_j)$ and $\omega_k'(A) = \omega_k(A)$ for $k \neq i$ or $k \neq j$ for all $A \in \mathfrak{B}_0$. Let $\omega' = \bigotimes_{k=1}^{\infty} \omega_k'$. Since $\lambda\omega' \leqslant \omega$ for some $\lambda > 0$ and since $\omega^G$ is pure we have $\omega'^G = \omega^G$. Hence,

$$\omega'(U_{\sigma(i,j)}) = \mathrm{tr}(e_je_i) = \omega(U_{\sigma(i,j)}).$$

We identify $\mathfrak{B}_0$ with $\mathfrak{B}(\mathscr{H}_0)$ where $\mathscr{H}_0$ is an $r$-dimensional complex Hilbert space. If $f_i$ and $f_j$ are unit vectors in the range of $e_i$ and $e_j$, respectively, then we have $\mathrm{tr}(e_je_i) = |(f_i, f_j)|^2$. Hence, if $f_i$ and $f_j$ are unit vectors in the range of $E_i$

and $E_j$, respectively, we have $|(f_i,f_j)|^2 = \omega(U_{\sigma(i,j)})$. If $E_iE_j \neq 0$ there are unit vectors $f_i$ and $f_j$ in the range of $E_i$ and $E_j$, respectively, so that $(f_i,f_j) \neq 0$. Since $\omega_i$ is not pure the rank of $E_i$ is at least two so there is another unit vector $f_i'$ in the range of $E_i$ so that $(f_i',f_j) = 0$. But this is a contradiction, since $|(f_i',f_j)|^2 = \omega(U_{\sigma(i,j)}) \neq 0$. Hence, we have $E_iE_j = 0$.

Now if $V \in R_0$ is unitary we have

$$\omega'(\gamma_i(V)U_{\sigma(i,j)}\gamma_i(V^{-1})) = \mathrm{tr}(Ve_jV^{-1}e_i) = \omega(\gamma_i(V)U_{\sigma(i,j)}\gamma_i(V^{-1})).$$

Thus, the same argument that shows $E_iE_j = 0$ shows that $VE_iV^{-1}E_j = 0$ for all unitary $V \in R_0$.

Let $e$ be the smallest projection in $\mathfrak{B}_0$ so that $e \geqslant VE_iV^{-1}$ for all unitary $V \in R_0$. Clearly, $e \in R_0'$, $\omega_i(e) = 1$ and from the last paragraph $\omega_k(e) = 0$ for $k \neq i$. As we saw in the discussion at the beginning of this section, it follows from the fact that $\omega_i(e) = 1$ and $\omega_k(e) = 0$ for $k \neq i$ that the dependence of $\omega^G$ on $\omega_i$ is completely determined by $\omega_i|R_0$. Since $\omega^G$ is pure it then follows that $\omega_i|R_0$ is pure (since decomposing $\omega_i|R_0$ yields a decomposition of $\omega^G$). Since $\omega$ is of minimal $G_\gamma$-support it follows that $\omega_i|R_0'$ is pure. Hence, $\omega_i$ is pure. But this contradicts our assumption that $\omega_i$ is not pure. Hence, we have shown $\omega_k$ is pure for all $k = 1, 2, \ldots$ .

THEOREM 4.16. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$ *of minimal* $G_\gamma$-*support. Suppose* $E$ *is the* $G_\gamma$-*support projection of* $\omega$. *Then* $\omega^G$ *is pure if and only if* $\omega_k$ *is pure for all* $k = 1, 2, \ldots$ *and*

$$(\#) \qquad \sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) = 0 \quad \text{or} \quad \infty$$

*for all projections* $e \in R_0'$ *with* $e \leqslant E$.

*Proof.* Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ is a product state of minimal $G_\gamma$-support and $E$ is the $G_\gamma$-support projection of $\omega$. Suppose $\omega^G$ is pure. Then $\omega^G$ is a factor state and by Theorem 4.13 the sum condition (#) is satisfied for all projections $e \in R_0'$ with $e \leqslant E$. By Lemma 4.15 we have $\omega_k$ is pure for each $k = 1, 2, \ldots$ . Hence, we have proved the implication ($\Rightarrow$).

Now suppose $\omega_k$ is pure for each $k = 1, 2, \ldots$ and the sum condition (#) is satisfied for all $e \in R_0'$ with $e \leqslant E$. Let $H = H^G(\omega)$ and $K = \{U \in H\,;\, UE = EU\}$. Note $K'$ is generated by $R^G(\omega)$ and $E$. Let $(\pi, \mathscr{H}, f_0\}$ be a cyclic $*$-representation of $\mathfrak{A}$ induced by $\omega$. Let $\mathscr{H}^G$, $\mathscr{H}^H$ and $\mathscr{H}^K$ be the closure of $\{\pi(\mathfrak{A}^G)f_0\}$, $\{\pi(\mathfrak{A}^H)f_0\}$ and $\{\pi(\mathfrak{A}^K)f_0\}$, respectively. Let $\{\pi^G, \mathscr{H}^G, f_0\}$, $\{\pi^H, \mathscr{H}^H, f_0\}$ and $\{\pi^K, \mathscr{H}^K, f_0\}$ be the restrictions of $\pi$ to $\mathfrak{A}^G$, $\mathfrak{A}^H$ and $\mathfrak{A}^K$ and $\mathscr{H}^G$, $\mathscr{H}^H$ and $\mathscr{H}^K$, respectively (e.g.,

$\pi^H(A)f = \pi(A)f$ for all $f \in \mathscr{H}^H$ and $A \in \mathfrak{A}(^H)$. By Lemma 3.7 and Theorem 3.8 we have $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)''$. Hence, $\mathscr{H}^G = \mathscr{H}^H$ and $\pi^G(\mathfrak{A}^G)'' = \pi^H(\mathfrak{A}^H)''$. We show that $\mathscr{H}^H = \mathscr{H}^K$ and $\pi^K(\mathfrak{A}^K)'' = \pi^H(\mathfrak{A}^H)''$.

Let $E_n = \pi^K(\gamma_1(E)\gamma_2(E)\ldots\gamma_n(E))$ and let $E_0$ be the strong limit of this decreasing sequence of projections. We have $E_0 f_0 = f_0$ since $\pi^K(\gamma_k(E))f_0 = f_0$ for all $k = 1, 2, \ldots$ . Clearly, we have $\pi^K(U_\sigma)E_0\pi^K(U_\sigma^{-1}) = E_0$ for all finite permutations $\sigma$. By the remark after Lemma 4.6 we have $E \in R^G(\omega)'$. Hence, $E \in \{R^G(\omega), E\}'$ so $E_0$ commutes with $\pi^K(\gamma_1(A))$ for all $A \in \{R^G(\omega), E\}''$. Since $\gamma_1(A)$ commutes with $A \in \{R^G(\omega), E\}''$ and the $U_\sigma$ generate $\mathfrak{A}^K$ we have $E_0 \in \pi^K(\mathfrak{A}^K)'$. Since $E_0 f_0 = f_0$ and $f_0$ is cyclic in $\mathscr{H}^K$ we have $E_0 = I$. Since $\pi^K(\gamma_1(E)) \geqslant E_0 = I$ we have $\pi^K(\gamma_1(E)) = I$. Since $\mathfrak{A}^K$ is generated by $\gamma_1(E)$ and $\mathfrak{A}^H$ we have $\mathscr{H}^H = \mathscr{H}^K$ and $\pi^H(\mathfrak{A}^H)'' = \pi^K(\mathfrak{A}^K)''$. Hence, we have $\mathscr{H}^G = \mathscr{H}^H = \mathscr{H}^K$ and $\pi^G(\mathfrak{A}^G)'' = \pi^H(\mathfrak{A}^H)'' = \pi^K(\mathfrak{A}^K)''$. Hence, $\omega^G$ is pure if and only if $\omega^K$ is pure. We show $\omega^K$ is pure.

Suppose $e \in \{R^G(\omega), E\}'$ and $U = I - 2e$. Since $U \in R^G(\omega)'$ we have by Lemma 3.7 and Theorem 3.8 that $U \in H^G(\omega)$. Hence, we have $\omega \underset{q}{\sim} \omega\circ\alpha_U$. Let $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ for $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Then we have

$$\sum_{k=1}^{\infty} \|\Omega_k^{1/2} - U^{-1}\Omega_k^{1/2}U\|_{\mathrm{HS}}^2 < \infty.$$

Since $\omega_k$ is pure for each $k$ we have $\Omega_k^{1/2} = \Omega_k$ and (since $U = I - 2e$)

$$\|\Omega_k^{1/2} - U^{-1}\Omega_k^{1/2}U\|_{\mathrm{HS}}^2 = \omega_k(e)(1 - \omega_k(e)).$$

Hence, $\sum_{k=1}^{\infty} \omega_k(e)(1 - \omega_k(e)) < \infty$. Since $\omega_k(Ee) = \omega_k(e)$ and $Ee \in R_0'$ and $Ee \leqslant E$, it follows from the sum condition (#), which $\omega$ is assumed to satisfy, that $\omega_k(Ee) = \omega_k(e) = 0$ or 1 for all $k$. Hence, we have shown that if $e$ is a projection in $\{R^G(\omega), E\}'$ then $\omega_k(e) = 0$ or 1 for all $k = 1, 2, \ldots$ . Hence, if $e \in \{R^G(\omega), E\}'$ we have $\omega_k(eA) = \omega_k(Ae)$ for all $A \in \mathfrak{B}_0$. Since each unitary $U \in K$ is a linear combination of projections in $\{R^G(\omega), E\}'$ we have $\omega_k(UA) = \omega_k(AU)$ or $\omega_k(UAU^{-1}) = \omega_k(A)$ for all $A \in \mathfrak{B}_0$, $U \in K$ and $k = 1, 2, \ldots$ . Hence, $\omega = \omega\circ\alpha_U$ for all $U \in K$.

Let $\Gamma$ be the conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}^K$ given by

$$\Gamma(A) = \int_{U\in K} \alpha_U(A)\mathrm{d}\nu(U)$$

where $\nu$ is Haar measure on $K$. Since $\omega$ is $\alpha_U$ invariant for $U \in K$ we have $\omega(A) = \omega(\Gamma(A)) = \omega^K(\Gamma(A))$ for all $A \in \mathfrak{A}$. Now suppose $\omega^K$ is not pure. Then we have $\omega^K = (1/2)(\rho_1 + \rho_2)$ with $\rho_1 \neq \rho_2$ states of $\mathfrak{A}^K$. But then $\omega = (1/2)(\tau_1 + \tau_2)$ where $\tau_i(A) = \rho_i(\Gamma(A))$ for $A \in \mathfrak{A}$. Since $\rho_1 \neq \rho_2$ we have $\tau_1 \neq \tau_2$ and hence $\omega$

is not pure. But this is a contradiction since $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ and the tensor product of pure states is well known to be pure. Hence, we have shown that $\omega^K$ is pure and, thus, $\omega^G$ is pure. ▨

THEOREM 4.17. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ *is a product state of* $\mathfrak{A}$ *so that* $\omega^G$ *is a factor state. Suppose* $\omega_k(A) = \mathrm{tr}(A\Omega_k)$ *for all* $A \in \mathfrak{B}_0$ *and* $k = 1, 2, \ldots$ . *Then,*

A) $\omega^G$ *is of type* I $\Leftarrow \sum_{k=1}^{\infty} 1 - \mathrm{tr}(\Omega_k^2) < \infty$*;*

B) $\omega^G$ *is of type* $\mathrm{II}_1 \Leftrightarrow$ *there is a positive trace one matrix* $\Omega_0 \in R_0'$ *(not of rank one) so that* $\sum_{k=1}^{\infty} \|\Omega_0^{1/2} - \Omega_k^{1/2}\|_{\mathrm{HS}}^2 < \infty$ *and the support of* $\Omega_k$ *is contained in the support of* $\Omega_0$ *for* $k = 1, 2, \ldots$ .

*Proof.* Suppose $\omega$, $\omega_k$ and $\Omega_k$ satisfy the hypothesis and notation of the theorem. Suppose the sum in the second part of (A) is finite. Let $E_k$ be a rank one projection in $\mathfrak{B}_0$ which maximizes $\omega_k(E)$ with $E$ rank one. Let $\rho = \bigotimes_{k=1}^{\infty} \rho_k$ with $\rho_k(A) = \mathrm{tr}(AE_k)$ for $A \in \mathfrak{B}_0$ and $k = 1, 2, \ldots$ . Since the some in (A) is finite a short computation shows that $\lambda\omega \geqslant \rho \geqslant 0$ for some $\lambda > 0$. Since $\omega^G$ is a factor state we have $\rho^G$ is a factor state and from Theorem 4.16 we have $\rho^G$ is pure. Since $\omega^G \underset{q}{\sim} \rho^G$ we have $\omega^G$ is of type I.

Suppose now that $\omega^G$ is of type $\mathrm{II}_1$. Then $\omega^G \underset{q}{\sim} \tau$, where $\tau$ is an extremal trace on $\mathfrak{A}^G$. Using a result of [14], in [9] it was recently shown that each extremal trace $\tau$ of $\mathfrak{A}^G$ is of the form $\tau = \rho^G$ with $\rho = \bigotimes_{k=1}^{\infty} \rho_k$ and $\rho_k(A) = \mathrm{tr}(A\Omega)$ for $A \in \mathfrak{B}_0$ for $k = 1, 2, \ldots$ with $\Omega \in R_0'$ and, conversely, each positive trace one $\Omega \in R_0'$ gives rise to an extremal trace on $\mathfrak{A}^G$. (This result can be derived for faithful extremal trace from [14] and Theorem II.4 of [2].) Since $\omega^G \sim \rho^G$ it follows from Theorem 4.14 that the implication ($\Leftarrow$) (B) of holds.

Conversely, if the second condition of (B) holds then by Theorem 4.14 $\omega^G$ is quasi-equivalent to an extremal trace. ▨

REMARK. We believe the implication ($\Leftarrow$) in statement (A) of the last theorem is actually ($\Leftrightarrow$).

We close by giving generalizations of Theorems 4.13 and 4.14 to arbitrary factor states of $\mathfrak{A}$.

THEOREM 4.18. *Suppose* $\omega$ *is a factor state of* $\mathfrak{A}$. *Let* $\Omega_k \in \mathfrak{B}_0$ *be given by* $\omega(\gamma_k(A)) = \mathrm{tr}(A\Omega_k)$ *for all* $A \in \mathfrak{B}_0$. *Let* $S$ *be the set of accumulation points of* *the sequence* $\{\Omega_k\}$. *Let* $G(S) = \{g \in G\ ;\ U_g \Omega U_g^{-1} = \Omega \text{ for all } \Omega \in S\}$. *Then if* $G(S) = \{\lambda I\}$, $\omega^G$ *is a factor state.*

*Proof.* Suppose the hypothesis of the theorem is valid. From Lemma 3.4 we have $S \subset R^G(\omega)$. Since $G(S) = \{\lambda I\}$ we have $H = G \cap R^G(\omega)' = \{\lambda I\}$. Suppose $\pi$ is a cyclic $*$-representation of $\mathfrak{A}$ induced by $\omega$. Then by Lemma 3.7 we have $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^H)'' = \pi(\mathfrak{A})''$. Hence, $\pi(\mathfrak{A}^G)''$ is a factor and this implies $\omega^G$ is a factor state.

Following a greatly simplified version of the argument of Theorem 4.14 one obtains:

THEOREM 4.19. *Suppose* $\omega$ *is a factor state of* $\mathfrak{A}$ *satisfying the hypothesis of Theorem* 4.18. *Suppose* $\omega'$ *is a factor state of* $\mathfrak{A}$. *Then* $\omega^G \underset{q}{\sim} \omega'^G$ *if and only if there is a* $g \in G$ *so that* $\omega \underset{q}{\sim} \omega' \circ \alpha_g$.

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