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NORMALITY IN TRACE IDEALS

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1. INTRODUCTION

The notion of normality for linear functionals and its utility is familiar in several contexts. For example, it is well known [6] that the predual of a von Neumann algebra may be identified with the closed subspace of its dual generated by the positive normal linear functionals. Similarly, a dominant role is played by normality in the duality theory of normed Köthe spaces [19] and, more generally, in the theory of abstract vector lattices [20]. It is therefore natural to expect that the concept of normality should play a useful role in the duality theory of trace ideals and it is our purpose in this paper to study duality theory for trace ideals from the standpoint initiated in [9] and to examine the relation of normality to questions of perfectness, weak sequential compactness and weak sequential completeness in the setting of arbitrary trace ideals.

It is well known, of course, that the Banach dual of a *minimal* symmetrically normed ideal $\mathscr{I}$ of operators in a Hilbert space may again be identified with an ideal of operators. On the other hand, this is not the case if the ideal is not minimal. Nonetheless, in the second section of this paper, we show that if $\mathscr{I}$ is an arbitrary symmetrically normed ideal of operators in a Hilbert space, then the Banach dual $\mathscr{I}^*$ admits a type of Yosida-Hewitt decomposition in that each element of $\mathscr{I}^*$ has a unique decomposition into normal and "singular" parts. Further, the normal linear functionals in $\mathscr{I}^*$ form a closed subspace which may be identified with the α-dual $\mathscr{I}^\times$ introduced by Garling [9]. This characterization of the α-dual ideal $\mathscr{I}^\times$ in purely order-theoretic terms is then exploited to give internal characterizations of $\sigma(\mathscr{I}, \mathscr{I}^\times)$-relatively compact subsets of an arbitrary perfect ideal $\mathscr{I}$ (Theorem 3.4). This extends to the setting of arbitrary trace ideals the normality-type criteria for relatively weakly compact subsets of the ideal of trace class operators obtained by specializing the results of Akemann [1] for relatively weakly compact subsets of the predual of a von Neumann algebra. Using this criterion, we then show that the trace ideal $\mathscr{I}$ is perfect if and only if $\mathscr{I}$ is $\sigma(\mathscr{I}, \mathscr{I}^\times)$-sequentially complete (Theorem 3.5). The motivation for this characterization is to be seen via the Calkin correspondence, for it is well known [12] that if μ is a symmetric Köthe sequence space, then μ is perfect if and only if μ is $\sigma(\mu, \mu^{\times})$-sequentially complete.

In the final section of the paper, the criteria developed in earlier sections are applied to the case of symmetrically normed ideals to give efficient characterizations of perfect ideals, ideals which are weakly sequentially complete, minimal ideals which have minimal dual, and reflexive norm ideals. While the characterizations given in this section are suggested by results familiar from the theory of Banach lattices, it is important to note that the structure of an arbitrary normed ideal is very different from that of any Banach lattice, as has been pointed out by Lewis [13].

We gather now some basic notation. Throughout the paper, $\mathcal{I}$ will denote an ideal of operators in the (arbitrary) complex Hilbert space $\mathcal{H}$. The ideal of all operators on $\mathcal{H}$ is denoted by $\mathcal{B}$ and the operator norm by $\|\cdot\|_\infty$. The ideal $\mathcal{I}$ will be called a symmetrically normed ideal, written s.n. ideal, if $\mathcal{I}$ is equipped with a norm $\|\cdot\|_{\mathcal{I}}$ with the following properties:

(i) $\mathcal{I}$ is a Banach space under $\|\cdot\|_{\mathcal{I}}$.

(ii) If $A \in \mathcal{I}$ and if U, $V \in \mathcal{B}$ then $\|UAV\|_{\mathcal{I}} \leqslant \|U\|_\infty \|V\|_\infty \|A\|_{\mathcal{I}}$.

(iii) $\|A\|_{\mathcal{I}} = \|A\|_\infty$ if A is of rank 1.

By $\mathcal{C}_\infty$ is denoted the ideal of compact operators, equipped with the operator norm and by $\mathcal{C}_1$ is denoted the ideal of trace class operators equipped with the usual trace norm $\|\cdot\|_1 = \operatorname{tr}(|\cdot|)$ where $|A|$ denotes $(A^*A)^{1/2}$ for any $A \in \mathcal{B}$. If f, $g \in \mathcal{H}$, then $f \otimes g$ will denote the operator $(\cdot, g)f$. If $\mathcal{I}$ is an ideal in $\mathcal{B}$, the *α-dual* $\mathcal{I}^{\times}$ of $\mathcal{I}$ is the set of all $A \in \mathcal{B}$ such that $AB \in \mathcal{C}_1$ for each $B \in \mathcal{I}$. Equivalently, $\mathcal{I}^{\times}$ is the set of all $A \in \mathcal{B}$ such that $BA \in \mathcal{C}_1$ for each $B \in \mathcal{I}$. $\mathcal{I}^{\times}$ is an ideal in $\mathcal{B}$ and if $A \in \mathcal{I}^{\times}$, $B \in \mathcal{I}$ then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. The ideal $\mathcal{I}$ is called *perfect* if $\mathcal{I} = \mathcal{I}^{\times\times}$. For basic properties of the α-dual $\mathcal{I}^{\times}$ of an ideal $\mathcal{I}$, the reader is referred to [9]. If $\mathcal{I}$ is a s.n. ideal then $\mathcal{I}^0$ will denote the $\|\cdot\|_{\mathcal{I}}$-closure in $\mathcal{I}$ of the ideal of finite rank operators $\mathcal{F}$. The s.n. ideal $\mathcal{I}$ is called *minimal* if $\mathcal{I} = \mathcal{I}^0$. Finally, if E is a Banach space then the Banach dual of E is denoted by E^*.

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2. NORMAL FUNCTIONALS AND A DECOMPOSITION THEOREM

This section is concerned with duality considerations related to normality and we establish a decomposition theorem for the Banach dual of an arbitrary s.n. ideal which is a non-commutative analogue of the well known Yosida-Hewitt decomposition [18] for finitely additive measures. This decomposition result will play a useful role in subsequent sections.

If $\mathcal{I}$ is an ideal in $\mathcal{B}$ and if φ is a linear functional on $\mathcal{I}$, the adjoint functional φ^* is defined by setting $\varphi^*(B) = \overline{\varphi(B^*)}$ for all $B \in \mathcal{I}$. The linear functional φ is called *self-adjoint* if $\varphi = \varphi^*$ and *positive*, written $\varphi \geqslant 0$, if $\varphi(A) \geqslant 0$ whenever $0 \leqslant A \in \mathscr{I}$. If $A \in \mathscr{I}^{\times}$, we denote by φ_A the linear functional $\operatorname{tr}(A(\cdot))$. It is a simple consequence of [10], Theorem III.8.3, that the linear functional φ_A is self-adjoint if and only if A is self-adjoint and $\varphi_A \geqslant 0$ if and only if $A \geqslant 0$.

We shall use the following terminology. If $\mathscr{I}$ is an ideal in $\mathscr{B}$, if $\{B_\tau\} \subseteq \mathscr{I}$ is an upwards (downwards) filtering system of self-adjoint elements of $\mathscr{I}$ and if $\{B_\tau\}$ is weak operator convergent to $B \in \mathscr{B}$, we will write $B_\tau \uparrow_\tau B$ ($B_\tau \downarrow_\tau B$).

We may now make the following definition.

DEFINITION 2.1. Let $\mathscr{I}$ be an ideal in B. The linear functional φ on $\mathscr{I}$ is said to be *normal* if and only if $\{B_\tau\} \subseteq \mathscr{I}$ and $0 \leqslant B_\tau \downarrow_\tau 0$ implies $\lim_\tau \varphi(B_\tau) = 0$.

We remark that if φ is a normal linear functional on $\mathscr{I}$ then the adjoint functional φ^* is also normal.

PROPOSITION 2.2. *If $\mathscr{I}$ is an ideal in $\mathscr{B}$ and if $A \in \mathscr{I}^{\times}$, then φ_A is a normal linear functional on $\mathscr{I}$.*

Proof. It is not difficult to see that $\operatorname{tr}(\cdot)$ is a normal linear functional on $\mathscr{C}_1$ If $\{B_\tau\} \subseteq \mathscr{I}$ satisfies $0 \leqslant B_\tau \downarrow_\tau 0$ and if $0 \leqslant A \in \mathscr{I}^{\times}$, then it follows from Lemma 2.3 below that $A^{1/2}B_\tau A^{1/2} \downarrow_\tau 0$ holds in $\mathscr{C}_1$. Consequently $\operatorname{tr}(AB_\tau) = \operatorname{tr}(A^{1/2}B_\tau A^{1/2}) \to 0$ and this completes the proof of the proposition.

LEMMA 2.3. *Let $\mathscr{I}$ be an ideal in $\mathscr{B}$. If $0 \leqslant B \in \mathscr{I}$ and if $0 \leqslant A \in \mathscr{I}^{\times}$, then $A^{1/2}BA^{1/2} \in \mathscr{C}_1$ and $\operatorname{tr}(A^{1/2}BA^{1/2}) = \operatorname{tr}(AB)$.*

Proof. Since $0 \leqslant B \in \mathscr{I}$, it follows that $0 \leqslant \varphi_B \in \mathscr{I}^{\times}$. Consequently, if $\{e_i\}_{i=1}^n$ is any finite orthonormal system in $\mathscr{H}$ and if $P = \sum_{i=1}^n e_i \otimes \overline{e}_i$, then

$$0 \leqslant \sum_{i=1}^n (A^{1/2}BA^{1/2}e_i, e_i) = \operatorname{tr}(PA^{1/2}BA^{1/2}) = \operatorname{tr}(A^{1/2}PA^{1/2}B) \leqslant \operatorname{tr}(AB).$$

It follows that $A^{1/2}BA^{1/2} \in \mathscr{C}_1$ and the final conclusion follows from [10], Theorem III.8.2.

Suppose now that $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ is a s.n. ideal. It is now our intention to identify those elements of the Banach dual $\mathscr{I}^*$ which arise from $\mathscr{I}^{\times}$. We observe first that if $A \in \mathscr{I}^{\times}$, then $\varphi_A \in \mathscr{I}^*$. This fact is an immediate consequence of the following simple observation.

PROPOSITION 2.4. *If $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ is a s.n. ideal, then each positive linear functional on $\mathscr{I}$ is continuous.*

Proof. If φ is a positive linear functional on the s.n. ideal $\mathscr{I}$, it clearly suffices to show that the restriction of φ to the positive cone of $\mathscr{I}$ is continuous.

If this is not so, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of positive elements of $\mathscr{I}$ for which $\|B_n\|_{\mathscr{I}} \leqslant 1$ and $\varphi(B_n) \geqslant 2^n$, $n = 1,2,\ldots$. Setting $A_n = 2^{-n}B_n$, $n = 1, 2,\ldots$, it follows that $\sum_{n=1}^{\infty} \|A_n\|_{\mathscr{I}} \leqslant 1$. Consequently, if A is the weak operator limit of the partial sum sequence $\left\{\sum_{j=1}^{n} A_j\right\}_{n=1}^{\infty}$, then $A \in \mathscr{I}$. However

$$\varphi(A) \geqslant \varphi\left(\sum_{j=1}^{n} A_n\right) \geqslant n, \quad n = 1, 2,\ldots$$

and this yields the desired contradiction.

From the preceding proposition, it follows that if $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ is a s.n. ideal and if $A \in \mathscr{I}^{\times}$, we may define

$$\|A\|_{\mathscr{I}^{\times}} = \|\varphi_A\|_{\mathscr{I}^*} = \sup\{|\mathrm{tr}(AB)| : B \in \mathscr{I}, \|B\|_{\mathscr{I}} \leqslant 1\}.$$

It is shown in Theorem 7 of [9] that $(\mathscr{I}^{\times}, \|\ \|_{\mathscr{I}^{\times}})$ is again a s.n. ideal. Suppose now that $A \in \mathscr{I}^{\times}$. Using the polar decomposition and the positivity of $\varphi_{|A|}$, it is not difficult to see directly that

$$\|A\|_{\mathscr{I}^{\times}} = \sup\{\mathrm{tr}(|A|B) : 0 \leqslant B \in \mathscr{I}, \|B\|_{\mathscr{I}} \leqslant 1\}.$$

Denote now by $\{E_\tau\}$ the system of projections with finite dimensional range, upwards directed by range inclusion. It is clear that $0 \leqslant B^{1/2}E_\tau B^{1/2} \uparrow_\tau B$ for each $0 \leqslant B \in \mathscr{I}$. From the normality of $\varphi_{|A|}$, it now follows that

$$\|A\|_{\mathscr{I}^{\times}} = \sup\{|\mathrm{tr}(AB)| : B \in \mathscr{I}^0, \|B\|_{\mathscr{I}} \leqslant 1\}.$$

From these remarks, it follows that under the mapping $A \to \varphi_A$ the s.n. ideal $(\mathscr{I}^{\times}, \|\ \|_{\mathscr{I}^{\times}})$ may be identified isometrically with a closed linear subspace of $(\mathscr{I}^0)^*$. This constitutes part of the proof of the following result; the remaining details follow the standard lines of the argument of [9], Proposition 11, and are accordingly omitted.

PROPOSITION 2.5. *If $\mathscr{I}$ is a* s.n. *ideal, then* $(\mathscr{I}^{\times}, \|\cdot\|_{\mathscr{I}^{\times}}) = (\mathscr{I}^0)^*$.

It is convenient to record here the following simple consequence of Proposition 2.5 above.

COROLLARY 2.6. *If $\mathscr{I}$ is a* s.n. *ideal, then*

$$(\mathscr{I}^0, \|\cdot\|_{\mathscr{I}}) = ((\mathscr{I}^{\times\times})^0, \|\cdot\|_{\mathscr{I}^{\times\times}}).$$

We now show that if $\mathscr{I}$ is a s.n. ideal, then $\mathscr{I}^\times$ is complemented in $\mathscr{I}^*$ and is the range of a positive projection of norm one.

PROPOSITION 2.7. *If $\mathscr{I}$ is a s.n. ideal, then $\mathscr{I}^* = \mathscr{I}^\times \oplus (\mathscr{I}^0)^\perp$ and each of the closed subspaces $\mathscr{I}^\times$, $(\mathscr{I}^0)^\perp$ is the range of a positive contractive projection.*

Proof. Since each positive element of $\mathscr{I}$ is the weak operator limit of an upwards directed system in $\mathscr{I}^0$, it follows directly from normality that $\mathscr{I}^\times \cap (\mathscr{I}^0)^\perp = \{0\}$. If now $\varphi \in \mathscr{I}^*$, let φ' denote the restriction of φ to $\mathscr{I}^0$. By Proposition 2.5, there exists $A \in \mathscr{I}^\times$ such that $\varphi'(C) = \mathrm{tr}(AC)$ for all $C \in \mathscr{I}^0$. Define $\varphi_1 \in \mathscr{I}^*$ by setting $\varphi_1(B) = \mathrm{tr}(AB)$ for all $B \in \mathscr{I}$. It is clear that $\varphi_1 \in \mathscr{I}^\times$ and that $\varphi_2 = \varphi - \varphi_1 \in (\mathscr{I}^0)^\perp$. Moreover

$$\|\varphi_1\|_{\mathscr{I}^*} = \|\varphi_A\|_{\mathscr{I}^*} = \|\varphi_A|\mathscr{I}^0\|_{(\mathscr{I}^0)^*} = \|\varphi|\mathscr{I}^0\|_{(\mathscr{I}^0)^*} \leqslant \|\varphi\|_{\mathscr{I}^*}.$$

We show also that $\|\varphi_2\|_{\mathscr{I}^*} \leqslant \|\varphi\|_{\mathscr{I}^*}$. Let $\varepsilon > 0$ be given and choose $D \in \mathscr{I}$ with $\|D\|_{\mathscr{I}} \leqslant 1$ and $|\varphi_2(D)| \geqslant \|\varphi_2\|_{\mathscr{I}^*} - \varepsilon/2$. Since $AD \in \mathscr{C}_1$ and since $\mathrm{tr}(AD(\cdot))$ is normal on $\mathscr{B}$, there exists a projection E with cofinite range such that $|\mathrm{tr}(ADE)| < \varepsilon/2$. Observe that

$$|\varphi(DE)| \geqslant |\varphi_2(DE)| - |\varphi_1(DE)| = |\varphi_2(D)| - |\mathrm{tr}(ADE)| \geqslant \|\varphi_2\|_{\mathscr{I}^*} - \varepsilon$$

and so $\|\varphi_2\|_{\mathscr{I}^*} \leqslant \|\varphi\|_{\mathscr{I}^*}$. Finally, if $\varphi \geqslant 0$, then it follows from

$$\varphi(f \otimes f) = \varphi_1(f \otimes f) = (Af, f),$$

for each $f \in \mathscr{H}$, that $A \geqslant 0$. Hence $\varphi_A \geqslant 0$ and so $\varphi_1 \geqslant 0$. To see also that $\varphi_2 \geqslant 0$, let $0 \leqslant B \in \mathscr{I}$ and let $\varepsilon > 0$ be given. From the normality of φ_1, there exists a projection E with co-finite range such that $\varphi_1(B^{1/2}EB^{1/2}) < \varepsilon$. It follows that

$$0 \leqslant \varphi(B^{1/2}EB^{1/2}) = \varphi_1(B^{1/2}EB^{1/2}) + \varphi_2(B) < \varepsilon + \varphi_2(B).$$

Hence $\varphi_2(B) \geqslant 0$ and this suffices to complete the proof of the proposition.

We remark that the preceding theorem is due to J. Dixmier [5] for the case that $\mathscr{I} = \mathscr{B}$.

We are now in a position to characterize $\mathscr{I}^\times$ as a subset of $\mathscr{I}^*$ in purely order-theoretic terms, in a manner familiar from the theory of normed Köthe spaces.

PROPOSITION 2.8. *Let $\mathscr{I}$ be an s.n. ideal and suppose that $\varphi \in \mathscr{I}^*$. The following statements are equivalent.*

(i) $\varphi \in \mathscr{I}^\times$.

(ii) φ *is normal.*

(iii) $\lim_\tau \varphi(BR_\tau) = 0$ *for every family* $\{R_\tau\}$ *of projections in* $\mathscr{B}$ *with* $R_\tau \downarrow_\tau 0$, *for each* $0 \leqslant B \in \mathscr{I}$.

In addition, if $I \subseteq \mathscr{C}_\infty$ *then each of the above conditions are equivalent to each of the following.*

(iv) $\varphi(B_n) \to 0$ *as* $n \to \infty$ *for all sequences* $\{B_n\} \subseteq \mathscr{I}$ *with* $0 \leqslant B_n \downarrow_n 0$.

(v) $\varphi(B_n) \to \varphi(B)$ *whenever* $0 \leqslant B \in \mathscr{I}$ *and* $\{B_n\}_{n=1}^\infty$ *is the partial sum sequence of the Schmidt expansion of* B.

Proof. That (i) $\Rightarrow$ (ii) is Proposition 2.2 above. The implication (i) $\Rightarrow$ (iii) is a simple application of the normality on $\mathscr{B}$ of $\mathrm{tr}(AB(\cdot))$ for each $A \in \mathscr{I}$, $B \in \mathscr{I}$. The implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious. We prove the implication (ii) $\Rightarrow$ (i). The implications (iii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) follow by a similar argument and so their detailed proof is omitted.

Assume then that φ is normal and let $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in \mathscr{I}^\times$ and $\varphi_2 \in (\mathscr{I}^0)^\perp$. If $0 \leqslant B \in \mathscr{I}$ and if $\{E_\tau\}$ is the family of all finite rank projections in $\mathscr{B}$, upwards directed by range inclusion, then $B^{1/2}E_\tau B^{1/2} \uparrow_\tau B$ holds in $\mathscr{I}$. Since φ_1 is normal, it follows also from (ii) that $\varphi_2 = \varphi - \varphi_1$ is normal. Consequently $\varphi_2(B) = \lim_\tau \varphi_2(B^{1/2}E_\tau B^{1/2}) = 0$. Thus $\varphi_2 = 0$ and so $\varphi = \varphi_1$.

The following remark will be useful. Since φ is normal if and only if the adjoint functional φ^* is normal, it follows that the statement "$\lim_\tau \varphi(BR_\tau) = 0$" in (iii) above can be replaced by the statement "$\lim_\tau \varphi(R_\tau B) = 0$".

We conclude this section by characterizing, in terms of normality properties of the norm, those s.n. ideals $\mathscr{I}$ which are minimal.

PROPOSITION 2.9. *If* $\mathscr{I}$ *is a* s.n. *ideal then the following statements are equivalent for* $A \in \mathscr{I}$.

(i) $A \in \mathscr{I}^0$.

(ii) $\|AP_n\|_\mathscr{I} \to 0$ *for every sequence* $\{P_n\}_{n=1}^\infty$ *of mutually disjoint projections on* $\mathscr{H}$.

(iii) *For every family* $\{R_\tau\}$ *of projections with* $R_\tau \downarrow_\tau 0$, *it follows that* $\lim_\tau \|AR_\tau\|_\mathscr{I} = 0$.

(iv) *For every family* $\{A_\tau\} \subseteq \mathscr{B}$ *with* $0 \leqslant A_\tau \leqslant |A|$ *and* $A_\tau \downarrow_\tau 0$, *it follows that* $\inf_\tau \|A_\tau\|_\mathscr{I} = 0$.

(v) *For every sequence* $\{A_n\} \subseteq \mathscr{B}$ *with* $0 \leqslant A_n \leqslant |A|$ *and* $A_n \downarrow_n 0$, *it follows that* $\inf_n \|A_n\|_\mathscr{I} = 0$.

Proof. (i) $\Rightarrow$ (ii). Let $\{P_n\}$ be a sequence of mutually disjoint projections. If $f, g \in \mathscr{H}$, then from

$$\|(f \otimes g)P_n\|_\mathscr{I} = \|f \otimes \overline{P_n g}\|_\mathscr{I} = \|f\|\,\|P_n g\|$$

for $n = 1, 2, \ldots$, it follows that $\|AP_n\|_{\mathscr{I}} \to 0$ for each $A \in \mathscr{F}$ and consequently for each $A \in \mathscr{I}^0$ since $\|P_n\|_\infty = 1$ for $n = 1, 2, \ldots$.

(ii) $\Rightarrow$ (i). Suppose that $A \in \mathscr{I}$ and that condition (ii) holds. It is clear that it may be assumed in addition that $A \geqslant 0$. If A is not compact, then it follows from the spectral theorem for self-adjoint operators that there exists a projection P with infinite dimensional range and $\lambda > 0$ such that $PAP \geqslant \lambda P$. Writing $P = \sum_{i=1}^{\infty} P_i$ with $\{P_i\}$ a sequence of mutually disjoint projections, observe that

$$P_iAP_i = P_iPAPP_i \geqslant \lambda P_i, \quad i = 1, 2, \ldots$$

so that

$$\|AP_i\|_{\mathscr{I}} \geqslant \|P_iAP_i\|_{\mathscr{I}} \geqslant \|P_iAP_i\|_\infty \geqslant \lambda > 0, \quad i = 1, 2, \ldots$$

which contradicts (ii), so that A is compact. Let $A = \sum_{i=1}^{\infty} \lambda_i \varphi_i \otimes \overline{\varphi}_i$ be the Schmidt expansion of A. If $A \notin \mathscr{I}^0$, there exists a sequence $\{n(k)\}_{k=1}^{\infty}$ of natural numbers and $\varepsilon > 0$ such that

$$\left\| \sum_{i=n(2k)}^{n(2k+1)} \lambda_i \varphi_i \otimes \overline{\varphi}_i \right\|_{\mathscr{I}} > \varepsilon, \quad k = 1, 2, \ldots .$$

Setting $P_k = \sum_{i=n(2k)}^{n(2k+1)} \varphi_i \otimes \overline{\varphi}_i$, $k = 1, 2 \ldots$, it follows that the sequence $\{P_k\}_{k=1}^{\infty}$ is mutually disjoint and $\|AP_k\|_{\mathscr{I}} > \varepsilon$ for $k = 1, 2, \ldots$ which is a contradiction and so the implication is proved.

(i) $\Rightarrow$ (iii). Suppose that $A \in \mathscr{I}^0$ and that $\{R_\tau\}$ is a downwards directed system of projections with $R_\tau \downarrow_\tau 0$ but that $\{\|AR_\tau\|_{\mathscr{I}}\}$ does not converge to 0. By passing to a cofinal system if necessary, it may be assumed that there exists $\varepsilon > 0$ with $\|AR_\tau\|_{\mathscr{I}} > \varepsilon$ for all τ. Since $A \in \mathscr{I}^0 = (\mathscr{I}^{\times\times})^0$, it follows that for each index τ,

$$\sup\{|\mathrm{tr}(AR_\tau B)| : B \in \mathscr{I}^\times, \|B\|_{\mathscr{I}^\times} \leqslant 1\} > \varepsilon.$$

Using Proposition 2.2, there exist sequences $\{R_k\} \subseteq \{R_\tau\}$ and $\{B_k\} \subseteq \mathscr{I}^\times$ such that $R_k \downarrow_k$, $\|B_k\|_{\mathscr{I}^\times} = 1$, $|\mathrm{tr}(AR_kB_k)| > \varepsilon$ and $|\mathrm{tr}(AR_{k+1}B_k)| < \varepsilon/2$ for each $k = 1, 2, \ldots$. Setting $P_k = R_k - R_{k+1}$, for $k = 1, 2, \ldots$, it follows that $|\mathrm{tr}(AP_kB_k)| > \varepsilon/2$ for $k = 1, 2, \ldots$, which implies that $\|AP_k\|_{\mathscr{I}^{\times\times}} > \varepsilon/2$ for $k = 1, 2, \ldots$. This contradicts the fact that $A \in (\mathscr{I}^{\times\times})^0$, by using the implication (i) $\Rightarrow$ (ii) applied to $\mathscr{I}^{\times\times}$.

(iii) $\Rightarrow$ (iv). Without loss of generality it may be assumed that $A \geqslant 0$. Let $\{A_\sigma\} \subseteq \mathscr{B}$ satisfy $0 \leqslant A_\sigma \leqslant A$ for each σ and $A_\sigma \downarrow_\sigma 0$. It is clear that $\{A_\sigma\} \subseteq \mathscr{I}$. Denote by $\{E_\tau\}$ the system of projections with co-finite range, downwards directed by range inclusion and observe that $E_\tau \downarrow_\tau 0$. Let $\varepsilon > 0$ be given. By the stated property of (iii), there exists E_{τ_0} such that $\|AE_{\tau_0}\| < \varepsilon/3$. For every index σ,

$$\|A_\sigma\|_{\mathscr{I}} \leqslant \|E_{\tau_0} A_\sigma E_{\tau_0}\|_{\mathscr{I}} + \|(I - E_{\tau_0})A_0 E_{\tau_0}\|_{\mathscr{I}} + \|A_\sigma(I - E_{\tau_0})\|_{\mathscr{I}} \leqslant$$

$$\leqslant \|AE_{\tau_0}\|_{\mathscr{I}} + 2\|A_\sigma(I - E_{\tau_0})\|_{\mathscr{I}}.$$

Since the range of $I - E_{\tau_0}$ is finite dimensional and since $\{A_\sigma\}$ is convergent to 0 for the strong operator topology, there exists σ_0 such that $\|A_{\sigma_0}(I - E_{\tau_0})\|_{\mathscr{I}} < \varepsilon/3$. It then follows that $\|A_\sigma\|_{\mathscr{I}} < \varepsilon$ for all $A_\sigma \leqslant A_{\sigma_0}$ and this proves the implication.

The implication (iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i). Suppose that $A \geqslant 0$ and that A satisfies the condition of (v). Observe that if P is any projection and $\lambda > 0$ is such that $\lambda P \leqslant A$, then also P satisfies the condition of (v). It is not difficult then to see that such a projection P has finite dimensional range and it follows from the spectral theorem that A is necessarily compact. That (i) follows from (v) is a consequence of the fact that $0 \leqslant A_n \uparrow_n A$ if $\{A_n\}$ denotes the partial sum sequence of the Schmidt expansion for A. By this, the proof of the proposition is complete.

3. WEAK COMPACTNESS AND PERFECT IDEALS

We begin with several order equicontinuity properties for conditionally $\sigma(\mathscr{I}, \mathscr{I}^\times)$-sequentially compact subsets of an arbitrary ideal $\mathscr{I}$.

PROPOSITION 3.1. *Let $\mathscr{I}$ be an ideal in $\mathscr{B}$ and suppose that $\mathscr{K} \subseteq \mathscr{I}$ has the property that each sequence in $\mathscr{K}$ contains a $\sigma(\mathscr{I}, \mathscr{I}^\times)$-Cauchy subsequence. Then $\mathscr{K}$ is $\sigma(\mathscr{I}, \mathscr{I}^\times)$-bounded and has the following properties.*

(i) $$\lim_\tau \sup\{|\mathrm{tr}(BA_\tau)| : B \in \mathscr{K}\} = 0$$

for every family $\{A_\tau\} \subseteq \mathscr{I}^\times$ with $0 \leqslant A_\tau \downarrow_\tau 0$.

(ii) $$\lim_\tau \sup\{|\mathrm{tr}(BR_\tau C)| : B \in \mathscr{K}\} = 0$$

for each $C \in \mathscr{I}^\times$, for every family $\{R_\tau\}$ of projections with $R_\tau \downarrow_\tau 0$.

(iii) $$\lim_\tau \sup\{|\mathrm{tr}(CR_\tau B)| : B \in \mathscr{K}\} = 0$$

for each $C \in \mathscr{I}^\times$, for every family $\{R_\tau\}$ of projections with $R_\tau \downarrow_\tau 0$.

Proof. It is trivial to see that $\mathscr{K}$ is $\sigma(\mathscr{I}, \mathscr{I}^\times)$-bounded. Assume that (i) fails to hold. By passing, if necessary, to a cofinal subnet, it may be assumed without loss of generality, that there exists $\varepsilon > 0$ such that

$$\sup\{|\mathrm{tr}(BA_\tau)| : B \in \mathscr{K}\} > \varepsilon$$

for all indices τ. Using the fact that $\mathscr{I} \subseteq (\mathscr{I}^\times)^\times$, it follows by induction that there exist sequences $\{B_n\} \subseteq \mathscr{K}$, $\{A_{\tau_n}\} \subseteq \{A_\tau\}$ such that $A_{\tau_n} \downarrow_n$, $|\mathrm{tr}(B_n A_{\tau_n})| > \varepsilon$ and $|\mathrm{tr}(B_j A_{\tau_{n+1}})| < \varepsilon/4$ for $1 \leqslant j \leqslant n$ and $n = 1, 2, \ldots$. By passing to subsequences if necessary, it may further be assumed that $\{B_n\}$ is $\sigma(\mathscr{I}, \mathscr{I}^\times)$-Cauchy. For $n = 1, 2, \ldots$ set $D_n = A_{\tau_n} - A_{\tau_{n+1}}$. Observe that if E is any finite subset of N, then $0 \leqslant \sum_{j \in E} D_j \leqslant A_{\tau_1}$. Consequently, $\sum_{j\in E} D_j \in \mathscr{I}^\times$ for each subset E of N. For $n \geqslant 2$ and $E \in 2^{\mathbf{N}}$, define

$$\mu_n(E) = \mathrm{tr}((B_n - B_{n-1}) \sum_{j \in E} D_j).$$

It is easily checked that $\{\mu_n\}$ is a sequence of bounded additive measures on the Boolean algebra $2^{\mathbf{N}}$ with the property that $\mu_n(E) \to 0$ as $n \to \infty$ for each $E \in 2^{\mathbf{N}}$. By Phillips' lemma [7] it follows that $\mu_n(\{n\}) \to 0$ as $n \to \infty$. However,

$$|\mu_n(\{n\})| = |\mathrm{tr}((B_n - B_{n-1})(A_{\tau_n} - A_{\tau_{n+1}}))| \geqslant |\mathrm{tr}(B_n A_{\tau_n})| -$$

$$- |\mathrm{tr}(B_n A_{\tau_{n+1}})| - |\mathrm{tr}(B_{n-1} A_{\tau_n})| - |\mathrm{tr}(B_{n-1} A_{\tau_{n+1}})| > \varepsilon/4 \quad \text{for } n \geqslant 2.$$

This is a contradiction, so (i) is established and (ii) and (iii) may be proved in exactly the same way.

It is convenient at this point to insert a sharpened form of property (ii) preceding.

LEMMA 3.2. *Let $\mathscr{I}$ be an ideal in $\mathscr{B}$ and suppose that $\mathscr{K}$ is a $\sigma(\mathscr{I}, \mathscr{I}^\times)$--bounded subset of $\mathscr{I}$. The following conditions are equivalent.*

(i) $$\lim_\tau \sup\{|\mathrm{tr}(BR_\tau C)| : B \in \mathscr{K}\} = 0$$

for every $C \in \mathscr{I}^\times$, for each family $\{R_\tau\}$ of projections with $R_\tau \downarrow_\tau 0$.

(ii) $$\lim_n \sup\{|\mathrm{tr}(BR_n C)| : B \in \mathscr{K}\} = 0$$

for each sequence $\{P_n\}$ of mutually disjoint projections.

Proof. (ii) $\Rightarrow$ (i). If (i) does not hold, then there exists $\varepsilon > 0$, $C \in \mathscr{I}^\times$, a sequence $\{R_n\}$ of projections with $R_n \downarrow_n$ and a sequence $\{B_n\} \subseteq \mathscr{K}$ such that

$$|\mathrm{tr}(B_n R_n C)| > \varepsilon \quad \text{and} \quad |\mathrm{tr}(B_n R_{n+1})| < \varepsilon/2$$

for $n = 1, 2, \ldots$. If $P_n = R_n - R_{n+1}$, $n = 1, 2, \ldots$, then $\{P_n\}$ is a sequence of mutually disjoint projections such that

$$|\mathrm{tr}(B_n P_n C)| > \varepsilon/2, \quad n = 1, 2, \ldots$$

and this contradicts (ii).

(i) $\Rightarrow$ (ii). Suppose that (ii) fails so that there exists $\varepsilon > 0$, a sequence $\{B_n\} \subseteq \mathscr{K}$ and a sequence $\{P_n\}$ of mutually disjoint projections such that

$$|\mathrm{tr}(B_n P_n C)| > \varepsilon, \quad n = 1, 2, \ldots .$$

Since $\mathscr{C}_1 = \mathscr{B}^\times$, there exists an increasing sequence $\{n(k)\}$, with $n(1) = 1$, such that

$$|\mathrm{tr}(B_{n(k)}(\sum_{j \geq n(k+1)} P_j) C)| < \varepsilon/2.$$

For $k = 1, 2, \ldots$ set $Q_k = P_{n(k)} + \sum_{j \geq n(k+1)} P_j$. The sequence of projections $\{Q_k\}$ satisfies $Q_k \downarrow_k 0$ but

$$|\mathrm{tr}(B_{n(k)} Q_k C)| > \varepsilon/2, \quad k = 1, 2, \ldots$$

so that (i) is not satisfied. This completes the proof of the lemma.

PROPOSITION 3.3. *Let $\mathscr{I}$ be a* s.n. *ideal. If $\mathscr{K} \subseteq \mathscr{I}^\times$ is $\sigma(\mathscr{I}^\times, \mathscr{I})$-bounded and if either*

(i) $$\lim_\tau \sup\{|\mathrm{tr}(BR_\tau C)| : B \in \mathscr{K}\} = 0$$

for each $C \in \mathscr{I}$ and each system $\{R_\tau\}$ of projections with $R_\tau \downarrow_\tau 0$ or

(ii) $$\lim_\tau \sup\{|\mathrm{tr}(BA_\tau)| : B \in \mathscr{K}\} = 0$$

for each system $\{A_\tau\} \subseteq \mathscr{I}$ with $A_\tau \downarrow_\tau 0$, then $\mathscr{K}$ is $\sigma(\mathscr{I}^\times, \mathscr{I})$-relatively compact.

Proof. By the Banach-Alaoglu theorem, it suffices to show that each $\sigma(\mathscr{I}^*, \mathscr{I})$--accumulation point of $\mathscr{K}$ in $\mathscr{I}^*$ belongs to $\mathscr{I}^\times$. In view of Proposition 2.8, it further suffices to show that each such accumulation point is normal. This, however, is an immediate consequence of either stated equicontinuity condition together with the corresponding characterization of normality given in Proposition 2.8.

We may now state an intrinsic characterization of $\sigma(\mathscr{I}, \mathscr{I}^\times)$-relatively compact subset of a perfect ideal $\mathscr{I}$, in terms of normality type equicontinuity conditions. For the special case that $\mathscr{I} = \mathscr{C}_1$, our result is due to Akemann [1].

THEOREM 3.4. *Let $\mathcal{I}$ be a perfect ideal in $\mathcal{C}_\infty$ and let $\mathcal{K} \subseteq \mathcal{I}$. The following statements are equivalent.*

(i) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively compact.*

(ii) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively sequentially compact.*

(iii) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively countably compact.*

(iv) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and*

$$\lim_\tau \sup\{|\mathrm{tr}(BA_\tau)| : B \in \mathcal{K}\} = 0$$

for all systems $\{A_\tau\} \subseteq \mathcal{I}^\times$ with $A_\tau \downarrow_\tau 0$.

(v) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and*

$$\lim_n \sup\{|\mathrm{tr}(BP_nC)| : B \in \mathcal{K}\} = 0$$

for each $C \in \mathcal{I}^\times$ and each sequence $\{P_n\}$ of mutually disjoint projections.

(vi) *$\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and*

$$\lim_n \sup\{|\mathrm{tr}(BR_\tau C)| : B \in \mathcal{K}\} = 0$$

for each $C \in \mathcal{I}^\times$ and each family $\{R_\tau\}$ of projections with $R_\tau \downarrow_\tau 0$.

Proof. The equivalence of conditions (i), (ii), (iii) has been established by Garling ([9], Theorem 11). The equivalence of (v) and (vi) is given in Lemma 3.2 above. That (ii) implies (iv) and (ii) implies (vi) is given in Proposition 3.1 above. As the proof that (vi) implies (i) is similar to the proof that (iv) implies (i), the theorem will be completely proved by establishing that (iv) implies (i). We observe first that if $\mathcal{I} = \mathcal{C}_1$, then $\mathcal{K}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively compact by Proposition 3.3 applied to $\mathcal{B}$. We may assume then that $\mathcal{I} \neq \mathcal{C}_1$. Since $\mathcal{I}$ is a perfect ideal, we have that $\mathcal{I} \neq \mathcal{C}_\infty$. Since $\mathcal{I}$ is an ideal of compact operators, we have by the Corollary to [9], Proposition 7, that $\mathcal{C}_1 \subsetneq \mathcal{I} \subsetneq \mathcal{C}_\infty$ and also that $\mathcal{C}_1 \subsetneq \mathcal{I}^\times \subsetneq \mathcal{C}_\infty$. For each $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$ we write

$$\|A\|_C = \sup\{\mathrm{tr}(|AUCV|) : U,\ V \in \mathcal{B},\ \|U\|_\infty \leqslant 1,\ \|V\|_\infty \leqslant 1\}.$$

If $\mathcal{I}_C$ denotes the collection of all $A \in \mathcal{B}$ for which $\|A\|_C < \infty$ then it is shown in [9] that $(\mathcal{I}_C, \|\cdot\|_C)$ is a perfect s.n. ideal. Observe that $\mathcal{I} \subseteq \mathcal{I}_C$ and that $\mathcal{I}_C^\times \subseteq \mathcal{I}^\times$ for each $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$. It follows from Proposition 3.3, that $\mathcal{K}$ is $\sigma(\mathcal{I}_C, \mathcal{I}_C^\times)$-relatively compact in $\mathcal{I}_C$ for each $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$. Let now $\{A_\alpha\}$ be a $\sigma(\mathcal{I}, \mathcal{I}^\times)$-Cauchy net. Since $\{A_\alpha\}$ is $\|\cdot\|_\infty$-bounded, it follows that there exists $A \in \mathcal{B}$ such that $A_\alpha \to A$ for the weak operator topology. For each $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$, there exists $A(C) \in \mathcal{I}_C$ such that $\{A_\alpha\}$ is $\sigma(\mathcal{I}_C, \mathcal{I}_C^\times)$-convergent to $A(C)$ and hence weak operator convergent to $A(C)$. It follows that $A(C) = A$ for all $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$. In particular, $A \in \mathcal{I}_C$ for all $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$ and it follows immediately that $A \in (\mathcal{I}^\times)^\times = \mathcal{I}$. Finally, by noting that $\mathcal{C}_1 \subseteq \mathcal{I}_C^\times$ for all $C \in \mathcal{I}^\times \setminus \mathcal{C}_1$, it follows that $\{A_\alpha\}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-convergent to A and by this the proof is complete.

We remark that the criterion given in part (v) above may be used to give a direct proof of the fact that if $\mathcal{I}$ is a perfect ideal with $\mathcal{C}_1 \subsetneq \mathcal{I} \subsetneq \mathcal{C}_\infty$, then the $\mathcal{U}$-invariant cover (see [9]) of each $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively compact subset of $\mathcal{I}$ is again $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively compact, where $\mathcal{U}$ denotes the $\|\cdot\|_\infty$-unit ball of $\mathcal{I}$. This result is established in [9], Theorem 12, and the proof given there is based, in part, on a separate study of weakly compact subsets of symmetric sequence spaces [8]. However, as it is not central to our present purposes, we shall not give the details.

If $\mathcal{I}$ is an ideal in $\mathcal{C}_\infty$ and if μ is the symmetric sequence space associated with $\mathcal{I}$ under the Calkin correspondence, then it is well known [12] that μ is perfect (i.e. $\mu = \mu^{\times\times}$) if and only if μ is $\sigma(\mu, \mu^\times)$-sequentially complete. We now show that this result remains valid if μ is replaced by $\mathcal{I}$. We recall first [9] that the BK-topology on an ideal $\mathcal{I} \subseteq \mathcal{B}$ is that defined by the family of norms

$$p_C(\cdot) = \{\mathrm{tr}(|CX(\cdot)Y|) : X, Y \in \mathcal{B}, \|X\|_\infty \leqslant 1, \|Y\|_\infty \leqslant 1\}, \quad 0 \neq C \in \mathcal{I}^\times.$$

It follows from [9], Theorem 10, that if $\mathcal{I}^\times \neq \mathcal{C}_1$ then the dual for $\mathcal{I}$ under the BK-topology is precisely $\mathcal{I}^\times$. We may now state our main characterization of perfectness.

THEOREM 3.5. *If $\mathcal{I}$ is an ideal in $\mathcal{C}_\infty$, then the following statements are equivalent.*

(i) *$\mathcal{I}$ is perfect.*

(ii) *$\mathcal{I}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-sequentially complete.*

(iii) *If $\{B_n\}_{n=1}^\infty \subseteq \mathcal{I}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and weak operator convergent to $B \in \mathcal{B}$, then $B \in \mathcal{I}$.*

(iv) *If $\{B_n\}_{n=1}^\infty$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and if $0 \leqslant B_n \uparrow_n$ then there exists $0 \leqslant \mathcal{B} \in \mathcal{I}$ with $B_n \to B$ for the weak operator topology.*

Proof. (i) $\Rightarrow$ (ii). It follows immediately from Theorem 3.4 and Proposition 3.1 above that each $\sigma(\mathcal{I}, \mathcal{I}^\times)$-Cauchy sequence in $\mathcal{I}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-relatively compact, and the implication follows.

(ii) $\Rightarrow$ (iv). If $\{B_n\}_{n=1}^\infty$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and $0 \leqslant B_n \uparrow_n$, then $\{B_n\}_{n=1}^\infty$ is $\|\cdot\|_\infty$-bounded and consequently there exists $B \in \mathcal{B}$ with $\{B_n\}$ weak operator convergent to B. However, from $0 \leqslant B_n \uparrow_n$ and the $\sigma(\mathcal{I}, \mathcal{I}^\times)$-boundedness of the sequence $\{B_n\}$, it follows that $\{B_n\}_{n=1}^\infty$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-Cauchy and the implication (ii) $\Rightarrow$ (iv) follows.

(i) $\Rightarrow$ (iii). Suppose that $\{B_n\}_{n=1}^\infty \subseteq \mathcal{I}$ is $\sigma(\mathcal{I}, \mathcal{I}^\times)$-bounded and that $\{B_n\}$ is weak operator convergent to $B \in \mathcal{B}$. Since $\mathcal{I}$ is perfect and $\mathcal{I} \subseteq \mathcal{C}_\infty$, it follows that $\mathscr{I}^{\times} \neq \mathscr{C}_1$ and consequently the sequence $\{B_n\}$ is bounded for the BK-topology on $\mathscr{I}$. If now $A \in \mathscr{I}^{\times} \setminus \{0\}$, if U, V are partial isometries in $\mathscr{B}$ and if $\{\varphi_j\}_{j=1}^{n}$ is an arbitrary finite orthonormal sequence in $\mathscr{H}$, observe that by [10], Lemma II.4.1

$$\sum_{j=1}^{n} |(UAB_kV\varphi_j, \varphi_j)| \leqslant \operatorname{tr}(|AB_k|) \leqslant \sup_k p_A(B_k) < \infty$$

holds for $n, k = 1, 2, \ldots$. It follows that also

$$\sum_{j=1}^{n} |(UABV\varphi_j, \varphi_j)| \leqslant \sup_k p_A(B_k)$$

for $n = 1, 2, \ldots$, and consequently from [10], Lemma II.4.1, it follows that $AB \in \mathscr{C}_1$ and so $B \in \mathscr{I}$, since $\mathscr{I}$ is perfect and $A \in \mathscr{I}^{\times} \setminus \{0\}$ is arbitrary.

The implication (iii) $\Rightarrow$ (iv) is clear so to complete the proof of the proposition it suffices to show that (iv) $\Rightarrow$ (i). To this end, assume that (iv) holds and observe first that this implies that $\mathscr{I}^{\times} \neq \mathscr{C}_1$. In fact, suppose that $\mathscr{I}^{\times} = \mathscr{C}_1$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $\mathscr{H}$. Setting $P = \sum_{i=1}^{\infty} \varphi_i \otimes \varphi_i$ and $P_n = \sum_{i=1}^{n} \varphi_i \otimes \bar{\varphi}_i$, $n = 1, 2, \ldots$, it follows that $\{P_n\}_{n=1}^{\infty}$ is $\sigma(\mathscr{I}, \mathscr{I}^{\times})$-bounded since $\mathscr{I}^{\times} = \mathscr{C}_1$. This implies by condition (iv) that $P \in \mathscr{I}$ and this contradicts the assumption that $\mathscr{I} \subseteq \mathscr{C}_{\infty}$. It now follows from [9], Proposition 7 (iii) that $\mathscr{I}^{\times\times} \subseteq \mathscr{C}_{\infty}$. Suppose now that $0 \leqslant B \in \mathscr{I}^{\times\times}$ and let $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{I}$ be the partial sum sequence of the Schmidt expansion of B. It is clear that $0 \leqslant B_n \uparrow_n$, that $\{B_n\}_{n=1}^{\infty}$ is $\sigma(\mathscr{I}, \mathscr{I}^{\times})$-bounded and that $\{B_n\}_{n=1}^{\infty}$ is weak operator convergent to B. Condition (iv) now guarantees that $B \in \mathscr{I}$ so that $\mathscr{I} = \mathscr{I}^{\times\times}$ and so $\mathscr{I}$ is perfect. By this, the proof is complete.

4. SOME TOPOLOGICAL PROPERTIES OF NORMED IDEALS

The results of the preceding section not only admit a certain sharpening for the case of s.n. ideals, but provide the basis for a unified approach to topological and order properties of arbitrary s.n. ideals that are related to weak sequential completeness, weak compactness and reflexivity. In order to consider perfectness, we shall need first a preliminary result which shows that, if $\mathscr{I}$ is an s.n. ideal, then $\mathscr{I}^{\times}$ is precisely a maximal ideal in the sense of [10]. For the definitions relevant to the following discussion, we refer to [10], Chapter III. Following [10], Φ will denote a symmetric norming function on c_{00} with maximal domain c_{Φ} and adjoint function Φ^*. By $\mathscr{C}_{\Phi}$ is denoted the s.n. ideal consisting of all $A \in \mathscr{C}_{\infty}$ for which the singular value sequence $\{s_j(A)\}_{j=1}^{\infty} \in c_{\Phi}$.

LEMMA 4.1. *Let $\mathscr{I}$ be a* s.n. *ideal and let Φ be the symmetric norming function on c_{00} defined by setting*

$$\Phi(\eta) = \|\sum_j \eta_j e_j \otimes \bar{e}_j\|_{\mathscr{I}}$$

for each $\eta \in c_{00}$, for some orthonormal sequence $\{e_j\}_{j=1}^{\infty}$ in $\mathscr{H}$. If $\mathscr{C}_1 \subsetneqq \mathscr{I} \subseteq \mathscr{C}_\infty$, then

$$(\mathscr{I}^{\times}, \|\ \|_{\mathscr{I}^{\times}}) = (\mathscr{C}_{\Phi^*}, \|\ \|_{\Phi^*}).$$

Proof. If μ is the symmetric sequence space corresponding to $\mathscr{I}$ under the Calkin correspondence, if μ is equipped with the norm $\|\cdot\|_\mu$ induced by $\mathscr{I}$ and if $\mu^{\times}$, the α-dual of μ is normed by setting

$$\|\eta\|_{\mu^{\times}} = \sup\left\{\sum_{i=1}^{\infty} |\eta_i \xi_i| : \xi \in \mu,\ \|\xi\|_\mu \leqslant 1\right\}$$

then it suffices, by Theorem 2 and Theorem 3 of [9] to show that $(\mu^{\times}, \|\cdot\|_{\mu^{\times}}) = (c_{\Phi^*}, \|\ \|_{\Phi^*})$. This however follows readily from the relevant definitions and so the details are omitted.

We may now characterize perfect s.n. ideals.

PROPOSITION 4.2. *If $\mathscr{I} \subseteq \mathscr{C}_\infty$ is a* s.n. *ideal, then the following statements are equivalent.*

(i) *$\mathscr{I}$ is perfect.*

(ii) *If $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{I}$ is $\|\cdot\|_{\mathscr{I}}$-bounded and weak operator convergent to $B \in \mathscr{B}$, then $B \in \mathscr{I}$.*

(iii) *$\mathscr{I}$ coincides pointwise with a* s.n. *ideal $\mathscr{C}_\Phi$ with symmetric norming function Φ not equivalent to the minimal one.*

(iv) *If $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{I}$ is $\|\cdot\|_{\mathscr{I}}$-bounded and if $0 \leqslant B_n \uparrow_n$, then there exists $0 \leqslant B \in \mathscr{I}$ with $0 \leqslant B_n \uparrow_n B$.*

Proof. The implication (iii) $\Rightarrow$ (ii) is proved in [10], Theorem III.5.1, while (ii) trivially implies (iv).

(i) $\Rightarrow$ (iii). It is clear that we may assume that $\mathscr{C}_1 \subsetneqq \mathscr{I} \subsetneqq \mathscr{C}_\infty$ so that $\mathscr{C}_1 \subsetneqq \mathscr{I}^{\times} \subsetneqq \mathscr{C}_\infty$, by the perfectness of $\mathscr{I}$ and [9], Proposition 7, since the implication (i) $\Rightarrow$ (iii) is clearly valid for $\mathscr{C}_1$. Let Φ denote the symmetric norming function induced by $\mathscr{I}$ on c_{00}. From Lemma 4.1 above, it follows that $\mathscr{I}^{\times} = \mathscr{C}_{\Phi^*}$ and hence

$$\mathscr{I} = (\mathscr{I}^{\times})^{\times} = \mathscr{C}_{\Phi^{**}} = \mathscr{C}_\Phi.$$

It is clear that Φ is not equivalent to the minimal one and by this the proof of the implication is complete.

It remains to prove the implication (iv) $\Rightarrow$ (i). As in the proof of Theorem 3.5, condition (iv) implies that $\mathscr{I}^{\times\times} \subseteq \mathscr{C}_\infty$. Let $0 \leqslant B \in \mathscr{I}^{\times\times}$ and let $\{B_n\}_{n=1}^\infty$ be the partial sum sequence of the Schmidt expansion of B. It is clear that the sequence $\{B_n\}_{n=1}^\infty$ is $\sigma(\mathscr{I}, \mathscr{I}^\times)$-bounded and hence $\|\cdot\|_{\mathscr{I}^{\times\times}}$ bounded in $\mathscr{I}^{\times\times}$. However, it follows from Corollary 2.6 that $\|B_n\|_{\mathscr{I}} = \|B_n\|_{\mathscr{I}^{\times\times}}$, $n = 1, 2, \ldots$, so that $\{B_n\}$ is $\|\cdot\|_{\mathscr{I}}$-bounded and so it now follows from (iv) that $B \in \mathscr{I}$. Thus $\mathscr{I}$ is perfect and by this, the proposition is completely proved.

We remark first that the equivalence (i) $\Leftrightarrow$ (iv) above can be viewed as a non--commutative analogue of a well-known characterization of perfect Banach lattices. See, for example [20], Theorem III.1. To facilitate some further remarks, a s.n. ideal $\mathscr{I}$ is said to have σ-Fatou norm if and only if whenever $\{A_n\}_{n=1}^\infty \subseteq \mathscr{I}$ and $0 \leqslant A_n \uparrow_n A$ with $A \in \mathscr{I}$, it follows that $\|A\|_{\mathscr{I}} = \sup_n \|A_n\|_{\mathscr{I}}$. In equivalent form, this property appears in [16]. It is not difficult to see that each Gohberg-Kreĭn ideal $\mathscr{C}_\Phi$ has a σ-Fatou norm. Moreover, in Proposition 4.2 preceding, if the ideal $\mathscr{I}$ is assumed to have σ-Fatou norm, the statement (iii) may be replaced by

(iii) $$(\mathscr{I}, \|\cdot\|_{\mathscr{I}}) = (\mathscr{C}_\Phi, \|\cdot\|_\Phi).$$

Let us now make one further remark. In [4], the authors consider a class of s.n. ideals with a property that is easily seen to be equivalent to the σ-Fatou property of the norm, together with the property given by statement (ii). It follows that the class of ideals considered in [4] are precisely the Gohberg-Kreĭn ideals $\mathscr{C}_\Phi$ and this answers a question raised in [4].

Before proceeding to our next result, we observe that if $\mathscr{I}$ is a s.n. ideal and if $A, B \in \mathscr{I}$ satisfy $0 \leqslant B \leqslant A$, then $\|B\|_{\mathscr{I}} \leqslant \|A\|_{\mathscr{I}}$. It follows (see, for example [2]) that each element of $\mathscr{I}^*$ is a linear combination of positive functionals.

We now characterize those s.n. ideals which are weakly sequentially complete.

PROPOSITION 4.3. *If $\mathscr{I}$ is a* s.n. *ideal, then the following statements are equivalent.*

(i) *$\mathscr{I}$ is $\sigma(\mathscr{I}, \mathscr{I}^*)$-sequentially complete.*

(ii) *$\mathscr{I}$ is perfect and minimal.*

(iii) *Each $\|\cdot\|_{\mathscr{I}}$-bounded increasing sequence of positive elements of $\mathscr{I}$ is $\|\cdot\|_{\mathscr{I}}$-convergent.*

(iv) *$\mathscr{I}$ contains no copy of c_0.*

Proof. (ii) $\Rightarrow$ (i). Since $\mathscr{I}$ is minimal, it follows that $\mathscr{I} \subseteq \mathscr{C}_\infty$ and that $\mathscr{I}^\times = \mathscr{I}^*$. The implication now follows from Theorem 3.5.

(i) $\Rightarrow$ (ii). We observe first that $\mathscr{I} \subseteq \mathscr{C}_\infty$. In fact if this is not so, then $\mathscr{I}^\times$ coincides with $\mathscr{C}_1$ by [9], Proposition 7, and consequently $\mathscr{I}^0 = (\mathscr{I}^{\times\times})^0$ coincides with $\mathscr{B}^0 = \mathscr{C}_\infty$. This contradicts (i). It now suffices to show that $\mathscr{I}^* = \mathscr{I}^\times$ and to this end, it suffices, in view of Proposition 2.8 to show that if $\{B_n\} \subseteq \mathcal{J}$ satisfies $B_n \downarrow_n 0$ then $\{B_n\}$ is $\sigma(\mathcal{J}, \mathcal{J}^*)$-convergent to 0. Since the dual cone generates $\mathcal{J}^*$, it follows that $\{B_n\}_{n=1}^{\infty}$ is $\sigma(\mathcal{J}, \mathcal{J}^*)$-Cauchy, hence $\sigma(\mathcal{J}, \mathcal{J}^*)$-convergent to an element $B \in \mathcal{J}$, by assumption. However, since $\{B_n\}_{n=1}^{\infty}$ is weak operator convergent to 0, it follows that $B = 0$ and the proof of the implication is complete.

(ii) $\Leftrightarrow$ (iii). This equivalence is easily seen via Proposition 4.2 and Proposition 2.9.

The implication (i) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (ii). Assume that $\mathcal{J}$ contains no copy of c_0. Observe first that $\mathcal{J}^{\times\times} \subseteq \mathcal{C}_\infty$, for if not, $\mathcal{J}^{\times\times} \supsetneq \mathcal{C}_\infty$ by [9], Proposition 1, so that $\mathcal{J}^{\times\times} = \mathcal{B}$ since $\mathcal{J}^{\times}$ is perfect. It follows that $\mathcal{J}^0 = (\mathcal{J}^{\times\times})^0 = \mathcal{C}_\infty$ and this is a contradiction. It now suffices to show that $\mathcal{J}^{\times\times} = (\mathcal{J}^{\times\times})^0$. To this end, suppose that $0 \leqslant A \in \mathcal{J}^{\times\times}$ but that $A \notin (\mathcal{J}^{\times\times})^0$. Let $A = \sum_{i=1}^{\infty} \lambda_i \varphi_i \otimes \varphi_i$ be the Schmidt expansion of A. Since $A \notin (\mathcal{J}^{\times\times})^0$, it follows that there exists $\varepsilon > 0$ and an increasing sequence $\{n(k)\}_{k=1}^{\infty}$ of natural numbers such that

$$\left\| \sum_{i=n(2k-1)}^{n(2k)} \lambda_i \varphi_i \otimes \varphi_i \right\|_{\mathcal{J}} = \left\| \sum_{i=n(2k-1)}^{n(2k)} \lambda_i \varphi_i \otimes \varphi_i \right\|_{\mathcal{J}^{\times\times}} > \varepsilon$$

for $k = 1, 2, \ldots$. Setting

$$A_k = \sum_{i=n(2k-1)}^{n(2k)} \lambda_i \varphi_i \otimes \varphi_i, \quad k = 1, 2, \ldots$$

it is not difficult to see that if $\xi \in c_{00}$, then

$$\varepsilon \|\xi\|_\infty \leqslant \|\sum_k \xi_k A_k\|_{\mathcal{J}} \leqslant \|\xi\|_\infty \|A\|_{\mathcal{J}}$$

and this clearly yields a contradiction. By this, the proposition is completely proved.

We remark first that the equivalence (i) $\Leftrightarrow$ (ii) of the preceding proposition contains as a special case the well-known fact ([1], Corollary III.1) that $\mathcal{C}_1$ is weakly sequentially complete. Further, Proposition 4.3 is a non-commutative analogue of well-known characterizations of Banach lattices which are weakly sequentially complete. See, for example, [15], Theorem 1.c.4. To make some additional remarks, suppose that $\mathcal{J}$ is a s.n. ideal with $\mathcal{J} \subseteq \mathcal{C}_\infty$ and let μ be the associated symmetrically normed sequence space. It follows from [9], Corollary 1, that $\mathcal{J}$ is perfect if and only if μ is perfect and it is not difficult to see that $\mathcal{J}$ is minimal if and only if the induced norm on μ is order continuous. Using standard facts from the theory of Banach lattices, it follows that each of the statements (i) — (iv) in Proposition 4.3 above are equivalent to each of the corresponding statements obtained by replacing $\mathcal{J}$ by μ. We omit the details. We note finally that if the s.n. ideal $\mathcal{J}$ is in addition assumed to be minimal, then a number of additional equivalent statements can be obtained by inspection from [3], Proposition 3.7 and [11], Theorem 3.4. The details however, we leave to the interested reader.

Our next result characterizes those *minimal* s.n. ideals $\mathcal{J}$ for which the dual ideal $\mathcal{J}^{\times} = \mathcal{J}^{*}$ is again minimal and is related to a question raised by Schatten [17]. Again, our motivation is drawn from analogous results from the theory of Banach lattices. See for example [14], Theorem 1.c.9.

PROPOSITION 4.4. *If $\mathcal{J}$ is minimal* s.n. *ideal, then the following statements are equivalent.*

(i) *$\mathcal{J}^{\times}$ is minimal.*

(ii) *The unit ball of $\mathcal{J}$ is conditionally $\sigma(\mathcal{J}, \mathcal{J}^{*})$-sequentially compact.*

(iii) *$\mathcal{J}$ contains no isomorphic copy of ℓ_1.*

Proof. The implication (ii) $\Rightarrow$ (i) follows immediately from Proposition 3.1 and Proposition 2.9.

Condition (i) implies via Proposition 3.3 that the unit ball of $\mathcal{J}$ is relatively $\sigma(\mathcal{J}^{\times\times}, \mathcal{J}^{\times})$-sequentially compact in $\mathcal{J}^{\times\times}$ and hence conditionally $\sigma(\mathcal{J}, \mathcal{J}^{\times})$-sequentially compact, which is (ii).

The implication (ii) $\Rightarrow$ (iii) is clear. The proof will be complete by showing that (iii) $\Rightarrow$ (i). If $\mathcal{J}^{\times}$ is not minimal, then by Proposition 2.9 (ii), there exists $0 \leqslant A \in \mathcal{J}^{\times}$ with $\|A\|_{\mathcal{J}^{\times}} = 1$, a sequence $\{P_n\}_{n=1}^{\infty}$ of mutually disjoint projections, a sequence $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{J}$ with $\|B_n\|_{\mathcal{J}} \leqslant 1$ for $n = 1, 2, \ldots$ and $\varepsilon > 0$ such that $|\mathrm{tr}(AP_nB_n)| > \varepsilon$ for $n = 1, 2, \ldots$. If $\xi \in c_{00}$, observe that

$$\|\sum_n \xi_n P_n B_n\|_{\mathcal{J}} \leqslant \|\xi\|_1 .$$

On the other hand, if $\lambda \in c_{00}$ and if $\|\lambda\|_{\infty} \leqslant 1$, then

$$\|\sum_n \xi_n P_n B_n\|_{\mathcal{J}} \geqslant |\mathrm{tr}((\sum_n \lambda_n A P_n)(\sum_n \xi_n P_n B_n))| =$$

$$= |\sum_n \lambda_n \xi_n \,\mathrm{tr}(AP_nB_n)| \geqslant \varepsilon \|\xi\|_1$$

by appropriate choice of λ. It follows that $\mathcal{J}$ contains an isomorphic copy of ℓ_1 and the proof of the proposition is complete.

Once again, we observe that if μ is the normed sequence space corresponding to the minimal s.n. ideal $\mathcal{J}$, then μ has order continuous norm. It is not difficult to see that $\mathcal{J}^{\times}$ is minimal if and only if $\mu^{\times}$ has order continuous norm. Using standard facts, it is then not difficult to show that each of the statements of the preceding proposition is equivalent to each of the statements obtained by replacing $\mathcal{I}$ by μ. We remark that the preceding result may be used as a basis for an alternative proof of [11], Theorem 3.5. It should be further remarked that the equivalence (ii) $\Leftrightarrow$ (iii) above is an immediate consequence of Rosenthal's ℓ_1 Theorem (see [7], Chapter XI). We have, however, preferred to indicate the direct and simple proof that is available in the present setting.

We conclude with a characterization of reflexive s.n. ideals, which is a simple consequence of Proposition 4.4 and Proposition 4.3 above.

PROPOSITION 4.5. *If* $\mathcal{I}$ *is a* s.n. *ideal, then the following statements are equivalent.*

(i) $\mathcal{I}$ *is reflexive.*

(ii) $\mathcal{I}$ *is perfect and both* $\mathcal{I}$ *and* $\mathcal{I}^{\times}$ *are minimal.*

(iii) *Every increasing bounded sequence in the positive cone of* $\mathcal{I}$ *is convergent and* $\mathcal{I}^{\times}$ *is minimal.*

(iv) $\mathcal{I}$ *contains no subspace isomorphic to* c_0 *or* ℓ_1.

We remark finally that, via Proposition 4.2, the implication (ii) $\Rightarrow$ (i) is given in [10], Theorem III.12.2.

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