J. OPERATOR THEORY
16(1986), 165–187

THE K-THEORY OF THE REDUCED C^*-ALGEBRA OF AN HNN-GROUP

JOEL ANDERSON and WILLIAM L. PASCHKE

1. INTRODUCTION

Let A denote a subgroup of a group G, and let $\theta : A \to G$ be a monomorphism. We write $\Gamma = \mathrm{HNN}(G, A, \theta)$ for the group derived from these ingredients by the Higman-Neumann-Neumann construction [3], [11]. (The group Γ is generated by a copy of G and an element s in Γ satisfing $sas^{-1} = \theta(a)$ for each a in A. It has the following universal property : if ρ is a homomorphism from G into a group K containing an element t such that $t\rho(a)t^{-1} = \rho(\theta(a))$ $\forall a$ in A, then ρ has an extension to Γ mapping s to t.) Our purpose in this paper is to show the existence of a cyclic exact sequence relating the K-groups of the reduced C^*-algebras of A, G, and Γ, viz.

$$\begin{array}{ccccc}
\mathrm{K}_0(C_r^*(A)) & \xrightarrow{\theta_* - i_*} & \mathrm{K}_0(C_r^*(G)) & \xrightarrow{j_*} & \mathrm{K}_0(C_r^*(\Gamma)) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(C_r^*(\Gamma)) & \xleftarrow[j_*]{} & \mathrm{K}_1(C_r^*(G)) & \xleftarrow[\theta_* - i_*]{} & \mathrm{K}_1(C_r^*(A)),
\end{array}$$

where i_*, θ_*, and j_* come from the injections induced at the C^*-algebra level by the the inclusion $i : A \to G$, by $\theta : A \to G$, and by the inclusion $j : G \to \Gamma$.

Our assumptions are that G is countable, and that either (G, A) or $(G, \theta(A))$ has property Λ of Lance-Natsume [4], [6]. The latter hypothesis is imposed solely so that we may exploit Natsume's result in [6] for the amalgamated product of two groups along a common subgroup. If his result remains valid without property Λ, then so does ours.

The basic idea of our computation is straightforward. The HNN-group Γ may be realized as a semidirect product $H \times \mathbf{Z}$, where

$$H = \ldots *_A G *_A G *_A G * \ldots,$$

the amalgam of a two-way infinite line of copies of G along copies of A injected left and right by θ and i, respectively, and $\mathbf{Z}$ acts on H by shifting. To take advantage of this, we establish analogues of the Natsume sequence, first for finite lines of groups, then for infinite lines. One then has an exact sequence relating $K_{\#}(C_r^*(H))$ to the given data. The rest of the work, in fact the bulk of our argument, consists in "feeding" this sequence into the Pimsner-Voiculescu sequence [9] relating $K_{\#}(C_r^*(\Gamma))$ to $K_{\#}(C_r^*(H))$. Of course, one wants the final sequence to come from a short exact sequence of C^*-algebras. To this end, we construct a "Toeplitz extension"

$$0 \to C_r^*(A) \otimes \mathscr{K} \to \mathscr{D} \to C_r^*(\Gamma) \to 0.$$

The algebra $\mathscr{D}$ lives on a certain subspace of $\ell^2(\Gamma)$, and is generated by a unitary copy of G and an isometry S that tries to play the role of s. There is a map from $C_r^*(G)$ into $\mathscr{D}$; the goal is to show that this map induces an isomorphism on K-theory. To this end, we realize $\mathscr{D}$ as the crossed product of a subalgebra $\mathscr{D}_0$ by an endomorphism σ in the sense of [7]. This $\mathscr{D}_0$ is an extension of $c_0(\mathbf{Z}) \otimes C_r^*(A) \otimes K$ by $C_r^*(H)$. Confronting the resulting exact sequence of K-groups with the sequence already obtained for $C_r^*(H)$, we show that $K_{\#}(\mathscr{D}_0)$ is the two-way infinite direct sum of copies of $K_{\#}(C_r^*(G))$, on which σ_* acts by shifting. The desired isomorphism of $K_{\#}(\mathscr{D})$ with $K_{\#}(C_r^*(G))$ then follows from [8].

There is a larger context which unites amalgamated products and HNN groups, namely the notion, developed in Serre's mononograph [11], of the fundamental group of a graph of groups. A graph of groups is simply a (connected) graph on which are placed a group G_P at each node P and a group A_y on each edge y, with given embeddings of each edge group A_y into the G's at the ends of y. The corresponding fundamental group may be constructed by choosing a maximal tree T in the graph, amalgamating the node groups along the edge groups coming from edges in T, and performing an HNN-construction for each edge not in T. (The isomorphism class of the result turns out to be independent of the choice of T.) The simplest cases are a graph with 2 nodes and 1 edge, which gives the amalgamated product of the node groups along the edge group, and a graph with 1 node and 1 edge, which corresponds to the HNN construction. Serre's fundamental groups are intimately related to group action on trees. The fundamental group of a graph of groups acts canonically on a tree constructed from the given data. Conversely, any group acting without inversion on a tree may be realized in a canonical way as the fundamental group of a certain graph of groups.

Suppose that Γ is constructed as above from node groups G_P and edge groups A_y. We believe that there is an exact sequence

$$\begin{array}{ccccc} \bigoplus_y K_0(C_r^*(A_y)) & \xrightarrow{(1)} & \bigoplus_P K_0(C_r^*(G_P)) & \xrightarrow{(2)} & K_0(C_r^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma)) & \xleftarrow{(2)} & \bigoplus_P K_1(C_r^*(G_p)) & \xleftarrow{(1)} & \bigoplus_y K_1(C_r^*(A_y)), \end{array}$$

where map (2) is the sum of the maps induced by the inclusions $G_p \to \Gamma$, and map (1) sends each y-summand to the two P-summands at the ends of y, via $A_y \hookrightarrow G_p$, positively at one end and negatively at the other. In the final section of this paper, we sketch a proof of this for the case of a loop of groups, assuming property Λ for the subgroup inclusions either clockwise or counterclockwise around the loop.

2. THE AMALGAM OF TWO GROUPS

Consider the amalgam $\Gamma = X *_A Y$ of two countable groups X and Y, both of which properly contain the amalgamated subgroup A. In [6] Natsume, building on the work of Lance [4], constructed an extension of $C_r^*(A) \otimes \mathscr{K}$ by $C_r^*(\Gamma)$. Let us begin by recalling that construction. Every element of $\Gamma \setminus A$ can be written as a finite product of x's in $X \setminus A$ alternating with y's in $Y \setminus A$. Let Γ_X^* be the set of all such words ending in $X \setminus A$, and let $\Gamma_X = \Gamma_X^* \cup A$. Denoting by λ the left regular representation of Γ on $\ell^2(\Gamma)$, we observe that $\lambda(X)\ell^2(\Gamma_X) = \ell^2(\Gamma_X)$ and $\lambda(Y)\ell^2(\Gamma_X^*) = \ell^2(\Gamma_X^*)$. For x in X and y in Y, define

$$\mu(x) = \lambda(x) \mid_{\ell^2(\Gamma_X)}$$

$$\nu(y) = \lambda(y) \mid_{\ell^2(\Gamma_X^*)},$$

where the latter is regarded as an operator on $\ell^2(\Gamma_X)$. Set $q_A = \mu(e) - \nu(e)$, the projection of $\ell^2(\Gamma_X)$ on $\ell^2(A)$. Write $\mathscr{T}$ for the C^*-algebra of operators on $\ell^2(\Gamma_X)$ generated by $\mu(X)$ and $\nu(Y)$. Let $\mathscr{J}$ be the closed ideal of $\mathscr{T}$ generated by q_A. There is then a short exact sequence

$$0 \to \mathscr{J} \to \mathscr{T} \xrightarrow{\pi} C_\mu^*(\Gamma) \to 0,$$

where $\pi \circ \mu = \lambda \mid_X$ and $\pi \circ \nu = \lambda \mid_Y$. Furthermore, the ideal $\mathscr{J}$ is isomorphic to $C_r^*(A) \otimes \mathscr{K}$; the isomorphism sends $\mu(a)q_A$ to $\lambda_A(a) \otimes e_0$, where $a \in A$, $\lambda_A(a)$ is its image in $C_r^*(A)$, and e_0 is a minimal projection in $\mathscr{K}$ (the algebra of compact operators). We shall refer to the above extension (resp. the C^*-algebra $\mathscr{T}$) as the *Toeplitz extension* (resp. *Toeplitz algebra*) for the amalgam $X *_A Y$. The construction depends upon the choice of one of the factors in the amalgam. Our convention will be always to choose the lefthand factor.

We need to know what happens to the extension when one or another of the factors is imbedded in a larger group.

2.1. PROPOSITION. *Suppose that X, Y, and A are as above, and that X is a subgroup of a larger group X'. Form $\Gamma' = X' *_A Y$ and the associated Toeplitz algebra $\mathscr{T}' = C^*(\mu'(X'), \nu'(Y))$ on $\ell^2(\Gamma'_{X'})$. There is a $*$-monomorphism φ from $\mathscr{T}$ to $\mathscr{T}'$*

sending $\mu(x)$ to $\mu'(x)$ (x in X), and $v(y)$ to $v'(y)$ (y in Y) and such that the following diagram commutes

$$\begin{array}{ccccc} \mathcal{S}' & \longrightarrow & \mathcal{T}' & \xrightarrow{\pi'} & C_r^*(\Gamma') \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} & & \uparrow \\ \mathcal{S} & \longrightarrow & \mathcal{T} & \xrightarrow[\pi]{} & C_r^*(\Gamma). \end{array}$$

Proof. We may regard Γ as a subgroup of Γ'. Under this identification, the set Γ_X becomes a subset of $\Gamma'_{X'}$. Let $\mathcal{B}$ be the C^*-subalgebra of $\mathcal{T}'$ generated by $\mu'(X)$ and $v'(Y)$, so $\ell^2(\Gamma_X)$ reduces $\mathcal{B}$. Notice also that $\Gamma'_{X'} \setminus \Gamma_X$ is invariant under left multiplication by Γ. Decompose $\Gamma'_{X'} \setminus \Gamma_X$ into Γ-orbits :

$$\Gamma'_{X'} \setminus \Gamma_X = \bigcup \{\Gamma\omega : \omega \in \Omega\},$$

where Ω is a set of orbit representatives. Restriction to $\ell^2(\Gamma_X)$ gives a map $\psi : \mathcal{B} \to \mathcal{T}$. Restriction to each $\ell^2(\Gamma\omega)$, on which $v'(y)$ acts unitarily for y in Y, just gives the left regular representation of Γ, i.e. $\pi \circ \psi$. This means that the identity representation of $\mathcal{B}$ on $\ell^2(\Gamma'_{X'})$ is the direct sum of ψ with several copies of $\pi \circ \psi$. It follows that ψ is an isomorphism. The desired map is $\varphi = \psi^{-1} : T \to \mathcal{B} \subseteq \mathcal{T}'$. ▨

There is an analogous result for the factor Y.

2.2. Proposition. *Suppose that X, Y, and A are as above, and that Y is a subgroup of a larger group Y''. Form $\Gamma'' = X *_A Y''$ and the associated Toeplitz algebra $\mathcal{T}'' = C^*(\mu''(X), v''(Y''))$ on $\ell^2(\Gamma'')$. Then there is a $*$-homomorphism from $\mathcal{T}$ to $\mathcal{T}''$ sending $\mu(x) \mapsto \mu''(x)$ (x in X), and $v(y) \mapsto v''(y)$ (y in Y) and an analogous commutative diagram.*

Proof. This is proved in the same manner as 2.1, after one notes that $\Gamma''_{X} \setminus \Gamma_X$ is invariant under left multiplication by Γ. ▨

The maps μ and v of X and Y into $\mathcal{T}$ are direct sums of copies of the left regular representations of these two groups, so they extend to $C_r^*(X)$ and $C_r^*(Y)$. There are also injections $i_X, i_Y : C_r^*(A) \to C_r^*(X), C_r^*(Y)$. As in [6], we have a commutative diagram

$$\begin{array}{ccc} K_\#(\mathcal{S}) & \longrightarrow & K_\#(\mathcal{T}) \\ \uparrow & & \uparrow \\ K_\#(C_r^*(A)) & \longrightarrow & K_\#(C_r^*(X)) \oplus K_\#(C_r^*(Y)). \end{array}$$

The map on top comes from inclusion. The lefthand map comes from the isomorphism of $\mathcal{S}$ with $C_r^*(A) \otimes \mathcal{K}$. The map on the bottom is $((i_X)_*, -(i_Y)_*)$; instead of giving it a symbolic designation, we will refer to it (and to its analogues to be encountered later on) as the *subgroup map*. The righthand map is μ_* on the first summand, ν_* on the second. We will call this map (and its subsequent analogues) the *group map*.

When the pair (X, A) has property A (roughly, there is a well-behaved homotopy joining the left regular representation of X on $\ell^2(X/A)$ to a representation with a non-zero fixed vector), it is shown in [6] that the group map is an isomorphism. This is the main step in obtaining the K-theoretic exact sequence of [6].

3. AMALGAM OF A LINE OF GROUPS

Let $G_1, \ldots, G_n$ be countable groups $(n \geqslant 2)$, and for $k = 1, \ldots, n-1$, let A_k be a proper subgroup of G_k and G_{k+1}. Let Γ denote the amalgam $G_1 *_{A_1} G_2 * \ldots *_{A_{n-1}} G_n$. We can "break" Γ at any one of the subgroups A_k and write

$$\Gamma = (G_1 *_{A_1} G_2 * \ldots *_{A_{k-1}} G_k) *_{A_k} (G_{k+1} * \ldots *_{A_{n-1}} G_n)$$

as the amalgam of two groups along A_k. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}_k & \longrightarrow & \mathscr{T}_k & \xrightarrow{\pi_k} & C_r^*(\Gamma) & \longrightarrow & 0 \\ & & \wr\wr & & & & & & \\ & & C_r^*(A_k) \otimes \mathscr{K} & & & & & & \end{array}$$

be the corresponding Toeplitz extension, as in Section 2. Borrowing a technique from [10], we let $(\mathscr{T}, \pi)$ be the pullback of $(\mathscr{T}_1, \pi_1), \ldots, (\mathscr{T}_{n-1}, \pi_{n-1})$. That is,

$$\mathscr{T} = \{(t_1, \ldots, t_{n-1}) \in \mathscr{T}_1 \oplus \ldots \oplus \mathscr{T}_{n-1} : \pi_1(t_1) = \pi_2(t_2) = \ldots = \pi_{n-1}(t_{n-1})\}$$

and $\pi : \mathscr{T} \to C_r^*(\Gamma)$ is the map to the common value. Plainly, $\ker \pi = \mathscr{I}_1 \oplus \ldots \oplus \mathscr{I}_{n-1}$. Each breaking of Γ as a 2-fold amalgam gives rise to a commuting diagram of maps between K-groups as at the end of Section 2. The maps involved can be put together to produce a single commuting diagram

$$\begin{array}{ccc} \bigoplus_{k=1}^{n-1} \mathrm{K}_\#(\mathscr{I}_k) & \longrightarrow & \mathrm{K}_\#(\mathscr{T}) \\ \uparrow & & \uparrow \\ \bigoplus_{k=1}^{n-1} \mathrm{K}_\#(C_r^*(A_k)) & \longrightarrow & \bigoplus_{j=1}^{n} \mathrm{K}_\#(C_r^*(G_j)). \end{array}$$

The subgroup map on the bottom sends the A_k component positively to the G_k component, and negatively to the G_{k+1} component. When restricted to the G_j component, the group map on the right has the form $(\nu_1, \ldots, \nu_{j-1}, \mu_j, \ldots, \mu_{n-1})_*$.

We are now ready to prove the following corollary of Natsume's result.

3.1. THEOREM. *In the situation described above, suppose that the pairs* $(G_1, A_1), \ldots, (G_{n-1}, A_{n-1})$ *all have property A. Then group map from* $\bigoplus_{j=1}^{n} \mathrm{K}_{\#}(C_r^*(G_j))$ *to* $\mathrm{K}_{\#}(\mathscr{T})$ *is an isomorphism.*

Proof. We use induction on n. The initial case $n = 2$ is of course taken care of in [6]. Take $n \geqslant 3$, and assume that 3.1 is true for the amalgam of a line of $n - 1$ groups. Consider $\Gamma^{\sim} = G_2 *_{A_2} G_3 * \ldots *_{A_{n-1}} G_n$. Breaking $\Gamma^{\sim}$ at A_k for $k = 2, \ldots, n - 1$ yields $\pi_k^{\sim} : \mathscr{T}_k^{\sim} \to C_r^*(\Gamma^{\sim})$ with pullback $\pi^{\sim} : \mathscr{T}^{\sim} \to C_r^*(\Gamma^{\sim})$. The induction hypothesis says that $\bigoplus_{j=2}^{n} \mathrm{K}_{\#}(C_r^*(G_j))$ is isomorphic to $\mathrm{K}_{\#}(\mathscr{T}^{\sim})$ via the group map for $\Gamma^{\sim}$.

We define a map $\alpha : \mathscr{T}^{\sim} \to \mathscr{T}$ as follows. Use 2.1 to obtain maps $\alpha_k : \mathscr{T}_k^{\sim} \to \mathscr{T}_k$ for $k = 2, \ldots, n - 1$. Let $v_1 : C_r^*(\Gamma^{\sim}) \to \mathscr{T}_1$ be as in Section 2 for the amalgam $\Gamma = G_1 *_{A_1} \Gamma^{\sim}$; recall that $\mathscr{T}_1$ comes from breaking Γ at A_1. For $(t^{\sim}) = (t_2^{\sim}, \ldots, t_{n-1}^{\sim})$ in $\mathscr{T}^{\sim}$, write

$$\alpha(t^{\sim}) = (v_1\pi^{\sim}(t^{\sim}),\ \alpha_2(t_2^{\sim}),\ \ldots,\ \alpha_{n-1}(t_{n-1}^{\sim})),$$

which lies in $\mathscr{T}_1 \oplus \mathscr{T}_2 \oplus \ldots \oplus \mathscr{T}_{n-1}$. Since $\pi_1 v_1$ is the natural injection $j : C_r^*(\Gamma^{\sim}) \to C_r^*(\Gamma)$, and since $\pi_k\alpha_k = j\pi_k^{\sim}$ for $k = 2, \ldots, n - 1$, it follows that α maps $\mathscr{T}^{\sim}$ to $\mathscr{T}$. We obtain a map between short exact sequences of C^*-algebras

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{k=2}^{n-1} \mathscr{J}_k & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{T}_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle v_1} & & \\
0 & \longrightarrow & \bigoplus_{k=2}^{n-1} \mathscr{J}_k^{\sim} & \longrightarrow & \mathscr{T}^{\sim} & \longrightarrow & C_r^*(\Gamma^{\sim}) & \longrightarrow & 0,
\end{array}$$

where $\mathscr{J}_k^{\sim} = \ker \pi_k^{\sim}$. Passing to K-theory, we have

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{K}_1(\mathscr{T}_1) & \longrightarrow & \bigoplus_{2}^{n-1} \mathrm{K}_0(C_r^*(A_k)) & \longrightarrow & \mathrm{K}_0(\mathscr{T}) & \longrightarrow & \mathrm{K}_0(\mathscr{T}_1) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\ldots & \longrightarrow & \mathrm{K}_1(C_r^*(\Gamma^{\sim})) & \longrightarrow & \bigoplus_{2}^{n-1} \mathrm{K}_0(C_r^*(A_k)) & \longrightarrow & \mathrm{K}_0(\mathscr{T}^{\sim}) & \longrightarrow & \mathrm{K}_0(C_r^*(\Gamma^{\sim})) & \longrightarrow & \ldots .
\end{array}$$

The sequences on top and bottom are exact. The squares commute because of functoriality of the cyclic exact sequence of K-theory. It is straightforward to check that the map between the "A" terms is the identity map.

Let $m : \mathrm{K}_\#(C_r^*(G_1)) \to \mathrm{K}_\#(\mathscr{T})$ be the restriction to the first component of the group map for Γ. We obtain another commuting "ladder" of K-groups

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{K}_1(\mathscr{T}_1) & \longrightarrow & \bigoplus_{2}^{n-1} \mathrm{K}_0(C_r^*(A_k)) & \longrightarrow & \mathrm{K}_0(\mathscr{T}) & \longrightarrow & \mathrm{K}_0(\mathscr{T}_1) & \longrightarrow \cdots \\
& \uparrow{\scriptstyle (\mu_1)_*} & & \uparrow & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle (\mu_1)_*} & \\
\cdots \xrightarrow[\mathrm{id}]{} & \mathrm{K}_1(C_r^*(G_1)) & \xrightarrow[0]{} & 0 & \longrightarrow & \mathrm{K}_0(C_r^*(G_1)) & \xrightarrow[\mathrm{id}]{} & \mathrm{K}_0(C_r^*(G_1)) & \xrightarrow[0]{} \cdots .
\end{array}$$

When this is superimposed on the previous ladder, the top sequence is unchanged, the bottom sequence remains exact, the rectangles still commute, and the vertical maps

$$\mathrm{K}_\#(C_r^*(G_1)) \oplus \mathrm{K}_\#(C_r^*(\Gamma^\sim)) \to \mathrm{K}_\#(\mathscr{T}_1)$$

are isomorphisms, since they are just the group maps corresponding to the decomposition $\Gamma = G_1 *_{A_1} \Gamma^\sim$. Applying the five lemma to the combined ladder, we conclude that the vertical maps from $\mathrm{K}_\#(C_r^*(G_1)) \oplus \mathrm{K}_\#(\mathscr{T}^\sim)$ to $\mathrm{K}_\#(\mathscr{T})$ are isomorphisms. When we use the inductive hypothesis to identify $\mathrm{K}_\#(\mathscr{T}^\sim)$ with $\bigoplus_{2}^{n-1} \mathrm{K}_\#(C_r^*(G_j))$ these vertical maps are seen to be the group maps for the n-fold amalgam Γ. This proves the theorem. ▨

As an immediate consequence we have

3.2. COROLLARY. *Under the hypotheses of the theorem above, there is an exact sequence*

$$\begin{array}{ccccc}
\bigoplus_{k=1}^{n-1} \mathrm{K}_0(C_r^*(A_k)) & \longrightarrow & \bigoplus_{j=1}^{n} \mathrm{K}_0(C_r^*(G_j)) & \longrightarrow & \mathrm{K}_0(C_r^*(\Gamma)) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(C_r^*(\Gamma)) & \longleftarrow & \bigoplus_{j=1}^{n} \mathrm{K}_1(C_r^*(G_j)) & \longleftarrow & \bigoplus_{k=1}^{n-1} \mathrm{K}_1(C_r^*(A_k)) .
\end{array}$$

The maps across the top (and likewise across the bottom) are the subgroup map discussed at the beginning of this section, and the sum of the maps induced by the natural injections $C_r^*(G_j) \to C_r^*(\Gamma)$.

3.3. REMARKS. (a) The cyclic exact sequence of K-groups produced by $\pi : \mathscr{T} \to C_r^*(\Gamma)$ maps to that produced by $\pi_k : \mathscr{T}_k \to C_r^*(\Gamma)$ $(1 \leqslant k \leqslant n-1)$ when we project $\mathscr{T}$ on $\mathscr{T}_k$. Using functoriality, it is easy to see that the kth component of each vertical map in 3.2 is the boundary map to $\mathrm{K}_\#(\mathscr{J}_k) = \mathrm{K}_\#(C_r^*(A_k))$ that comes from $\pi_k : \mathscr{T}_k \to C_r^*(\Gamma)$.

(b) The apparatus we have introduced in this section behaves well when the groups G_j are lumped segmentally along the line. For example, fix a subgroup A_k and write $\Gamma = X *_{A_k} Y$. The pair (X, A_k) might not have property A, but the result in [6] for two groups is still valid provided $(G_1, A_1), \ldots, (G_{n-1}, A_{n-1})$ have property A. To see this, form the $(k-1)$-fold pullback $\mathcal{T}_X$ for X, and likewise $\mathcal{T}_Y$ for Y. Use 2.1 and 2.2 to define maps $\psi_X : \mathcal{T}_X \to \mathcal{T}, \psi_Y : \mathcal{T}_Y \to \mathcal{T}$ making the following diagrams commute:

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_k \\ {\scriptstyle \psi_X}\uparrow & & \uparrow{\scriptstyle \mu} \\ \mathcal{T}_X & \longrightarrow & C_r^*(X) \end{array} \qquad \begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_k \\ {\scriptstyle \psi_Y}\uparrow & & \uparrow{\scriptstyle \nu} \\ \mathcal{T}_Y & \longrightarrow & C_r^*(Y). \end{array}$$

It follows readily from 3.1 that $(\psi_X)_* + (\psi_Y)_* : \mathrm{K}_\#(\mathcal{T}_X) \oplus \mathrm{K}_\#(\mathcal{T}_Y) \to \mathrm{K}_\#(\mathcal{T})$ is an isomorphism. Superimpose the ladders of K-groups that come from the diagrams and apply the five-lemma to show the group map $\mathrm{K}_\#(C_r^*(X)) \oplus \mathrm{K}_\#(C_r^*(Y)) \to \mathrm{K}_\#(\mathcal{T}_k)$ is an isomorphism.

By taking inductive limits in 3.2, we obtain an exact sequence for the case of an infinite line of groups.

3.4. THEOREM. *Let $\Gamma = \ldots *_{A_{k-1}} G_k *_{A_k} G_{k+1} * \ldots$, where the line of groups extends infinitely to the left, or to the right, or in both directions. Assume that all of the pairs (G_k, A_k) have property A. There is an exact sequence exactly like that of* 3.2, *except that the direct sums are infinite. The boundary maps are as in* 3.3(a), *that is, their A_k-component comes from $\pi_k : \mathcal{T}_k \to C_r^*(\Gamma)$, the Toeplitz extension defined in Section* 2 *for the amalgam $\Gamma = X *_{A_k} Y$.*

Proof. The inductive system one needs to look at is indexed by finite segments of the set of integers that indexes the G's. Given such a segment $I = \{m, m+1, \ldots, n\}$, one has the amalgam $\Gamma_I = G_m *_{A_m} * \ldots *_{A_{n-1}} G_n$ and the Toeplitz extension $\mathcal{T}_I \to C_r^*(\Gamma_I)$, which is the pullback of $n-m$ extensions of the sort discussed in Section 2. From this comes the exact sequence of K-groups in 3.2. If J is a larger segment, we can use 2.1 and 2.2 to obtain a well-behaved map from $\mathcal{T}_I$ to $\mathcal{T}_J$. This gives a map between the corresponding sequences of K-groups. The point of these observations is that the boundary maps cohere in the proper fashion as we move through the inductive system. Now one really can just pass to the limit. ∎

4. TOEPLITZ EXTENSION FOR THE REDUCED C^*-ALGEBRA OF AN HNN-GROUP

Let G be a countable group, let A be a subgroup of G, and let $\theta : A \to G$ be a monomorphism. Let $\Gamma = \mathrm{HNN}(G, A, \theta)$ as in Section 1, so Γ is the universal group generated by G and an outside element s satisfying $sas^{-1} = \theta(a)$ (a in A).

Words in Γ are reduced so that no element of A (resp. $\theta(A)$) lies between s and s^{-1} (resp. s^{-1} and s); see Section 5 of [11]. Let Γ_1 be the set of reduced words in Γ ending in sA. Notice that $\ell^2(\Gamma_1)$ reduces $\lambda(G)$ and is invariant for $\lambda(s)$, where λ is the left regular representation of Γ on $\ell^2(\Gamma)$. Let S and $\beta(g)$ (g in G) denote respectively the restrictions of $\lambda(s)$ and $\lambda(g)$ to $\ell^2(\Gamma_1)$, and consider the C^*-algebra $\mathcal{D}$ on $\ell^2(\Gamma_1)$ generated by S and $\beta(G)$. This is our Toeplitz algebra. We must map it onto $C_r^*(\Gamma)$ and identify the kernel of the map.

The arguments here are similar to those in [10], so we shall proceed briskly. Let R be the unitary on $\ell^2(\Gamma)$ corresponding to multiplication on the right by s. Observe that for any x in Γ, we have $xs^n \in \Gamma_1$ for sufficiently large n. If we momentarily think of $\mathcal{D}$ as acting on $\ell^2(\Gamma)$ this means that $R^{-n}SR^n \to \lambda(s)$ and $R^{-n}\beta(g)R^n \to \lambda(g)$, ($g$ in G) in the strong operator topology. Thus we obtain a $*$-homomorphism $\pi : \mathcal{D} \to C_r^*(\Gamma)$ such that $\pi(S) = \lambda(s)$ and $\pi \circ \beta = \lambda|_G$. Let $q = 1 - SS^*$, the projection of $\ell^2(\Gamma_1)$ onto $\ell^2(sA)$. As in [10], it is not difficult to see that the kernel of π is the closed ideal of $\mathcal{D}$ generated by q. Call this ideal Q. We claim that Q is isomorphic to $C_r^*(A) \otimes \mathcal{K}$.

For this, let $\{g_0, g_1, \ldots\}$ (resp. $\{h_0, h_1, \ldots\}$) be coset representatives for G/A (resp. $G/\theta(A)$), with $g_0 = e = h_0$. Each element of Γ_1 can be written uniquely in the form

$$\left.\begin{matrix} h_{i_1}s \\ \\ g_{j_1}s^{-1} \end{matrix}\right\} \left.\begin{matrix} h_{i_2}s \\ \\ g_{j_2}s^{-1} \end{matrix}\right\} \cdots \left.\begin{matrix} h_{i_n}s \\ \\ g_{j_n}s^{-1} \end{matrix}\right\} h_{i_{n+1}}sa,$$

where $n \geqslant 0$, $a \in A$, and g_0 (resp. h_0) is not allowed to lie between s and s^{-1} (resp. s^{-1} and s). We denote by Ω_0 the set of products of this form in which $a = e$. Let $\Omega = \Omega_0 s^{-1} = G \bigcup \{\text{words ending in } sG\} \bigcup \{\text{words ending in } s^{-1}(G \setminus \theta(A))\} = \Gamma \setminus \{\text{words ending in } s^{-1}\theta(A)\}$. We have $\Gamma_1 = \bigcup\{\omega sA : \omega \in \Omega\}$. For ω in Ω, let V_ω be the corresponding product in $\mathcal{D}$ of S's, S^*'s, $\beta(g_j)$'s and $\beta(h_i)$'s. The desired isomorphism of $C_r^*(A) \otimes \mathcal{K}$ with Q, where $\mathcal{K}$ is the algebra of compact operators on $\ell^2(\Omega)$, is implemented spatially by identifying $\ell^2(A) \otimes \ell^2(\Omega)$ with $\ell^2(\Gamma_1)$ via the bijection $(a, \omega) \mapsto \omega sa$ from $A \times \Omega$ to Γ_1. It sends $\lambda_A(a) \otimes E_{\omega',\omega}$ to $V_{\omega'}\, \beta(\theta(a))qV_\omega^*$, where $a \in A$ and λ_A is the left regular representation of A. (Notice that $\beta(\theta(A))$ commutes with q.)

Since Γ_1 is the disjoint union of right translates of G, the map β extends to a $*$-monomorphism, which we will also call β, from $C_r^*(G)$ into $\mathcal{D}$. Likewise extending $\theta : A \to G$ and the inclusion $i : A \to G$ to $*$-monomorphisms $C_r^*(A) \to C_r^*(G)$, we claim that the diagram

$$\begin{array}{ccc} K_\#(Q) & \longrightarrow & K_\#(\mathcal{D}) \\ \uparrow & & \uparrow \beta_* \\ K_\#(C_r^*(A)) & \xrightarrow[\theta_* - i_*]{} & K_\#(C_r^*(G)) \end{array}$$

commutes. To see this, consider the map $a \mapsto \beta(\theta(a))$ of $C_r^*(A)$ into $\mathscr{D}$. It is the sum of $a \mapsto \beta(\theta(a))q$ and $a \mapsto \beta(\theta(a))(1-q)$. Because $\beta(\theta(a))(1-q) = S\beta(a)S^*$, the second map induces the same map from $K_\#(C_r^*(A))$ to $K_\#(\mathscr{D})$ that $a \mapsto \beta(a)$ induces.

5. THE TOEPLITZ ALGEBRA $\mathscr{D}$ AS A CROSSED PRODUCT BY AN ENDOMORPHISM

(We will assume in this section that A and $\theta(A)$ are both *proper* subgroups of G. The case of our final result in which one or the other is equal to G will be dealt with in a separate argument.)

By the universal property of our HNN-group Γ, there is a homomorphism $\varphi : \Gamma \mapsto \mathbf{Z}$ annihilating G with $\varphi(s) = 1$. For ξ in $\mathbf{T}$ (the unit circle in $\mathbf{C}$), we obtain a unitary operator U_ξ on $\ell^2(\Gamma_1)$ multiplying the basis element corresponding to γ in Γ_1 by $\xi^{\varphi(\gamma)}$. Conjugation by the U_ξ's gives a continuous action $\alpha : \mathbf{T} \mapsto \mathrm{Aut}(\mathscr{D})$ fixing $\beta(G)$ and spinning S. Let $\mathscr{D}_0$ be the subalgebra of $\mathscr{D}$ fixed by this action. Define $\sigma_0 : \mathscr{D}_0 \to (1-q)\mathscr{D}_0(1-q) \subseteq \mathscr{D}_0$ by $\sigma_0(x) = SxS^*$.

5.1 PROPOSITION. *$\mathscr{D}$ is the crossed product (as in* [7]*) of $\mathscr{D}_0$ by the endomorphism σ_0.*

Proof. Let $\mathscr{D}^\sim$ be the veritable crossed product of $\mathscr{D}_0$ by σ_0, so $\mathscr{D}^\sim$ is generated by $\mathscr{D}_0$ and an outside isometry $S^\sim$ with $S^\sim x(S^\sim)^* = \sigma_0(x)$ (x in $\mathscr{D}_0$), and $\mathscr{D}^\sim$ has the obvious universal property with respect to this arrangement. Because of the universal property, there is an action $\alpha^\sim$ of $\mathbf{T}$ on $\mathscr{D}^\sim$ fixing $\mathscr{D}_0$ and spinning $S^\sim$, and there is a map $\psi : \mathscr{D}^\sim \to \mathscr{D}$ sending $S^\sim$ to S whose restriction to $\mathscr{D}_0$ is the identity map. Clearly $\psi \circ \alpha_\xi^\sim = \alpha_\xi \circ \psi$ for each $\xi \in \mathbf{T}$. Integrating $\alpha^\sim$ and α yields faithful conditional expectations $E^\sim : \mathscr{D}^\sim \to \mathscr{D}_0$ and $E : \mathscr{D} \to \mathscr{D}_0$. We have $\psi \circ E^\sim = E \circ \psi$. If $y \in \mathscr{D}^\sim$ and $\psi(y) = 0$, then $\psi(E^\sim(y^*y)) = E(\psi(y^*y)) = 0$. But $E^\sim(y^*y)$ lies in $\mathscr{D}_0$, where ψ is the identity, so $E^\sim(y^*y) = 0$. As $E^\sim$ is faithful, $y = 0$. Thus ψ is an isomorphism. ▨

Let H denote the kernel of $\varphi : \Gamma \to \mathbf{Z}$. Using φ, we obtain an action of $\mathbf{T}$ on $C_r^*(\Gamma)$ spinning $\lambda(s)$ whose fixed algebra is $C_r^*(H)$, regarded as a subalgebra of $C_r^*(\Gamma)$. This action is intertwined by $\pi : \mathscr{D} \to C_r^*(\Gamma)$ with the action α on $\mathscr{D}$ defined above. We thus have $\pi(\mathscr{D}_0) = C_r^*(H)$. Consider now $\ker(\pi|_{\mathscr{D}_0}) = Q \cap \mathscr{D}_0$. Recall from Section 4 that we may regard Q as acting on $\ell^2(A) \otimes \ell^2(\Omega)$. Break Ω into blocks $\Omega_k = \Omega \cap \varphi^{-1}(k)$ (k in $\mathbf{Z}$). The Ω_k's are all non-empty because of our assumptions that $A \neq G \neq \theta(A)$. If $j \neq k$, the $\Omega_k \times \Omega_j$ piece of Q meets $\mathscr{D}_0$ trivially. Hence $Q \cap \mathscr{D}_0$ consists of the $\Omega_k \times \Omega_k$ pieces only. For each Q, let Q_k be the closed subalgebra of $\mathscr{D}_0$ generated by

$$\{V_{\omega'}\beta(\theta(a))qV_\omega^* : \omega, \omega' \in \Omega_k,\ a \in A\}.$$

This is an ideal of $\mathscr{D}_0$, isomorphic to $C_r^*(A) \otimes \mathscr{K}(\ell^2(\Omega_k))$. We have $Q \cap \mathscr{D}_0 \simeq \bigoplus_{-\infty}^{\infty} Q_k$, and thus an exact sequence of K-groups:

$$\ldots \to K_1(C_r^*(H)) \to \bigoplus_{-\infty}^{\infty} K_0(C_r^*(Q_k)) \to K_0(\mathscr{D}_0) \to K_0(C_r^*(H)) \to \ldots .$$

Call this sequence 1.

We will confront sequence 1 with another sequence, obtained as follows. The group H may be realized as the amalgam of a two-way infinite line of copies of G along copies of A, viz.

$$H = \ldots * G_{-1} *_{A_0} G_0 *_{A_1} G_1 * \ldots ,$$

where $G_k = s^k G s^{-k}$ and $A_k = s^k A s^{-k} = s^{k-1}\theta(A)s^{1-k}$. (See [3], [11], [5].) If we assume that $(G, \theta(A))$ has property Λ, then Theorem 3.4 above gives us an exact sequence

$$\ldots \to K_1(C_r^*(H)) \to \bigoplus_{-\infty}^{\infty} K_0(C_r^*(A_k)) \to \bigoplus_{-\infty}^{\infty} K_0(C_r^*(G_k)) \to K_0(C_r^*(H)) \to \ldots$$

which we call sequence 2.

In the next section, we will define a map from $\bigoplus_{-\infty}^{\infty} K_\#(C_r^*(G))$ to $K_\#(\mathscr{D}_0)$ which will be shown to be an isomorphism when $(G, \theta(A))$ has property Λ.

6. MAPPING SEQUENCE 2 TO SEQUENCE 1

We continue to assume that $A \neq G \neq \theta(A)$.

For $k \geqslant 0$, define $\Phi_k : C_r^*(G) \to \mathscr{D}_0$ by $\Phi_k(g) = S^{k'}\beta(g)(S^*)^k$. Since S is an isometry, Φ_k is a $*$-monomorphism. We need a similar map for $k < 0$ also.

6.1. Lemma. *Given $j \geqslant 1$, there exist x_j and y_j in $\mathscr{D}_0$ such that*

$$x_j S^j (S^*)^j x_j^* + y_j S^j (S^*)^j y_j^* = 1.$$

Proof. Notice that $1 - S^j(S^*)^j$ is the projection of $\ell^2(\Gamma_1)$ on $\ell^2(sA \cap s^2A \cup \ldots \ldots \cup s^jA)$. Take g in $G \setminus \theta(A)$. Then $1 - \beta(g)S^j(S^*)^j\beta(g^{-1})$ is the projection on $l^2(gsA \cup gs^2A \cup \ldots gs^jA)$, and hence is orthogonal to $1 - S^j(S^*)^j$. It follows that $S^j(S^*)^j + \beta(g)S^j(S^*)^j\beta(g^{-1}) = 2 - p$, where p is a projection in $\mathscr{D}_0$. We may take $x_j = (2-p)^{-1/2}$ and $y_j = x_j\beta(g)$. ▨

6.2. Remark. With $j = 1$, the lemma shows that $(1-q)\mathscr{D}_0(1-q)$ is a full corner of $\mathscr{D}_0$.

We define $\Phi_{-j} : C_r^*(G) \to \mathscr{D}_0 \otimes M_2$ for $j \geqslant 1$ as follows. Let x_j and y_j be as in 6.1 and let $R_j = \begin{pmatrix} x_j S^j & y_j S^j \\ 0 & 0 \end{pmatrix}$ in $\mathscr{D}_0 \otimes M_2$. Notice that $R_j R_j^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Write $\Phi_{-j}(g) = R_j^* \begin{pmatrix} \beta(g) & 0 \\ 0 & 0 \end{pmatrix} R_j$ for g in $C_r^*(G)$, so Φ_{-j} is a $*$-monomorphism into $\mathscr{D}_0 \otimes M_2$.

6.3. LEMMA. *For all $k \in \mathbf{Z}$, we have*

$$(\sigma_0)_*(\Phi_k)_* = (\Phi_{k+1})_* : \mathrm{K}_\#(C_r^*(G)) \to \mathrm{K}_\#(\mathscr{D}_0).$$

Proof. This is obvious for $k \geqslant 0$. For $k = -1$, we have

$$\sigma_0 \Phi_{-1} = (S \otimes 1_2) R_1^* \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} R_1 (S^* \otimes 1_2).$$

Notice that $(S \otimes 1_2) R_1^* \in \mathscr{D}_0 \otimes M_2$, and

$$R_1 (S^* \otimes 1_2) \sigma_0 \Phi_{-1} (S \otimes 1_2) R_1^* = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix},$$

which is the same on $\mathrm{K}_\#$ as Φ_0. Since $(S \otimes 1_2) R_1^*$ is a partial isometry with the correct initial and final projections, we have $(\Phi_0)_* = (\sigma_0)_*(\Phi_{-j})_*$. For $k < -1$, the argument is similar, except that the equivalence between $\sigma_0 \Phi_{-j-1}$ and Φ_{-j} is implemented by the partial isometry $(S \otimes 1_2) R_{j+1}^* R_j$, which lies in $\mathscr{D}_0 \otimes M_2$. ▨

6.4. LEMMA. *The diagram below commutes for every k in $\mathbf{Z}$.*

$$\begin{array}{ccc} \mathrm{K}_\#(Q_k) & \longrightarrow & \mathrm{K}_\#(\mathscr{D}_0) \\ \uparrow & \nearrow_{(\Phi_k \circ \theta)_* - (\Phi_{k-1} \circ i)_*} & \\ \mathrm{K}_\#(C_r^*(A)) & & \end{array} .$$

(The horizontal arrow comes from inclusion, the vertical arrow from the isomorphism of $C_r^*(A) \otimes \mathscr{K}$ with Q_k described in Section 5.)

Proof. First consider the case $k = -j$, where $j \geqslant 2$. Decompose $\Phi_{-j} \circ \theta$ as $\psi_q + \psi_{1-q}$, where

$$\psi_q = R_j^* \begin{pmatrix} (\beta \circ \theta) q & 0 \\ 0 & 0 \end{pmatrix} R_j$$

and similarly for ψ_{1-q}. Recalling that $\beta(a) = S^* \beta(\theta(a)) S$ for a in $C_r^*(A)$, one checks that the partial isometry $R_j^*(S \otimes 1_2) R_{j-1}$, which lies in $\mathscr{D}_0 \otimes M_2$, conjugates $\Phi_{-j+1} \circ i$

to ψ_{1-q}, so $(\Phi_{-j+1}\circ i)_* = (\psi_{1-q})_*$. The map from $K_\#(C_r^*(A))$ to $K_\#(\mathscr{D}_0)$ obtained by moving up and then right in the diagram is induced by

$$\begin{pmatrix} V_\omega(\beta\circ\theta)qV_\omega^* & 0 \\ 0 & 0 \end{pmatrix},$$

where $\omega \in \Omega_{-j}$. This map is equivalent to ψ_q via the $\mathscr{D}_0 \otimes M_2$-partial isometry $(V_\omega q \otimes 1_2)R_j$. The lemma is thus proved for $k \leqslant -2$.

The case $k = -1$ follows similarly after replacing Φ_0 by $\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}$. The argument for $k \geqslant 0$ is easy, like that at the end of Section 4. ▨

We define $\Phi : \bigoplus_{-\infty}^{\infty} K_\#(C_r^*(G_k)) \to K_\#(\mathscr{D}_0)$ to be the map obtained by summing the maps $(\Phi_k)_*$, after identifying each G_k with G.

6.5. Remark. We have the following commuting diagram, linking part of sequence 2 with part of sequence 1.

$$\begin{array}{ccc} \bigoplus_{-\infty}^{\infty} K_\#(Q_k) & \longrightarrow & K_\#(\mathscr{D}_0) \\ \uparrow & & \uparrow \Phi \\ \bigoplus_{-\infty}^{\infty} K_\#(C_r^*(A_k)) & \longrightarrow & \bigoplus_{-\infty}^{\infty} K_\#(C_r^*(G_k)). \end{array}$$

The bottom arrow maps $K_\#(C_r^*(A_k))$ via $(\theta_*, -i_*)$ to $K_\#(C_r^*(G_{k-1}))\oplus K_\#(C_r^*(G_k))$, while the lefthand arrow maps $K_\#(C_r^*(A_k))$ isomorphically to $K_\#(Q_{k-1})$.

Our project in this section will be finished once we show that the diagram

$$\begin{array}{ccc} K_0(C_r^*(H)) & \longrightarrow & \bigoplus_{-\infty}^{\infty} K_1(Q_k) \\ & \searrow & \uparrow \\ & & \bigoplus_{-\infty}^{\infty} K_1(C_r^*(A_k)) \end{array}$$

commutes (and likewise with indices reversed). For this, we need to relate the extension $\mathscr{D}_0 \to C_r^*(H)$ to the Toeplitz extension for $C_r^*(H)$ that results from realizing H as a 2-fold amalgam $X_k *_{A_k} Y_k$ (k in $\mathbf{Z}$). Here,

$$X_k = \ldots * G_{k-2} *_{A_{k-1}} G_{k-1}$$

$$Y_k = G_k *_{A_{k+1}} G_{k+1} * \ldots ,$$

and as in Section 2 we have the associated Toeplitz algebra $\mathcal{T}_{X_k}$ on $\ell^2(H_{X_k})$, generated by the images of the maps $\mu_{X_k} : X_k \to \mathcal{T}_{X_k}$ and $\mu_{X_k} : Y_k \to \mathcal{T}_{X_k}$. We proceed to define $*$-homomorphisms from $\mathcal{D}_0$ to each $\mathcal{T}_{X_k}$.

For k in $\mathbf{Z}$, write $\Gamma_1^{(k)} = \Gamma_1 \cap \varphi^{-1}(k)$, where $\varphi : \Gamma \to \mathbf{Z}$ is as at the beginning of Section 5. Notice that each subspace $\ell^2(\Gamma_1^{(k)})$ of $\ell^2(\Gamma_1)$ reduces $\mathcal{D}_0$. For $T \in \mathcal{D}_0$, let $\rho_k(T)$ denote the restriction of T to $\ell^2(\Gamma_1^{(k)})$. Now we must get from $\Gamma_1^{(k)}$ to $H_{X\,k}$. The lemma that accomplishes this is obvious at the experimental level, but we feel obliged to give a complete proof.

6.6. Lemma. *Regarding H as a subgroup of Γ, we have $H_{X_k} s^k = \Gamma_1^{(k)}$ for $k \geq 1$.*

Proof. It will be convenient to concentrate on the case $k = 1$ for most of the argument. To simplify notation, write

$$X = X_1 = \ldots * G_{-2} *_{A_{-1}} G_{-1} *_{A_0} G_0$$

and

$$Y = Y_1 = G_1 *_{A_2} G_2 *_{A_3} G_3 * \ldots ,$$

so $H = X *_{A_1} Y$. Write H as the disjoint union $H_X^* \cup A_1 \cup H_Y^*$, where, as in Section 2, H_X^* (resp. H_Y^*) consists of reduced words in X and Y ending in $X \setminus A_1$ (resp. $Y \setminus A_1$), and $H_X = H_X^* \cup A_1$. We observe that $A_1 s = sA$. To show that $H_X s = \Gamma_1^{(1)}$, it will suffice to prove that $H_X^* s \subseteq \varphi^{-1}(1)$ and $H_Y^* s \cap \Gamma_1 = \emptyset$. (Notice that $Hs \subseteq \varphi^{-1}(1)$ and $\Gamma_1^{(1)} s^{-1} \subseteq H$.)

For ω in H, we can write ω (in many ways) as a product of conjugates of elements of G by powers of s. We define

$$\operatorname{size}(\omega) = \min\{k + \sum |r_j| : \omega = (s^{r_1} g_1 s^{-r_1}) \ldots (s^{r_k} g_k s^{-r_k}), \quad g_1, \ldots, g_k \in G\}.$$

Claim 1. $H_X^* s \subseteq \Gamma_1$. [Suppose not. Let ω be an element of minimal size in $\{v \in H_X^* : vs \notin \Gamma_1\}$, with size-realizing factorization

$$\omega = (s^{r_1} g_1 s^{-r_1}) \ldots (s^{r_k} g_k s^{-r_k}).$$

Set $\omega' = s^{r_1} g_1^{-1} s^{-r_1} \omega$. If $r_1 > 0$, then $\omega' \in H_X^*$, since $Y H_X^* \subseteq H_X^*$. By minimality, $\omega' s \in \Gamma_1$. We have $s^{r_1} g_1 s^{-r_1} \omega' s = \omega s \notin \Gamma_1$. Since $r_1 > 0$, this forces $s^{-r_1} \omega' s \notin \Gamma_1$, so $\omega' s \in \Gamma_1 \setminus s^{r_1} \Gamma_1 = sA \cup \ldots \cup s^{r_1} A$. Since $\varphi(\omega' s) = 1$, we have $\omega' s = sa$, where $a \in A$. Thus $\omega = s^{r_1} g_1 s^{-r_1} \theta(a)$, which lies in $H_Y^* \cup A_1$, contradicting $\omega \in H_X^*$. We must therefore have $r_1 \leq 0$, and $\omega' \in X H_X^* \subseteq H_X = H_X^* \cup A_1$. If $\omega' \in A_1$, then $\omega s = s^{r_1} g_1 s^{1-r_1} a \in \Gamma_1$, which is not allowed. If on the other hand $\omega' \in H_X^*$, then $\omega' s \in \Gamma_1$ by minimality. Also $\omega s = s^{r_1} g_1 s^{-r_1} \omega' s \notin \Gamma_1$, which forces $r_1 < 0$ and $g_1 s^{-r_1} \omega' s \in$ $\in sA \cup \ldots \cup s^{-r_1} A$, which is impossible because $\varphi(g_1 s^{-r_1} \omega' s) = 1 - r_1 > - r_1$. We have proved Claim 1.]

Claim 2. $H_Y^{*}s \cap \Gamma_1 = \emptyset$. [Suppose not. Let ω be an element of minimal size in $\{v \in H_Y^{*} : vs \in \Gamma_1\}$, with size-realizing factorization as in the proof of Claim 1. Set $\omega' = s^{r_1} g_1^{-1} s^{-r_1}\omega$. If $r_1 > 0$, then $\omega' \in H_Y^{*} \cup A_1$. If $\omega' \in A_1$, then $\omega s = s^{r_1} g_1 s^{1-r_1} a$ (where $a \in A$), which can't belong to Γ_1. (Here notice that minimality forces $g_1 \notin A$, so the case $r_1 = 1$ causes no trouble, and for larger r_1, the g_1 can't move past s^{1-r_1}.) If $\omega' \in H_Y^{*}$, then minimality forces $\omega' s \notin \Gamma_1$. The only way left multiplication by $s^{r_1} g_1^{-1} s^{-r_1}$ can move ωs out of Γ_1 is if $\omega s \in sA$, so we have $\omega \in sAs^{-1} = A_1$, contradicting $\omega \in H_Y^{*}$. If $r_1 \leqslant 0$, then $\omega' \in H_Y^{*}$, and minimality forces $\omega' s \notin \Gamma_1$. Since however $\omega s \in \Gamma_1$, we must have $r_1 < 0$ and $g_1^{-1} s^{-r_1} \omega s \in sA \cup \ldots \cup s^{-r_1}A$, which is impossible because $\varphi(g_1^{-1} s^{-r_1}\omega s) = 1 - r_1$. We have proved Claim 2.]

Thus $H_{X_1}s = \Gamma_1^{(1)}$. To prove the lemma for $k > 1$, we observe that $H_{X_k} = s^{k-1} H_{X_1} s^{1-k}$. Hence $H_{X_k} s^k = s^{k-1} H_{X_1} s = s^{k-1}\Gamma_1^{(1)} = (s^{k-1}\Gamma_1) \cap \Gamma_1^{(k)} = (\Gamma_1 \setminus (sA \cup \ldots \cup s^{k-1}A)) \cap \Gamma_1^{(k)} = \Gamma_1^{(k)}$. ▨

For $k \geqslant 1$, the lemma gives us a unitary $U_k : \ell^2(\Gamma_1^{(k)}) \to \ell^2(H_{X_k})$ corresponding to the map $\gamma \mapsto \gamma s^{-k}$ of $\Gamma_1^{(k)}$ onto H_{X_k}. We define $\psi_k(T)$ on $\ell^2(H_{X_k})$ for T in $\mathscr{D}_0$ by $\psi_k(T) = U_k \rho_k(T) U_k^{*}$. Routine checking establishes the following formulas, with r in $\mathbf{Z}^+$, g in G, and a in A.

$$\psi_k(S^r \beta(g)(S^*)^r) = \begin{cases} \nu_{X_k}(s^r g s^{-r}) & r \geqslant k \\ \mu_{X_k}(s^r g s^{-r}) & 0 \leqslant r < k \end{cases}$$

$$\psi_k((S^*)^r \beta(g) s^r) = \mu_{X_k}(s^{-r} g s^r)$$

$$\psi_k(Q_j) = 0 \quad \text{for } j \neq k-1$$

$$\psi_k(S^{k-1}\beta(\theta(a)) q (S^*)^{k-1}) = q_{X_k}\mu_{X_k}(s^k a s^{-k}).$$

(Here, $q_{X_k} = 1 - \mu_{X_k}(e)$.) We have observed earlier that π maps $\mathscr{D}_0$ onto $C_r^*(H)$; from this, it follows readily that $\mathscr{D}_0$ is generated by operators of the type appearing in the first two formulas above, together with the ideals Q_j. The essential features of this situation are summarized in our next remark.

6.7. Remark. For $k \geqslant 1$, we have a map between exact sequences of C^*-algebras, namely

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{-\infty}^{\infty} Q_j & \longrightarrow & \mathscr{D}_0 & \xrightarrow{\pi} & C_r^*(H) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi_k} & & \| & & \\ 0 & \longrightarrow & C_r^*(A_k) \otimes \mathscr{K} & \longrightarrow & \mathscr{T}_{X_k} & \xrightarrow{\pi_{X_k}} & C_r^*(H) & \longrightarrow & 0. \end{array}$$

The map on the left annihilates all summands except Q_{k-1}, which is sent isomorphically to $C_r^*(A_k) \otimes \mathcal{K}$.

We also wish to define $\psi_{-j} : \mathcal{D}_0 \to \mathcal{T}_{X_{-j}}$ for $j \geqslant 0$. Write $V_j : \ell^2(H_{X_1}) \to \ell^2(H_{X_{-j}})$ for the unitary corresponding to conjugation by s^{-j-1}, and for T in $\mathcal{D}_0$, define $\psi_{-j}(T)$ on $\ell^2(H_{X_{-j}})$ by

$$\psi_{-j}(T) = V_j \psi_1(S^{j+1}T(S^*)^{j+1})V_j^*.$$

It is easily seen that

$$\psi_{-j}(S^r\beta(g)(S^*)^r) = \nu_{X_{-j}}(s^r g s^{-r}) \quad \text{for } r \geqslant 0$$

$$\psi_{-j}((S^*)^r\beta(g)S^r) = \begin{cases} \nu_{X_j}(s^{-r}gs^r) & \text{for } j \geqslant r \geqslant 0 \\ (1 - q_{X_{-j}})\mu_{X_{-j}}(s^{-r}gs^r)(1 - q_{X_{-j}}) & \text{for } r \geqslant j + 1 \end{cases}$$

for g in G, and further that ψ annihilates Q_i for $i \neq -j-1$, and maps Q_{-j-1} isomorphically to the kernel of $\pi_{-j} : \mathcal{T}_{X_{-j}} \to C_r^*(H)$.

6.8. REMARK. The previous remark is valid for all k in $\mathbf{Z}$. The following diagram commutes:

$$\begin{array}{ccc} K_0(C_r^*(H)) & \xrightarrow{\partial} & \bigoplus_{-\infty}^{\infty} K_1(C_r^*(A)) \\ & \searrow^{\delta_k} & \downarrow \\ & & K_1(C_r^*(A)) \end{array}$$

(and likewise with indices reversed.) Here ∂ is the boundary map on K-theory arising from $\mathcal{D}_0 \xrightarrow{\pi} C_r^*(H)$, δ_k is the boundary map from $\mathcal{T}_{X_k} \to C_r^*(H)$, and the vertical arrow is projection on the $(k-1)^{\text{st}}$ summand.

Before proceeding on to the main result, we pause for the treatment of a special case.

7. THE CASE $\theta(A) = G$

The arguments and apparatus of the two preceding sections are for the situation in which $A \neq G \neq \theta(A)$. When $\theta(A) = G$, we can proceed more directly to show that $\beta : C_r^*(G) \to \mathcal{D}$ induces an isomorphism on K-theory. What follows is our modification of W. Arveson's version of the proof of the crucial step in [9]. (See also [2].) We thank Professor Arveson for permitting us to use his unpublished lecture material.

Throughout this section, $\mathscr{B}$ will denote a unital C^*-algebra and ρ a unital $*$-monomorphism of $\mathscr{B}$ into itself. (For instance, in case $\theta(A) = G$, we could have $\rho = \theta^{-1} : C_r^*(G) \to C_r^*(A) \subseteq C_r^*(G)$.) We call an isometry V a $(\mathscr{B}, \rho)$*-isometry* if V acts on a Hilbert space on which $\mathscr{B}$ is faithfully represented and satisfies $bV = V\rho(b)$ for all b in B. Notice that this condition forces VV^* to commute with $\mathscr{B}$. We say that V is *faithful* if $b \mapsto b(1 - VV^*)$ is injective on $\mathscr{B}$.

Let $\mathscr{K} = \mathscr{K}(l^2(\mathbf{Z}^+))$ and define $\Delta : \mathscr{B} \to \mathscr{M}(\mathscr{B} \otimes \mathscr{K})$ (the multiplier algebra) by

$$\Delta(b) = b \oplus \rho(b) \oplus \rho^2(b) \oplus \dots \ .$$

Let v be the forward shift on $l^2(\mathbf{Z}^+)$, so $1_{\mathscr{B}} \otimes v \in \mathscr{M}(\mathscr{B} \otimes \mathscr{K})$. When we identify $\mathscr{B}$ with $\Delta(\mathscr{B})$, we see that $1_{\mathscr{B}} \otimes v$ is a faithful $(\mathscr{B}, \rho)$-isometry.

7.1. LEMMA. *Let V be a faithful $(\mathscr{B}, \rho)$-isometry. For any $(\mathscr{B}, \rho)$-isometry W, there is a $*$-homomorphism $\varphi : C^*(\mathscr{B}, V) \to C^*(\mathscr{B}, W)$ such that $\varphi(b) = b$ for all b in $\mathscr{B}$ and $\varphi(V) = W$.*

Proof. Let $\mathscr{B}$ and V act on the Hilbert space $\mathscr{H}$. Let $\mathscr{H}_0 = (1 - VV^*)\mathscr{H}$. Then the subspaces $\mathscr{H}_0$, $V\mathscr{H}_0$, $V^2\mathscr{H}_0, \dots$ are pairwise orthogonal, and their direct sum reduces $C^*(\mathscr{B}, V)$. (Each one of them reduces $\mathscr{B}$ because $bV^k = V^k\rho^k(b)$ for b in $\mathscr{B}$.) Since we are seeking a homomorphism from $C^*(\mathscr{B}, V)$, there is no loss of generality in assuming that $\mathscr{H} = \mathscr{H}_0 \oplus V\mathscr{H}_0 \oplus V^2\mathscr{H}_0 \oplus \dots$, i.e. that V is completely non-unitary. It follows from the faithfulness of V that $C^*(\mathscr{B}, V)$ is isomorphic to the C^*-subalgebra of $\mathscr{M}(\mathscr{B} \otimes \mathscr{K})$ generated by $\Delta(\mathscr{B})$ and $1_{\mathscr{B}} \otimes v$. Consider now $C^*(\mathscr{B} \otimes 1, W \otimes v) \subseteq C^*(\mathscr{B}, W) \otimes C^*(v)$. The $(\mathscr{B}, \rho)$-isometry $W \otimes v$ is faithful because its defect projection majorizes $1_B \otimes (1 - vv^*)$ and completely non-unitary because $(W^* \otimes v^*)^n$ tends strongly to 0 as $n \to \infty$. We thus have a $*$-isomorphism $\psi : C^*(\mathscr{B}, V) \to C^*(\mathscr{B} \otimes 1, W \otimes v)$, via $C^*(\Delta(\mathscr{B}), 1_{\mathscr{B}} \otimes v)$, sending b in $\mathscr{B}$ to $b \otimes 1$ and V to $W \otimes v$. Let $f : C^*(v) \to \mathbf{C}$ be the multiplicative linear functional with $(v) = 1$. We have a $*$-homomorphism id $\otimes f : C^*(\mathscr{B}, W) \otimes C^*(v) \to C^*(\mathscr{B}, W)$. Following ψ by the restriction of id $\otimes f$ to $C^*(\mathscr{B} \otimes 1, W \otimes v)$ yields the desired map φ. ▨

7.2. THEOREM. *If V is a faithful $(\mathscr{B}, \rho)$-isometry, then the inclusion $j : \mathscr{B} \to C^*(\mathscr{B}, V)$ induces an isomorphism on $\mathscr{K}$-theory.*

Proof. This is well known when ρ is an automorphism of $\mathscr{B}$. Our argument really consists in noticing that the proof for the automorphism case does not require the existence of ρ^{-1}. We will work in the setting of quasihomomorphisms as described in [2].

Let $\mathscr{T} = C^*(\mathscr{B}, V)$. Using 7.1, we obtain $*$-homomorphisms $\varphi, \overline{\varphi} : \mathscr{T} \to \mathscr{M}(\mathscr{B} \otimes \mathscr{K})$ such that $\varphi(b) = \Delta(b)$ and $\overline{\varphi}(b) = \Delta(b)(1_{\mathscr{B}} \otimes vv^*)$ for b in $\mathscr{B}$, $\varphi(V) = 1_{\mathscr{B}} \otimes v$, and $\overline{\varphi}(V) = 1_B \otimes v^2v^*$. Since $(\varphi - \overline{\varphi})(\mathscr{T}) \subseteq \mathscr{B} \otimes \mathscr{K}$, the pair

$(\varphi, \overline{\varphi})$ induces a map $(\varphi, \overline{\varphi})_* : \mathrm{K}_\#(\mathscr{T}) \to \mathrm{K}_\#(\mathscr{B})$. It is clear that $(\varphi, \overline{\varphi})_* j_*$ is the identity map on $\mathrm{K}_\#(\mathscr{B})$. To analyze $j_*(\varphi, \overline{\varphi})_*$, use 7.1 to define $\tilde{j} : C^*(\Delta(\mathscr{B}), 1_\mathscr{B} \otimes v) \to \mathscr{M}(\mathscr{T} \otimes \mathscr{K})$ by $\tilde{j}(\Delta(b)) = \Delta^\sim(b)$, where the latter is the same infinite matrix as $\Delta(b)$ but now regarded as a multiplier of $\mathscr{T} \otimes \mathscr{K}$, and $\tilde{j}(1_\mathscr{B} \otimes v) = 1_\mathscr{T} \otimes v$. Notice that $\varphi(\mathscr{T})$ and $\overline{\varphi}(\mathscr{T})$ are both contained in $C^*(\Delta(\mathscr{B}), 1_\mathscr{B} \otimes v)$, so $\tilde{j}\varphi$ and $\tilde{j}\overline{\varphi}$ are maps from $\mathscr{T}$ to $\mathscr{M}(\mathscr{T} \otimes \mathscr{K})$. They differ by $\mathscr{T} \otimes \mathscr{K}$, and we have $(\tilde{j}\varphi, \tilde{j}\overline{\varphi})_* = j_*(\varphi, \overline{\varphi})_*$. (See the product construction in [2].)

Consider now the self-adjoint unitary $U_1 = \begin{pmatrix} 1_\mathscr{T} - VV^* & V \\ V^* & 0 \end{pmatrix}$ in $M_2(\mathscr{T})$, which commutes with the C^*-subalgebra $(\mathrm{id} \oplus \rho)(\mathscr{B})$ of $M_2(\mathscr{T})$. There is a path $\{U_t\}$ of unitaries in $M_2(\mathscr{T})$ joining U_1 to $U_0 = 1_\mathscr{T} \otimes 1_2$ such that each U_t commutes with $(\mathrm{id} \oplus \rho)(\mathscr{B})$. Define $U_t^\sim$ in $M(\mathscr{T} \otimes \mathscr{K})$ to be the infinite matrix obtained by replacing the upper left 2×2 block of $1_\mathscr{T} \otimes 1$ by U_t. Then $U_t^\sim$ commutes with $\Delta^\sim(\mathscr{B})$, and when we identify $\mathscr{B}$ with $\Delta^\sim(\mathscr{B})$, we see that $U_t^\sim(1_\mathscr{T} \otimes v)$ is a $(\mathscr{B}, \rho)$-isometry. For each t, Lemma 7.1 yields a map $\varphi_t : \mathscr{T} \to \mathscr{M}(\mathscr{T} \otimes \mathscr{K})$ such that $\varphi_t(b) = \Delta^\sim(b)$ for b in $\mathscr{B}$ and $\varphi_t(V) = U_t^\sim(1_\mathscr{T} \otimes v)$. Notice that $\psi_0 = \tilde{j}\varphi$. Since $U_t^\sim - 1_\mathscr{T} \otimes 1 \in \mathscr{T} \otimes \mathscr{K}$, we have $(\varphi_t - \tilde{j}\overline{\varphi})(\mathscr{T}) \subseteq \mathscr{T} \otimes \mathscr{K}$ for all t. The quasihomomorphisms $(\tilde{j}\varphi, \tilde{j}\overline{\varphi})$ and $(\varphi_1, \tilde{j}\overline{\varphi}_1)$ are thus homotopic. To conclude the proof, observe that $\varphi_1 = \nu \oplus \tilde{j}\overline{\varphi}$, where $\nu(x) = x \otimes (1 - vv^*)$ for x in $\mathscr{T}$, so $(\tilde{j}\varphi, \tilde{j}\overline{\varphi})_* = \nu_*$, which is the identity map on $\mathscr{K}_\#(\mathscr{T})$. ▨

7.3. COROLLARY. *With reference to Section 4 above, the map $\beta : C_r^*(G) \to \mathscr{T}$ induces an isomorphism on K-theory whenever $G = \theta(A)$.*

Proof. The isometry S is a $(C_r^*(G), \rho)$-isometry, where $\rho : C_r^*(G) \to C_r^*(G)$ comes from $\theta^{-1} : G \to A \subseteq G$. The defect space of S is $\ell^2(sA) = \ell^2(Gs)$, on which $\beta(C_r^*(G))$ acts faithfully. ▨

8. THE MAIN RESULT FOR HNN-GROUPS

It is at this point that we shall need to impose property A.

8.1. LEMMA. *Suppose that $A \neq G \neq \theta(A)$ and $(G, \theta(A))$ has property A. Then the map $\Phi : \bigoplus_{-\infty}^{\infty} \mathrm{K}_\#(C_r^*(G)) \to \mathrm{K}_\#(\mathscr{D}_0)$ defined in Section 6 is an isomorphism intertwining the forward shift on the direct sum with $(\sigma_0)_*$ on $\mathrm{K}_\#(\mathscr{D}_0)$.*

Proof. Consider

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{K}_1(C_r^*(H)) & \longrightarrow & \bigoplus_{-\infty}^{\infty} \mathrm{K}_0(C_r^*(A)) & \longrightarrow & \mathrm{K}_0(\mathscr{D}_0) & \longrightarrow & \mathrm{K}_0(C_r^*(H)) & \longrightarrow \ldots \\
& \| & (1) & \uparrow & (2) & \uparrow \Phi & (3) & \| & \\
\ldots \longrightarrow & \mathrm{K}_1(C_r^*(H)) & \longrightarrow & \bigoplus_{-\infty}^{\infty} \mathrm{K}_0(C_r^*(A)) & \longrightarrow & \bigoplus_{-\infty}^{\infty} \mathrm{K}_0(C_r^*(G)) & \longrightarrow & \mathrm{K}_0(C_r^*(H)) & \longrightarrow \ldots .
\end{array}$$

The top and bottom sequences are exact. The one on top comes from $\mathscr{D}_0 \xrightarrow{\pi} C_r^*(H)$, after identifying the kernel of this map with $\bigoplus_{-\infty}^{\infty} C_r^*(A) \otimes \mathscr{K}$ as in Section 5. The sequence on the bottom comes from Theorem 3.4. The vertical map between the A-terms is the backward shift. Rectangle (1) commutes because of Remark 6.8 and the description of the boundary maps given in Theorem 3.4. Rectangle (2) commutes by Remark 6.5. The commutativity of rectangle (3) is checked componentwise by referring to the definition of the C^*-algebra maps Φ_k at the beginning of Section 6. (For negative indices, notice that $\begin{pmatrix} \pi(x_j) & \pi(y_j) \\ 0 & 0 \end{pmatrix}$ in $C_r^*(H) \otimes M_2$ conjugates $\pi \circ \Phi_{-j}$ to $(\mathrm{ad}(\lambda(s^{-j})) \circ i_G) \oplus 0$, where $j \geq 1$ and $i_G : C_r^*(G) \to C_r^*(H)$ is the natural injection.) The desired isomorphism now follows from the five-lemma, and the intertwining is a consequence of Lemma 6.3. ▨

8.2. THEOREM. *Let A be a subgroup of a countable group G, and let $\theta : A \to G$ be a monomorphism. Let $\Gamma = \mathrm{HNN}(G, A, \theta)$ be the corresponding* HNN-*group. If at least one of the pairs (G, A), $(G, \theta(A))$ has Natsume's relative property Λ* [6], *there is an exact sequence*

$$\begin{array}{ccccc} K_0(C_r^*(A)) & \xrightarrow{\theta_* - i_*} & K_0(C_r^*(G)) & \xrightarrow{j_*} & K_0(C_r^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma)) & \xleftarrow{j_*} & K_1(C_r^*(G)) & \xleftarrow{\theta_* - i_*} & K_1(C_r^*(A)), \end{array}$$

where i_*, j_*, and θ_* come from the maps induced on C^*-algebras by the inclusions $i : A \to G$, $j : G \to \Gamma$, and $\theta : A \to G$.

Proof. We begin with the case in which $A \neq G \neq \theta(A)$ and $(G, \theta(A))$ has property Λ. Our Toeplitz algebra $\mathscr{D}$ is the crossed product of $\mathscr{D}_0$ by the endomorphism $\sigma_0 : \mathscr{D}_0 \to (1-q)\mathscr{D}_0(1-q)$, whose range is a full corner of $\mathscr{D}_0$ (Proposition 5.1 and Remark 6.2). By [8], we have an exact sequence

$$\ldots \longrightarrow K_1(\mathscr{D}) \longrightarrow K_0(\mathscr{D}_0) \xrightarrow{(\sigma_0)_* - \mathrm{id}} K_0(\mathscr{D}_0) \longrightarrow K_0(\mathscr{D}) \longrightarrow \ldots .$$

When we identify $K_\#(\mathscr{D}_0)$ with $\bigoplus_{-\infty}^{\infty} K_\#(C_r^*(G))$ as in Lemma 8.1, $(\sigma_0)_*$ becomes the forward shift. Hence $(\sigma_0)_* - \mathrm{id}$ has trivial kernel, and moreover the image of $K_\#(C_r^*(G))$ in $K_\#(\mathscr{D}_0)$ under $(\Phi_0)_*$ is mapped isomorphically onto $K_\#(\mathscr{D})$. This means that $\beta_* : K_\#(C_r^*(G)) \to K_\#(\mathscr{D})$ is an isomorphism. Using observations made in Section 4, the sequence in the statement of the theorem is now seen to be the exact sequence of K-groups produced by

$$0 \to Q \to \mathcal{D} \to C_r^*(\Gamma) \to 0.$$

If $G = \theta(A)$, in which case property A is superfluous, β_* is again an isomorphism by Corollary 7.3. The remaining cases of the theorem are obtained by interchanging the roles of A and $\theta(A)$. ▨

9. FUNDAMENTAL GROUP OF A LOOP OF GROUPS

In this final section, we indicate how our methods and results can be extended to treat a construction in combinatorial group theory that generalizes the HNN--construction.

Consider first two (countable) groups X and Y. Suppose A and B are subgroups of both X and Y, via imbeddings $i_A : A \to Y$, $\theta_A : A \to X$, $i_B : B \to X$, and $\theta_B : B \to Y$. We may regard A as a subgroup of the amalgamated product $X *_B Y$ via i_A, and think of θ_A as an imbedding of this subgroup into $X *_B Y$; let $\Delta = \mathrm{HNN}(X *_B Y, A, \theta_A)$ be the resulting HNN-group, so Δ is generated by $X *_B Y$ and an additional element s with $si_A(a)s^{-1} = \theta_A(a)$ (a in A).

9.1. REMARK. (cf. § 5, Proposition 20 of [11]). By appealing to the universal properties of the groups involved, it is straightforward to show that there is an isomorphism $\psi : \mathrm{HNN}(X *_A Y, B, \theta_B) \to \Delta$ such that $\psi(x) = x$, $\psi(y) = sys^{-1}$ (x in X, y in Y), and $\psi(t) = s$, where t is the canonical θ_B-implementing element of $\mathrm{HNN}(X *_A Y, B, \theta_B)$.

Form the Toeplitz algebra $\mathcal{D} = C^*(\beta(X *_B Y), S)$ for the HNN-group Δ as in Section 4. Write $\Gamma = X *_A Y$, and let $\mathcal{T} = C^*(\mu(X), \nu(Y))$ be its Toeplitz algebra as in Section 2.

9.2. LEMMA. *There is a $*$-monomorphism $\mathcal{T} \to \mathcal{D}$ sending $\mu(x) \mapsto \beta(x)$, $\nu(y) \mapsto S\beta(y)S^*$ (x in X, y in Y).*

Proof. Recall that $\mathcal{T}$ acts on $\ell^2(\Gamma_X)$, where $\Gamma_X = A \cup \{$words in Γ ending in $X \setminus A\}$, and $\mathcal{D}$ acts on $\ell^2(\Delta_1)$, where $\Delta_1 = \{$words in Δ ending in $sA\}$. The map ψ in 9.1 gives an imbedding of Γ in Δ, and one checks that $\psi(\Gamma_X)s \subseteq \Delta_1$. Let $\mathcal{E}$ be the C^*-subalgebra of $\mathcal{D}$ generated by $\beta(X)$ and $S\beta(Y)S^*$. Then $\ell^2(\psi(\Gamma_X)s)$ reduces $\mathcal{E}$, and the map $\gamma \mapsto \psi(\gamma)s$ induces a spatial isomorphism of $\mathcal{T}$ with $\mathcal{E}|_{\ell^2(\psi(\Gamma_X)s)}$. We thus have a $*$-homomorphism $\tau : \mathcal{E} \to \mathcal{T}$ sending $\beta(x) \mapsto \mu(x)$, $S\beta(y)S^* \mapsto \nu(y)$. Furthermore, it is not hard to see that $\Delta_1 \setminus \psi(\Gamma_X)s$ is invariant under left multiplication by $\psi(\Gamma)$. As in the proof of Proposition 2.1, we conclude that the identity representation of $\mathcal{E}$ on $\ell^2(\Delta_1)$ is unitarily equivalent to the direct sum of τ with several copies of $\pi\circ\tau$, where π is the Toeplitz map from $\mathcal{T}$ to $C_r^*(\Gamma)$. Thus τ is an isomorphism.

The situation so far is that of a length 2 loop of groups. For the length n case, suppose we have (countable) groups $G_1, G_2, \ldots, G_n$, and $A_1, A_2, \ldots, A_n$ with imbeddings $i_j : A_j \to G_j$, $\theta_j : A_j \to G_{j-1}$ for $j = 1, \ldots, n$. Indices are treated circularly here; n is identified with 0, so θ_1 maps A_1 into G_n. For each j, break the loop at A_j and form $\Gamma_j = G_j *_{A_{j+1}} G_{j+1} * \ldots *_{A_{j-1}} G_{j-1}$. Thus A_j is a subgroup of Γ_j (via $i_j : A_j \to G_j$), and we have $\theta_j : A_j \to \Gamma_j$ (via G_{j-1}). Let $\Delta_j = \mathrm{HNN}(\Gamma_j, A_j, \theta_j)$. The groups Δ_j are all isomorphic. To see this, take $j > 1$ and write

$$\Gamma_j = (G_j *_{A_{j+1}} G_{j+1} * \ldots *_{A_n} G_n) *_{A_1} (G_1 *_{A_2} G_2 * \ldots *_{A_{j-1}} G_{j-1})$$

$$\Gamma_1 = (G_1 *_{A_2} G_2 * \ldots *_{A_{j-1}} G_{j-1}) *_{A_j} (G_j *_{A_{j+1}} G_{j+1} * \ldots *_{A_n} G_n).$$

Remark 9.1 now gives an isomorphism of Δ_j with Δ_1. In this way we identify Δ_j for $j = 1, \ldots, n$ with a single group which we will call simply Δ. In the terminology of [11], Δ is the *fundamental group* of the given loop of groups.

9.3. THEOREM. *Let $(G_1, A_2, G_2, A_3, \ldots, A_n, G_n, A_1, G_1)$ be a loop of groups as above, with fundamental group Δ. Assume that each G_j is countable and that the pairs $(G_1, A_2), (G_2, A_3), \ldots, (G_n, A_1)$ all have property A. Then there is an exact sequence*

$$\begin{array}{ccccc} \bigoplus_1^n \mathrm{K}_0(C_r^*(A_j)) & \longrightarrow & \bigoplus_1^n \mathrm{K}_0(C_r^*(G_j)) & \longrightarrow & \mathrm{K}_0(C_r^*(\Delta)) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(C_r^*(\Delta)) & \longleftarrow & \bigoplus_1^n \mathrm{K}_1(C_r^*(G_j)) & \longleftarrow & \bigoplus_1^n \mathrm{K}_1(C_r^*(A_j)). \end{array}$$

The map from $\bigoplus_1^n \mathrm{K}_\#(C_r^*(A_j))$ to $\bigoplus_1^n \mathrm{K}_\#(C_r^*(G_j))$ takes the summand $\mathrm{K}_\#(C_r^*(A_j))$ to $\mathrm{K}_\#(C_r^*(G_{j-1})) \oplus \mathrm{K}_\#(C_r^*(G_j))$ via $(-(\theta_j)_*, (i_j)_*)$. The map from $\bigoplus_1^n \mathrm{K}_\#(C_r^*(G_j))$ to $\mathrm{K}_\#(C_r^*(\Delta))$ sums the maps induced on K-theory by the natural injections $C_r^*(G_j) \to C_r^*(\Delta)$.

Proof. (sketch). For $j = 1, \ldots, n$, we identify Δ with $\Delta_j = \mathrm{HNN}(\Gamma_j, A_j, \theta_j)$ and form $\pi_j : \mathcal{D}_j \to C_r^*(\Delta)$ as in Section 4. Let $(\pi, \mathcal{D})$ be the pullback of the $(\pi_j, \mathcal{D}_j)$'s. Our strategy is to obtain the sequence in the theorem as the exact sequence of K--groups produced by $\pi : \mathcal{D} \to C_r^*(\Delta)$. The main thing, as usual, is to exhibit an isomorphism of $\mathrm{K}_\#(\mathcal{D})$ with $\bigoplus_1^n \mathrm{K}_\#(C_r^*(G_j))$.

Focus now on $\Gamma_1 = G_1 *_{A_2} G_2 * \ldots *_{A_n} G_n$. Breaking this amalgam at A_j $(j = 2, \ldots, n)$ gives us Toeplitz extensions $\mathcal{T}_1^{(j)} \to C_r^*(\Gamma_1)$ as in Section 2. These pull back as in Section 3 to give $\mathscr{T}_1 \to C_r^*(\Gamma_1)$. We note that $\bigoplus_{j=1}^{n} K_\#(C_r^*(G_j))$ is isomorphic via what we call the "group map" to $K_\#(\mathscr{T}_1)$, by Theorem 3.1. We proceed to construct a map from $\mathscr{T}_1$ to $\mathscr{D}$ which will turn out to induce an isomorphism on K-theory.

For the time being, fix j between 2 and n. Let $X = G_1 *_{A_2} G_2 * \ldots *_{A_{j-1}} G_{j-1}$ and $Y = G_j *_{A_{j+1}} G_{j+1} * \ldots *_{A_n} G_n$. We are in the situation of Lemma 9.2 and the discussion preceding it, with $A = A_j$, $B = A_1$, $\Gamma = \Gamma_1 = X *_{A_j} Y, \Delta = \Delta_j = = \mathrm{HNN}(X *_{A_1} Y, A_j, \theta_j)$. Let $\alpha_j : \mathscr{T}_1^{(j)} \to \mathscr{D}_j$ be the map whose existence is asserted in 9.2. Let $q_j : \mathscr{T}_1 \to \mathscr{T}_1^{(j)}$ be the natural projection. (Recall that $\mathscr{T}_1$ is a subalgebra of $\bigoplus_{2}^{n} \mathscr{T}_1^{(k)}$.) Thus $\alpha_j q_j$ maps $\mathscr{T}_1$ to $\mathscr{D}_j$. We further have $\beta_1 : C_r^*(\Gamma_1) \to \mathscr{D}_1$ as in Section 4. Preceding β_1 by the map from $\mathscr{T}_1$ to $C_r^*(\Gamma_1)$ yields $\beta^\sim : \mathscr{T}_1 \to \mathscr{D}_1$. Set $\alpha = (\beta^\sim, \alpha_2 q_2, \ldots, \alpha_n q_n) : \mathscr{T}_1 \to \bigoplus_{1}^{n} \mathscr{D}_j$. Chasing through the definitions, one checks hat $\pi_1 \beta^\sim = \pi_j \alpha_j q_j$ $(j = 2, \ldots, n)$, so $\alpha(\mathscr{T}_1) \subseteq \mathscr{D}$. It now follows easily that the map $p_1 : \mathscr{D} \to \mathscr{D}_1$ coming from projection on the first summand is surjective.

The maps $\mathscr{T}_1 \to C_r^*(\Gamma_1)$ and $p_1 : \mathscr{D} \to \mathscr{D}_1$ give rise to exact sequences of diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & K_1(C_r^*(\Gamma_1)) & \longrightarrow & \bigoplus_{2}^{n} K_0(C_r^*(A_j)) & \longrightarrow & K_0(\mathscr{T}_1) & \longrightarrow & K_0(C_r^*(\Gamma_1)) & \longrightarrow \ldots \\
& \downarrow (\beta_1)_* & & \| & & \downarrow \alpha_* & & \downarrow (\beta_1)_* & \\
\ldots \longrightarrow & K_1(\mathscr{D}_1) & \longrightarrow & \bigoplus_{2}^{n} K_0(C_r^*(A_j)) & \longrightarrow & K_0(\mathscr{D}) & \longrightarrow & K_0(\mathscr{D}_1) & \longrightarrow \ldots .
\end{array}$$

We claim that $(\beta_1)_*$ is an isomorphism. This would follow (as in the proof of Theorem 8.2) from Lemma 8.1 if we knew that the pair $(\Gamma_1, \theta(A_1))$ had property Λ. But all 8.1 really needs is Theorem 3.4 for the two-way infinite amalgam $\ldots *_A G *_A G * \ldots$. The extension of this result to the present situation ($G = \Gamma_1$, $A = A_1$) is readily accomplished by appealing to Remark 3.3(b).

Thus α_* is an isomorphism by the five-lemma. We have already observed the isomorphism of $\bigoplus_{2}^{n} K_\#(C_r^*(G_j))$ with $K_\#(\mathscr{T}_1)$, so from the short exact sequence $0 \to \bigoplus_{1}^{n} \ker \pi_j \to \mathscr{D} \to C_r^*(\Delta) \to 0$ we obtain an exact sequence of K-groups whose terms are as in the sequence announced in the theorem. We omit the routine but somewhat tedious verification that the statement of the theorem also correctly identifies the maps in the sequence. ▨

Supported in part by a grant from the National Science Foundation.

REFERENCES

1. BLACKADAR, B., *K-theory for operator algebras*, monograph in preparation.
2. CUNTZ, J., K-theory and C^*-algebras, in *Springer Lecture Notes in Mathematics*, **1046**, 1984, 55–79.
3. HIGMAN, G.; NEUMANN, B. H.; NEUMANN, H., Embedding theorems for groups, *J. London Math. Soc.*, **24**(1949), 247–254.
4. LANCE, E. C., K-theory for certain group C^*-algebras, *Acta Math.*, **151**(1983), 209–230.
5. LYNDON, R. C.; SCHUPP, P. E., *Combinatorial group theory*, Springer, 1977.
6. NATSUME, T., On $K_*(C^*(SL_2(Z)))$, *J. Operator Theory*, **13**(1985), 103–118.
7. PASCHKE, W., The crossed product of a C^*-algebra by an endomorphism, *Proc. Amer. Math. Soc.*, **80**(1980), 113–118.
8. PASCHKE, W., K-theory for actions of the circle group on C^*-algebras, *J. Operator Theory*, **6**(1981), 125–133.
9. PIMSNER, M.; VOICULESCU, D., Exact sequences for K-groups and Ext-groups of certain cross--products C^*-algebras, *J. Operator Theory*, **4**(1980), 93–118.
10. PIMSNER, M.; VOICULESCU, D., K-groups of reduced crossed products by free groups, *J. Operator Theory*, **8**(1982), 131–156.
11. SERRE, J.-P., *Trees*, Springer, 1980.
12. TAYLOR, J., Banach algebras and topology, in *Algebras in Analysis* (ed. J. H. Williamson), Academic Press, 1975, pp. 118–186.

JOEL ANDERSON
Department of Mathematics,
Pennsylvania State University,
University Park, PA 16802,
U.S.A.

WILLIAM L. PASCHKE
Department of Mathematics,
University of Kansas,
Lawrence, KS 66045,
U.S.A.

Received December 18, 1985.