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STRONG MORITA EQUIVALENCE, SPINORS AND SYMPLECTIC SPINORS

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INTRODUCTION

Some of the fundamental equations of mathematical physics are expressed in terms of spinor fields on spacetime. We have, for example, the Dirac equation, the Dirac-Weyl neutrino equation, Maxwell's equations, and the linearized Einstein equation [20, p. 359]. At the same time, spinor fields associated to certain vector bundles play a crucial role in the Thom isomorphism in K-theory [1, p. 30], and in K-homology and index theory [2]. In addition, symplectic spinors play an important role in geometric quantization [3] and in the spectral theory of Toeplitz operators [4].

The purpose of this article is to show that spinors, Dirac spinors and symplectic spinors admit a uniform formulation in terms of the operator-theoretic concept of A-B-equivalence bimodule, where A and B are C^*-algebras. In the present context, the C^*-algebra B is abelian. The spinors we construct all come from spin^c-structures or from the corresponding structure in the symplectic case. A spin^c-structure is the simplest case of a generalized spin structure in the sense of Hawking and Pope [12]. In the context of fermion fields, a spin^c-structure corresponds to coupling the fermions to an electromagnetic field. One can consistently define charged spinors in such a context.

In Section 1, we give a concise account of equivalence bimodules and strong Morita equivalence. In the course of this account, we refine the classical discussion of Dixmier-Douady [7]. We show that, whenever the Dixmier-Douady class vanishes, a new invariant $\varkappa$ emerges in $H^2(X; \mathbf{Z}_2)$ which is the mod 2 reduction of an integral class.

In Section 2, we give a uniform account of spinors, Dirac spinors and symplectic spinors in terms of equivalence bimodules.

In Section 3, we give a detailed proof of the two-out-of-three lemma. This, together with Section 2, lays the foundations of spin^c-structures within K-homology [2]. In the symplectic case, the two-out-of-three lemma is useful in the spectral theory of Toeplitz operators [4].

In Section 4, we have assembled some examples which we hope are interesting and unusual.

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This article replaces an unpublished IHES preprint called "On the invariants of Serre and Dixmier-Douady" and an unpublished MSRI preprint with the title of the present article.

1. STRONG MORITA EQUIVALENCE

1.1. Let B be a separable C^*-algebra. Let $\mathscr{E}$ be a complex vector space which is also a right B-module. That is, $\lambda(xb) = (\lambda x)b = x(\lambda b)$ for all $x \in \mathscr{E}$, $\lambda \in \mathbf{C}$, $b \in B$. $\mathscr{E}$ is a pre-Hilbert B-module if there exists a map $(\ ,\ ) : \mathscr{E} \times \mathscr{E} \to B$ such that for all $x, y, z \in \mathscr{E}$, $b \in B$, $\lambda \in \mathbf{C}$,

(1) $(x, y + z) = (x, y) + (x, z)$; $(x, \lambda y) = \lambda(x, y)$

(2) $(x, yb) = (x, y)b$

(3) $(y, x) = (x, y)^*$

(4) $(x, x) \geqslant 0$; if $(x, x) = 0$ then $x = 0$.

Then $\|x\| = \|(x, x)\|^{1/2}$ defines a norm on $\mathscr{E}$. If $\mathscr{E}$ is complete with respect to this norm, we say that $\mathscr{E}$ is a Hilbert B-module. We shall consider only modules $\mathscr{E}$ which are separable.

We define $L(\mathscr{E})$ as the set of linear B-module maps $T: \mathscr{E} \to \mathscr{E}$ such that there exists $T^*: \mathscr{E} \to \mathscr{E}$ satisfying

$$(Tx, y) = (x, T^*y) \quad \text{for all } x, y \in \mathscr{E}.$$

T^* is called the adjoint of T, and is well-defined. Every T in $L(\mathscr{E})$ is bounded. For $x, y \in \mathscr{E}$, define $\theta_{x,y}(z) = x(y, z)$. Then $\theta_{x,y} \in L(\mathscr{E})$. Let $K(\mathscr{E})$ denote the closure of the linear span of the $\theta_{x,y}$. An element of $K(\mathscr{E})$ is called a compact operator on $\mathscr{E}$.

1.2. DEFINITION. A Hilbert B-module $\mathscr{E}$ is *full* if the closure of the linear span of $\{(x, y) : x, y \in \mathscr{E}\}$ is B.

1.3. Definition. Let A, B be separable C^*-algebras. Then A, B are *strongly Morita equivalent* if there is a full Hilbert B-module $\mathscr{E}$ with $A \simeq K(\mathscr{E})$. In this case, $\mathscr{E}$ is called an A-B-equivalence bimodule.

1.4. Suppose that $B = C_0(X)$ where X is a locally compact Hausdorff space. Let $\mathscr{E}$ be a Hilbert B-module, and let $t \in X$. Define

$$\mathscr{E}_t = \{x \in \mathscr{E} : (x, x)(t) = 0\}$$

$$H_t = \mathscr{E}/\mathscr{E}_t.$$

Then H_t is a Hilbert space with inner product

$$(\bar{x}, \bar{y})_t = (x, y)(t),$$

where $\bar{x}$ is the image of x in the projection $\mathscr{E} \to \mathscr{E}/\mathscr{E}_t$. In this way, $\mathscr{E}$ determines a continuous field $((H_t), \mathscr{E})$ of Hilbert spaces on X.

1.5. Let H be a complex Hilbert space. Let $\mathscr{A}$ be a locally trivial field of C^*-algebras on X such that each fibre is isomorphic to $K(H)$. Isomorphism classes of such bundles are represented by elements in the sheaf cohomology group $H^1(X; \underline{PU(H)})$. Here, $U(H)$ is the unitary group of H in the strong operator topology, and $PU(H)$ is the quotient of $U(H)$ by its centre $\mathbf{T}$. Shifting dimension twice in sheaf cohomology, we obtain a map

$$\delta \colon H^1(X; \underline{PU(H)}) \to H^2(X; \underline{\mathbf{T}}) \simeq H^3(X; \mathbf{Z}).$$

The element $\delta(\mathscr{A})$ in $H^3(X; \mathbf{Z})$ is the *Dixmier-Douady class* of $\mathscr{A}$. Let $B = C_0(X)$, $A = C_0(\mathscr{A}) = C^*$-algebra of continuous sections of $\mathscr{A}$ which vanish at infinity. Then $\delta(\mathscr{A}) = 0$ if and only if there is a full Hilbert B-module $\mathscr{E}$ with $A \simeq K(\mathscr{E})$. Suppose in addition that X has finite Lebesgue covering dimension, and that the associated field $((H_t), \mathscr{E})$ of Hilbert spaces has constant dimension $\aleph_0$. Then $((H_t), \mathscr{E})$ is trivial [6, p. 243]. Therefore $\mathscr{E} \simeq C_0(X, H)$ and

$$A \simeq K(\mathscr{E}) \simeq C_0(X, K(H)).$$

1.6. Definition. A C^*-algebra which is isomorphic to $K(H)$ for some Hilbert space H is called *elementary*.

1.7. Theorem. *Let $\mathscr{A}$ be a bundle of elementary C^*-algebras on X such that $\delta(\mathscr{A}) = 0$. Let $A = C_0(\mathscr{A})$, $B = C_0(X)$. Then*

(i) *The group $H^2(X; \mathbf{Z})$ acts simply transitively on the set of A-B-equivalence bimodules.*

(ii) *There is a characteristic class $\varkappa(\mathscr{A})$ of $\mathscr{A}$ in $H^2(X; \mathbf{Z}_2)$ which is the reduction of an integral class.*

Proof. Since $\delta(\mathscr{A}) = 0$, there exists a Hilbert B-module $\mathscr{E}$ with $A \simeq K(\mathscr{E})$. Let S be the associated Hilbert bundle $((H_t), \mathscr{E})$. We call S an irreducible $\mathscr{A}$-module, since the $\mathscr{A}$-action is pointwise irreducible. There is a bijection of the set of irreducible $\mathscr{A}$-modules S onto the set of A-B-equivalence bimodules.

The set of isomorphism classes of complex Hermitian line bundles on X, with the operation of tensor product, is an abelian group isomorphic under the first Chern class c_1 with $H^2(X; \mathbf{Z})$. We form the tensor product $S \otimes L$, with L a complex Hermitian line bundle, with $\mathscr{A}$-module structure given by $a(s \otimes l) = (as) \otimes l$. The $\mathscr{A}$-module $S \otimes L$ is irreducible. The action of $H^2(X; \mathbf{Z})$ on the set of irreducible $\mathscr{A}$-modules is given by

$$S \cdot L = S \otimes L.$$

This is an $H^2(X; \mathbf{Z})$-action since $(S \otimes L) \otimes M \cong S \otimes (L \otimes M)$, where L, M are complex Hermitian line bundles.

There is a canonical isomorphism α of complex Hermitian line bundles as follows:

$$\alpha: L \cong \mathrm{Hom}(S, S \otimes L).$$

The map α is defined as follows: $(\alpha(l))(s) = s \otimes l$. Now α is surjective by irreducibility of the $\mathscr{A}$-modules S and $S \otimes L$ (Schur's Lemma). Let now L, M be complex Hermitian line bundles. Then

$$S \otimes L \cong S \otimes M \quad \text{as irreducible } \mathscr{A}\text{-modules}$$

$$\Rightarrow \quad \mathrm{Hom}(S, S \otimes L) = \mathrm{Hom}(S, S \otimes M)$$

$$\Rightarrow \quad L \cong M \quad \text{as complex Hermitian line bundles.}$$

This proves that $H^2(X; \mathbf{Z})$ acts freely.

To prove that $H^2(X; \mathbf{Z})$ acts transitively we follow Dixmier-Douady [7, Theorem 9]. Let S be fixed and let T be an irreducible $\mathscr{A}$-module. Consider the map

$$\varphi: S \otimes \mathrm{Hom}(S, T) \to T$$

given by $\varphi(s \otimes h) = h(s)$. Now $\mathrm{Hom}(S, T)$ is a complex Hermitian line bundle by irreducibility of S and T. It is clear that φ is surjective. The $\mathscr{A}$-module structure on $S \otimes \mathrm{Hom}(S, T)$ is given by $a \cdot (s \otimes h) = (a \cdot s) \otimes h$. Then φ is an $\mathscr{A}$-module isomorphism $\Leftrightarrow \varphi(a \cdot (s \otimes h)) = a \cdot \varphi(s \otimes h) \Leftrightarrow \varphi(as \otimes h) = a(h(s)) \Leftrightarrow h(as) = a(h(s))$ which is so since $h \in \mathrm{Hom}(S, T)$. This proves that $H^2(X; \mathbf{Z})$ acts transitively.

(ii) Let S be an irreducible $\mathscr{A}$-module and let S' be the dual of S with the dual $\mathscr{A}$-module structure. By (i), there exists uniquely a complex Hermitian line bundle L such that $S = S' \otimes L$. We shall call L the line bundle associated to S, and shall denote it by $\lambda(S)$; it is characterized by the equation

$$S = S' \otimes \lambda(S). \tag{1}$$

Let T be an irreducible $\mathscr{A}$-module. Then $T = S \otimes K$ for some uniquely determined K in $H^2(X; \mathbf{Z})$. Now

$$\begin{aligned} T' \otimes (L \otimes K \otimes K) &\cong (S' \otimes K') \otimes (K \otimes L \otimes K) \cong \\ &\cong S' \otimes (K' \otimes K) \otimes L \otimes K \cong S' \otimes L \otimes K \cong \\ &\cong S \otimes K \cong T \qquad \text{as } \mathscr{A}\text{-modules} \end{aligned}$$

since $K' \otimes K$ is trivial via the map $k' \otimes k \to k'(k)$. Hence

$$\lambda(S \otimes K) = \lambda(S) \otimes K \otimes K. \tag{2}$$

Define $\varkappa(\mathscr{A})$ to be the mod 2 reduction of the first Chern class of $\lambda(S)$. Then $\varkappa(\mathscr{A})$ is, by (2), a well-defined element in $H^2(X; \mathbf{Z}_2)$.

1.8. Suppose that S is a Hilbert bundle of finite rank n. Taking the first Chern class of each side of equation (1), we obtain

$$c_1(S) = c_1(S' \otimes \lambda(S)) = c_1(S') + nc_1(\lambda(S)) = -c_1(S) + nc_1(\lambda(S))$$

so that

$$c_1(\lambda(S))/2 = c_1(S)/n \quad \text{in } H^2(X; Q).$$

The element $c_1(\lambda(S))/2$, which occurs in the Atiyah-Hirzebruch version of the Riemann-Roch theorem, is thus expressed directly in terms of S.

2. SPINORS, DIRAC SPINORS AND SYMPLECTIC SPINORS

2.1. Let V be a real vector bundle on X. Then V admits a metric φ, i.e. a symmetric bilinear form on V such that $\varphi_x(v, v) > 0$ for every non-zero vector v of V_x.

Let E denote the Euclidean vector bundle (V, φ). We can form the Clifford bundle $\tilde{E}$. This will be a bundle of algebras whose fibre at x is the complexified Clifford algebra $\mathrm{Cliff}(E_x) \otimes_{\mathbf{R}} \mathbf{C}$. We recall that the Clifford algebra $\mathrm{Cliff}(E_x)$ is the quotient of the tensor algebra $T(E_x)$ by the two-sided ideal generated by elements of the form $v \otimes v + \varphi_x(v, v) \cdot 1$, where $v \in E_x$. Let A be the C^*-algebra $C_0(\tilde{E})$.

If φ' is another metric on V, then there exists an automorphism f of the vector bundle V such that $\varphi(u, v) = \varphi'(f(u), f(v))$ (take f to be the positive square root of the composite of the bundle maps $V \to V^*$, $V^* \to V$ induced by φ', φ). Let E' denote the Euclidean vector bundle (V, φ'). The bundle map f induces an isomorphism

$$\tilde{f} : \tilde{E} \to \tilde{E}'.$$

Thus the C^*-algebra A is determined up to isomorphism by the vector bundle V. In other words, the isomorphism class of A is independent of the choice of Euclidean structure on V.

2.2. DEFINITION. Let E be a Euclidean vector bundle on X of rank $2n$. Let $A = C_0(\tilde{E})$, $B = C_0(X)$.

(i) Then E *admits a spin*^{c}*-structure* if and only if E is orientable and $\delta(\tilde{E}) = 0$.

(ii) In that case, a *spin*^{c}*-structure on* E is a pair $(\varepsilon, \mathscr{E})$ where ε is an orientation on E and $\mathscr{E}$ is an A-B-equivalence bimodule.

When E is of rank $2n + 1$, replace $\tilde{E}$ in the above definition by $(\tilde{E})^{\text{ev}}$ the even part of the Clifford bundle $\tilde{E}$.

2.3. THE LOCAL SITUATION. Let E be a Euclidean vector bundle on X. We recall from [1] that

$$\text{Cliff}(E_x) \otimes_{\mathbf{R}} \mathbf{C} \cong \begin{cases} M(2^n) & \text{if } E \text{ has rank } 2n \\ M(2^n) \oplus M(2^n) & \text{if } E \text{ has rank } 2n+1 \end{cases}$$

$$(\text{Cliff}(E_x) \otimes_{\mathbf{R}} \mathbf{C})^{\text{ev}} \cong M(2^n) \quad \text{if } E \text{ has rank } 2n+1$$

$$e^* = -e \quad \text{if } e \in E_x$$

$$\|e\| = (\varphi_x(e, e))^{1/2} \quad \text{if } e \in E_x$$

where $M(2^n)$ is the C^*-algebra of complex $2^n \times 2^n$ matrices. Suppose that E admits a spin^c-structure, that $\mathscr{E}$ is the corresponding equivalence bimodule, and that S is the corresponding complex Hermitian vector bundle. The requirements

$$\tilde{E} \cong \text{End}(S) \quad \text{if } E \text{ of even rank}$$

$$(\tilde{E})^{\text{ev}} \cong \text{End}(S) \quad \text{if } E \text{ of odd rank}$$

can always be met locally; the Dixmier-Douady class $\delta(\tilde{E})$ (resp. $\delta(\tilde{E})^{ev}$) is the global obstruction.

2.4. TERMINOLOGY. Let $(\varepsilon, \mathscr{E})$ be a spin^c-structure on E. Let S be the complex Hermitian vector bundle determined by the equivalence bimodule $\mathscr{E}$. Then S is called a *spinor bundle*. The sections of S, which are elements of $\mathscr{E}$, are called *spinor fields*. Spinor fields are sometimes simply called *spinors*.

2.5. HALF-SPINORS. Let $(\varepsilon, \mathscr{E})$ be a spin^c-structure on E, and suppose that E has rank $2n$. The orientation ε determines local frame fields $e_1, e_2, \ldots, e_{2n}$ which in turn determine the Clifford orientation

$$\omega = \mathrm{i}^n e_1 e_2 \ldots e_{2n}.$$

Concerning the Clifford orientation ω, we have

$$\omega \in \tilde{E} \cong \mathrm{End}(S)$$

$$\omega^2 = 1, \quad \omega^* = \omega.$$

The bundle map ω determines an orthogonal splitting

$$S = S^+ \oplus S^-$$

into the $+1$ and -1 eigenbundles. If $s \in S^+$ then

$$\omega(es) = (\omega e)s = (-e\omega)s = -es$$

so that $es \in S^-$. Similarly, if $s \in S^-$ then $es \in S^+$. Thus S is automatically a $\mathbf{Z}_2$-graded $\tilde{E}$-module. The bundles S^+, S^- are the *half-spinor bundles*. The sections of S^+ are called *positive spinors*, the sections of S^- are called *negative spinors*.

2.6. CLIFFORD MULTIPLICATION. Let $(\varepsilon, \mathscr{E})$ be a spin^c-structure on E. Let e be a vector field, i.e. a section of E, and let s be a spinor field, i.e. a section of the spinor bundle S.

Suppose first that E is of even rank. Then $\tilde{E} \cong \mathrm{End}(S)$, vector fields act on spinor fields, and es is well-defined: this is Clifford multiplication.

Suppose now that E is of rank $2n+1$. In this case the Clifford orientation is

$$\omega = \mathrm{i}^{n+1} e_1 e_2 \ldots e_{2n+1}$$

where $e_1, e_2, \ldots, e_{2n+1}$ is a local frame field. The Clifford orientation enters into the definition of Clifford multiplication. For let e be a vector field, s a spinor field. We have

$$(\tilde{E})^{\text{ev}} \cong \text{End}(S).$$

If $v \in (\tilde{E})^{\text{odd}}$ then define

$$v \cdot s = (v\omega)s.$$

We are exploiting the fact that $v\omega \in (\tilde{E})^{\text{ev}}$. Clifford multiplication is then defined by the formula

$$e \cdot s = (e\omega)s.$$

For the Clifford orientation ω, we have $\omega = \omega^*$, $\omega^2 = 1$, and so

$$\omega \cdot s = \omega^2 s = s.$$

The Clifford orientation acts trivially on the spinor fields.

2.7. THE REVERSED SPIN^c-STRUCTURE. Let $(\varepsilon, \mathscr{E})$ be a spin^c-structure on E. Denote the reversed orientation by $-\varepsilon$. Then the reversed spin^c-structure is $(-\varepsilon, \mathscr{E})$. Thus to reverse a spin^c-structure on E, we simply reverse the orientation, just as in singular homology. Note that a spin^c-structure and its reverse share the same equivalence bimodule.

If we reverse the spin^c-structure $(\varepsilon, \mathscr{E})$ then the Clifford orientation ω is replaced by $-\omega$. This has the following effect:

(i) E is of even rank. The positive and negative spinor bundles are interchanged.

(ii) E is of odd rank. Clifford multiplication $e \cdot s$ is replaced by its negative $-e \cdot s$.

Let E be an oriented Euclidean vector bundle. Let $W_3(E)$ be the third integral Stiefel-Whitney class of E.

2.8. THEOREM.

$$W_3(E) = \begin{cases} \delta(\tilde{E}) & \text{if } E \text{ has even rank} \\ \delta((\tilde{E})^{\text{ev}}) & \text{if } E \text{ has odd rank.} \end{cases}$$

Proof. (i) We take first the case when E has rank $2n$. Let $R \in SO(2n)$. There exists a unitary element v in the even part of $\text{Cliff}(\mathbf{R}^{2n}) \otimes_{\mathbf{R}} \mathbf{C}$ such that

$$Rx = vxv^{-1} \quad \text{all } x \text{ in } \mathbf{R}^{2n}.$$

The group $\mathrm{Spin}^c(2n)$ may be realized as the set of all such v. The map $\mathrm{Spin}^c(2n) \to$ $\to \mathrm{SO}(2n)$ sends v to R. If we realize $\mathrm{Cliff}(\mathbf{R}^{2n}) \otimes_{\mathbf{R}} \mathbf{C}$ as $M(2^n)$ then the spin representation $\mathrm{Spin}^c(2n) \to U(2^n)$ is the inclusion; the spin representation induces a projective unitary representation $\mathrm{SO}(2n) \to PU(2^n)$ and we have the following commutative diagram of Lie groups with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & \mathrm{Spin}^c(2n) & \longrightarrow & \mathrm{SO}(2n) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T & \longrightarrow & U(2^n) & \longrightarrow & PU(2^n) & \longrightarrow & 1.
\end{array}$$

At the level of Čech cohomology we therefore have a commutative triangle:

$$\begin{array}{ccc}
H^1(X; \mathrm{SO}(2n)) & \overset{d^1}{\searrow} & \\
\tau^1 \downarrow & & H^2(X; \mathbf{T}) \overset{\delta^2}{\cong} H^3(X; \mathbf{Z}) \\
H^1(X; PU(2^n)) & \underset{\delta^1}{\nearrow} &
\end{array}$$

where τ^1 is the map induced by the homomorphism $\tau\colon \mathrm{SO}(2n) \to PU(2^n)$. If E is represented by the 1-cocycle x, then $\tilde{E}$ is represented by the 1-cocycle $\tau^1(x)$. Since the triangle is commutative, we have

$$\delta^1(\tau^1(x)) = d^1(x).$$

But $\delta^2(d^1(x))$, the obstruction to a $\mathrm{Spin}^c(2n)$-lifting, is equal to $W_3(E)$; see, for example, [15, p. 373]. Therefore $\delta(\tilde{E}) = W_3(E)$.

(ii) Let $R \in \mathrm{SO}(2n+1)$. There exists a unitary element v in the even part of $\mathrm{Cliff}(\mathbf{R}^{2n+1}) \otimes_{\mathbf{R}} \mathbf{C}$ such that $Rx = vxv^{-1}$ for all x in $\mathbf{R}^{2n+1}$. The even part of $\mathrm{Cliff}(\mathbf{R}^{2n+1}) \otimes_{\mathbf{R}} \mathbf{C}$ realizes itself as $M(2^n)$ and the spin representation is

$$\mathrm{Spin}^c(2n+1) \to U(2^n).$$

The argument in (i) now shows that $\delta((\tilde{E})^{\mathrm{ev}}) = W_3(E)$.

2.9. Let $\mathrm{Spin}(n)$ be the double cover of the special orthogonal group $\mathrm{SO}(n)$ and let ε be the generator of the kernel $\mathbf{Z}_2$ of the covering map $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. Then $\mathbf{Z}_2$ acts on $\mathrm{Spin}(n) \times \mathbf{T}$ as follows:

$$\varepsilon(v, z) = (\varepsilon v, -z).$$

Define

$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times \mathbf{T})/\mathbf{Z}_2.$$

This definition is equivalent to the one given in 2.8; see [1, p. 9]. The homomorphism $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ is induced by the map which sends (v, z) to $\pi(v)$.

2.10. Definition. Let E be an oriented Euclidean vector bundle of rank k on X. A *spin^c-structure on E* (in the sense of Atiyah-Bott-Shapiro) is a pair (η, β) where

(i) η is a principal $\mathrm{Spin}^c(k)$-bundle over X;

(ii) β is an isomorphism of $\eta \times_{\mathrm{Spin}^c(k)} \mathbf{R}^k$ onto E.

2.11. Theorem. *Let E be an oriented Euclidean vector bundle of rank $2k$. Then there is a canonical bijection of the set of spin^c-structures on E in the sense of Atiyah-Bott-Shapiro onto the set of irreducible $\tilde{E}$-modules.*

Proof. The spin representation $\sigma\colon \mathrm{Spin}^c(2k) \to U(2^k)$ is faithful and determines a pull-back square in the category of compact Lie groups and smooth homomorphisms:

$$\begin{array}{ccc} \mathrm{Spin}^c(2k) & \xrightarrow{\sigma} & U(2^k) \\ \uparrow & & \uparrow \\ \mathrm{SO}(2k) & \xrightarrow{\tilde{\sigma}} & PU(2^k). \end{array}$$

(i) Let (η, β) be a spin^c-structure on E. The commutativity of the pull-back square determines a principal $U(n)$-bundle ξ and an isomorphism of $\xi \times_{U(n)} M_n(\mathrm{C})$ onto $\tilde{E}$. Then $\xi \times_{U(n)} \mathrm{C}^n$ is an irreducible $\tilde{E}$-module.

(ii) Let S be an irreducible $\tilde{E}$-module, so that we have a definite isomorphism θ of $\mathrm{End}(S)$ onto $\tilde{E}$. Let ξ be the principal $U(n)$-bundle of orthonormal frames of S. Then we have an isomorphism α:

$$\xi \times_{U(n)} M_n(\mathrm{C}) = \mathrm{End}(S) \overset{\theta}{\cong} \tilde{E}.$$

The principal $PU(n)$-bundle which underlies $\tilde{E}$ is the prolongation (determined by $\tilde{\sigma}$) of the principal $\mathrm{SO}(2k)$-bundle which underlies E. Then the pull-back of (ξ, α) determines a spin^c-structure on E. Clearly the correspondence is one-one.

The case when E is an oriented Euclidean vector bundle of *odd* rank is similar.

Theorem 2.11 shows that the definition of spin^c-structure in Atiyah-Bott-Shapiro [1] is compatible with Definition 2.2.

2.12. Dirac spinors. We assume that spacetime is a connected, non-compact, oriented, time-oriented 4-manifold on which is defined a Lorentz metric

g; see Hawking and Ellis [11, Section 6.1], Wald [20, p. 60], Penrose and Rindler [17, p. 55—56]. We use the convention that g has metric signature $+ - - -$ in order to conform with Feynman [9, p. 24], Wald [20, Chapter 13], Penrose and Rindler [17, p. 235].

Since spacetime M is oriented and time-oriented, there exists a nowhere-vanishing future-timelike vector field v on M. This allows us to "replace" g by a Riemannian metric φ in the following way. Define

$$\varphi_x(v_x) = + g_x(v_x)$$

$$\varphi_x(u_x) = - g_x(u_x)$$

whenever u is a space-like vector field on M. Then φ has metric signature $+ + + +$.

Let $(TM, g)^{\sim}$ be the complex Clifford bundle of TM with respect to g, and let $(TM, \varphi)^{\sim}$ be the complex Clifford bundle of TM with respect to φ.

In $(TM, g)^{\sim}$ we have

$$v_x^2 = -g_x(v_x) = -\varphi_x(v_x)$$

$$(iu_x)^2 = -u_x^2 = g_x(u_x) = -\varphi_x(u_x).$$

The map $(v, u) \mapsto (v, iu)$ therefore determines an isomorphism:

$$(TM, \varphi)^{\sim} \cong (TM, g)^{\sim}.$$

It follows that $(TM, g)^{\sim}$ is a bundle $\mathscr{A}$ of 4×4 matrix algebras. If $\delta(\mathscr{A}) = 0$ then there exists a complex vector bundle S such that $\mathscr{A} \cong \mathrm{End}(S)$. The sections of S are called *Dirac spinors.*

The spacetime M admits Dirac spinors if and only if $W_3(M) = 0$, by 2.1. In that case, $H^2(M; \mathbf{Z})$ acts freely and transitively on the set of spinor bundles, by 1.7(i). The Robertson-Walker cosmological models [20, p. 96] certainly satisfy the condition $W_3 = 0$.

Let $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ be a local frame field on (M, g) such that γ_0 is timelike and $\gamma_1, \gamma_2, \gamma_3$ are spacelike. Then $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ is a local frame field on (M, φ) which we shall denote e_0, e_1, e_2, e_3. We now proceed as in 1.13 and consider the Clifford orientation

$$\omega = -e_0e_1e_2e_3 \quad \text{in } (TM, \varphi)^{\sim}.$$

Then

$$\omega = -e_0 \cdot ie_1 \cdot ie_2 \cdot ie_3 = \qquad \text{in } (TM, g)^{\sim}$$

$$= i\gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where we take $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5$ as in Feynman [9, p. 114, 120, 158] and I is the 2×2 identity matrix.

Feynman refers to the positive Dirac spinors as *cospinors*, and to the negative Dirac spinors as *contraspinors* [9, p. 111]. The contraspinors occur in the Dirac-Weyl neutrino equation [9, p. 111], [17, p. 220—223]. The role of the spacetime orientation is to split the Dirac spinors into cospinors and contraspinors, as noted by Wald [20, p. 367].

2.13. SYMPLECTIC SPINORS. Let $Q_1, \ldots, Q_n$, $P_1, \ldots, P_n$ be the standard operators on $L^2(\mathbf{R}^n)$ which obey the canonical commutation relations in quantum mechanics:

$$[Q_j, Q_k] = 0$$

$$[P_j, P_k] = 0$$

$$[Q_j, P_k] = i\delta_{jk}I.$$

Let $\pi(x)$ be the unitary operator on $L^2(\mathbf{R}^n)$ given by

$$\pi(x) = \exp i\{x_1Q_1 + \ldots + x_nQ_n + y_1P_1 + \ldots + y_nP_n\}$$

where $x = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n}$. The Fourier-Weyl transform of the L^1-function f is defined by

$$\pi(f) = \int f(x)\pi(x)\,dx.$$

The C^*-algebra $\overline{L^1(\mathbf{R}^{2n})}$ is, by definition, generated by the Fourier-Weyl transforms of functions in $L^1(\mathbf{R}^{2n})$. The L^1-C^*-algebra $\overline{L^1(\mathbf{R}^{2n})}$ is a separable, infinite-dimensional C^*-algebra which has a unique irreducible representation (up to unitary equivalence): consequently, we have

$$\overline{L^1(\mathbf{R}^{2n})} = k$$

the C^*-algebra of compact operators on $L^2(\mathbf{R}^n)$.

Let (F, ω) be a symplectic vector bundle on X. This means that F is a real vector bundle, and that each fibre F_x has a non-degenerate skew-symmetric bilinear form ω_x. Then F necessarily has even rank $2n$, and F is orientable. The bundle P of symplectic frames is a principal Sp(n)-bundle, where $Sp(n)$ is the symplectic group. Now the maximal compact subgroup of Sp(n) is the unitary group $U(n)$. We reduce the structure group of P from Sp(n) to $U(n)$, and let H be the corresponding complex Hermitian vector bundle. The underlying real vector bundle F has a canonical preferred orientation.

The symplectic group Sp(n) preserves the canonical commutation relations and acts on the L^1-C^*-algebra. We form the associated bundle of elementary C^*-algebras:

$$\dot{F} = P \times_{\mathrm{Sp}(n)} \overline{L^1(\mathbf{R}^{2n})}.$$

This is called the *Weyl bundle*. We proved in [18] that $\delta(\dot{F}) = 0$.

Let $A = C_0(\dot{F})$, $B = C_0(X)$.

2.14. Definition. An *Mp^c-structure* is an A-B-equivalence bimodule.

Since F has a canonical preferred orientation, we do not need to specify the orientation in the definition. The notation Mp^c-structure comes about as follows. The symplectic group Sp(n) admits a double cover $Mp(n)$, called the metaplectic group. By analogy with Spin^c(n), define

$$Mp^c(n) = Mp(n) \times_{\mathbf{Z}_2} \mathbf{T}.$$

Let $\mathscr{E}$ be an Mp^c-structure on the symplectic vector bundle F, and let S be the Hilbert bundle determined by $\mathscr{E}$. The bundle S is called a bundle of symplectic spinors. Symplectic spinors were introduced by Kostant in [22].

3. THE TWO-OUT-OF-THREE LEMMA

3.1. Two-out-of-three lemma. *Let E and F be oriented Euclidean vector bundles on X. Given spin^c-structures on two of the three bundles E, F and $E \oplus F$, there is a uniquely determined spin^c-structure on the third.*

Proof. We have

$$w_2(E \oplus F) = w_2(E) + w_1(E)w_1(F) + w_2(F) = w_2(E) + w_2(F)$$

since $w_1(E) = w_1(F) = 0$. Therefore

$$W_3(E \oplus F) = W_3(E) + W_3(F).$$

It follows from Theorem 2.8 and Definition 2.2 that, if two of the three bundles admit a spin^c-structure, then so does the third.

We consider 4 cases separately.

(i) E of rank $2m$, F of rank $2n$. Let S (resp. T) be a spin^c-structure on E (resp. F). Make (e,f) act on $S \otimes T$ as

$$e \otimes 1 + \omega_1 \otimes f$$

where ω_1 is the Clifford orientation on E. Recall that $\omega_1 s = +s$ if $s \in S^+$ and $\omega_1 s = -s$ if $s \in S^-$. Now $(e \otimes 1 + \omega_1 \otimes f)^2 = e^2 \otimes 1 + e\omega_1 \otimes f + \omega_1 e \otimes f + \omega_1^2 \otimes f^2 = e^2 \otimes 1 + 1 \otimes f^2 = -Q(e) - Q(f) = -Q(e \oplus f)$ since $\omega_1^2 = 1$ and $e\omega_1 = -\omega_1 e$. The given action lifts to a homomorphism

$$(E \oplus F)^{\sim} \to \mathrm{End}(S \otimes T)$$

which is an isomorphism by a dimension count. Then $S \otimes T$ is a spin^c-structure on $E \oplus F$.

(ii) E of rank $2m$, F of rank $2n+1$. Let S (resp. T) be a spin^c-structure on E (resp. F). This means that S is an irreducible $\tilde{E}$-module, and T is an irreducible $(\tilde{F})^{\mathrm{ev}}$-module. Let ω_1 be the Clifford orientation on E, and let ω_2 be the Clifford orientation on F. Make (e,f) act on $S \otimes T$ as

$$e \otimes 1 + \omega_1 \otimes f\omega_2 .$$

Note that $f\omega_2 \in (\tilde{F})^{\mathrm{ev}}$. Then $(e \otimes 1 + \omega_1 \otimes f\omega_2)^2 = e^2 \otimes 1 + e\omega_1 \otimes f\omega_2 + \omega_1 e \otimes f\omega_2 + \omega_1^2 \otimes f\omega_2 f\omega_2 = -Q(e) + f\omega_2\omega_2 f = -Q(e) - Q(f) = -Q(e \oplus f)$ since $\omega_2^2 = 1$. This determines a homomorphism

$$\widetilde{(E \oplus F)} \to \mathrm{End}(S \otimes T)$$

which restricts to an isomorphism

$$\widetilde{(E \oplus F)}^{\mathrm{ev}} \to \mathrm{End}(S \otimes T)$$

by a dimension count. Hence $S \otimes T$ is a spin^c-structure on $E \oplus F$.

(iii) E of rank $2m+1$, F of rank $2n$. Make (e,f) act on $S \otimes T$ as

$$e\omega_1 \otimes \omega_2 + 1 \otimes f.$$

Then, as in (ii), $S \otimes T$ is a spin^c-structure on $E \oplus F$.

(iv) E of rank $2m+1$, F of rank $2n+1$. Make (e,f) act on $S \otimes T \oplus \oplus S \otimes T$ as

$$e\sigma_2 + f\sigma_1$$

where σ_1, σ_2 are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

and we have the understanding that $e = e\omega_1 \otimes 1$ and $f = 1 \otimes f\omega_2$. Now $(e\sigma_2 + f\sigma_1)^2 = e^2 + f^2 + ef\sigma_2\sigma_1 + fe\sigma_1\sigma_2 = -Q(e \oplus f)$. The action lifts to an isomorphism, again by a dimension count:

$$(E \oplus F)^{\sim} \cong \mathrm{End}(S \otimes T \oplus S \otimes T).$$

Hence $S \otimes T \oplus S \otimes T$ is a spin^c-structure on $E \oplus F$.

REMARK. The formula $e\sigma_2 + f\sigma_1$ is an intimation of the Dirac operator.

Suppose now that S is a spin^c-structure on E, and that U is a spin^c-structure on $E \oplus F$. Since $W_3(F) = 0$, there exists a spin^c-structure T on F.

In cases (i) — (iii), let L be the complex Hermitian line bundle $\mathrm{Hom}(S \otimes T, U)$. Then

$$S \otimes (T \otimes L) \cong (S \otimes T) \otimes L \cong U.$$

Therefore $T \otimes L$ is the unique spin^c-structure determined by S and U.

In case (iv), let $L = \mathrm{Hom}(S \otimes T \oplus S \otimes T, U)$. Then

$$S \otimes (T \otimes L) \oplus S \otimes (T \otimes L) \cong$$

$$\cong (S \otimes T \oplus S \otimes T) \otimes L \cong U.$$

Therefore $T \otimes L$ is the unique spin^c-structure determined by S and U.

We can summarize this discussion in the following way. Here we must specify that the direct sum $E_x \oplus F_x$ of two oriented vector spaces is to be oriented by taking an oriented basis for E_x followed by an oriented basis for F_x. If ε is an orientation on E, and ε' is an orientation on F, then $\varepsilon \oplus \varepsilon'$ shall denote the direct sum orientation on $E \oplus F$. Let $(\varepsilon, \mathscr{E})$ be a spin^c-structure on E and let $(\varepsilon', \mathscr{E}')$ be a spin^c-structure on F. Then the uniquely determined spin^c-structure on the direct sum $E \oplus F$ is

$$(\varepsilon \oplus \varepsilon', \mathscr{E} \otimes \mathscr{E}')$$

unless E and F are of odd rank, in which case the spin^c-structure is

$$(\varepsilon \oplus \varepsilon', \mathscr{E} \otimes \mathscr{E}' \oplus \mathscr{E} \otimes \mathscr{E}').$$

3.2. STABILIZING AND DE-STABILIZING. θ^n denotes the trivial vector bundle $X \times \mathbf{R}^n$. There is a one-one correspondence between spin^c-structures on E and spin^c-structures on $E \oplus \theta^n$. This follows immediately from the two-out-of-three lemma.

3.3. TWO-OUT-OF-THREE LEMMA (Mp^c-structures). *Let* (F_1, ω_1), (F_2, ω_2) *be symplectic vector bundles on* X, *and let* (F, ω) *be the direct sum. If two of these bundles have* Mp^c-*structures, then the third has a uniquely determined* Mp^c-*structure.*

Proof. The obstructions all vanish:

$$\delta(\dot{F}_1) = \delta(\dot{F}_2) = \delta(\dot{F}) = 0.$$

Suppose that F_1 (resp. F_2) has an Mp^c-structure S_1 (resp. S_2). S_1 (resp. S_2) is a Hilbert bundle of infinite rank, hence is necessarily trivial. Let f be a function on $F_1 \oplus F_2$ integrable on each fibre with respect to $\omega \wedge \ldots \wedge \omega$. The restriction f_1 (resp. f_2) of f to each fibre of F_1 (resp. F_2) is integrable with respect to $\omega_1 \wedge \ldots \wedge \omega_1$ (resp. $\omega_2 \wedge \ldots \wedge \omega_2$) by Fubini's lemma. Define

$$\pi(f)(s_1 \otimes s_2) = \pi_1(f_1)s_1 \otimes \pi_2(f_2)s_2 \qquad s_j \in S_j$$

where π_j is the Schrödinger representation of $\overline{L^1(F_j, \omega_j)}$ on $k(S_j)$, the C^*-algebra of compact operators on S_j. Then π induces an isomorphism

$$\overline{L^1(F, \omega)} \cong \overline{L^1(F_1, \omega_1)} \otimes \overline{L^1(F_2, \omega_2)} \tag{1}$$

following the local isomorphism in [14, p. 38]. The tensor product in (1) is the minimal (spatial) tensor product. Therefore $F_1 \oplus F_2$ has Mp^c-structure $S_1 \otimes S_2$.

Suppose now that S (resp. U) is an Mp^c-structure on F_1 (resp. F). If T is a chosen Mp^c-structure on F_2, then T modified by the line bundle $\mathrm{Hom}(S \otimes T, U)$ is the uniquely determined Mp^c-structure on F_2, exactly as in 3.1.

4. EXAMPLES

4.1. STIEFEL'S THEOREM [21, p. 148]. *Every compact orientable* 3-*manifold* M *is parallelizable.*

The tangent bundle $E = TM$ is trivial and admits a spin^c-structure.

4.2. WHITNEY'S THEOREM [13, p. 169]. *Every compact orientable* 4-*manifold* M *has* $W_3(M) = 0$.

Since $W_3(E) = W_3(M) = 0$, it is immediate that E admits a spin^c-structure.

4.3. THE DOLD 5-MANIFOLD. Let M be the product manifold $S^1 \times \mathbf{C}P^2$ with the identification $(x, z) = (-x, \bar{z})$ where $x \in S^1 \subset \mathbf{R}^2$ and $z \in \mathbf{C}P^2$. Then M is the Dold 5-manifold. Dold proves in [8] that its cohomology is given by

$$H^*(M; \mathbf{Z}_2) = \mathbf{Z}_2[c, d]/c^2 = d^3 = 0$$

and its total Stiefel-Whitney class is given by

$$w(M) = (1 + c)(1 + c + d)^3 = 1 + d + cd + \dots \ .$$

It is immediate that

$$w_1 = 0 \quad w_2 = d \quad w_3 = cd.$$

Since $w_3 = \rho(W_3)$ the mod 2 reduction of W_3, it follows that $W_3 \neq 0$. So M is *not* a spin^c-manifold.

4.4. The Dold 5-manifold M is *not cobordant* to a spin^c-manifold because the Stiefel-Whitney number $w_2w_3[M]$ is non-zero. This is because

$$w_2w_3 = d \cdot cd = cd^2 \neq 0 \quad \text{in } \mathbf{Z}_2[c, d]/c^2 = d^3 = 0.$$

4.5. Let $\mathscr{A} = \widetilde{(TM)}^{ev}$ the even part of the complex Clifford bundle of TM, where M is the Dold 5-manifold. Then $\mathscr{A}$ is a bundle of 4×4 matrix algebras on a compact 5-manifold M with $W_3(M) \neq 0$. Therefore $\delta(\mathscr{A}) \neq 0$. Let $A = C(\mathscr{A})$. Then A has the following properties:

(i) A is a unital 4-homogeneous C^*-algebra;

(ii) A has compact 5-dimensional dual;

(iii) A is not strongly Morita equivalent to the abelian C^*-algebra $C(M)$.

4.6. Let $X = \mathbf{C}P^2 \setminus \{\text{point}\}$. Then X is a complex 2-manifold, orientable and admits spin^c-structures. Since $H^2(X; \mathbf{Z}) = \mathbf{Z}$, X has countably many spin^c-structures. Since X is non-compact, X admits a Lorentz metric. Now $w_2(X) \neq 0$. Then X has the following properties:

(i) X is an orientable spacetime;

(ii) X has countably many spin^c-structures;

(iii) X is not parallelizable.

If we insist that orientable spacetime Y admit a spin-structure, then Y is parallelizable, by a well-known result of Geroch [10] and [17, p. 55]. The manifold X is a non-compact, orientable, non-parallelizable spacetime which admits spinors.

4.7. THE DOLD MANIFOLDS $P(m,n)$. The Dold manifold $P(m,n)$ is the product manifold $S^m \times CP^n$ with the identification $(x,z) = (-x,\bar{z})$ where $x \in S^m \subset \mathbf{R}^{m+1}$ and $z \in \mathbf{C}P^n$. Note that

$$P(m,0) = \mathbf{R}P^m \quad P(0,n) = \mathbf{C}P^n$$

so that $P(m,n)$ is a blend of real and complex projective space. We have already seen in 6.3 that the Dold manifold $P(1,2)$ is not a spin^c-manifold. We have the formulae of Dold [8]:

$$H^*(P(m,n);\mathbf{Z}_2) = \mathbf{Z}_2[c,d]/c^{m+1} = d^{n+1} = 0$$

$$w(P(m,n)) = (1+c)^m(1+c+d)^{n+1}.$$

It is immediate that

$$w_1 = (m+n+1)c, \quad w_3 = Bc^3 + m(n+1)cd$$

where B is a $\mathbf{Z}_2$-coefficient. If m is odd and n is even, then

$$w_1 = 0, \quad w_3 = Bc^3 + cd.$$

But c^3 and cd are independent, therefore $w_3 \neq 0$. Thus we have $w_1 = 0$, $W_3 \neq 0$. We have shown the following:

If m is odd and n is even, then the Dold manifold $P(m,n)$ is a compact orientable $(m+2n)$-manifold which is *not* a spin^c-manifold.

Among the Dold manifolds, half are orientable, namely those for which m and n have opposite parity. Among the orientable Dold manifolds, at least half are not spin^c-manifolds, namely those for which m is odd and n is even. This should dispel once for all the idea that non-spin^c-manifolds are pathological.

4.8. THE INVARIANT $\varkappa(A)$. Let E be a Euclidean vector bundle with spin^c-structure $(\varepsilon, \mathscr{E})$. Let S be the complex Hermitian vector bundle determined by the equivalence bimodule $\mathscr{E}$. Let $\lambda(S)$ be the associated complex Hermitian line bundle which occurs in the proof of Theorem 1.7; let (η, β) be the spin^c-structure corresponding to S (Theorem 2.11). The homomorphism

$$\mathrm{Spin}^c(n) \to \mathbf{T}$$

sending (v,z) to z^2, associates to η a line bundle $l(\eta)$. Now $w_2(E)$ is the mod 2 reduction of the first Chern class $c_1(l(\eta))$ and $\varkappa(\tilde{E})$ is the mod 2 reduction of the first Chern class $c_1(\lambda(S))$; since $\lambda(S) = l(\eta)$ we have

$$\varkappa(\tilde{E}) = w_2(E).$$

There is an exact parallel in the case of symplectic spinors. Let $\dot{F}$ be a symplectic vector bundle, and let $\dot{F}$ be the Weyl bundle. It is shown in [18] that

$$\varkappa(\dot{F}) = w_2(F).$$

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Note added in proof. In 1.7(ii), we must specify an anti-automorphism of the bundle A (so that S' has an A-module structure). The Clifford bundle (and the Weyl bundle) has a canonical anti-automorphism.

In 2.12 we describe Dirac spinors in terms of a certain Clifford module. The precise link between this Clifford module and the Penrose theory of 2-component spinors is set out in [23].