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ORDERED GROUPS AND TOEPLITZ ALGEBRAS

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INTRODUCTION

The classical theory of Toeplitz operators and their associated C^*-algebras is an elegant and important area of modern mathematics. For this reason many authors (e.g. Douglas, Singer, Howe, Devinatz) have sought to extend this theory to a more general setting. In this paper a new extension is given principally with the objective of presenting a certain new class of C^*-algebras which have very interesting properties — to each partially ordered group G we associate a C^*-algebra $\mathscr{T}(G)$, its Toeplitz algebra. $\mathscr{T}(G)$ has a certain universal property which may be useful in general C^*-algebra theory, particularly K-theory. Often $\mathscr{T}(G)$ contains a simple C^*-subalgebra as a closed ideal and this is analyzable in terms of G.

The classical Toeplitz algebra $\mathscr{T}(\mathbf{Z})$ associated to the ordered group of integers $\mathbf{Z}$ appears in two guises in the literature:

1. $\mathscr{T}(\mathbf{Z})$ is the C^*-algebra generated by the Toeplitz operators with continuous symbol.

2. $\mathscr{T}(\mathbf{Z})$ is the C^*-algebra (unique up to $*$-isomorphism) generated by a non-unitary isometry (Coburn [3]).

It is principally in its second guise that we are interested in generalizing $\mathscr{T}(\mathbf{Z})$. However in analyzing the Toeplitz algebra of a general partially ordered group we need to extend many results of the classical Toeplitz operator theory.

Here is a brief outline of what we do in each section: In Section 1 we construct $\mathscr{T}(G)$ and show that the functor $G \to \mathscr{T}(G)$ is "continuous", a result very important for the sequel. In Section 2 we specialize to the case where the ordering on G is total. In this situation $\mathscr{T}(G)$ is representable as a hereditary C^*-subalgebra of a certain crossed product C^*-algebra got by an action α of G on an abelian C^*-algebra $\mathscr{S}_1(G)$. By showing that $\mathscr{S}_1(G)$ is G-prime and calculating the Connes' spectrum $\Gamma(\alpha)$ in the case that G is finitely generated, and then extending by "continuity" to the general case, we show that $\mathscr{T}(G)$ is prime, and is isomorphic to any C^*-algebra generated by a non-unitary semigroup of isometries over G.

We then identify a certain simple ideal in $\mathcal{T}(G)$ ("usually" not type I) which plays a somewhat similar role in the general theory to that played by the ideal of compact operators in the classical case. A number of the results here generalize some results of Douglas [4]. However Douglas confines himself to ordered subgroups of $\mathbf{R}$, and our methods are completely different from his (his techniques do not appear to be extendable to this generality). In Section 3 we return to general partially ordered groups. Here we extend many results of the classical theory, for example we show that generalized analytic Toeplitz operators have connected spectra. For each partially ordered group G we exhibit a very explicit and useful irreducible representation of $\mathcal{T}(G)$ (faithful if G is totally ordered). Finally in Section 4 we show that a number of our results are best possible, and that for a totally ordered group G, $\mathcal{T}(G)$ has simple commutator ideal iff G is (ismorphic to) an ordered subgroup of $\mathbf{R}$ (sufficiency is due to Douglas [4]).

A brief word concerning terminology. What we call a partially ordered group is sometimes referred to as a directed partially ordered (abelian) group. There is little consistency in the literature in this area (compare Effros [6], Rudin [14], Goodearl [7], etc.).

Finally the author would like to thank R. Douglas for drawing his attention to his paper [4].

1. THE GENERAL TOEPLITZ ALGEBRA

Let G be a discrete abelian group and $\leqslant$ a partial ordering on G. For $S \subseteq G$ we write S^+ for the set of all x in S such that $0 \leqslant x$. We call $(G, \leqslant)$ a *partially ordered group* if $G = G^+ - G^+$ and $x \leqslant y$ implies that $x + z \leqslant y + z$ $(x, y, z \in G)$. If I is a subgroup of G such that $I = I^+ - I^+$ we call I a *partially ordered subgroup* of G. Then of course $(I, \leqslant)$ is itself a partially ordered group. If $\leqslant$ is a total ordering on G (i.e. for all $x, y \in G$ we have $x \leqslant y$ or $y \leqslant x$) then we refer to $(G, \leqslant)$ simply as an *ordered group*. In this case any subgroup I of G is a (partially) ordered subgroup, since $G = G^+ \cup (-G^+)$ implies that $I = I^+ \cup (-I^+) = I^+ - I^+$. Thus $(I, \leqslant)$ is an ordered group.

Suppose now only that G is a discrete abelian group and M is a subset of G such that

$$0 \in M, \quad M + M \subseteq M, \quad M \cap (-M) = 0 \quad \text{and} \quad G = M - M.$$

In this case we call M a *cone* in G and we define $x \leqslant_M y$ to mean that $y - x \in M$ for x, y in G. It is easily checked that $(G, \leqslant_M)$ is a partially ordered group with $G^+ = M$. We shall often use (G, M) to refer to $(G, \leqslant_M)$. If $(G, \leqslant)$ is a partially ordered group then G^+ is a cone and $(G, \leqslant_{G^+}) = (G, \leqslant)$. Moreover $(G, \leqslant)$ is an ordered group iff $G = G^+ \cup (-G^+)$.

If H is a closed subspace of a Hilbert space K we shall let S_H denote the compression to H of the bounded linear operator S on K.

If G is a partially ordered group and B a unital C^*-algebra, a *semigroup of isometries* in B (relative to G) is a map $\beta : G^+ \to B$ such that each $\beta(x)$ is an isometry, i.e. $\beta(x)^*\beta(x) = 1$ for all $x \in G^+$, and $\beta(x + y) = \beta(x)\beta(y)$ for all $x, y \in G^+$. (This implies that $\beta(0) = 1$.) If $B = B(H)$, the C^*-algebra of all bounded linear operators on the Hilbert space H, we call a pair (K, π) a *unitary lifting* of β if K is a Hilbert space containing H as a closed subspace, $\pi: G \to B(K)$ is a homomorphism of G into the group of unitaries of $B(K)$, H is invariant for all $\pi(x)$, $x \in G^+$, and $\beta(x) = (\pi(x))_H$ for such x. The following is the basic result concerning unitary liftings and will be used a number of times below.

THEOREM 1.1 (Ito). *Let G be a partially ordered group and $\beta : G^+ \to B(H)$ a semigroup of isometries on the Hilbert space H. Then β admits a unitary lifting (K, π).*

For a proof, see Suciu [15], p. 221. (Note that there it is only shown that $\beta(x)$ is a compression of $\pi(x)$—i.e. invariance of H is not stated. However an elementary 2×2 operator matrix argument shows that if an isometry is the compression of a unitary it is in fact a restriction of the unitary. Hence we can conclude that H above is invariant for all $\pi(x)$, $x \in G^+$.)

Here is another result that we will be using a number of times. It is well known and follows easily from the von Neumann inequality, see e.g. Suciu [15], p. 213.

(For G a discrete abelian group, $G^\wedge$ will denote the dual group of G considered as a compact abelian group. If T denotes the circle group and $x \in G$, we let ε_x or $\varepsilon(x)$ denote the evaluation homomorphism from $G^\wedge$ to T defined by $\varepsilon_x(\gamma) = \gamma(x)$, $\gamma \in G^\wedge$.)

LEMMA 1.2. *If $\pi : G \to B$ is a homomorphism from an abelian group into the group of unitaries of a unital C^*-algebra B then there is a unique $*$-homomorphism $\beta : C(G^\wedge) \to B$ such that $\beta(\varepsilon_x) = \pi(x)$, $x \in G$.*

We need one more preliminary concept for the construction we are about to undertake: Let A be a C^*-algebra with identity element 1 and let A_1, A_2 be C^*--subalgebras such that $1 \in A_1 \cap A_2$. We say that A is a *free product* of A_1 and A_2 and we write $A = A_1 * A_2$ if for every unital C^*-algebra B each pair of unital $*$-homomorphisms $\beta_j : A_j \to B$ $(j = 1, 2)$ have a unique extension to a $*$-homomorphism $\beta : A \to B$. Any two unital C^*-algebras admit a free product (Brown [1]). By the way, since any two free products of A_1 and A_2 are canonically $*$-isomorphic, we can talk about *the* free product. $A_1 \cup A_2$ generates $A_1 * A_2$.

We can now define the Toeplitz algebra and show it has a certain universal property.

Let G be a partially ordered group. Let p denote the projection $(1, 0)$ in $\mathbf{C}^2$, and let I be the closed ideal in $\mathbf{C}^2 * C(G^\wedge)$ generated by all $\varepsilon_x p - p\varepsilon_x p$, $x \in G^+$. If π denotes the quotient map from $\mathbf{C}^2 * C(G^\wedge)$ to $\mathbf{C}^2 * C(G^\wedge)/I$ then we set $\mathscr{T}(G) = \pi(p)(\mathrm{Im}(\pi))\pi(p)$. Thus $\mathscr{T}(G)$ is a unital C^*-algebra ($\pi(p)$ is the identity element). We call $\mathscr{T}(G)$ the *Toeplitz algebra* of G. We define the *canonical semigroup of isometries* $V = V^G : G^+ \to \mathscr{T}(G)$ by $V_x = \pi(\varepsilon_x)\pi(p)$. It is readily verified that V is in fact a semigroup of isometries generating $\mathscr{T}(G)$.

THEOREM 1.3. *Let G be a partially ordered group and $\beta : G^+ \to B$ a semigroup of isometries in a unital C^*-algebra B. Then there is a unique $*$-homomorphism $\beta^* : \mathscr{T}(G) \to B$ such that $\beta^* V = \beta$.*

Proof. We may assume without loss of generality that B is a C^*-subalgebra of $B(H)$ for some Hilbert space H such that $1 = \mathrm{id}_H \in B$. By Theorem 1.1 the semigroup of isometries $\beta : G^+ \to B(H)$ has a unitary lifting (K, π). There exists γ_2 a unique unital $*$-homomorphism from $C = C(G^\wedge)$ to $B(K)$ such that $\gamma_2(\varepsilon_x) = \pi(x)$, $x \in G$, by Lemma 1.2. Let $Q \in B(K)$ be the projection onto H and $\gamma_1 : \mathbf{C}^2 \to B(K)$ the unique unital $*$-homomorphism such that $\gamma_1(p) = Q$ where $p = (1, 0) \in \mathbf{C}^2$. We let γ denote the unique $*$-homomorphism extending γ_1 and γ_2 to $\mathbf{C}^2 * C \to B(K)$. Since $\gamma(p) = Q$ and $\gamma(\varepsilon_x) = \pi(x)$ we have $\gamma(\varepsilon_x p - p\varepsilon_x p) = \pi(x)Q - Q\pi(x)Q = 0$ for all $x \in G^+$ (H is invariant for $\pi(x)$, $x \in G^+$). Thus $\gamma(I) = 0$ where I is the closed ideal in $\mathbf{C}^2 * C$ generated by all $\varepsilon_x p - p\varepsilon_x p$, $x \in G^+$. It now follows that the map $\beta^* : \mathscr{T}(G) \to B(H)$ defined by $\beta^*(a + I) = \gamma(a)_H$ for $a + I \in \mathscr{T}(G)$, is a well-defined $*$-homomorphism. Also $\beta^*(V_x) = \beta^*(\varepsilon_x p + I) = \gamma(\varepsilon_x p)_H = \pi(x)_H = \beta(x)$ for all $x \in G^+$, so $\mathrm{Im}(\beta^*) \subseteq B$. Thus $\beta^* : \mathscr{T}(G) \to B$ is a $*$-homomorphism such that $\beta^* V = \beta$.

Uniqueness of β^* is trivial, since the V_x ($x \in G^+$) generate $\mathscr{T}(G)$. ∎

If A is a C^*-algebra we let $K(A)$ denote its *commutator ideal*, i.e. the closed ideal generated by all $ab - ba$ ($a, b \in A$). $K(A)$ is the smallest closed ideal J in A such that A/J is abelian. If $\beta : A \to B$ is a $*$-homomorphism of C^*-algebras then $\beta(K(A)) \subseteq K(B)$, with equality if β is surjective.

If $\varphi : G_1 \to G_2$ is a homomorphism of partially ordered groups we say that φ is *positive* if $\varphi(G_1^+) \subseteq G_2^+$ (equivalently $x \leqslant y$ in $G_1 \Rightarrow \varphi(x) \leqslant \varphi(y)$ in G_2). We let $\varphi^\sim : G_1^+ \to G_2^+$ be the restriction of φ. There is a unique $*$-homomorphism $\varphi^* : \mathscr{T}(G_1) \to \mathscr{T}(G_2)$ such that $\varphi^* V^{G_1} = V^{G_2}\varphi^\sim$ (simply take $\varphi^* = (V^{G_2}\varphi^\sim)^*$ — it is clear that $V^{G_2}\varphi^\sim : G_1^+ \to \mathscr{T}(G_2)$ is a semigroup of isometries). We thus get covariant functors $G \to \mathscr{T}(G)$ and $G \to K(\mathscr{T}(G))$.

For G a partially ordered group we define $q_x = q_x^G = 1 - V_x V_x^*$ ($x \in G^+$). Since V_x is an isometry, q_x is a projection (in $K(\mathscr{T}(G))$). If $x \leqslant y$ then $q_x \leqslant q_y$ (because: $y = x + z$ for some $z \in G^+ \Rightarrow V_y = V_x V_z \Rightarrow V_y V_y^* = V_x V_z V_z^* V_x^* \leqslant V_x V_x^*$ since $V_z V_z^* \leqslant 1$. Hence $q_y = 1 - V_y V_y^* \geqslant 1 - V_x V_x^* = q_x$). We are now going to show that $q_x \leqslant q_y \Rightarrow x \leqslant y$. To do this we consider briefly a certain representation of

$\mathscr{T}(G)$ that will be very important later. Before doing this, a useful remark: If $\varphi: G_1 \to G_2$ is a positive homomorphism of partially ordered groups then $\varphi^*(q_x) = q_{\varphi(x)}$ for all $x \in G_1^+$.

PROPOSITION 1.4. *If G is a partially ordered group and $x, y \in G^+$ then $x \leqslant y$ if and only if $q_x \leqslant q_y$.*

Proof. Let $H^2 = H^2(G)$ be the closed linear span in $L^2(G^\wedge)$ of all ε_x ($x \in G^+$). Define $U_x \in B(H^2)$ by $U_x f = \varepsilon_x f$, $x \in G^+$. The map $U : G^+ \to B(H^2)$, $x \mapsto U_x$, is easily seen to be a semigroup of isometries: thus U^* maps $\mathscr{T}(G)$ to $B(H^2)$ and $U^*V = U$. Now the projections $Q_x = 1 - U_x U_x^*$ on H^2 satisfy the relations $Q_x(\varepsilon_y) = 0$ for $x \leqslant y$ and $Q_x(\varepsilon_y) = \varepsilon_y$ for $x \nleqslant y$ ($x, y \in G^+$). Hence $x \leqslant y$ iff $Q_x \leqslant Q_y$. But $U^* q_x = Q_x$, so $x \leqslant y \Rightarrow q_x \leqslant q_y \Rightarrow Q_x \leqslant Q_y \Rightarrow x \leqslant y$. ▨

Note in passing that it is now easy to see that $V : G^+ \to \mathscr{T}(G)$ is injective ($V_x = V_y \Rightarrow q_x = q_y \Rightarrow x = y$).

Given any partially ordered group G the map $\varepsilon : G^+ \to C(G^\wedge)$, $x \mapsto \varepsilon_x$, is a semigroup of isometries (actually of course the ε_x are unitaries) so we have the induced map $\varepsilon^* : \mathscr{T}(G) \to C(G^\wedge)$. Since ε_x ($x \in G^+$) generate $C(G^\wedge)$ (by the Stone-Weierstrass theorem), ε^* is surjective. Here is the full story:

THEOREM 1.5. *If G is a partially ordered group then* $\ker(\varepsilon^*) = K(\mathscr{T}(G))$ *and the map $\mathscr{T}(G)/K(\mathscr{T}(G)) \to C(G^\wedge)$, $a + K(\mathscr{T}(G)) \mapsto \varepsilon^*(a)$, is a $*$-isomorphism.*

Proof. All we have to show is that $K(\mathscr{T}(G)) = \ker(\varepsilon^*)$. Since $\mathscr{T}(G)/\ker(\varepsilon^*)$ is abelian, $\ker(\varepsilon^*) \supseteq K(\mathscr{T}(G))$. The map $\pi : G \to \mathscr{T}(G)/K(\mathscr{T}(G))$, $x - y \mapsto V_y^* V_x + K(\mathscr{T}(G))$, (for $x, y \in G^+$) is a well-defined homomorphism into the unitaries of $\mathscr{T}(G)/K(\mathscr{T}(G))$, so by Lemma 1.2 there exists a unique $*$-homomorphism $\gamma : C(G^\wedge) \to \mathscr{T}(G)/K(\mathscr{T}(G))$ such that $\gamma(\varepsilon_{x-y}) = \pi(x - y) = V_y^* V_x + K(\mathscr{T}(G))$ ($x, y \in G^+$). If δ is the $*$-homomorphism $\mathscr{T}(G)/K(\mathscr{T}(G)) \to C(G^\wedge)$, $a + K(\mathscr{T}(G)) \mapsto \varepsilon^*(a)$ then, $\gamma\delta(V_x + K(\mathscr{T}(G))) = \gamma\varepsilon^*(V_x) = \gamma(\varepsilon_x) = \pi(x) = V_x + K(\mathscr{T}(G))$ ($x \in G^+$) $\Rightarrow \gamma\delta = \mathrm{id}$ (since $V_x + K(\mathscr{T}(G))$ generate $\mathscr{T}(G)/K(\mathscr{T}(G))$). Hence $a \in \ker(\varepsilon^*) \Rightarrow \delta(a + K(\mathscr{T}(G))) = \varepsilon^*(a) = 0 \Rightarrow a + K(\mathscr{T}(G)) = \gamma\delta(a + K(\mathscr{T}(G))) = 0 \Rightarrow a \in K(\mathscr{T}(G))$. Thus $K(\mathscr{T}(G)) = \ker(\varepsilon^*)$. ▨

Our next result, showing that the functor $G \to \mathscr{T}(G)$ is "continuous", i.e. preserves direct limits, is interesting in its own right and plays a crucial role in the development of the theory. First we need to make a somewhat technical remark about direct limits in the category of partially ordered groups. Let $(\varphi_{ij} : G_i \to G_j)_{i \leqslant j}$ be a direct system of partially ordered groups (indexed by I) with direct limit G and natural maps $(\varphi^i : G_i \to G)_i$. If $x \in G_i$, $y \in G_j$, and $\varphi^i(x) = \varphi^j(y)$ then there exists $k \in I$, $k \geqslant i, j$ and $\varphi_{ik}(x) = \varphi_{jk}(y)$. This detail is needed in the proof that follows. The way to see it is to construct one example of a direct limit G and natural maps $(\varphi^i)_i$ satisfying it. Then it follows from an elementary diagram chase that every direct limit and system of natural maps for $(\varphi_{ij} : G_i \to G_j)_{i \leqslant j}$ has the above property. Here is a sketch of how to construct the required limit: Define an equivalence relation $\sim$ on the disjoint union of the sets G_i $(i \in I)$ by setting $(i, x) \sim (j, y)$ if there exists $k \in I$, $k \geqslant i, j$, such that $\varphi_{ik}(x) = \varphi_{jk}(y)$. Let $[i, x]$ denote the equivalence class of (i, x) and G be the set of all equivalence classes. Define the map $\varphi^i : G_i \to G$ by $\varphi^i(x) = [i, x]$. There is a unique operation on G making G an abelian group and all the maps φ^i homomorphisms. Define G^+ to be the union of all the sets $\varphi^i(G_i^+)$ $(i \in I)$. Then G^+ is a cone in G and it is easily checked that the partially ordered group (G, G^+) is a direct limit of the direct system $(\varphi_{ij} : G_i \to \to G_j)_{i \leqslant j}$ in the category of all partially ordered group (with the maps φ^i as natural maps). Clearly $x \in G_i$, $y \in G_j$, and $\varphi^i(x) = \varphi^j(y)$ imply that there exists $k \in I$, $k \geqslant i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$.

THEOREM 1.6. *Let the partially ordered group G be the direct limit of the direct system of partially ordered groups $(\varphi_{ij} : G_i \to G_j)_{i \leqslant j}$. Then $\mathscr{T}(G)$ is the direct limit of the direct system of C^*-algebras $((\varphi_{ij})^* : \mathscr{T}(G_i) \to \mathscr{T}(G_j))_{i \leqslant j}$.*

More explicitly if $(\varphi^i : G_i \to G)_i$ are natural maps for G then $((\varphi^i)^* : \mathscr{T}(G_i) \to \to \mathscr{T}(G))_i$ are natural maps for $\mathscr{T}(G)$.)

Proof. Let I be the index set for the direct system $(\varphi_{ij} ; G_i \to G_j)_{i \leqslant j}$. Given B a C^*-algebra and $\beta^i : \mathscr{T}(G_i) \to B$ $*$-homomorphisms such that $\beta^j(\varphi_{ij})^* = \beta^i$ for all $i \leqslant j$ in I, we must show that there is a unique $*$-homomorphism $\beta : \mathscr{T}(G) \to \to B$ such that $\beta(\varphi^i)^* = \beta^i$ $(i \in I)$. By replacing B by the unital C^*-subalgebra $C = (\bigcup \{\beta^i(\mathscr{T}(G_i)) : i \in I\})^-$ if necessary, we may assume without loss of generality that B is unital and that all the maps β^i are unital. Let $V^i : G_i^+ \to \mathscr{T}(G_i)$ and $V : G^+ \to \mathscr{T}(G)$ be the canonical maps and recall that $(\varphi_{ij})^\sim$ and $(\varphi^i)^\sim$ are the restrictions to the positive cones of φ_{ij} amd φ^i respectively. We have

$$(1) \qquad (\varphi_{ij})^* V^i = V^j (\varphi_{ij})^\sim \qquad (i \leqslant j)$$

$$(2) \qquad (\varphi^i)^* V^i = V(\varphi^i)^\sim \qquad (i \in I)$$

$$(3) \qquad \beta^j(\varphi_{ij})^* = \beta^i \qquad (i \leqslant j).$$

Let $x \in G_i$, $y \in G_j$ and suppose that $\varphi^i(x) = \varphi^j(y)$. Then there exists $k \in I$, $k \geqslant i$, j, such that $\varphi_{ik}(x) = \varphi_{jk}(y)$ implies $\beta^i V_x^i = \beta^k(\varphi_{ik})^* V_x^i$ (by (3)) $=$ $= \beta^k V^k(\varphi_{ik}(x))$ (by (1)) $= \beta^k V^k(\varphi_{jk}(y)) = \beta^k(\varphi_{jk})^* V_y^j$ (by (1) again) $= \beta^j V_y^j$ (by (3) again). Thus $\varphi^i(x) = \varphi^j(y)$ implies $\beta^i V_x^i = \beta^j V_y^j$.

We define the map $\beta : G^+ \to B$ by setting $\beta(\varphi^i(x)) = \beta^i V_x^i$. This is well defined from the above calculation and from the fact that G^+ is the union of all the sets $\varphi^i(G_i^+)$ $(i \in I)$. Now β is clearly a semigroup of isometries, so $\beta^* : \mathscr{T}(G) \to B$ is a $*$-homomorphism and $\beta^* V = \beta$. Also, by (2), $\beta^*(\varphi^i)^* V^i = \beta^* V(\varphi^i)^\sim = \beta(\varphi^i)^\sim =$

$= \beta^i V^i$ (by the definition of β), which implies $\beta^*(\varphi^i)^* V^i = \beta^i V^i$, so $\beta^*(\varphi^i)^* = \beta^i$ (since V^i_x $(x \in G_i^+)$ generate $\mathscr{T}(G_i)$).

Finally suppose that $\gamma : \mathscr{T}(G) \to B$ were another $*$-homomorphism such that $\gamma(\varphi^i)^* = \beta^i$ $(i \in I)$. We must show that $\gamma = \beta^*$. But $\gamma V(\varphi^i)^\sim = \gamma(\varphi^i)^* V^i$(by (2)) $= \beta^i V^i = \beta(\varphi^i)^\sim$, so $\gamma V = \beta$ (since G^+ is the union of the sets $(\varphi^i)(G_i^+)$ $(i \in I)$). Thus $\gamma V = \beta = \beta^* V \Rightarrow \gamma = \beta^*$ (since V_x $(x \in G^+)$ generate $\mathscr{T}(G)$). ▨

2. THE TOEPLITZ ALGEBRA OF AN ORDERED GROUP

To prove deeper results about the Toeplitz algebra $\mathscr{T}(G)$ one needs to specialize G. Specifically one needs to assume that G is totally ordered. That this assumption is not merely convenient, but actually necessary to get our results, is shown in a later section below. Our technique is to represent $\mathscr{T}(G)$ as a hereditary C^*-subalgebra of a certain crossed product C^*-algebra, and to use some powerful results of the theory of crossed products to analyse $\mathscr{T}(G)$.

Let G be an ordered group. We define an action α of G on $\ell^\infty(G)$ by setting $(\alpha_x f)(y) = f(y - x)$ for all $f \in \ell^\infty(G)$ and all $x, y \in G$. It is clear that $\alpha_x \in \mathrm{Aut}(\ell^\infty(G))$ and that the map $G \to \mathrm{Aut}(\ell^\infty(G))$, $x \mapsto \alpha_x$, is a homomorphism. We define p_0 to be the characteristic function of the set G^+ as a subset of G and $p_x = \alpha_x(p_0)$ for each $x \in G$. Thus p_x is the characteristic function of the set $x + G^+$. We call p_x the projection *determined* by x. Clearly $x \leqslant y$ iff $p_x \geqslant p_y$. If $x \vee y$ denotes the maximum of x and y then $p_x p_y = p_{x \vee y}$. It follows that the closed linear span $\mathscr{S}(G)$ of all the projections p_x $(x \in G)$ is a C^*-subalgebra of $\ell^\infty(G)$. Put $\mathscr{S}_1(G) = \mathscr{S}(G) + \mathbf{C}1$ $(1 \in \mathscr{S}(G) \Leftrightarrow G = 0)$ and let $\mathscr{S}_0(G)$ denote the closed linear span of all $p_x - p_y$ $(x, y \in G)$. $\mathscr{S}_0(G)$ is clearly a closed ideal in $\mathscr{S}(G)$. Since each α_x maps $\mathscr{S}_1(G)$ into itself we get by restriction a homomorphism $\alpha\colon G \to \mathrm{Aut}(\mathscr{S}_1(G))$, $x \mapsto \alpha_x$, i.e. $(\mathscr{S}_1(G), G, \alpha)$ is a C^*-dynamical system. Clearly $\mathscr{S}(G)$ and $\mathscr{S}_0(G)$ are G-invariant ideals in $\mathscr{S}_1(G)$. Recall that one can regard $\mathscr{S}_1(G)$ as a C^*-subalgebra of the crossed product C^*-algebra $\mathscr{S}_1(G) \times_\alpha G$ and that $\mathscr{S}_1(G)$ contains the identity element of $\mathscr{S}_1(G) \times_\alpha G$. Also if $\delta : G \to \mathscr{S}_1(G) \times_\alpha G$ is the canonical homomorphism into the unitaries then we have $\alpha_x(f) = \delta_x f \delta_x^*$ $(f \in \mathscr{S}_1(G), x \in G)$. $\mathscr{S}_1(G) \times_\alpha G$ is generated by $\mathscr{S}_1(G)$ and $\delta(G)$. If J is a closed G-invariant ideal of $\mathscr{S}_1(G)$ then the C^*-subalgebra of $\mathscr{S}_1(G) \times_\alpha G$ generated by all $f\delta_x$ $(f \in J,\ x \in G)$ is in fact a closed ideal of $\mathscr{S}_1(G) \times_\alpha G$ which is $*$-isomorphic to the crossed product $J \times_\alpha G$. We can, and do, therefore regard $J \times_\alpha G$ as this ideal in $\mathscr{S}_1(G) \times_\alpha G$. Note that $J \subseteq J \times_\alpha G$.

Now we define $\mathscr{A}(G) = p_0(\mathscr{S}_1(G) \times_\alpha G)p_0$. Thus $\mathscr{A}(G)$ is a hereditary C^*-subalgebra of $\mathscr{S}_1(G) \times_\alpha G$ with identity element p_0. Define $W : G^+ \to \mathscr{A}(G)$ by setting $W_x = p_0 \delta_x p_0$. W is a semigroup of isometries. This is immediate from the fact that $p_0 \delta_x p_0 = \delta_x p_0$ for all $x \in G^+$ $(p_0 \delta_x p_0 \delta_x^* = (\chi_{G^+})(\chi_{x+G^+}) = \chi_{G^+ \cap (x+G^+)} = \chi_{x+G^+} = \delta_x p_0 \delta_x^* \Rightarrow p_0 \delta_x p_0 = \delta_x p_0)$. Thus we have the induced $*$-homomorphism

$W^* : \mathcal{T}(G) \to \mathcal{A}(G)$ with $W^* V = W$. Since $\mathcal{S}_1(G) \times_\alpha G$ is generated by p_0 and all δ_x ($x \in G^+$) one can easily show that W_x ($x \in G^+$) generate $\mathcal{A}(G)$. This implies that W^* is surjective. We are going to see in a moment that W^* is in fact a $*$-isomorphism, but first we need a lemma which shows that $\mathcal{S}_1(G)$ has an interesting universal property.

LEMMA 2.1. *Let G be a non-zero ordered group, $\mu : G \to B$ a homomorphism into the unitaries of a unital C^*-algebra B, and q a projection in B such that $\mu(x)q = q\mu(x)q$ for all x in G^+. Then there is a unique unital $*$-homomorphism $\gamma : \mathcal{S}_1(G) \to B$ such that $\gamma(p_x) = \mu(x)q\mu(x)^*$ for all x in G.*

Proof. Uniqueness is obvious, we show existence.

Let Γ denote the linear span of all p_x ($x \in G$), so Γ is a dense $*$-subalgebra of $\mathcal{S}(G)$. Put $q_x = \mu(x)q\mu(x)^*$ and note that if $x \leqslant y$ then $q_x \geqslant q_y$, since $q_x q_y = \mu(x)q\mu(y-x)q\mu(y)^* = \mu(x)\mu(y-x)q\mu(y)^*$ (as $y - x \in G^+$) $= q_y$. Thus $q_x q_y = q_{x \vee y}$. Now let $x^1, \ldots, x^n \in G$, and let $p_i = p_{x^i}$, and $q_i = q_{x^i}$ ($i = 1, \ldots, n$). We show that $\|\lambda_1 p_1 + \ldots + \lambda_n p_n\| \geqslant \|\lambda_1 q_1 + \ldots + \lambda_n q_n\|$ ($\lambda_1, \ldots, \lambda_n \in \mathbf{C}$). We may assume (by re-indexing if necessary) that $x^1 \leqslant \ldots \leqslant x^n$, and hence that $p_1 \geqslant \ldots \geqslant p_n$ and $q_1 \geqslant \ldots \geqslant q_n$. Therefore the projections $p_1 - p_2, p_2 - p_3, \ldots, p_{n-1} - p_n, p_n$ are pairwise orthogonal, as are the projections $q_1 - q_2, q_2 - q_3, \ldots, q_{n-1} - q_n, q_n$. Let $\nu_i = \lambda_1 + \ldots + \lambda_i$ ($i = 1, \ldots, n$). Then we have $\lambda_1 p_1 + \ldots + \lambda_n p_n = \nu_1(p_1 - p_2) + \ldots + \nu_{n-1}(p_{n-1} - p_n) + \nu_n p_n$ and correspondingly $\lambda_1 q_1 + \ldots + \lambda_n q_n = \nu_1(q_1 - q_2) + \ldots + \nu_{n-1}(q_{n-1} - q_n) + \nu_n q_n$. Since $p_x = p_y$ implies $x = y$, and so $q_x = q_y$, we now deduce that

$$\|\lambda_1 p_1 + \ldots + \lambda_n p_n\| = \max\{|\nu_i| : p_i - p_{i+1} \neq 0 \ (i = 1, \ldots, n-1) \ \text{ or } p_n \neq 0\} \geqslant$$

$$\geqslant \max\{|\nu_i| : q_i - q_{i+1} \neq 0 \ (i = 1, \ldots, n-1) \ \text{ or } \ q_n \neq 0\} = \|\lambda_1 q_1 + \ldots + \lambda_n q_n\|.$$

It is now routine algebra to check that the map $\gamma : \Gamma \to B$ defined by setting $\gamma(\lambda_1 p_1 + \ldots + \lambda_n p_n) = \lambda_1 q_1 + \ldots + \lambda_n q_n$ is a (norm-decreasing) $*$-homomorphism, and so extends to a $*$-homomorphism $\gamma : \mathcal{S}(G) \to B$. Finally we extend γ to unital $*$-homomorphism $\gamma : \mathcal{S}_1(G) \to B$ by setting $\gamma(1) = 1$. ▨

THEOREM 2.2. *If G is an ordered group then the canonical map $W^* : \mathcal{T}(G) \to \mathcal{A}(G)$ is a $*$-isomorphism.*

Proof. We already know that W^* is surjective, so we just have to show injectivity.

Now we can regard $\mathcal{T}(G)$ as a C^*-subalgebra of $B(H)$ for some Hilbert space H with $\mathrm{id}_H = 1 \in \mathcal{T}(G)$. Also we may assume $G \neq 0$. By Theorem 1.1 the semigroup of isometries $V : G^+ \to B(H)$, $x \mapsto V_x$, admits a unitary lifting (K, π). Thus π is a homomorphism from G into the unitary operarors on the Hilbert space K, and if Q denotes the projection of K onto its subspace H we have $\pi(x)Q = Q\pi(x)Q$ for all $x \in G^+$, since H is invariant for such $\pi(x)$. Also $V_x = \pi(x)_H$ $(x \in G^+)$. By Lemma 2.1 there exists a unique unital $*$-homomorphism $\gamma : \mathscr{S}_1(G) \to B(K)$ such that $\gamma(p_x) = \pi(x)Q\pi(x)^*$ $(x \in G)$.

We now claim that (γ, π, K) is a covariant representation of the C^*-dynamical system $(\mathscr{S}_1(G), G, \alpha)$. All we need to do to see this is to show that $\gamma(\alpha_x(f)) = \pi(x)\gamma(f)\pi(x)^*$ for $f \in \mathscr{S}_1(G)$ and $x \in G$. By using the fact that 1 and all the projections $p_x (x \in G)$ have closed linear span $\mathscr{S}_1(G)$, it clearly suffices to show the above equation for f of the form $f = p_y$. But $\gamma(\alpha_x(p_y)) = \gamma(p_{x+y}) = \pi(x+y)Q\pi(x+y)^* = \pi(x)\pi(y)Q\pi(y)^*\pi(x)^* = \pi(x)\gamma(p_y)\pi(x)^*$.

Thus since (γ, π, K) is a covariant representation it induces a unique $*$-homomorphism $\gamma^\sim : \mathscr{S}_1(G) \times_\alpha G \to B(K)$ extending γ such that $\gamma^\sim(\delta_x) = \pi(x) (x \in G)$. It is now easily verified that the map $\mu : \mathscr{A}(G) \to B(H)$, $a \to \gamma^\sim(a)_H$, is a $*$-homomorphism. However $\mu(W_x) = \mu(\delta_x p_0) = (\gamma^\sim(\delta_x)\gamma(p_0))_H = (\pi(x)Q)_H = V_x \in \mathscr{T}(G)$ $(x \in G^+)$, so $\mathrm{Im}(\mu) \subseteq \mathscr{T}(G)$. Thus we can regard μ as a $*$-homomorphism from $\mathscr{A}(G)$ to $\mathscr{T}(G)$. Again since $\mu(W_x) = V_x$ we have $\mu W^*(V_x) = V_x$ $(x \in G^+)$, so $\mu W^* = \mathrm{id}_{\mathscr{T}(G)}$, thus W^* is injective. ▨

We state now a result of Power [13] that we will need for the next theorem Power defines a C^*-algebra C of operators on the Hilbert space K to be *inner* with respect to a closed subspace H of K if $\mathrm{id}_K \in C$ and C is generated by its elements T such that $T(H) \subseteq H$. If this is the case and C is commutative, and B is the C^*-subalgebra of $B(H)$ generated by all T_H $(T \in C)$ then $T \in K(C^*(C \cup \{Q\}))$ (Q is the projection of K on H) implies $T_H \in K(B)$ (see [13], proof of Theorem 4.2).

THEOREM 2.3. *If G is an ordered group then $K(\mathscr{A}(G)) = p_0(K(\mathscr{S}_1(G) \times_\alpha G))p_0$ is a full hereditary C^*-subalgebra of $K(\mathscr{S}_1(G) \times_\alpha G)$.*

Proof. Let $Z = \mathscr{S}_1(G) \times_\alpha G$. Since $\mathscr{A}(G)$ is a C^*-subalgebra of Z, $K(\mathscr{A}(G)) \subseteq K(Z)$, and since $\mathscr{A}(G) = p_0 Z p_0$, $K(\mathscr{A}(G)) \subseteq p_0 K(Z) p_0$.

Now regard Z as a C^*-subalgebra of $B(K)$ for some Hilbert space K with $\mathrm{id}_K = 1 \in Z$, and let $H = p_0(K)$. Since $\delta_x p_0 = p_0 \delta_x p_0$ $(x \in G^+)$, H is an invariant subspace for these δ_x, so the commutative C^*-subalgebra C of $B(K)$ generated by all $\delta_x (x \in G^+)$ is inner with respect to H. Let B be the C^*-subalgebra of $B(H)$ generated by all T_H $(T \in C)$. By Power's result mentioned above $T \in K(Z)$ implies $T_H \in K(B)$ (since $Z = C^*(C \cup \{p_0\})$). Now the map $\beta : \mathscr{A}(G) \to B$, $T \mapsto T_H$, is easily seen to be a $*$-isomorphism. Thus $T \in p_0 K(Z) p_0$ implies $T \in K(Z)$, and $T \in \mathscr{A}(G)$ implies $T_H \in K(B)$, which implies $\beta^{-1}(T_H) = T \in K(\mathscr{A}(G))$. We have therefore $p_0 K(Z) p_0 = K(\mathscr{A}(G))$, so $K(\mathscr{A}(G))$ is a hereditary C^*-subalgebra of $K(Z)$.

Finally we show $K(\mathscr{A}(G))$ is full in $K(Z)$, i.e. the closed ideal J in $K(Z)$ generated by $K(\mathscr{A}(G))$ is $K(Z)$ itself. This is because J contains $p_0 - p_x = p_0 - W_x W_x^*$ $(x \in G^+)$, and therefore $p_0\delta_x - \delta_x p_0 = (p_0 - \delta_x p_0 \delta_x^*)\delta_x = (p_0 - p_y)\delta_x \in J$. Hence Z/J is abelian (it is generated by commutating normal elements), so $J \supseteq K(Z) \Rightarrow J = K(Z)$. ▨

As a consequence of a theorem of Brown [2] and Theorem 2.3 above it follows that if $K(\mathscr{S}_1(G)\times_\alpha G)$ is separable (e.g. if G is countable) then $K(\mathscr{A}(G))$ and $K(\mathscr{S}_1(G)\times_\alpha G)$ are stably isomorphic.

Although we shall not be using it, we record here the interesting fact that for G an ordered group $K(\mathscr{S}_1(G)\times_\alpha G) = \mathscr{S}_0(G)\times_\alpha G$. (Proof: Let $Z = \mathscr{S}_1(G)\times_\alpha G$ and $J = \mathscr{S}_0(G)\times_\alpha G$. J is a closed ideal of Z generated as a C^*-algebra by all $f\delta_x$ ($f \in \mathscr{S}_0(G)$, $x \in G$). Now $(p_0 - p_x)\delta_x = (p_0 - \delta_x p_0 \delta_x^*)\delta_x = p_0\delta_x - \delta_x p_0 \in K(Z)$, so $(p_x - p_y)\delta_z = (p_0 - p_y)\delta_z - (p_0 - p_x)\delta_z \in K(Z)$, thus $f\delta_z \in K(Z)$ for all $f \in \mathscr{S}_0(G)$, and all $z \in G$. Hence $J \subseteq K(Z)$. Also $p_0\delta_x - \delta_x p_0 = (p_0 - p_x)\delta_x \in J$ implies Z/J is abelian, so $J \supseteq K(Z)$.)

Recall that a subgroup I of a partially ordered group G is an *ideal* of G if $I = I^+ - I^+$ and $0 \leqslant x \leqslant y \in I$ implies $x \in I$ ($x \in G$). G is said to be *simple* if 0 and G are its only ideals. All ordered subgroups of $\mathbf{R}$ with the usual order relation are simple. For $n = 2, 3, \ldots$ the group $\mathbf{Z}^n$ with the *lexicographic* order ($(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if $a_1 < b_1$ or if $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} < b_{i+1}$) is a non-simple ordered group.

If I is an ideal in a partially ordered group G, and φ is the quotient map from G to G/I then $\varphi(G^+)$ is a cone in the quotient group G/I. We call the partially ordered group $(G/I, \varphi(G^+))$ the *quotient partially ordered group* of G by I. Of course φ is a positive homomorphism from G to G/I. If G is totally ordered, so are I and G/I.

LEMMA 2.4. *If G is a finitely generated non-zero ordered group then G contains a non-zero simple ideal I.*

Proof. Let k be the rank of G. Note that an ordered group is necessarily torsion-free. Thus if I_1 is a proper ideal in G, then G/I_1 is non-zero and so has positive rank. This implies $\operatorname{rank}(I_1) = \operatorname{rank}(G) - \operatorname{rank}(G/I_1) < \operatorname{rank}(G) = k$.

We show the result by induction on k. Suppose it is true for all ranks $< k$. If G has no proper ideals then there is nothing to prove (take $I = G$). Otherwise G contains a non-zero ideal I_1 with rank $(I_1) < k$. By the induction hypothesis I_1 contains a non-zero simple ideal I. I is then an ideal in G, thus completing the induction. ▨

If G is any ordered group let $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$ ($x \in G$). Let $F(G) = \{x \in G : \text{for all } y \in G,\ y > 0, \text{ there exists } n \in \mathbf{N},\ |x| \leqslant ny\}$. Using the triangle inequality $|x + y| \leqslant |x| + |y|$ one easily sees that $F(G)$ is a simple ideal in G. Hence if I is any non-zero ideal of G, $F(G) \subseteq I$ (since if $F(G)$ is non-zero then there exists $x \in F(G)$, $x > 0$, and there exists $y \in I$, $y > 0$, so that if z is their minimum, then $0 < z \in I \cap F(G)$ implies $I \cap F(G)$ is a non-zero ideal of $F(G)$, so $I \cap F(G) = F(G)$, thus $F(G) \subseteq I$). Of course $F(G)$ might be just the zero ideal. In fact it is for $G = \mathbf{Z}^\infty$, the direct sum of countably infinitely many copies of $\mathbf{Z}$ with the *lexicographic* order: $(a_1, a_2, \ldots) < (b_1, b_2, \ldots)$ if $a_1 < b_1$ or if $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} < b_{i+1}$ for some integer i.

The point of Lemma 2.4 can be rephrased as follows: If G is a non-zero finitely generated ordered group, then $F(G)$ is non-zero. (Proof: By Lemma 2.4, G contains a non-zero simple ideal I. Let $x \in G$, $x > 0$, and let I_x be the set of y in G such that for some positive integer n, $|y| \leqslant nx$. Then I_x is a non-zero ideal in G ($I_x \neq 0$ as $x \in I_x$). Now $I \cap I_x$ is a non-zero ideal in I, so $I \cap I_x = I$, thus $I \subseteq I_x$. Hence $y \in I$ implies $|y| \leqslant nx$ for some $n \in \mathbf{N}$. Thus we have shown $I \subseteq F(G)$, and since $F(G) \subseteq I$ by our earlier remarks, $F(G) = I \neq 0$.)

If G is a non-zero finitely generated ordered group we let $F\mathscr{S}(G)$ denote the closed linear span of all $p_x - p_y$ ($x, y \in G$, $x - y \in F(G)$).

LEMMA 2.5. *If G is a non-zero finitely generated ordered group then $F\mathscr{S}(G)$ is the smallest non-zero G-invariant closed ideal in $\mathscr{S}_1(G)$.*

Proof. If $x, y, z \in G$ with $x - y \in F(G)$ then $z \vee x - z \vee y \in F(G)$. Hence $p_z(p_x - p_y) = p_{z \vee x} - p_{z \vee y} \in F\mathscr{S}(G)$, so it is clear that $F\mathscr{S}(G)$ is an ideal in $\mathscr{S}_1(G)$. Since $\alpha_z(p_x - p_y) = p_{x+z} - p_{y+z}$, it is trivial that $F\mathscr{S}(G)$ is G-invariant. As $F(G)$ is non-zero, $p_0 - p_x \neq 0$ for some $x \in F(G)$, so $F\mathscr{S}(G)$ is non-zero.

Now let J be a non-zero G-invariant closed ideal in $\mathscr{S}_1(G)$. We have to show that $F\mathscr{S}(G) \subseteq J$. By replacing J by $J \cap \mathscr{S}_0(G)$ if necessary, we may assume that $J \subseteq \mathscr{S}_0(G)$ (the reason that $J \cap \mathscr{S}_0(G)$ is non-zero is the easily checked fact that $\mathscr{S}_0(G)$ is an essential closed ideal in $\mathscr{S}_1(G)$).

Put $I = \{x \in G : p_0 - p_x \in J\}$. If $x, y \in I$ then $p_0 - p_x$ and $\alpha_x(p_0 - p_y) \in J$ implies $p_0 - p_x + p_x - p_{x+y} = p_0 - p_{x+y} \in J$, so $x + y \in I$. Also $x, y \in G$ and $0 \leqslant x \leqslant y \in I$ implies $0 \leqslant p_0 - p_x \leqslant p_0 - p_y \in J$, so $p_0 - p_x \in J$, thus $x \in I$. Thus I is an ideal of G.

We define Γ to be the linear span of 1 and all p_x ($x \in G$). In the terminology of Goodearl [7], Γ is a dense ultramatricial $*$-subalgebra of the AF-algebra $\mathscr{S}_1(G)$. Hence $(J \cap \Gamma)^- = J$ and $J \cap \Gamma$ is the linear span of its projections (see [7], p. 121, 16D, 16E). Since J is non-zero, there is a non-zero projection p in $J \cap \Gamma$. Hence there exists $x^1, \ldots, x^n \in G$ determining projections $p_1, \ldots, p_n$ in $\mathscr{S}_1(G)$ such that $p = \lambda_1 p_1 + \ldots + \lambda_n p_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$. Recalling a detail from the proof of Lemma 2.1 we may assume that $x^1 \leqslant \ldots \leqslant x^n$ (by re-indexing if necessary) and then we have $p = \nu_1(p_1 - p_2) + \ldots + \nu_{n-1}(p_{n-1} - p_n) + \nu_n p_n$ for some $\nu_i \in \mathbf{C}$. Of course the projections $p_1 - p_2, \ldots, p_{n-1} - p_n, p_n$ are pairwise orthogonal. Now if ν_n is non-zero then $p_n \in \mathscr{S}_0(G)$ (since $p, p_1 - p_2, \ldots, p_{n-1} - p_n \in \mathscr{S}_0(G)$) and this is easily seen to be impossible. Thus we have $\nu_n = 0$ and of course each $\nu_i = 0$ or 1. In short p is a sum of k pairwise orthogonal projections $p_{u^j} - p_{v^j}$ with $u^j \leqslant v^j$ in G. Thus $p \geqslant p_{u^j} - p_{v^j} \geqslant 0$ and $p \in J$ implies $p_{u^j} - p_{v^j} \in J$, so $\alpha(-u^j)(p_{u^j} - p_{v^j}) = p_0 - p_{v^j - u^j} \in J$, so $v^j - u^j \in I$. Since $p \neq 0$, $v^j \neq u^j$ for some j, we have $I \neq 0$. Hence $F(G) \subseteq I$, so if $x \in F(G)$ then $p_0 - p_x \in J$ as $x \in I$. More generally if $x, y \in G$ and $x - y \in F(G)$ then $p_0 - p_{x-y} \in J$ implies $\alpha_y(p_0 - p_{x-y}) = p_y - p_x \in J$. Hence $F\mathscr{S}(G) \subseteq J$. ▨

We now need some definitions and results from the theory of crossed products. We do not state the strongest possible forms of these results, just versions sufficient for our purposes.

Let (A, G, α) be a C^*-dynamical system with A a separable abelian C^*-algebra and G a countable discrete abelian group. A is *G-prime* if every two non-zero G-invariant closed ideals of A have non-zero intersection. A is *G-simple* if 0 and A are its only G-invariant closed ideals. The *Arveson spectrum* $\mathrm{sp}(\alpha)$ of (A, G, α) is the set of all $\rho \in G^\wedge$ for which there exists a sequence of unit vectors f_n in A such that $\|\alpha_x(f_n) - \rho(x)f_n\|$ converges to 0 as $n \to \infty$ $(x \in G)$. A useful fact: the annihilator $\mathrm{sp}(\alpha)^\perp = \{x \in G : \alpha_x = \mathrm{id}_A\}$. The *Connes spectrum* $\Gamma(\alpha)$ of (A, G, α) is the intersection of all $\mathrm{sp}(\alpha|J)$ where J runs over all non-zero G-invariant closed ideals of A, and $\mathrm{sp}(\alpha|J)$ is the Arveson spectrum of the C^*-dynamical system (J, G, α) got by restriction of α_x to J $(x \in G)$. $\Gamma(\alpha)$ is a closed subgroup of $G^\wedge$, so if $\Gamma(\alpha)^\perp = 0$ then $\Gamma(\alpha) = G^\wedge$.

The following two important results will be needed:

Result 1. If A is G-prime and $\Gamma(\alpha) = G^\wedge$ then $A \times_\alpha G$ is primitive.

Result 2. If A is G-simple and $\Gamma(\alpha) = G^\wedge$ then $A \times_\alpha G$ is simple.

Useful references for these results, and crossed product theory in general, are Pedersen [11] and [12].

LEMMA 2.6. *Let G be a finitely generated ordered group. Then $\mathcal{S}_1(G) \times_\alpha G$ is primitive and $F\mathcal{S}(G) \times_\alpha G$ is simple.*

Proof. If $G = 0$ then $\mathcal{S}_1(C) \times_\alpha G = \mathbf{C}$ and $F\mathcal{S}(G) \times_\alpha G = 0$, so there is nothing to prove. So we may suppose that G is non-zero, and hence $F(G)$ is non-zero. If J_1 and J_2 are non-zero G-invariant closed ideals of $\mathcal{S}_1(G)$, then by Lemma 2.5 $F\mathcal{S}(G) \subseteq J_1 \cap J_2$ and $F\mathcal{S}(G)$ is a non-zero G-invariant closed ideal in $\mathcal{S}_1(G)$. Thus $\mathcal{S}_1(G)$ is G-prime, and since $J_1 \subseteq J_2$ implies $\mathrm{sp}(\alpha|J_1) \subseteq \mathrm{sp}(\alpha|J_2)$ we have $\Gamma(\alpha) = \mathrm{sp}(\alpha|F\mathcal{S}(G))$. Hence $\Gamma(\alpha)^\perp = \mathrm{sp}(\alpha|F\mathcal{S}(G))^\perp = \{x \in G : \alpha_x = \mathrm{id}\}$. Let $x \in \Gamma(\alpha)^\perp$ and $y \in F(G)$, $y > 0$. Then $\alpha_x(p_0 - p_y) = p_x - p_{x+y} = p_0 - p_y$ (since $p_0 - p_y \in F\mathcal{S}(G)$). Now if $z < t$ in G and $[z, t) = \{u \in G : z \leqslant u < t\}$ then $p_z - p_t = \chi_{[z, t)}$. Thus $\chi_{[x, x+y)} = \chi_{[0, y)}$, so $x = 0$, and therefore $\Gamma(\alpha)^\perp = 0$, implying that $\Gamma(\alpha) = G^\wedge$. By Result 1 above, $\mathcal{S}_1(G) \times_\alpha G$ is primitive. Of course from what we have just shown, it is clear that $(F\mathcal{S}(G), G, \alpha)$ is G-simple (since any G-invariant closed ideal of $F\mathcal{S}(G)$ is one of $\mathcal{S}_1(G)$ also) and the Connes spectrum $\Gamma(\alpha) = G^\wedge$ for $(F\mathcal{S}(G), G, \alpha)$ also. Hence by Result 2 above $F\mathcal{S}(G) \times_\alpha G$ is simple. (Note that $F\mathcal{S}(G) \times_\alpha G$ is non-zero since it contains $F\mathcal{S}(G)$, and this is non-zero.) ▨

If G is an ordered group we let $\mathcal{F}\mathcal{T}(G)$ denote the closed ideal in $\mathcal{T}(G)$ generated by all $q_x = 1 - V_x V_x^*$ $(x \in F(G)^+)$. Clearly $\mathcal{F}\mathcal{T}(G) \subseteq K(\mathcal{T}(G))$, and $\mathcal{F}\mathcal{T}(G) = K(\mathcal{T}(G))$ if $F(G) = G$.

LEMMA 2.7. *If G is a finitely generated ordered group then $\mathcal{T}(G)$ is primitive and $\mathcal{F}\mathcal{T}(G)$ is simple.*

Proof. $\mathcal{A}(G)$ is a hereditary C^*-subalgebra of $\mathcal{S}_1(G)\times_\alpha G$, so $\mathcal{A}(G)$ is primitive as $\mathcal{S}_1(G)\times_\alpha G$ is. Let $J = p_0(F\mathcal{S}(G)\times_\alpha G)p_0$. Then J is a closed ideal in $\mathcal{A}(G) = p_0(\mathcal{S}_1(G)\times_\alpha G)p_0$ (since $F\mathcal{S}(G)\times_\alpha G$ is a closed ideal in $\mathcal{S}_1(G)\times_\alpha G$), and J is a hereditary C^*-subalgebra of the simple C^*-algebra $F\mathcal{S}(G)\times_\alpha G$, so J is simple.

Now let $W^*\colon \mathcal{T}(G) \to \mathcal{A}(G)$ be the canonical $*$-isomorphism. Since $\mathcal{A}(G)$ is primitive, so is $\mathcal{T}(G)$. Also $W^*(\mathcal{F}\mathcal{T}(G)) = J$, so $\mathcal{F}\mathcal{T}(G)$ is simple. (To see that $W^*(\mathcal{F}\mathcal{T}(G)) = J$, note $W^*(q_x) = W^*(1 - V_xV_x^*) = p_0 - W_xW_x^* = p_0 - \delta_x p_0 \delta_x^* = p_0 - p_x$. Thus $x \in F(G)^+$ implies $W^*(q_x) \in p_0F\mathcal{S}(G)p_0 \subseteq J$ which implies $W^*(\mathcal{F}\mathcal{T}(G)) \subseteq J$. If $G = 0$ then $F\mathcal{S}(G) = 0$, so $J = 0$, thus $W^*(\mathcal{F}\mathcal{T}(G)) = J$. If G is non-zero, then $F(G)$ has a positive element x, so $q_x \neq 0$, so $W^*(q_x) \neq 0$, so $W^*(\mathcal{F}\mathcal{T}(G)) = J$ by simplicity of J.) ▨

Recall that a C^*-algebra A is *prime* if every two non-zero closed ideals of A have non-zero intersection. Every primitive C^*-algebra is prime (we are about to use this fact in a moment) and the converse holds for separable C^*-algebras. (The non-separable case is an open question, see Pedersen [11].)

Let $\beta\colon G^+ \to B$ be a semigroup of isometries in the unital C^*-algebra B, over the partially ordered group G. We say that β is *nonunitary* if $\beta(x)$ is non-unitary for all $x > 0$, $x \in G$. The following lemma will be generalized immediately in Theorem 2.9 below.

LEMMA 2.8. *Let G be a finitely generated ordered group and $\beta\colon G^+ \to B$ a nonunitary semigroup of isometries in a unital C^*-algebra B. Then the unique $*$-homomorphism $\beta^*\colon \mathcal{T}(G) \to B$ such that $\beta^*V = \beta$ is injective.*

Proof. Let $J = \ker(\beta^*)$. If J is non-zero then G is non-zero ($G = 0$ implies $\mathcal{T}(G) = 0$, so $J = 0$), so $F(G)$ is non-zero, thus $\mathcal{F}\mathcal{T}(G)$ is non-zero. Hence $J \cap \mathcal{F}\mathcal{T}(G)$ is non-zero (as $\mathcal{T}(G)$ is primitive and therefore prime). As $\mathcal{F}\mathcal{T}(G)$ is simple, $J \cap \mathcal{F}\mathcal{T}(G) = \mathcal{F}\mathcal{T}(G)$, so $\mathcal{F}\mathcal{T}(G) \subseteq J$. Now there exists $x \in F(G)$, $x > 0$, so we have $q_x \in J$, thus $0 = \beta^*(q_x) = \beta^*(1 - V_xV_x^*) = 1 - \beta(x)\beta(x)^*$, which implies that $\beta(x)$ is unitary. Since β is nonunitary this is impossible, so J cannot be non-zero. Thus β^* is injective. ▨

THEOREM 2.9. *Let G be an ordered group and $\beta\colon G^+ \to B$ a nonunitary semigroup of isometries in a unital C^*-algebra B. Then $\beta^*\colon \mathcal{T}(G) \to B$ is injective.*

Proof. Let I be the set of finite non-empty subsets of G, ordered by set inclusion (i.e. $i \leqslant j$ iff $i \subseteq j$). Thus I is a directed set. For $i \in I$ let G_i be the subgroup of G generated by i, and let $\varphi^i\colon G_i \to G$ be the inclusion homomorphism. Likewise for $i \leqslant j$ in I let $\varphi_{ij}\colon G_i \to G_j$ be the inclusion homomorphism. Of course all the G_i

are ordered groups and the maps φ^i and φ_{ij} are positive. Since G is the union of all G_i ($i \in I$) it is easily checked that G is the direct limit (in the category of all partially ordered groups) of the direct system $(\varphi_{ij} : G_i \to G_j)_{i \leqslant j}$ with the maps φ^i as natural maps. By Theorem 1.6 $\mathscr{T}(G)$ is the direct limit (in the category of C^*-algebras) of the direct system $((\varphi_{ij})^* : \mathscr{T}(G_i) \to \mathscr{T}(G_j))_{i \leqslant j}$ with the maps $(\varphi^i)^* : \mathscr{T}(G_i) \to \mathscr{T}(G)$ as natural maps. Let $A_i = (\varphi^i)^* (\mathscr{T}(G_i))$ ($i \in I$). Then $\mathscr{T}(G) = (\bigcup\{A_i : i \in I\})^-$, since $\mathscr{T}(G)$ is the direct limit.

Let $\psi^i : G_i^+ \to G^+$ be the restriction of φ^i. Now $\beta\psi^i : G_i \to B$ is a semigroup of isometries over G_i and $\beta\psi^i$ is nonunitary, since if $\beta\psi^i(x)$ were a unitary then $\psi^i(x) = 0$ implies $x = 0$ ($\psi^i(x) = x$). Hence by Lemma 2.8, $(\beta\psi^i)^*$ is injective. Let $V^i : G_i^+ \to \mathscr{T}(G_i)$ and $V : G^+ \to \mathscr{T}(G)$ be the canonical maps. Then $\beta^*(\varphi^i)^* V^i = \beta^* V \psi^i = \beta\psi^i$, so $\beta^*(\varphi^i)^* = (\beta\psi^i)^*$. Thus β is an isometry on each A_i ($i \in I$), implying that β^* is an isometry on $\mathscr{T}(G) = (\bigcup\{A_i : i \in I\})^-$. ▨

THEOREM 2.10. *Let G be an ordered group. Then $\mathscr{T}(G)$ is prime.*

Proof. We retain the notation of the proof of Theorem 2.9. Let J be a non-zero closed ideal of $\mathscr{T}(G)$. Then $J \cap A_i$ is non-zero for some $i \in I$. (For otherwise letting π be the quotient map from $\mathscr{T}(G)$ to $\mathscr{T}(G)/J$, π is isometric on each C^*-algebra A_i, so π is isometric on $\mathscr{T}(G) = (\bigcup\{A_i : i \in I\})^-$, thus $J = \ker(\pi) = 0$.) Thus if J_1 and J_2 are non-zero closed ideals of $\mathscr{T}(G)$ then (since I is directed) $J_1 \cap A_i$ and $J_2 \cap A_i$ are non-zero closed ideals in some A_i. Now $(\varphi^i)^* : \mathscr{T}(G_i) \to \mathscr{T}(G)$ is injective since $(\varphi^i)^* = (V\psi^i)^*$ and $V\psi^i : G_i^+ \to \mathscr{T}(G)$ is a nonunitary semigroup of isometries (which implies $(V\psi^i)^*$ is injective by Theorem 2.9). Hence $A_i = (\varphi^i)^*(\mathscr{T}(G_i))$ is $*$-isomorphic to $\mathscr{T}(G_i)$, so A_i is primitive (by Lemma 2.7), and therefore prime. It follows that $(J_1 \cap A_i) \cap (J_2 \cap A_i)$ is non-zero, so $J_1 \cap J_2$ is non-zero. Thus $\mathscr{T}(G)$ is prime. ▨

We included Theorem 2.10 here since one can derive it so easily given one has set up the machinery to prove Theorem 2.9. Actually however we will show in the next section that $\mathscr{T}(G)$ is primitive (for G an ordered group) by exhibiting explicitly a faithful irreducible representation of $\mathscr{T}(G)$.

THEOREM 2.11. *If G is an ordered group then $\mathscr{F}\mathscr{T}(G)$ is simple.*

Proof. Let J be a non-zero closed ideal of $\mathscr{F}\mathscr{T}(G)$ and let π be the quotient map from $\mathscr{T}(G)$ to $\mathscr{T}(G)/J$. Let the map $\beta: G^+ \to \mathscr{T}(G)/J$ be defined by setting $\beta(x) = \pi(V_x)$ (i.e. $\beta = \pi V$). β is clearly a semigroup of isometries and $\beta^* = \pi$. Suppose β were nonunitary. Then $\pi = \beta^*$ is injective, so $J = 0$. Thus β is not nonunitary, and so there is an element $x \in G$, $x > 0$, such that $\beta(x)$ is a unitary. If $y \in F(G)^+$ then $y \leqslant nx$ for some $n \in \mathbb{N}$, so $nx = y + z$ for some $z \in G^+$. Hence $\beta(x)^n = \beta(y)\beta(z) = \beta(z)\beta(y)$, so $\beta(y)$ is invertible as $\beta(x)^n$ is. Thus $\pi(q_y) = \pi(1 - V_y V_y^*) = 1 - \beta(y)\beta(y)^* = 0$, so $\mathscr{F}\mathscr{T}(G) \subseteq \ker(\pi)$ thus $\mathscr{F}\mathscr{T}(G) = J$. ▨

Of course $\mathscr{F}\mathscr{T}(G)$ is non-zero iff $F(G)$ is non-zero.

COROLLARY 2.12. (Douglas, [4]). *If G is an ordered subgroup of* $\mathbf{R}$ *(usual order) then* $K(\mathscr{T}(G))$ *is simple.*

Proof. In this case $F(G) = G$. Hence $\mathscr{T}(G)/\mathscr{F}\mathscr{T}(G)$ is abelian (as $1 - V_x V_x^* \in \mathscr{F}\mathscr{T}(G)$ for all $x \in G^+$), so $\mathscr{F}\mathscr{T}(G) \supseteq K(\mathscr{T}(G))$ and we know already that $\mathscr{F}\mathscr{T}(G) \subseteq K(\mathscr{T}(G))$, so $\mathscr{F}\mathscr{T}(G) = K(\mathscr{T}(G))$. ▨

This result is attributed to Douglas because for G an ordered subgroup of $\mathbf{R}$, $V: G^+ \to \mathscr{T}(G)$ is a nonunitary one-parameter semigroup of isometries in his terminology, and the corollary follows from [4]. The techniques used by Douglas to prove this result are completely different from ours.

3. A GENERALIZED THEORY OF TOEPLITZ OPERATORS

We return in this section to partially ordered groups. We exhibit an irreducible representation of the Toeplitz algebra as a C^*-algebra of generalized "Toeplitz" operators (this representation is faithful for ordered groups). This involves our deriving a theory of such operators. The results and many of the proofs are closely analogous to the classical special case $G = \mathbf{Z}$, although there are some interesting differences. Perhaps the most remarkable fact here is that so much of the classical theory extends in such generality.

Let G be a partially ordered group, T the circle group, and recall that $\varepsilon(x) : G^\wedge \to T$ is the evaluation homomorphism $\varepsilon(x)(\gamma) = \gamma(x)$ $(x \in G, \gamma \in G^\wedge)$. As is well known $(\varepsilon(x))_{x \in G}$ forms an orthonormal basis for the Hilbert space $L^2 = L^2(G^\wedge)$, and letting P_G denote their linear span, it follows from the Stone-Weierstrass theorem that this $*$-subalgebra of $C(G^\wedge)$ is dense in $C(G^\wedge)$ in the sup-norm topology. The elements of P_G are called the *trigonometric polynomials* (relative to G). Denote by $H^2 = H^2(G)$ the closed subspace of L^2 having orthonormal basis $(\varepsilon(x))_{x \in G^+}$, and let $P \in B(L^2)$ be the projection onto H^2. If $\varphi \in L^\infty = L^\infty(G^\wedge)$ we define $T_\varphi \in B(H^2)$ by setting $T_\varphi(f) = P(\varphi f)$. T_φ is the *Toeplitz operator* with *symbol* φ (relative to G). The map $L^\infty \to B(H^2)$, $\varphi \mapsto T_\varphi$, is easily seen to be linear and norm-decreasing. Also $T_{\varphi^*} = T_{\varphi^-}$.

If $G = \mathbf{Z}$ (with the usual ordering) then of course H^2 is the usual Hardy space and we get the classical Toeplitz operators.

If G is a partially ordered group and F is a finite non-empty subset of G, then there exists $x \in G^+$ such that $x \geqslant y$ $(y \in F)$. (Proof: If $F = \{x^1, \ldots, x^n\}$ then each $x^i = y^i - z^i$ with $y^i, z^i \in G^+$. Take $x = y^1 + \ldots + y^n$.) This is used in the next easy but useful lemma. (Both this result and the next lemma will be often used tacitly.)

LEMMA 3.1. *If G is a partially ordered group and* $\varphi \in P_G$ *then* $\varepsilon(x)\varphi \in H^2(G)$ *for some x in* G^+.

Proof. $\varphi = \lambda_1\varepsilon(y^1) + \ldots + \lambda_n\varepsilon(y^n)$ for some $y^1, \ldots, y^n \in G$, and some $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$. Choose $x \in G^+$ such that $x \geqslant -y^1, \ldots, -y^n$. Then $\varepsilon(x)\varphi = \lambda_1\varepsilon(x + y^1) + \ldots + \lambda_n\varepsilon(x + y^n) \in H^2$. ▨

As in the proof above we shall often drop explicit reference to G when referring to the spaces $L^2(G^\wedge)$, $H^2(G)$ and $L^\infty(G^\wedge)$.

If E is a subset of $\mathbf{C}$ then hull(E) denotes its closed convex hull.

THEOREM 3.2. *Let G be a partially ordered group. If $\varphi \in L^\infty(G^\wedge)$ then $\|T_\varphi\| = \|\varphi\|_\infty$ and $\mathrm{Sp}(\varphi) \subseteq \mathrm{Sp}(T_\varphi) \subseteq \mathrm{hull}(\mathrm{Sp}(\varphi))$. ($\mathrm{Sp}(\varphi)$ is the essential range of φ.)*

Proof. Let $M_\varphi \in B(L^2)$ be the multiplication defined by $M_\varphi(f) = \varphi f$. Now the map $L^\infty \to B(L^2)$, $\varphi \mapsto M_\varphi$, is an isometric $*$-homomorphism, so $\mathrm{Sp}(M_\varphi) = \mathrm{Sp}(\varphi)$ Let $S = \{\overline{\varepsilon(x)}f : x \in G^+, f \in H^2\}$. Then $S^- = L^2$, since $(P_G)^- = L^2$ and $P_G \subseteq S$. Suppose that T_φ is bounded below, so for some $\mu > 0$, $\|T_\varphi\| \geqslant \mu\|f\|$ $(f \in H^2)$. Then $\|M_\varphi f\varepsilon(x)^-\| = \|\varphi f\| \geqslant \|P(\varphi f)\| = \|T_\varphi(f)\| \geqslant \mu\|f\| = \mu\|\overline{\varepsilon(x)}f\|$. Hence $\|M_\varphi(g)\| \geqslant \mu\|g\|$ $(g \in L^2)$, as $S^- = L^2$. Thus for any $\varphi \in L^\infty$, $\mathrm{Sp}(\varphi) = \mathrm{Sp}(M_\varphi) \subseteq \mathrm{Sp}(T_\varphi)$. Hence $\|T_\varphi\| \geqslant r(T_\varphi) \geqslant r(M_\varphi) = \|\varphi\|_\infty$, so $\|T_\varphi\| = \|\varphi\|_\infty$. But an isometric $*$-linear map $\rho : A \to B$ from an abelian C^*-algebra A to another C^*-algebra B has the property that $\mathrm{Sp}(\rho(a)) \subseteq \mathrm{hull}(\mathrm{Sp}(a))$ for all $a \in A$ (Douglas [5], p. 203). Hence $(A = L^\infty,\ B = B(H^2),\ \rho(\varphi) = T_\varphi)$ $\mathrm{Sp}(T_\varphi) \subseteq \mathrm{hull}(\mathrm{Sp}(\varphi))$ $(\varphi \in L^\infty)$. ▨

For G a partially ordered group let $H^\infty = H^\infty(G)$ be the set of all $\varphi \in L^\infty$ such that $\varphi \in H^2$. Then H^∞ is a closed subalgebra of the Banach algebra L^∞ (since for $\varphi \in L^\infty$ we have $\varphi \in H^\infty$ iff $\varphi H^2 \subseteq H^2$).

PROPOSITION 3.3. *Let G be a partially ordered group and $\varphi, \psi \in L^\infty(G^\wedge)$. If φ^- or $\psi \in H^\infty(G)$ then $T_{\varphi\psi} = T_\varphi T_\psi$.*

Proof. If $\psi \in H^\infty$ then $\psi H^2 \subseteq H^2$ implies $T_\varphi T_\psi(f) = T_\varphi P(\psi f) = T_\varphi(\psi f) = P(\varphi\psi f) = T_{\varphi\psi}(f)$ $(f \in H^2)$, so $T_{\varphi\psi} = T_\varphi T_\psi$. If on the other hand $\varphi^- \in H^\infty$ then $(T_{\varphi\psi})^* = T_{\psi^-\varphi^-} = T_{\psi^-}T_{\varphi^-}$ (by what we have just shown) $= T_{\psi^*}T_{\varphi^*}$, thus $T_{\varphi\psi} = T_\varphi T_\psi$. ▨

If G is a partially ordered group then we denote by $\mathscr{T}^r(G)$ the C^*-subalgebra of $B(H^2)$ generated by all T_φ $(\varphi \in C(G^\wedge))$. We call $\mathscr{T}^r(G)$ the *reduced Toeplitz algebra* of G. For $x \in G^+$, let U_x be the isometry $T_{\varepsilon(x)}$ and Q_x be the projection $1 - U_xU_x^*$, and let U denote the map $G^+ \to \mathscr{T}^r(G)$, $x \mapsto U_x$. U is a semigroup of isometries and $x \leqslant y$ in G^+ is equivalent to $Q_x \leqslant Q_y$ in $K(\mathscr{T}^r(G))$ (as we saw already in the proof of Proposition 1.4). These projections Q_x commute. Finally since $(P_G)^- = C(G^\wedge)$ it is easily checked (using Lemma 3.1) that U_x $(x \in G^+)$ generate $\mathscr{T}^r(G)$.

LEMMA 3.4. *Let G be a partially ordered group and let J be the linear span in $\mathscr{T}^r(G)$ of all $T_{\varphi_1}T_{\varphi_2}\ldots T_{\varphi_n} - T_{\varphi_1\cdots\varphi_n}$ $(\varphi_1, \ldots, \varphi_n \in P_G)$. Then $S \in J$ implies that $S = SQ_x$ for some $x \in G^+$.*

Proof. If S_1 and S_2 are in J and $S_j = S_jQ_{x_j}$; then $(S_1 + \lambda S_2)Q_{x_1+x_2} = S_1Q_{x_1}Q_{x_1+x_2} + \lambda S_2Q_{x_2}Q_{x_1+x_2} = S_1Q_{x_1} + \lambda S_2Q_{x_2} = S_1 + \lambda S_2$ $(\lambda \in \mathbf{C})$. This calculation shows that is suffices to prove the theorem for S of the form $S = T_{\varphi_n}T_{\varphi_{n-1}}\dots T_{\varphi_1} - T_{\varphi_n\dots\varphi_1}$ $(\varphi_1,\dots,\varphi_n \in P_G)$. However since P_G is the linear span of all $\varepsilon(x)$ $(x \in G)$ it follows that we may, again without loss of generality, assume each $\varphi_i = \varepsilon(y^i)$ for some y^i in G. In this case choose $x \in G^+$ such that $x \geqslant$ all the elements $-y^1$, $-(y^1 + y^2), \dots, -(y^1 + \dots + y^n)$. Then

$$SU_x = T_{\varepsilon(y^n)}T_{\varepsilon(y^{n-1})}\dots T_{\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} =$$
$$= T_{\varepsilon(y^n)}T_{\varepsilon(y^{n-1})}\dots T_{\varepsilon(y^2)\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} = \dots =$$
$$= T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} = 0$$

(true by Proposition 3.3 and since $\varepsilon(y^i)\dots\varepsilon(y^1)\,\varepsilon(x) \in H^\infty$, $i = 1,\dots,n$). Thus $SU_x = 0$, so $SU_xU_x^* = S(1 - Q_x) = 0$, implying $S = SQ_x$. ▨

THEOREM 3.5. *If G is a partially ordered group then*

1. $(Q_x)_{x\in G^+}$ *is an approximate unit for* $K(\mathscr{T}^r(G))$.
2. *If* $\varphi \in L^\infty(G^\wedge)$, *then* $T_\varphi \in K(\mathscr{T}^r(G))$ *if and only if* $\varphi = 0$.

Proof. 1. $(G^+, \leqslant)$ is a directed set, so $(Q_x)_{x\in G^+}$ is a net. Let J be defined as in Lemma 3.4. If $\varphi, \varphi_1, \dots, \varphi_n \in P_G$ then $T_\varphi(T_{\varphi_1}T_{\varphi_2}\dots T_{\varphi_n} - T_{\varphi_1\dots\varphi_n}) = T_\varphi T_{\varphi_1}\dots T_{\varphi_n} - T_{\varphi\varphi_1\dots\varphi_n} + T_{\varphi\varphi_1\dots\varphi_n} - T_\varphi T_{\varphi_1\dots\varphi_n}$ is in J. Hence J^- is a closed ideal in $\mathscr{T}^r(G)$. By Lemma 3.4, $S \in J$ implies $S = SQ_x$ for some $x \in G^+$, so we have $\lim_y TQ_y = T$ $(T \in J^-)$. Thus $(Q_x)_{x\in G^+}$ is an approximate unit for J^-. Since all $Q_x \in K(\mathscr{T}^r(G))$, $J^- \subseteq K(\mathscr{T}^r(G))$, and since all $Q_x \in J^-$, $\mathscr{T}^r(G)/J^-$ is abelian, implying that $J^- \supseteq K(\mathscr{T}^r(G))$. Thus $J^- = K(\mathscr{T}^r(G))$ and $(Q_x)_{x\in G^+}$ is an approximate unit for $K(\mathscr{T}^r(G))$.

2. Let $\varphi \in L^\infty$ and $T_\varphi \in K(\mathscr{T}^r(G))$. Then $T_\varphi = \lim_x T_\varphi Q_x$, so $0 = \lim_x T_\varphi U_xU_x^*$, thus $0 = \lim_x \|T_\varphi U_x\| = \lim_x \|T_{\varphi\varepsilon(x)}\| = \lim_x \|\varphi\varepsilon(x)\|_\infty = \|\varphi\|_\infty$, so $0 = \varphi$. ▨

Part 2 of the above theorem generalizes the classical result that 0 is the only compact Toeplitz operator (relative to $G = \mathbf{Z}$). $K(\mathscr{T}^r(\mathbf{Z}))$ is $K(H^2(\mathbf{Z}))$, the ideal of all compact operators on $H^2(\mathbf{Z})$.

COROLLARY 3.6. *If* $\varphi, \psi \in C(G^\wedge)$ *then* $T_\varphi T_\psi - T_{\varphi\psi} \in K(\mathscr{T}^r(G))$.

Proof. Since $(P_G)^- = C(G^\wedge)$ it suffices to show the result for $\varphi, \psi \in P_G$. But this case is obvious from the proof of Theorem 3.5. ▨

THEOREM 3.7. *Let G be a partially ordered group.*

1) *The map*

$$C(G^\wedge) \to \mathscr{T}^r(G)/K(\mathscr{T}^r(G)) \quad \varphi \mapsto T_\varphi + K(\mathscr{T}^r(G))$$

is a $*$-isomorphism.

2). *If* $S \in \mathcal{T}^{r}(G)$ *then there exists unique* $\varphi \in C(G^{\wedge})$ *and unique* $K \in K(\mathcal{T}^{r}(G))$ *such that* $S = T_{\varphi} + K$.

Proof. Let ρ denote the map in 1). Then ρ is clearly $*$-linear and by Corollary 3.6 ρ is multiplicative. ρ is injective by Theorem 3.5 and surjective since T_{φ} ($\varphi \in C(G^{\wedge})$) generate $\mathcal{T}^{r}(G)$. This proves 1), and 2) follows immediately from 1). ▨

LEMMA 3.8. *Let* H_0 *be a dense linear submanifold of a Hilbert space* H, *and* $(S_{\lambda})_{\lambda \in \Lambda}$ *a net in* $B(H)$. *Suppose that* $\lim_{\lambda} (S_{\lambda} f, g)$ *exists for all* $f, g \in H_0$ *and that there is a positive number* μ *such that* $|(S_{\lambda} f, g)| \leqslant \mu \|f\| \|g\|$ $(\lambda \in \Lambda,\ f,\ g \in H)$. *Then there exists* $S \in B(H)$ *such that* $S = \lim_{\lambda} S_{\lambda}$ *in the weak operator topology on* $B(H)$.

For a proof, see Halmos [8].

LEMMA 3.9. *Let* G *be a discrete abelian group and suppose that the matrix* $(a_{x,y})_{x,y \in G}$ *of* $S \in B(L^2(G^{\wedge}))$ *with respect to the orthonormal basis* $(\varepsilon(x))_{x \in G}$ *is a Laurent matrix* (i.e. $a_{x+z,y+z} = a_{x,y}$ $(x, y, z \in G)$). *Then* S *is a multiplication,* $S = M_{\varphi}$ *for some* $\varphi \in L^{\infty}(G^{\wedge})$.

(Explicitly: $a_{x,y} = (S(\varepsilon(y)), \varepsilon(x))$.)
For a proof, see Murphy [10].

THEOREM 3.10. *Let* G *be a partially ordered group, and let* $S \in B(H^2(G))$. *Then* S *is a Toeplitz operator (relative to* G*) if and only if* $U_x^* S U_x = S$ $(x \in G^+)$.

Proof. If $S = T_{\varphi}$ for some $\varphi \in L^{\infty}$ then $U_x^* S U_x = T_{\overline{\varepsilon(x)}} T_{\varphi} T_{\varepsilon(x)} = T_{\overline{\varepsilon(x)}\varphi\varepsilon(x)} = T_{\varphi} = S$ (as $\varepsilon(x) \in H^{\infty}$). Conversely suppose that $U_x^* S U_x = S$ $(x \in G^+)$. Define $S_x \in B(L^2)$ by setting $S_x(f) = \overline{\varepsilon(x)} S P \varepsilon(x) f$, for $x \in G^+$, and note that $\|S_x\| \leqslant \|S\|$. Also for $f, g \in H^2$, $(S_x f, g) = (\overline{\varepsilon(x)} S P \varepsilon(x) f, g) = (U_x^* S U_x f, g) = (Sf, g)$.

Now let $\varphi_1, \varphi_2 \in P_G$ and put $\mu_x = (S_x \varphi_1, \varphi_2)$. We show that the net $(\mu_x)_{x \in G^+}$ converges by showing that there exists $x_0 \in G^+$ such that $\mu_x = \mu_{x_0}$ for $x \geqslant x_0$. Certainly there exists $x_0 \in G^+$ such that $\varphi_1, \varphi_2 \in \overline{\varepsilon(x_0)} H^2$. Let $\psi_j = \varepsilon(x_0)\varphi_j$, so $\psi_j \in H^{\infty}$. Now if $x \geqslant x_0$ then $\mu_x = (S_x \overline{\varepsilon(x_0)} \psi_1,\ \overline{\varepsilon(x_0)} \psi_2) = (S \varepsilon(x - x_0) \psi_1,\ \varepsilon(x - x_0) \psi_2) = (S_{x-x_0} \psi_1,\ \psi_2) = (S \psi_1,\ \psi_2)$ (as $\psi_1, \psi_2 \in H^2$) $= \mu_{x_0}$. Since $(P_G)^- = L^2$ it follows from Lemma 3.8 that there exists $T \in B(L^2)$ such that $T = \lim_x S_x$ in the weak operator topology. Let $(a_{x,y})_{x,y \in G}$ be the matrix of T relative to the basis $(\varepsilon(x))_{x \in G}$ of L^2. If $y, z \in G$ and $x \in G^+$ then $a_{y+x,z+x} = (T\varepsilon(z)\varepsilon(x), \varepsilon(y)\varepsilon(x)) = \lim_t (S_t \varepsilon(x)\varepsilon(z), \varepsilon(x)\varepsilon(y)) = \lim_t (S_{t+x}\varepsilon(z), \varepsilon(y)) = \lim_t (S_t \varepsilon(z), \varepsilon(y))$ (since $\lim_t \alpha_{t+x} = \lim_t \alpha_t$). Thus $a_{y+x,z+x} = a_{y,z}$, and one can now immediately extend this equation to arbitrary $x \in G$ since $G = G^+ - G^+$. Hence by Lemma 3.9, $T = M_{\varphi}$ for some $\varphi \in L^{\infty}$.

Clearly, for $f, g \in H^2$, $(T_\varphi f, g) = (\varphi f, g) = (Tf, g) = \lim_x (S_x f, g) = (Sf, g)$ (as $(S_x f, g) = (Sf, g)$). Thus $S = T_\varphi$. ▨

The next proposition is important — it shows that $H^\infty(G)$ displays "analytic behaviour".

PROPOSITION 3.11. *Let G be a partially ordered group. If φ and $\bar{\varphi} \in H^\infty(G)$ then $\varphi \in \mathbf{C}1$.* ▨

Proof. If $x \in G^+$ and $\overline{\varepsilon(x)} \in H^2$ then $-x \in G^+$, so $x = 0$. Now $\varphi, \bar{\varphi} \in H^\infty$ and $x \in G^+$, $x > 0$ implies $0 = (\bar{\varphi}, \overline{\varepsilon(x)}) = \int \bar{\varphi}\varepsilon(x)$ (as $\overline{\varepsilon(x)} \in (H^2)^\perp$). Hence $\int \overline{\varphi \varepsilon(x)} = 0$, i.e. $(\varphi, \varepsilon(x)) = 0$. But $(\varphi, \varepsilon(x)) = 0$ for $x \in G \setminus G^+$ also, since $\varphi \in H^\infty$. Thus $\varphi \in \mathbf{C}\varepsilon(0) = \mathbf{C}1$. ▨

If G is a partially ordered group and $\varphi \in H^\infty(G)$ we say that T_φ is an *analytic* Toeplitz operator (relative to G). Of course T_φ is subnormal (it is the restriction of M_φ). All analytic Toeplitz operators commute. The map $H^\infty \to B(H^2)$, $\varphi \mapsto T_\varphi$, is an isometric algebra isomorphism onto the closed subalgebra of all analytic Toeplitz operators.

THEOREM 3.12. *Let G be a partially ordered group.*

1) *If $S \in B(H^2(G))$ then S is an analytic Toeplitz operator (relative to G) if and only if $U_x S = S U_x$ ($x \in G^+$).*

2) *The analytic Toeplitz operators relative to G form a maximal commutative subalgebra of $B(H^2(G))$.*

3) *If $\varphi \in H^\infty(G)$ then $\mathrm{Sp}(T_\varphi) = \mathrm{Sp}_{H^\infty}(\varphi)$.*

4) *Every analytic Toeplitz operator has connected spectrum.*

Proof. 1) If S is an analytic Toeplitz operator then $SU_x = U_x S$ since the U_x are analytic Toeplitz operators. Conversely if $SU_x = U_x S$ ($x \in G^+$) then $U_x^* S U_x = S$, so $S = T_\varphi$ for some $\varphi \in L^\infty$ by Theorem 3.10. Now for $x, y \in G^+$, $(\varphi, \varepsilon(x - y)) = (\varphi\varepsilon(y), \varepsilon(x)) = (T_\varphi U_y \varepsilon(0), \varepsilon(x)) = (U_y T_\varphi \varepsilon(0), \varepsilon(x)) = (T_\varphi \varepsilon(0), \varepsilon(x - y))$. Thus if $x - y \notin G^+$, then $(\varphi, \varepsilon(x - y)) = 0$. So $\varphi \in H^\infty$. This proves 1), and 2) follows immediately from this.

3) Let A be the maximal commutative subalgebra of $B(H^2)$ of all analytic Toeplitz operators. Then $\mathrm{Sp}_A(T_\varphi) = \mathrm{Sp}(T_\varphi)$ for $\varphi \in H^\infty$. But $\mathrm{Sp}_A(T_\varphi) = \mathrm{Sp}_{H^\infty}(\varphi)$ since the map $H^\infty \to A$, $\varphi \mapsto T_\varphi$, is an isomorphism. This proves 3).

4) Let X be the character space of H^∞. If $\varphi \in H^\infty$ is an idempotent then $\varphi = \varphi^2$, thus $\varphi = \bar{\varphi} \in H^\infty$, so $\varphi \in \mathbf{C}1$ by Proposition 3.11. Thus $\varphi = 0$ or 1. Since H^∞ thus has no non-trivial idempotents it follows from the Shilov Idempotent Theorem that X is connected. Now if $\varphi \in H^\infty$ and $\varphi^\wedge$ denotes its Gelfand transform then $\mathrm{Sp}_{H^\infty}(\varphi) = \varphi^\wedge(X)$ is connected, i.e. $\mathrm{Sp}(T_\varphi)$ is connected. ▨

THEOREM 3.13. *If G is a partially ordered group then $\mathcal{T}^r(G)$ and $K(\mathcal{T}^r(G))$ are irreducible algebras on $H^2(G)$. Moreover if $G \neq 0$ then $\lim_x Q_x = 1$ in the strong operator topology on $B(H^2(G))$.*

Proof. $\mathcal{T}^r(G)$ is irreducible iff its commutant $B = \mathbf{C}1$ iff 0,1 are the only projections in B (since B is a von Neumann algebra). Now for $Q \in B$, $QU_x = U_xQ$ $(x \in G^+)$, so Q is an analytic Toeplitz operator, thus, $\mathrm{Sp}(Q)$ is connected. Thus if Q is a projection, then $\mathrm{Sp}(Q) = \{0\}$ or $\{1\}$, so $Q = 0$ or 1. Hence $B = \mathbf{C}1$ and $\mathcal{T}^r(G)$ is irreducible on H^2.

If $G = 0$ then $\dim(H^2) = 1$, so $K(\mathcal{T}^r(G))$ is irreducible on H^2. So we suppose that G is non-zero. Let $M = (K(\mathcal{T}^r(G))H^2)^-$. If $M = 0$ then $K(\mathcal{T}^r(G)) = 0$, so $Q_x = 0$ $(x \in G^+)$, thus $G^+ = 0$ which implies $G = 0$. Thus $M \neq 0$. Since M reduces $\mathcal{T}^r(G)$, $M = H^2$. If $f \in K(\mathcal{T}^r(G))H^2$ then $f = T_1f_1 + \ldots + T_nf_n$ for some $T_1, \ldots, T_n \in K(\mathcal{T}^r(G))$ and some $f_1, \ldots, f_n$ in H^2. Thus $\lim_x Q_xf = \lim_x Q_xT_1f_1 + \ldots + \lim_x Q_xT_nf_n = T_1f_1 + \ldots + T_nf_n = f$, because $T_j = \lim_x Q_xT_j$ $(j = 1, \ldots, n)$. Hence $f = \lim_x Q_xf$ $(f \in H^2)$ as $K(\mathcal{T}^r(G))H^2$ is dense in H^2, so $\lim_x Q_x = 1$ in the strong operator topology on $B(H^2)$.

Now suppose that N is an invariant closed subspace of H^2 for $K(\mathcal{T}^r(G))$, and $f \in N$, $T \in \mathcal{T}^r(G)$. Then $Tf = \lim_x TQ_xf$ is in N, since $TQ_xf \in N$ $(x \in G^+)$. Thus N is an invariant subspace for $\mathcal{T}^r(G)$, so $N = 0$ or H^2. We have thus shown $K(\mathcal{T}^r(G))$ is irreducible on H^2. ▨

Recall that if G is a partially ordered group then the map $U: G^+ \to \mathcal{T}^r(G)$ is a semigroup of isometries, so it induces a unique $*$-homomorphism $U^*: \mathcal{T}(G) \to \mathcal{T}^r(G)$. Since U_x $(x \in G^+)$ generate $\mathcal{T}^r(G)$, U^* is onto. We can thus regard U^* as an irreducible representation of $\mathcal{T}(G)$ on $H^2(G)$. This representation is not always faithful as the next example shows:

EXAMPLE. $M = N \setminus \{1\}$ is a cone on $\mathbf{Z}$. Thus $(\mathbf{Z}, M)$ is a partially ordered group. Note that 2 and 3 are not comparable for the partial order $\leqslant_M$. Let $G_1 = (\mathbf{Z}, M)$ and $G_2 = (\mathbf{Z}, \mathbf{N})$. The identity map $\varphi : G_1 \to G_2$ is a positive homomorphism, so it induces a $*$-homomorphism $\varphi^* : \mathcal{T}(G_1) \to \mathcal{T}(G_2)$ and this is surjective since φ is surjective. Hence the restriction $\varphi^* : K(\mathcal{T}(G_1)) \to K(\mathcal{T}(G_2))$ is surjective, and thus non-zero. If $K(\mathcal{T}(G_1))$ were simple then this restriction map φ^* would be a $*$-isomorphism, so $2 \leqslant_{\mathbf{N}} 3$ implies $q_2 \leqslant q_3$ in $K(\mathcal{T}(G_2))$, so $\varphi^*(q_2) \leqslant \varphi^*(q_3)$, implying $q_2 \leqslant q_3$ in $K(\mathcal{T}(G_1))$, so $2 \leqslant_M 3$, which is false. Thus $K(\mathcal{T}(G_1))$ is not simple. However it is easily seen that all the Q_x^1 are of finite rank $(x \in M)$, so $K(\mathcal{T}^r(G_1)) \subseteq K(H^2(G_1))$ (as $(Q_x^1)_{x \in M}$ are an approximate unit for $K(\mathcal{T}^r(G_1))$, and since $K(\mathcal{T}(G_1))$ is irreducible on $H^2(G_1)$ we therefore deduce that $K(\mathcal{T}^r(G_1)) = K(H^2(G_1))$. In particular $K(\mathcal{T}^r(G_1))$ is simple. Since $K(\mathcal{T}(G_1))$ is not simple, the map $U^* : \mathcal{T}(G_1) \to T^r(G_1)$ is not injective.

THEOREM 3.14. *If G is an ordered group then the canonical map $U^* : \mathscr{T}(G) \to \mathscr{T}^r(G)$ is a faithful irreducible representation of $\mathscr{T}(G)$ on $H^2(G)$.*

Proof. The map $U : G^+ \to \mathscr{T}^r(G)$ is a nonunitary semigroup of isometries, so by Theorem 2.9 U^* is injective. ▨

4. CONVERSES OF SOME EARLIER RESULTS

The idea of this section is to show that a number of the stronger results we proved earlier are in fact "best possible". For example if G is a torsion-free partially ordered group for which $K(\mathscr{T}(G))$ is simple we show G is isomorphic to an ordered subgroup of $\mathbf{R}$ (cf. Corollary 2.12). One can interpret Theorem 2.9 as saying there is essentially only one candidate for the title "Toeplitz algebra" if G is an ordered group. More specifically it implies that if B is any C^*-algebra generated by a nonunitary semigroup of isometries β over G then $\beta^* : \mathscr{T}(G) \to B$ is a $*$-isomorphism. We show that this result characterizes the ordered groups amongst the torsion-free partially ordered groups.

If G is an abelian group we call a cone M in G *maximal* if M is not contained in any other cone of G. A simple application of Zorn's Lemma implies that every cone of G is contained in a maximal cone of G. The following elementary result is probably known, but we include a proof for the sake of completeness.

LEMMA 4.1. *If G is a torsion-free abelian group and M a cone of G then M is a maximal cone of G if and only if $(G, \leqslant_M)$ is an ordered group (*i.e. *$\leqslant_M$ is a total ordering).*

Proof. It is trivial that if $(G, \leqslant_M)$ is totally ordered, then M is maximal (this does not require G to be torsion-free). Suppose conversely that M is maximal. First, let $x \in G\setminus\{0\}$ such that $nx \in M$ for some positive integer n. We show that $x \in M$: Define $N = \{y + mx : y \in M,\ m \in \mathbf{N}\}$. Clearly $0 \in N$, $N + N \subseteq N$ and $G = N - N$. Suppose that $z \in N \cap (-N)$, so that $z = y_1 + m_1 x = -y_2 - m_2 x$ for some $y_1, y_2 \in M$, and some $m_1, m_2 \in \mathbf{N}$. Thus $0 \leqslant_M n(y_1 + y_2) = -(m_1 + m_2)nx \leqslant_M 0$, so $n(y_1 + y_2) = 0 = -(m_1 + m_2)nx$ implying that $y_1 + y_2 = 0 = (m_1 + m_2)x$ (since G is torsion-free), thus $y_1 = y_2 = 0$ (since $y_1, y_2 \geqslant_M 0$), and $m_1 + m_2 = 0$ implies $m_1, m_2 = 0$. Hence $z = 0$, implying that $N \cap (-N) = 0$. Thus N is a cone, and $N \supseteq M$ implies $N = M$, so $x \in M$.

Now suppose only that $x \in G\setminus(-M)$. Again let $N = \{y + mx : y \in M,\ m \in \mathbf{N}\}$ and again we have $0 \in N$, $N + N \subseteq N$ and $G = N - N$. If $z \in N \cap (-N)$ then $z = y_1 + m_1 x = -y_2 - m_2 x$ for some $y_1, y_2 \in M$, and some $m_1, m_2 \in \mathbf{N}$, so $y_1 + y_2 = -(m_1 + m_2)x$, thus $n(-x) \in M$ where $n = m_1 + m_2$. If $n > 0$ then by the earlier part of this proof, $-x \in M$, so $x \in -M$, which is false. Thus $n = 0$ implies $m_1 = m_2 = 0$, so $y_1 + y_2 = 0$, which implies $y_1, y_2 = 0$ (since $0 \leqslant_M y_1, y_2$), so $z = 0$. We therefore have $N \cap (-N) = 0$, thus N is a cone, and since $M \subseteq N$, $M = N$. Thus $x \in M$. We have shown that $G = M \cup (-M)$, i.e. $(G, \leqslant_M)$ is totally ordered. ▨

Of course the hypothesis that G is torsion-free is necessary in Lemma 4.1, since ordered groups are torsion-free.

THEOREM 4.2. *Let $(G, \leqslant)$ be a torsion-free partially ordered group such that for every unital C^*-algebra B and every nonunitary semigroup of isometries $\beta : G^+ \to B$, β^* is injective. Then $(G, \leqslant)$ is an ordered group.*

Proof. G^+ is contained in a maximal cone M, so if φ denotes id_G then φ is a positive homomorphism from $G_1 = (G, G^+)$ to $G_2 = (G, M)$, and so induces a $*$-homomorphism φ^* from $\mathscr{T}(G_1)$ to $\mathscr{T}(G_2)$. Since φ is surjective, so is φ^*. The semigroup of isometries $\beta : G_1^+ \to T(G_2)$, $x \to \varphi^*(V_x)$, is nonunitary (for if $\beta(x)$ is unitary then $\varphi^*(V_x) = V_{\varphi(x)}$ is unitary, so $\varphi(x) = 0$, i.e. $x = 0$). Hence $\beta^* = \varphi^*$ is injective. Thus if $x, y \in G^+$ then we may suppose $x \leqslant_M y$ (since (G, M) is totally ordered). Hence $q_x \leqslant q_y$ in $K(\mathscr{T}(G_2))$, so $\varphi^*(q_x) \leqslant \varphi^*(q_y)$, implying that $q_x \leqslant q_y$ in $K(\mathscr{T}(G_1))$, so $x \leqslant y$. More generally if $x, y \in G$ then there exists $z \in G^+$ such that $x, y \leqslant z$ imply $z - y$, $z - x \in G^+$, so $z - y$, $z - x$ are comparable in $(G, \leqslant)$, thus x, y are comparable in $(G, \leqslant)$. Thus $(G, \leqslant)$ is totally ordered. ▨

THEOREM 4.3. *If G is a torsion-free partially ordered group for which $K(\mathscr{T}(G))$ is simple then G is order isomorphic to an ordered subgroup of* **R**.

Proof. (Two partially ordered groups G_1, G_2 are *order isomorphic* if there exists a bijective map $\psi : G_1 \to G_2$ such that ψ and ψ^{-1} are positive homomorphisms.)

First we show that G is simple: Let I be an ideal of G, and let $i : I \to G$ and $\varphi : G \to G/I$ be the inclusion and quotient homomorphisms respectively. Since φ is surjective, so is $\varphi^* : \mathscr{T}(G) \to \mathscr{T}(G/I)$, and hence also the restriction map $\varphi^* : K(\mathscr{T}(G)) \to K(\mathscr{T}(G/I))$. As $K(\mathscr{T}(G))$ is simple this restriction map φ^* is zero or injective. In the first case we have $K(\mathscr{T}(G/I)) = 0$, so $G/I = 0$, so $G = I$. In the second case for $x \in I^+$, $\varphi^*(q_x) = q_{\varphi(x)} = q_0 = 0$, so $q_x = 0$, so $x = 0$. Thus, $I^+ = 0$ implying that $I = 0$. This shows G is simple.

Now we show G is totally ordered: Using the same trick as in the proof of Theorem 4.2, there is a maximal cone M containing G^+. Let $\psi = \mathrm{id}_G$, $G_1 = (G, G^+)$, and $G_2 = (G, M)$. Thus ψ is a positive homomorphism from G_1 to G_2. As ψ is surjective, so is $\psi^* : \mathscr{T}(G_1) \to \mathscr{T}(G_2)$, implying that the restriction map $\psi^* : K(\mathscr{T}(G_1)) \to K(\mathscr{T}(G_2))$ is zero or injective (again we are using the simplicity of $K(\mathscr{T}(G_1))$). In the first case $K(\mathscr{T}(G_2)) = 0$ which implies $G = 0$, so G is order isomorphic to the ordered subgroup 0 of **R**. In the second case if $x, y \in G^+$ we may suppose that $x \leqslant_M y$, so $q_x \leqslant q_y$ in $K(\mathscr{T}(G_2))$, i.e. $\varphi^*(q_x) \leqslant \varphi^*(q_y)$, so $q_x \leqslant q_y$, in $K(\mathscr{T}(G_1))$, thus $x \leqslant y$. This implies that $(G, \leqslant)$ is totally ordered.

Thus G is a simple ordered group, so $G = F(G) = \{x \in G :$ for all $y > 0$ $|x| \leqslant ny$ for some $n \in \mathbf{N}\}$. (For if $x \in G$ and $x > 0$ then $I_x = \{z \in G : |z| \leqslant nx$ for some $n \in \mathbf{N}\}$ is a non-zero ideal of G, so $I_x = G$.) Thus G is an archimedean group in the terminology of Rudin [14], and by a well known result G is order isomorphic to an ordered subgroup of $\mathbf{R}$ ([14], p. 194). ▨

One can thus summarize Theorem 4.3 and Douglas' result (Corollary 2.12) as follows: For G a torsion-free partially ordered group $K(\mathscr{T}(G))$ is simple iff G is a simple ordered group iff G is order isomorphic to an ordered subgroup of $\mathbf{R}$.

Recall that a C^*-algebra A is *elementary* if there is a Hilbert space H such that A is $*$-isomorphic to $K(H)$.

THEOREM 4.4. *Let G be a torsion-free partially ordered group. Then $K(\mathscr{T}(G))$ is elementary if and only if G is order isomorphic to 0 or $\mathbf{Z}$.*

Proof. If $G = 0$ then $\mathscr{T}(G) = \mathbf{C}$, so $K(\mathscr{T}(G)) = 0 = K(H)$ for $H = 0$. If $G = \mathbf{Z}$ then by Theorem 3.14, $K(\mathscr{T}(G))$ is $*$-isomorphic to $K(\mathscr{T}^{\mathrm{r}}(G))$. But $K(\mathscr{T}^{\mathrm{r}}(G)) = K(H^2(G))$ since all Q_x $(x \in G^+)$ are finite rank, and $K(\mathscr{T}^{\mathrm{r}}(G))$ is irreducible on $H^2(G)$.

Conversely, suppose that $K(\mathscr{T}(G))$ is elementary and that $\beta : K(\mathscr{T}(G)) \to K(H)$ is a $*$-isomorphism for some Hilbert space H. Then in particular $K(\mathscr{T}(G))$ is simple, so we may assume that G is an ordered subgroup of $\mathbf{R}$, and without loss of generality suppose also $G \neq 0$. Since all $\beta(q_x)$ $(x \in G^+)$ have finite rank in $K(H)$ we may choose $x \in G$, $x > 0$, such that the rank of $\beta(q_x)$ is minimal. Then x is a smallest positive element of G. Hence $G = \mathbf{Z}x$, so G is order isomorphic to $\mathbf{Z}$. ▨

We finish with a few remarks and questions. The author showed in [9] that for G an ordered subgroup of $\mathbf{Q}$, we have $K(\mathscr{T}(G))$ is an AF-algebra and the group $K_0(K(\mathscr{T}(G)))$ is isomorphic to the group G. Is the converse true, i.e. if $K(\mathscr{T}(G))$ is an AF-algebra is G isomorphic to an ordered subgroup of $\mathbf{Q}$? These commutator ideals are not AF-algebras in general, nor is $K_0(K(\mathscr{T}(G))) = G$ always true. The author hopes to deal with the K-theory of these algebras more fully in a forthcoming paper.

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Added in proofs. The author has recently learnt that R. Ji and J. Xia have calculated the K-theory of the algebras $\mathscr{T}(G)$ and $K(\mathscr{T}(G))$ for G an ordered group **R** ("On the classification of commutator ideals", to appear in *J. Funct. Anal.*).