J. OPERATOR THEORY
19(1988), 381–386

SPACES IN WHICH FREDHOLM OPERATORS HAVE WELL DEFINED TRACE

JACOB JACOBSEN and HENRIK STETKAER

INTRODUCTION

In his studies on topological tensor products Grothendieck introduced the condition on a locally convex space that every Fredholm operator on it possesses a well defined trace [3; § 5]. For Banach spaces this is equivalent to the so called approximation property. Grothendieck verified that the condition holds for some important Banach spaces [3; § 5], but counterexamples, the first one due to Enflo [2], show that it does not hold for all. So a natural problem is to find criteria ensuring its validity on important classes of locally convex space.

Furthermore, the condition is not just of interest in its own right, but in other connections as well. It plays a role in the theory of representations of groups and algebras on topological vector spaces. To mention an example it enters in a crucial way in Litvinov's and Lomonosov's joint papers [6], [7] (see Corollary 1 of [7]), in particular in their proof of ultra-irreducibility of the Schrödinger representations on $C(\mathbf{R}^n)$, $C^\infty(\mathbf{R}^n)$ and $C_0^\infty(\mathbf{R}^n)$, where they need that these locally convex spaces satisfy the condition. They prove it by noting that each of them has a Schauder basis, which in turn implies the condition [5, Corollary].

The main result of the present note (Theorem 1) is another criterion for a locally convex space, consisting of functions or distributions on a homogeneous space, to satisfy the condition. Our theorem implies that e.g. the following often occuring spaces of distributions satisfy the condition:

$$L_p(\mathbf{R}^n),\quad L_p^{\mathrm{loc}}(\mathbf{R}^n),\quad L_p^{\mathrm{c}}(\mathbf{R}^n)\quad \text{for } 1 \leqslant p < \infty$$

$$C^r(\mathbf{R}^n),\quad C_0^r(\mathbf{R}^n)\quad \text{for } 0 \leqslant r \leqslant \infty$$

$$\mathscr{D}'(\mathbf{R}^n),\quad \mathscr{S}'(\mathbf{R}^n),\quad \mathscr{E}'(\mathbf{R}^n)$$

$$H_{(s)},\quad H_{(s)}^{\mathrm{loc}},\quad H_{(s)}^{\mathrm{c}}\quad \text{for } -\infty < s < \infty.$$

ON THE INTEGRATED REPRESENTATION

Let M be a C^∞-manifold on which a Lie group G acts smoothly, and let τ denote the corresponding action on functions on M. Let dm be a volume element on M, i.e. a smooth everywhere strictly positive 1-density on M (cf. [9; VII 2.5]). Any two volume elements differ by a strictly positive smooth factor. By help of the given volume element dm we imbed functions on M into the distributions on M, i.e. into the strong dual $\mathscr{D}'(M)$ of $\mathscr{D}(M)$. Extending the action τ from functions to distributions we get a continuous representation of G on $\mathscr{D}'(M)$.

In the proof of our main theorem below we shall need the corresponding integrated representation — again denoted by τ — of the group algebra $\mathscr{D}(G)$ on $\mathscr{D}'(M)$, given by

$$\langle\tau(\varphi)u, \psi\rangle := \int_G \varphi(g)\langle\tau(g)u, \psi\rangle \mathrm{d}g \quad \text{for } \varphi \in \mathscr{D}(G),\ u \in \mathscr{D}'(M),\ \psi \in \mathscr{D}(M),$$

where dg denotes a fixed left Haar measure on G. We shall in particular need the fact that $\tau(\varphi)$ for any $\varphi \in \mathscr{D}(G)$ is a continuous linear map of $\mathscr{D}'(M)$ into $C^\infty(M)$, provided G acts transitively on M.

THE MAIN RESULT

DEFINITION (cf. [8; Definition 28.1]). A *normal space of distributions on M* is a subspace E of $\mathscr{D}'(M)$, equipped with its own locally convex topology, such that E contains $\mathscr{D}(M)$ as a dense subspace and such that the inclusions $\mathscr{D}(M) \hookrightarrow E \hookrightarrow \mathscr{D}'(M)$ are continuous.

We remind the reader that a locally convex space E is said to be *semi-complete* if every Cauchy sequence in E converges in E.

We can now state and prove our main result, that extends Proposition 43 of [3].

THEOREM 1. *Let G be a Lie group with a countable basis for the topology acting transitively as a Lie transformation group on a C^∞-manifold M. Let* dm *be a volume element on M by means of which we identify functions with distributions.*

Let E be a τ-invariant semi-complete normal space of distributions on M with the property that the restriction of τ to E is a strongly continuous representation of G on E.

Assume furthermore that there exists a sequence $\{\psi_n\} \subseteq \mathscr{D}(M)$ such tha $\psi_n u \xrightarrow[n\to\infty]{} u$ for each $u \in E \cap C^\infty(M)$ in $(E, \sigma(E, E'))$.

Then every Fredholm operator on E possesses a well defined trace.

A Fredholm operator on a semi-complete locally convex space E is a linear map of the form $x \mapsto \sum_{j=1}^{\infty} \lambda_j \langle x'_j, x \rangle x_j$, where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbf{N})$ and where $\{x_j\}_{j=1}^{\infty}$ and $\{x'_j\}_{j=1}^{\infty}$ are bounded sequences in E and E' respectively. In general more is required of the sequences $\{x_j\}$ and $\{x'_j\}$ (see p. 80 of [3]), but that is irrelevant for our treatment. Also note that the different notions of boundedness in E' coincide since E is semi-complete [1; III, § 4, no. 3, Corollaire]. That the Fredholm operators have well defined trace means that

$$(1) \qquad \sum_{j=}^{\infty} \lambda_j \langle x'_j, x \rangle x_j = 0 \quad \text{for all } x \in E$$

implies

$$(2) \qquad \sum_{j=1}^{\infty} \lambda_j \langle x'_j, x_j \rangle = 0.$$

Proof of the theorem. We assume (1) is the case and shall derive (2).

The series (1) converges in E and hence in $\mathscr{D}'(M)$, E being a space of distributions. $\tau(\varphi)$ is for any $\varphi \in \mathscr{D}(M)$ a continuous linear map of $\mathscr{D}'(M)$ into $C^\infty(M)$ as remarked above, so

$$\sum_{j=1}^{\infty} \lambda_j \langle x'_j, x \rangle \tau(\varphi) x_j = 0$$

with convergence in $C^\infty(M)$, and hence in particular

$$(3) \qquad \sum_{j=1}^{\infty} \lambda_j [\tau(\varphi)x_j](m) \langle x'_j, i\psi \rangle = 0 \quad \text{for all } m \in M \text{ and } \psi \in \mathscr{D}(M),$$

where $i : \mathscr{D}(M) \hookrightarrow E$ denotes the inclusion map. Since the sequence $\{x'_j\}_{j=1}^{\infty}$ is bounded in E', the sequence $\{i^t(x'_j)\}_{j=1}^{\infty}$ is bounded in $\mathscr{D}'(M)$, so the series $\sum_{j=1}^{\infty} \lambda_j [\tau(\varphi)x_j](m) i^t(x'_j)$ converges in $\mathscr{D}'(M)$. By (3) its sum is 0. Applying $\tau(\varphi')$ to it, where $\varphi' \in \mathscr{D}(G)$, we get

$$\sum_j \lambda_j [\tau(\varphi)x_j](m)[\tau(\varphi')i^t(x'_j)](m') = 0 \quad \text{for all } m, m' \in M.$$

Putting $m' = m$, multiplying the result by ψ_n and integrating over M we find

$$(4) \qquad \sum_j \lambda_j \langle \tau(\varphi')i^t(x'_j), \psi_n \tau(\varphi) x_j \rangle_{\mathscr{D}' \times \mathscr{D}} = 0.$$

Here we let φ' range over a δ-sequence $\{\varphi_k'\}_{k=1}^{\infty} \subseteq \mathscr{D}(G)$. Then $\{\tau(\varphi_k')\}_{k=1}^{\infty}$ is a sequence of continuous linear operators on $\mathscr{D}'(M)$ which converges pointwise to the identity operator and which hence is pointwise bounded. By the Banach-Steinhaus theorem — $\mathscr{D}'(M)$ is barrelled — $\{\tau(\varphi_k') \mid k \in \mathbf{N}\}$ is an equicontinuous set of operators in $\mathscr{D}'(M)$ so that the set $\{\tau(\varphi_k')i^t(x_j') \mid k, j \in \mathbf{N}\}$ is bounded in $\mathscr{D}'(M)$. From the dominated convergence theorem for sequences we get from (4) that

$$\sum_j \lambda_j \langle x_j', \psi_n \tau(\varphi) x_j \rangle_{E' \times E} = 0. \tag{5}$$

Consider now for fixed $e' \in E'$ the linear functional

$$e_n' : x \mapsto \langle e', \psi_n \tau(\varphi) x \rangle_{E' \times E} \quad \text{on } E.$$

Since $\psi_n \tau(\varphi) \in \mathscr{L}(\mathscr{D}'(M), \mathscr{D}(M)) \subseteq \mathscr{L}(E)$, we see that $e_n' \in E'$. The sequence $\{e_n'\}_{n=1}^{\infty}$ is pointwise bounded by the assumption on $\{\psi_n\}$, and then even strongly bounded by the semi-completeness [1; III, § 4, no. 3, Corollaire]. So for any bounded set $B \subseteq E$, the set $\bigcup_n \psi_n \tau(\varphi) B$ is bounded in $(E, \sigma(E, E'))$ and hence also bounded in E. In particular, $\{\psi_n \tau(\varphi) x_j \mid n, j \in \mathbf{N}\}$ is bounded in E, so by the dominated convergence theorem for sequences we get for $n \to \infty$ from (5) that

$$\sum_j \lambda_j \langle x_j', \tau(\varphi) x_j \rangle_{E' \times E} = 0,$$

i.e. that

$$\sum_j \lambda_j \int_G \varphi(g) \langle x_j', \tau(g) x_j \rangle_{E' \times E} \, dg = 0.$$

The series $g \mapsto \sum_{j=1}^{\infty} \lambda_j \langle x_j', \tau(g) x_j \rangle_{E' \times E}$ converges in $C(G)$, since τ is strongly continuous and E semi-complete, so the summation and integration signs may be interchanged which implies that

$$\int_G \varphi(g) \sum_{j=1}^{\infty} \lambda_j \langle x_j', \tau(g) x_j \rangle_{E' \times E} \, dg = 0.$$

This holds for all $\varphi \in \mathscr{D}(G)$, so

$$\sum_{j=1}^{\infty} \lambda_j \langle x_j', \tau(g) x_j \rangle_{E' \times E} = 0 \quad \text{for all } g \in G.$$

The desired relation (2) is the special case of this in which g equals the identity element of G. ▨

EXAMPLES

Let $G = \mathbf{R}^n$ act on $M = \mathbf{R}^n$ by left translations:

$$g \cdot m = m - g \quad \text{for } g,\ m \in \mathbf{R}^n.$$

Choose $\psi \in C_0^\infty(\mathbf{R}^n, [0,1])$ such that

$$\psi(t) = \begin{cases} 1 & \text{for } |t| \leqslant 1 \\ 0 & \text{for } |t| > 2, \end{cases}$$

and put

$$\psi_k(t) := \psi(t/k) \quad \text{for } t \in \mathbf{R}^n,\ k \in \mathbf{N}.$$

Then the assumptions of the theorem are fulfilled for the spaces E below:

$$E = C_0^r(\mathbf{R}^n) \quad \text{for } 0 \leqslant r \leqslant \infty$$

$$E = \mathscr{D}'(\mathbf{R}^n), \quad \mathscr{S}'(\mathbf{R}^n), \quad \mathscr{E}'(\mathbf{R}^n) \quad \text{(strong duals)}$$

$$E = \mathscr{B}_{p,k}(\mathbf{R}^n), \quad \mathscr{B}_{p,k}^{\mathrm{loc}}(\mathbf{R}^n), \quad \mathscr{B}_{p,k}^{\mathrm{c}}(\mathbf{R}^n),$$

$$L_p(\mathbf{R}^n, k\,\mathrm{d}x),\ L_p^{\mathrm{loc}}(\mathbf{R}^n, k\,\mathrm{d}x),\ L_p^{\mathrm{c}}(\mathbf{R}^n, k\,\mathrm{d}x)$$

where $1 \leqslant p < \infty$ and where k is a temperate weight function. See Section 10.1 of [4], especially Theorems 10.1.7 and 10.1.16. In particular the Sobolev spaces

$$E = H_{(s)}(\mathbf{R}^n), \quad H_{(s)}^{\mathrm{loc}}(\mathbf{R}^n), \quad H_{(s)}^{\mathrm{c}}(\mathbf{R}^n) \quad \text{for } -\infty < s < \infty$$

fulfil the assumptions.

REFERENCES

1. BOURBAKI, N., *Espaces vectoriels topologiques*, Chap. 1–5, Masson, 1981.
2. ENFLO, P., A counterexample to the approximation problem in Banach spaces, *Acta Math.*, **130** (1973), 309–317.
3. GROTHENDIECK, A., *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., **16** (1955).

4. HÖRMANDER, L., *The analysis of linear partial differential operators. II*, Springer-Verlag, 1983.
5. LITVINOV, G. L., Traces of linear operators in locally convex spaces, *Functional Anal. Appl.*, **13** (1979), 60–62.
6. LITVINOV, G. L.; LOMONOSOV, V. I., Density theorems in locally convex spaces and their applications (Russian), *Trudy Sem. Vektor. Tenzor. Anal.*, **20** (1981), 210–227.
7. LITVINOV, G. L.; LOMONOSOV, V. I., Density theorems in locally convex spaces and irreducible representations, *Soviet Math. Dokl.*, **23** (1981), 372–376.
8. TREVES, F., *Topological vector spaces, distributions and kernels*, Academic Press, New York and London, 1967.
9. TREVES, F., *Introduction to pseudodifferential and Fourier integral operators*, *Vol.* 2, Plenum Press, New York and London, 1980.

JACOB JACOBSEN and HENRIK STETKAER
Matematisk Institut,
Aarhus Universitet,
Ny Munkegade, Bygning 530,
DK-8000 Aarhus C.,
Denmark.

Received November 16, 1987.