J. OPERATOR THEORY
22(1989), 233–241

COMPACT COMPOSITION OPERATORS ON SOME WEIGHTED HARDY SPACES

NINA ZORBOSKA

1. INTRODUCTION

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ when $n \to \infty$. The set $H^2(\beta)$ of complex formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ is a Hilbert space of functions analytic in the unit disc $\mathbf{D}$ with the inner product

$$(f, g)_\beta = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2 ;$$

here $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. A detailed study of spaces $H^2(\beta)$ is given in [14].

The function φ in $H^2(\beta)$ that maps the unit disc into itself induces a composition operator C_φ on $H^2(\beta)$ defined by $C_\varphi f = f \circ \varphi$. The study of composition operators began with the work of Nordgren [9] and Schwartz [10] on classical Hardy space H^2 (space $H^2(\beta)$ with $\beta_n = 1$ for all n).

One of the questions that was open for many years was characterizing the compact composition operators on the space H^2. There were many partial results (see for example [10], [13], [2] and [3]) and finally the problem was solved in 1987 by J. Shapiro. He estimated the essential norm of composition operators on spaces $H^2(\beta)$ with $\beta_n = (n+1)^a$ and $a \leqslant 1/2$, using versions of Nevanlinna counting function (see [12]). When $a > 1/2$, the situation is more elementary: the function φ in such $H^2(\beta)$ induces a compact composition operator if and only if the supnorm of φ is strictly smaller than 1 which is, furthermore, equivalent to C_φ being in every Schatten p-class, $p > 0$, of $H^2(\beta)$ (see [16]).

The spectrum of compact composition operators on H^2 was described by Caughran and Schwartz in [2] which also contains the fact that if C_φ is compact then φ has a fixed point in $\mathbf{D}$.

In this article we consider compact composition operators on some general classes of spaces $H^2(\beta)$. We investigate spectra of such C_φ and fixed points of the inducing functions.

These results are a part of the author's Ph. D. thesis written at the University of Toronto under the supervision of Professor Peter Rosenthal.

2. THE DENJOY-WOLFF POINT OF A FUNCTION THAT INDUCES A COMPACT COMPOSITION OPERATOR

If φ is a function analytic in the unit disc $\mathbf{D}$ and mapping $\mathbf{D}$ into itself, then there is a special "fixed point" for φ in $\bar{\mathbf{D}}$. The theorem that shows the exact properties of that point, which we are going to call the Denjoy-Wolff theorem, actually contains the results obtained by Denjoy [4] and Wolff [15] and a part of the Julia-Carathéodory theorem [8, p. 57].

THEOREM. (Denjoy-Wolff). *Suppose φ is not a disc automorphism, φ is analytic in* $\mathbf{D}$ *and maps* $\mathbf{D}$ *into* $\mathbf{D}$. *Let* $\varphi^{(0)}(z) = z$ *and* $\varphi^{n+1}(z) = \varphi^n(\varphi(z))$ *for* $n \geqslant 1$. *Then there exists a point α in* $\bar{\mathbf{D}}$ *such that the sequence of iterates of φ,* $\{\varphi^{(n)}\}_{n=0}^{\infty}$, *converges to the constant function α uniformly on compact subsets of* $\mathbf{D}$. *Moreover,* $|\alpha| = 1$, *then* $\lim\limits_{r\to 1}\varphi(r\alpha) = \alpha$ *and* $\lim\limits_{r\to 1^-}\varphi'(r\alpha)$, *exists and is in* $(0, 1]$.

We say that α is the *Denjoy-Wolff point* of φ.

We already mentioned that if C_φ is compact on H^2, then the Denjoy-Wolff point of φ must be in $\mathbf{D}$. But this is not true in all spaces $H^2(\beta)$. Shapiro gave an example in [11] of a function that induces a compact composition operator on some "small" spaces $H^2(\beta)$ and has the point 1 as its Denjoy-Wolff point. The interesting fact is that in these spaces disc automorphisms that are not rotations induce unbounded composition operators. This connection continues to be important in some other $H^2(\beta)$ spaces, (i.e., in the case when the sequence β is bounded), as we shall see below. But first we would like to state an interesting property of compact composition operators on the small $H^2(\beta)$ spaces mentioned above.

The spaces that we are going to be working with will be the $H^2(\beta)$ spaces with sequences β such that $\sum\limits_{n=0}^{\infty}\frac{1}{\beta_n^2} < \infty$. In this case functions in $H^2(\beta)$ have absolutely convergent Taylor series on $\partial\mathbf{D}$, and there exists a constant c such that if $f \in H^2(\beta)$, then $\|f\|_\infty \leqslant c \cdot \|f\|_\beta$. This follows easily from the Cauchy-Schwarz inequality for the series involved in the β norm of f.

There is one more reason why we will consider sequences β such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$. Let β be any sequence, $\omega \in \mathbf{D}$ and

$$k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \bar{\omega}^n z^n.$$

Then $k_\omega^\beta \in H^2(\beta)$ and k_ω^β are point evaluations for $H^2(\beta)$, i.e. for $f \in H^2(\beta)$, we have

$$(f, k_\omega^\beta)_\beta = f(\omega).$$

If $\omega \in \partial\mathbf{D}$, then $k_\omega^\beta \in H^2(\beta)$ if and only if $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$.

Now we are ready to proceed with the following proposition. The idea of the proof is taken from the proof of the Theorem 2.1 in [11].

PROPOSITION 1. *Let the sequence β be such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$, and let the function φ in $H^2(\beta)$ induce a compact composition operator on $H^2(\beta)$. Then φ has exactly one fixed point in $\bar{\mathbf{D}}$.*

Proof. The functions in $H^2(\beta)$ are continuous on $\bar{\mathbf{D}}$, and therefore can be evaluated on the boundary without any excuses. Let α be the Denjoy-Wolff point of φ and

$$H_\alpha^2(\beta) = \{f : f \in H^2(\beta) \text{ and } f(\alpha) = 0\}.$$

If a sequence $\{f_n\}_{n=0}^{\infty}$ of functions in $H_\alpha^2(\beta)$ converges to f in the β-norm, then $f \in H_\alpha^2(\beta)$, because $(f_n, k_\alpha^\beta)_\beta$ conveges to $(f, k_\alpha^\beta)_\beta$ when $\alpha \in \mathbf{D}$ and also when $\alpha \in \partial\mathbf{D}$. So, $H_\alpha^2(\beta)$ is a closed subspace of $H^2(\beta)$. If $f \in H_\alpha^2(\beta)$ then, because α is a fixed point of φ, we have

$$f(\varphi(\alpha)) = f(\alpha) = 0;$$

i.e., $H_\alpha^2(\beta)$ is an invariant subspace for C_φ. Let T be the restriction of C_φ to $H_\alpha^2(\beta)$. Then T is a compact operator too, and the spectral radius $r(T)$ is equal to the maximum of $\{|\lambda| : \lambda \in \Pi_0(T)\}$ where $\Pi_0(T)$ is the point spectrum of T. Let the function $f \neq 0$ from $H_\alpha^2(\beta)$ be such that $Tf = \lambda f$ for some $\lambda \in \mathbf{C}$. Then there exists a point $z_0 \in \mathbf{D}$ such that $f(z_0) \neq 0$. Also

$$\lambda^n f(z_0) = (T^n f)(z_0) = f(\varphi^{(n)}(z_0)).$$

As $n \to \infty$, $\varphi^{(n)}(z_0) \to \alpha$; i. e.,

$$\lambda^n f(z_0) \to f(\alpha) = 0.$$

So $|\lambda| < 1$ and $r(T) < 1$.

On the other hand

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

and we have $\|T^n\| \to 0$ as $n \to \infty$. Let $e_1(z) = z$ and $e_0(z) = 1$. Then $e_1 - \alpha e_0 \in H^2_\alpha(\beta)$ and so

$$\|T^n(e_1 - \alpha e_0)\|_\beta \to 0 \quad \text{when } n \to \infty.$$

But

$$\|T^n(e_1 - \alpha e_0)\|_\beta = \|\varphi^{(n)} - \alpha e_0\|_\beta$$

and, because there exists a constant c such that

$$\|\varphi^{(n)} - \alpha e_0\|_\infty \leqslant c\|\varphi^{(n)} - \alpha e_0\|_\beta,$$

we have

$$|\varphi^{(n)}(z) - \alpha| \to 0 \quad \text{when } n \to \infty, \text{ for all } z \in \mathbf{D}.$$

Now if z_1 is another fixed point of φ in $\mathbf{D}$, we have

$$|\varphi^{(n)}(z_1) - \alpha| = |z_1 - \alpha| = 0;$$

i. e., $z_1 = \alpha$. ∎

We say that the space $H^2(\beta)$ is *disc-automorphism invariant* if disc automorphisms induce bounded composition operators on $H^2(\beta)$.

The following result was obtained in [11] in a more general context and by a slightly different approach.

COROLLARY 1. *Let the space $H^2(\beta)$ be disc-automorphism invariant and* $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$. *If the operator C_φ is compact on $H^2(\beta)$, then φ has a fixed point in* $\mathbf{D}$, *and the supremum of φ is strictly smaller than* 1.

Proof. Suppose that φ has a Denjoy-Wolff point α in $\partial\mathbf{D}$. Composing φ with a disc-automorphism ψ such that $\psi(\alpha) = \alpha$ and $\psi(\varphi(0)) = 0$, we get a new function $\varphi_1 = \psi \cdot \varphi$ with two fixed points in $\overline{\mathbf{D}}$. But C_{φ_1} is a compact composition operator on $H^2(\beta)$ and that is a contradiction to Proposition 1.

The supremum norm of φ must be strictly smaller than 1, because otherwise composing φ with suitable rotations we shall get a function with a fixed point on $\partial\mathbf{D}$. ▨

What is happening with the Denjoy-Wolff point of φ if C_φ is compact on the "big spaces" $H^2(\beta)$ such that $H^2 \subset H^2(\beta)$? Note, that by Proposition in [16], this is the case when the sequence β is bounded. Then we can use the following result ([3], Corollary 4.4): if the Denjoy-Wolff point α of φ is in $\partial\mathbf{D}$ and $\varphi'(\alpha) < 1$, then every eigenvalue of C_φ on H^2 has infinite multiplicity. But then every eigenvalue of C_φ on $H^2(\beta)$ is also going to have infinite multiplicity, and so C_φ cannot be compact on $H^2(\beta)$ since 1 is always an eigenvalue for C_φ. We still do not know what happens if α is in $\partial\mathbf{D}$ and $\varphi'(\alpha) = 1$. We can avoid this problem if we impose more restrictions on the space $H^2(\beta)$. We have the following:

PROPOSITION 2. *Suppose that the space $H^2(\beta)$ is disc-automorphism invariant and $H^2 \subset H^2(\beta)$. If the operator C_φ is compact on $H^2(\beta)$ then the Denjoy-Wolff point α of φ is in $\mathbf{D}$.*

Proof. By the previous discussion we already know that if C_φ is compact and $\alpha \in \partial\mathbf{D}$, then $\varphi'(\alpha) \geqslant 1$; i.e., by the Denjoy-Wolff theorem, $\varphi'(\alpha) = 1$. Let ψ be a disc-automorphism such that $\psi(\alpha) = \alpha$ and $\psi'(\alpha) < 1$. Then $(\psi \circ \varphi)(\alpha) = \alpha$ and

$$(\psi\circ\varphi)'(\alpha) = \psi'(\varphi(\alpha)) \cdot \varphi'(\alpha) = \psi'(\alpha) \cdot \varphi'(\alpha) < 1$$

and so α is the Denjoy-Wolff point of $\psi\circ\varphi$. But the operator $C_{\psi\circ\varphi}$ is compact on $H^2(\beta)$ because C_φ is compact and C_ψ is bounded. This contradicts the previous discussion since $(\psi\circ\varphi)'(\alpha) < 1$. So, we must have $\alpha \in \mathbf{D}$. ▨

Besides the cases $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$ and β bounded (i.e., $H^2 \subset H^2(\beta)$) there are still a lot of cases of spaces $H^2(\beta)$ for which we do not know if the compactness of C_φ on $H^2(\beta)$ implies that the Denjoy-Wolff point of φ is in $\mathbf{D}$, even if we suppose that $H^2(\beta)$ is disc-automorphism invariant. It would be helpful to know the answer because then, as we shall see below, we would be able to describe the spectrum of such compact composition operators.

3. SPECTRUM OF COMPACT COMPOSITION OPERATORS

If C_φ is a compact composition operator on H^2 and α is its Denjoy-Wolff point, Caughran and Schwartz have proved in [2] that α is in $\mathbf{D}$ and the spectrum of C_φ is the set $\{0,1\} \subseteq \{(\varphi'(\alpha))^k : k = 1, 2, \ldots\}$.

PROPOSITION 3. *Suppose that the function φ induces a compact composition operator on the space $H^2(\beta)$, and the Denjoy-Wolff point α of φ is in* $\mathbf{D}$. *Then $\sigma_{H^2(\beta)}(C_\varphi)$ (the spectrum of C_φ on $H^2(\beta)$) is the set $\{0,1\} \cup \{(\varphi'(\alpha))^n : n = 1, 2, \ldots\}$.*

Proof. Let

$$k_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \bar{\alpha}^n z^n$$

be the point evaluation at α, and for $k \geq 1$, let $k_\alpha^{(k)} = \frac{d^k}{d\bar{\alpha}^k} k_\alpha$. Then

$$k_\alpha^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{\beta_n^2} \bar{\alpha}^{n-k} z^n$$

is a function in $H^2(\beta)$ and for f in $H^2(\beta)$ we have that $(f, k_\alpha) = f^{(k)}(\alpha)$. Thus

$$(f,\ C_\varphi^* k_\alpha^{(k)}) = (C_\varphi f, k_\alpha^{(k)}) = (f \circ \varphi)^{(k)}(\alpha) =$$

$$= f^{(k)}(\alpha)(\varphi'(\alpha))^k + \ldots + f'(\alpha)\varphi^{(k)}(\alpha) =$$

$$= (f, \overline{(\varphi'(\alpha))}^k k_\alpha^{(k)}) + \ldots + (f, \overline{\varphi^{(k)}(\alpha)} k_\alpha^{(1)}).$$

The functions $k_\alpha, k_\alpha^{(1)} \ldots, k_\alpha^{(k)}$ are a basis for a finite dimensional space invariant for C_φ^* and with respect to this basis the restriction of C_φ^* has an upper triangular matrix representation with diagonal $\overline{(\varphi'(\alpha))}^s$, $0 \leq s \leq k$. Because the operator C_φ^* is compact, we have that the points $(\varphi'(\alpha))^s$, $s \geq 0$ are eigenvalues for the operator C_φ.

From the other side, if f in $H^2(\beta)$ is such that there exists λ in $\mathbf{C}$ with $f \cdot \varphi = \lambda f$, then $f(\varphi(\alpha)) = f(\alpha) = \lambda f(\alpha)$ and either $\lambda = 1$ and $f = \text{const}$, or $f(\alpha) = 0$. If α is a zero of f of order s, taking the $(s+1)$ derivative on the both sides of the equation $f \cdot \varphi = \lambda f$ and evaluating at α, we get that

$$f^{(s+1)}(\alpha)(\varphi'(\alpha))^{s+1} = \lambda f^{(s+1)}(\alpha),$$

i.e., that $\lambda = (\varphi'(\alpha))^{s+1}$. Thus

$$\sigma_{H^2(\beta)}(C^\varphi) = \{0,\ 1\} \cup \{(\varphi'(\alpha))^k : k = 1,\ 2, \ldots\}.$$ ▨

There is a special interesting case when we can also describe the spectrum of some compact composition operators.

For example, let the space $H^2(\beta)$ be given by $\beta_n = \exp(n^a)$ for some a in $[1/2, 1)$, and $\mu < 1/2$ be such that the function $\varphi_\mu(z) = 1 - \mu + \mu z$ induces a compact composition operator on $H^2(\beta)$ spaces are spaces of quasi-analytic functions on $\overline{\mathbf{D}}$ (see [1]). We will show that $\sigma_{H^2(\beta)}(C_{\varphi_\mu})$ (the spectrum of C_{φ_μ} on $H^2(\beta)$) is $\{0,1\} \cup \cup \{(\varphi_\mu'(1))^n : n = 1, 2, \ldots\}$, while by Corollary 4.8 in [3], the spectrum of C_{φ_μ} on H^2 is $\{z : |z| \leqslant \mu^{-1/2}\}$, and every point of its interior is an eigenvalue of infinite multiplicity ([3], p. 97). First we shall prove that $\{0,1\} \cup \{(\varphi_\mu'(1))^n ; n = 1, 2, \ldots\} \subset \subset \sigma_{H^2(\beta)}(C_{\varphi_\mu})$. Let $f_n(z) = (1 - z)^n$, $n = 0, 1, 2, \ldots$. Then

$$(C_{\varphi_\mu} f_n)(z) = f_n(\varphi_\mu(z)) = f_n(1 - \mu + \mu z) = (\mu - \mu z)^n = \mu^n(1 - z)^n = (\mu^n f_n)(z).$$

So, for $n = 0, 1, 2, \ldots$ the functions f_n are eigenvectors of C_{φ_μ} corresponding to the eigenvalues μ^n. But $\varphi_\mu'(1) = \mu$ and so we got the needed inclusion. Note that up to now we did not use any of the restrictions imposed on the space $H^2(\beta)$ and that the functions f_n belong to every $H^2(\beta)$. So the set $\{0,1\} \cup \{(\varphi_\mu'(1))^n ; n = 1, 2, \ldots\}$ is a subset of the spectrum of C_{φ_μ} on all $H^2(\beta)$ spaces on which C_{φ_μ} is bounded.

Next, we will show that there are no other eigenvalues of C_{φ_μ} on the given $H^2(\beta)$ space except the ones given above. For suppose that $f \in H^2(\beta)$ and $f \circ \varphi_\mu = = \lambda f$ for some $\lambda \in \mathbf{C}$, $\lambda \neq 1$. The functions in $H^2(\beta)$ are continuous on $\overline{\mathbf{D}}$ and we can evaluate them on the boundary. So $(f \circ \varphi_\mu)(1) = f(1)$; i.e., $f(1) = \lambda f(1)$, and, since $\lambda \neq 1$, we get $f(1) = 0$.

The functions in $H^2(\beta)$ are such that every derivative has an absolutely convergent power series in $\overline{\mathbf{D}}$, every function that is nonzero can have only finitely many zeroes in $\mathbf{D}$ and every zero in $\overline{\mathbf{D}}$ has finite order (see [1] and [14], p. 103). Suppose that 1 is a zero of the function f of order k; i.e., $f^{[s]}(1) = 0$ for $s \leqslant k$ (where $f^{[s]}$ denotes the derivative of order s of f) and $f^{[k+1]}(1) \neq 0$. Then, by taking the $(k + 1)$ derivative on the both sides of the equation $f \circ \varphi_\mu = \lambda f$ and by evaluating at the point $z = 1$, we get

$$f^{[k+1]}(1) \cdot (\varphi_\mu'(1))^{k+1} = \lambda \cdot f^{[k+1]}(1);$$

i.e., $\lambda = (\varphi_\mu'(1))^{k+1}$.

Because the operator C_{φ_μ} is compact on $H^2(\beta)$, every nonzero point in its spectrum has to be an eigenvalue. So, finally, we get that

$$\sigma_{H^2(\beta)}(C_{\varphi_\mu}) = \{0,1\} \cup \{(\varphi_\mu'(1))^n ; n = 1, 2, \ldots\}.$$

In some parts of the proof of the last statement, we actually proved some more general results; we give them as remarks.

REMARK 1. If the function φ is not "of the type φ_μ" but still induces a compact composition operator on one of the quasi-analytic $H^2(\beta)$ spaces discussed above and the Denjoy-Wolff point of φ is 1, then

$$\sigma_{H^2(\beta)}(C_\varphi) \subset \{0,1\} \cup \{(\varphi'(1))^n \; ; \; n = 1, 2, \ldots\}.$$

REMARK 2. If the functions in $H^2(\beta)$ are continuous on $\bar{\mathbf{D}}$, and φ is not a disc automorphism, then $\Pi_{H^2(\beta)}(C_\varphi)$ (the point spectrum of C_φ on $H^2(\beta)$) is subset of $\bar{\mathbf{D}}$ and $\lambda = 1$ is the only eigenvalue of C_φ on $\partial\mathbf{D}$. (If α is the Denjoy-Wolff point of φ; $f \in H^2(\beta)$ such that $f \circ \varphi = \lambda f$, then either $\lambda = 1$, or $f(\alpha) = 0$. But for $z_0 \in \mathbf{D}$ with $f(z_0) \neq 0$

$$\lambda^n f(z_0) = f(\varphi^{(n)}(z_0)) \to f(\alpha) \quad \text{for } n \to \infty,$$

and if $f(\alpha) = 0$, then $|\lambda| < 1$.)

The cases mentioned above suggest the following conjecture: if C_φ is a compact composition operator on $H^2(\beta)$ and α is the Denjoy-Wolff point of φ, then

$$\sigma_{H^2(\beta)}(C_\varphi) = \{0,1\} \cup \{(\varphi'(\alpha))^n : n = 1, 2, \ldots\}.$$

From the above discussion we know that if the Denjoy-Wolff point of φ (φ not a disc automorphism) is in $\mathbf{D}$, or if the functions in $H^2(\beta)$ are continuous on $\bar{\mathbf{D}}$, then every eigenvalue of C_φ different than 1 has absolute value strictly smaller than 1. Is that true for all other points in the interior of the spectrum of C_φ in these two cases? We know that in H^2, if the Denjoy-Wolff point of φ is in $\mathbf{D}$, then the spectral radius of C_φ is equal to 1 (see [3]), but that is all that is known about this problem.

We thank the referee for several suggestions, including the removal of an additional hypothesis in our original version of Proposition 3.

REFERENCES

1. CARLESON, L., Sets of uniqueness for functions regular in the unit circle, *Acta Math.*, 87(1952), 325–345.
2. CAUGHRAN, J. G.; SCHWARTZ, H. J., Spectra of compact composition operators, *Proc. Amer. Math. Soc.*, 51(1975), 127–130.
3. COWEN, C., Composition operator on H^2, *J. Operator Theory*, 9(1983), 77–106.
4. DENJOY, A., Sur l'itération des fonctions analytiques, *C.R. Acad. Sci. Paris*, **182**(1926), 255–257.
5. KOENIGS, G., Recherches sur les intégrales de certaines equation fonctionnelles, *Ann. Sci. École Norm. Sup. (3)*, 1(1884), supplement, 3–41.

6. MacCluer, B., Composition operators on S^P, *Houston J. of Math.*, **13**(1987), 245–254.
7. MacCluer, B.; Shapiro, J., Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.*, **38**(1986), 878–907.
8. Nevanlinna, R., *Analytic function*, Springer-Verlag, Berlin, 1970.
9. Nordgren, E. A., Composition operators, *Canad. J. Math.*, **20**(1968), 442–449.
10. Schwartz, H. J., Composition operator on H^p, Thesis, U. of Toledo, 1969.
11. Shapiro, J. H., Compact composition operators on spaces of bundary-regular holomorphic functions, *Proc. Amer. Math. Soc.*, **100**(1987), 49–57.
12. Shapiro, J. H., The essential norm of a composition operator, *Ann. of Math.*, **125**(1987), 375–404.
13. Shapiro, J. H.; Taylor, P. D., Compact, nuclear and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.*, 23(1973), 471–496.
14. Shields, A. L., Weighted shift operators and analytic function theory, in *Topics in Operator Theory*, Math. Surveys, No. **13**, Amer. Math. Soc., Providence, 1974.
15. Wolff, J., Sur l'itération des fonctions, *C.R. Acad. Sci. Paris*, **182**(1926), 42–43, 200–201.
16. Zorboska, N., Composition operators on spaces S_a, preprint.

NINA ZORBOSKA
Department of Mathematics,
University of Toronto,
Toronto, Ontario,
Canada, M5S 1A1.

Received March 6, 1988; first revision July 25, 1988; second revision November 11, 1988.