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INTEGRATION WITH RESPECT TO A COMMUTATIVE SUBSPACE LATTICE

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1. INTRODUCTION

The concept of the integral of triangular truncation with respect to a nest (a totally ordered complete set of closed subspaces) was introduced by M. S. Brodskii [1] as an extension of the upper-triangular representation of an operator on a finite-dimensional space. This concept has proved to be very fruitful in the study of non-self adjoint operators and operator algebras having a nest of invariant subspaces, and was developed by many authors (see [4], [5], [7]).

The set of all operators for which the "triangular" integrals are well-defined (see Section 2 for definitions) was first characterized by Erdos-Longstaff [5], while Macaev [12] characterized the "universally truncatable" operators, i.e. those for which the "triangular" integrals with respect to every nest exist.

In addition to its use in the structure theory of nest algebras, this concept has found applications in the theory of integral equations [7] and in mathematical systems theory [6]. These applications have prompted Porter and De Santis [13], [14] to explore a more general notion of integration with respect to suitable families of projections that are not totally ordered, and to develop a theory of factorization of operators with respect to such finite families.

In this paper we define a notion of integration of operators with respect to a commutative subspace lattice (CSL), which generalizes the integral of triangular truncation with respect to a nest, while retaining several of its properties. This allows us to reformulate, in a more transparent way, a conjecture of Hopenwasser and Larson [8], [10] regarding the radical of a CSL-algebra. The problem of factorization of operators with respect to a CSL will be considered in a subsequent paper.

2. THE NOTION OF INTEGRATION IN A C.S.L.

Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. A commutative subspace lattice (C.S.L.) $\mathscr{L}$ is a lattice of commuting projections in

$\mathcal{H}$, containing 0 and the identity 1, which is closed in the strong operator topology. By Alg $\mathcal{L}$ we denote the (weakly closed, unital) subalgebra of $B(\mathcal{H})$, consisting of all operators A leaving the range of each projection P in $\mathcal{L}$ invariant, i.e. $AP = PAP$ for all P in $\mathcal{L}$. A nest algebra is an algebra of the form Alg $\mathcal{N}$, where $\mathcal{N}$ is a totally ordered C.S.L., i.e. a nest.

Let $\mathcal{F}$ be a finite sublattice of $\mathcal{L}$, containing 0 and 1. If $P \in \mathcal{F}$, we define

$$E_{\mathcal{F}}(P) = \sup\{Q \in \mathcal{F};\ Q < P\},$$

$$P_{-}^{\mathcal{F}} = \sup\{Q \in \mathcal{F};\ Q \not\geqslant P\}$$

(observe that $E_{\mathcal{F}}(P) = P_{-}^{\mathcal{F}}$ in case $\mathcal{F}$ is a nest),

$$\Delta_{\mathcal{F}}(P) = P - E_{\mathcal{F}}(P).$$

Given any function $f\colon \mathcal{L} \to B(\mathcal{H})$, we form the sum

$$S(f,\ \mathcal{F},\ \{Q_P\}) = \sum_{P \in \mathcal{F}} f(Q_P)\Delta_{\mathcal{F}}(P)$$

where $Q_P \in \mathcal{L}$, $E_{\mathcal{F}}(P) \leqslant Q_P \leqslant P$. Observe that the set of all finite sublattices is directed by set inclusion, because if $\mathcal{F}_1$, $\mathcal{F}_2$ are finite sublattices of $\mathcal{L}$, then the lattice $\mathcal{F}_0$ generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ is finite (this is due to the fact that $\mathcal{L}$ is commutative). So if the norm limit of the net $\{S(f, \mathcal{F}, \{Q_P\})\}$, indexed by the finite sublattices of $\mathcal{L}$, exists and is independent of the choice of $\{Q_P\}$, we say that f is integrable and write

$$\int f(P)\mathrm{d}(P) = \|\cdot\|\text{-}\lim_{\mathcal{F}} S(f,\ \mathcal{F},\ \{Q_P\}).$$

For $A \in B(\mathcal{H})$, we define the "triagular" integrals of A as:

$$\mathcal{L}(A) = \|\cdot\|\text{-}\lim \mathcal{L}_{\mathcal{F}}(A) \equiv \|\cdot\|\text{-}\lim \sum_{P \in \mathcal{F}} E_{\mathcal{F}}(P)A\Delta_{\mathcal{F}}(P),$$

$$\mathcal{D}(A) = \|\cdot\|\text{-}\lim \mathcal{D}_{\mathcal{F}}(A) \equiv \|\cdot\|\text{-}\lim \sum_{P \in \mathcal{F}} \Delta_{\mathcal{F}}(P)A\Delta_{\mathcal{F}}(P)$$

(the diagonal integral of A),

$$\mathcal{U}(A) = \|\cdot\|\text{-}\lim \mathcal{U}_{\mathcal{F}}(A) \equiv \|\cdot\|\text{-}\lim \sum_{P \in \mathcal{F}} PA\Delta_{\mathcal{F}}(P)$$

if the limit exists.

These definitions generalize the triangular integrals with respect to a nest [5]. However, they are only formal analogues of these, in that the finite sums $\mathscr{U}_{\mathscr{F}}(A)$, $\mathscr{L}_{\mathscr{F}}(A)$ are not "block upper triangular", since $\mathscr{L}$ is not totally ordered.

3. THE FUNCTION $\Delta_{\mathscr{F}}$

We explore the properties of the function $\Delta_{\mathscr{F}}$, where $\mathscr{F}$ is a finite sublattice of a C.S.L.

LEMMA 3.1. (Davidson [2]). *Let $\mathscr{L}$ be a C.S.L. and $\mathscr{F}$ a finite sublattice of $\mathscr{L}$. Then, for P, Q in $\mathscr{F}$ we have:*

a) *If $P \neq Q$, $\Delta_{\mathscr{F}}(P)$ and $\Delta_{\mathscr{F}}(Q)$ are orthogonal,*

b) *$\Delta_{\mathscr{F}}(P) \neq 0$ if and only if P has a unique immediate predecessor in $\mathscr{F}$,*

c) *$\sum_{S \leqslant P} \Delta_{\mathscr{F}}(S) = P$, and,*

d) *$\Delta_{\mathscr{F}}(P)Q \neq 0$ if $Q \geqslant P$ and $\Delta_{\mathscr{F}}(P) \neq 0$.*

When $\mathscr{L}$ is a nest, any finite partition $\mathscr{F} = \{0 = P_0 < P_1, \ldots, P_n = 1\}$ is a sublattice and $\Delta_{\mathscr{F}}(P_k) = P_k - P_{k-1}$. If $\mathscr{F}_1$ is a refinement to a partition $\mathscr{F}_1 \supseteq \mathscr{F}$, then for each $\Delta_{\mathscr{F}_1}(P)$, $P \in \mathscr{F}_1$, there is a $\Delta_{\mathscr{F}}(Q)$, $Q \in \mathscr{F}$, such that $\Delta_{\mathscr{F}_1}(P) \leqslant \Delta_{\mathscr{F}}(Q)$. The following lemma generalises this to arbitrary C.S.L.'s.

DEFINITION. Let $\mathscr{F}_0 = \{E_i, F_i\,;\, i \in I\}$ be a finite subset of a C.S.L. $\mathscr{L}$. We say that $\mathscr{F}_0$ is a *partition of* $\mathscr{L}$ if

a) $E_i \geqslant F_i$ for all $i \in I$,

b) $\sum_{i \in I}(E_i - F_i) = 1$ (in particular the $\{E_i - F_i\}_{i \in I}$ are mutually orthogonal).

With this definition, we have

LEMMA 3.2. *Let $\mathscr{F}_0 = \{E_i, F_i\,;\, i \in I\}$ be a finite partition of $\mathscr{L}$ and $\mathscr{F}$ a finite sublattice containing $\mathscr{F}_0$. Then for each $P \in \mathscr{F}$ there exists $i \in I$ such that $\Delta_{\mathscr{F}}(P) \leqslant E_i - F_i$.*

Proof. Let $P \in \mathscr{F}$ such that $\Delta_{\mathscr{F}}(P) \neq 0$. Then $\sum (E_i - F_i) = 1$ implies $\sum \Delta_{\mathscr{F}}(P)(E_i - F_i) = \Delta_{\mathscr{F}}(P) \neq 0$. Each term $\Delta_{\mathscr{F}}(P)(E_i - F_i)$ is a projection and thus, there exists an $i_0 \in I$ such that $\Delta_{\mathscr{F}}(P)(E_{i_0} - F_{i_0}) \neq 0$.

Since $\Delta_{\mathscr{F}}(P) \neq 0$, $\Delta_{\mathscr{F}}(P) = P - R$ where $R = \sup\{Q \in \mathscr{F},\ Q < P\} < P$. Notice that $0 < \Delta_{\mathscr{F}}(P)(E_{i_0} - F_{i_0}) = (PR^{\perp})(E_{i_0}F_{i_0}^{\perp}) \leqslant PF_{i_0}^{\perp}$. Thus $P > PF_{i_0}$, and since $PF_{i_0} \in \mathscr{F}$, we have $PF_{i_0} \leqslant R$, hence

$$PF_{i_0}^{\perp} \geqslant PR^{\perp}. \tag{1}$$

Similarly $0 < (PR^{\perp})(E_{i_0}F_{i_0}^{\perp}) \leqslant (PE_{i_0})R^{\perp}$ so $PE_{i_0} > R$, hence

$$P \leqslant E_{i_0}. \tag{2}$$

Thus $\Delta_{\mathscr{F}}(P) = PR^{\perp} \leqslant E_{i_0}(PR^{\perp}) \leqslant E_{i_0}(PF_{i_0}^{\perp}) \leqslant E_{i_0} - F_{i_0}$ (the first inequality holds by (2) and the second by (1)).

COROLLARY 3.3. *If $\mathscr{F}_0$, $\mathscr{F}$ are finite sublattices of $\mathscr{L}$ such that $\mathscr{F}_0 \subseteq \mathscr{F}$, then:*
a) *for each $P \in \mathscr{F}$, there is a $P_0 \in \mathscr{F}_0$ such that $\Delta_{\mathscr{F}}(P) \leqslant \Delta_{\mathscr{F}_0}(P_0)$,*
b) *for each $P_0 \in \mathscr{F}_0$, there exist $P_1, P_2, \ldots, P_n$ in $\mathscr{F}$, such that*

$$\sum_{i=1}^{n} \Delta_{\mathscr{F}}(P_i) = \Delta_{\mathscr{F}_0}(P_0).$$

In what follows, we will develop a theory of integration using only property (c) (Lemma 3.1). Thus if there exists a different projection valued function, say $\mathfrak{A}_{\mathscr{F}}: \mathscr{F} \to \mathscr{L}''$, satisfying this property, the results of the following paragraphs would yield distinct "integrals" with respect to this function. The next proposition shows that this cannot happen and lends support to the conjecture of Hopenwasser that the radical of a C.S.L. algebra can be characterized in terms of the diagonal integral (see Section 4).

PROPOSITION 3.4. *Let $\mathscr{F}$ be a finite sublattice of a* C.S.L. *$\mathscr{L}$. If $\mathfrak{A}_{\mathscr{F}} : \mathscr{F} \to \mathscr{L}''$ is a projection valued function, such that for every $P \in \mathscr{F}$*

$$\sum_{S \leqslant P} \mathfrak{A}_{\mathscr{F}}(S) = P$$

then $\mathfrak{A}_{\mathscr{F}}(P) = \Delta_{\mathscr{F}}(P)$, for every $P \in \mathscr{F}$.

Proof. The proof follows by induction. If $P \in \mathscr{F}$ and $E_{\mathscr{F}}(P) = 0$, then, it is obvious that $\mathfrak{A}_{\mathscr{F}}(P) = P = \Delta_{\mathscr{F}}(P)$.

Suppose that for every $S < P$ in $\mathscr{F}$, the proposition holds. Then $\mathfrak{A}_{\mathscr{F}}(P) = P - \sum_{S < P} \mathfrak{A}_{\mathscr{F}}(S) = P - \sum_{S < P} \Delta_{\mathscr{F}}(S) = \Delta_{\mathscr{F}}(P)$.

4. THE RADICAL OF A C.S.L. ALGEBRA

If $\mathscr{A}$ is a Banach algebra, then we define:

$$\begin{aligned}\mathrm{Rad}(\mathscr{A}) &= \{\mathrm{Ker}\,\pi : \pi \text{ is an irreducible representation of } \mathscr{A}\} = \\ &= \{\alpha \in \mathscr{A} : \alpha\beta \text{ is quasinilpotent for every } \beta \in \mathscr{A}\} = \\ &= \{\alpha \in \mathscr{A} : \beta\alpha \text{ is quasinilpotent for every } \beta \in \mathscr{A}\}.\end{aligned}$$

Ringrose's characterization [16] of the radical of a nest algebra motivates the following definition.

DEFINITION. (Hopenwasser [8], Hopenwasser and Larson [10]). If $\mathscr{L}$ is a C.S.L. we define as $J_{\mathscr{L}}$ the set of all A in $\operatorname{Alg}\mathscr{L}$ such that for every $\varepsilon > 0$, there exist a partition $\{E_i, F_i ; i \in I\}$ of $\mathscr{L}$, such that

$$\|(E_i - F_i)A(E_i - F_i)\| < \varepsilon \quad \text{for all } i \in I.$$

Unfortunately there is no known characterization of the radical of a C.S.L. algebra analogous to Ringrose characterization for nest algebras.

THEOREM 4.1. (Hopenwasser [8]). *For every* C.S.L. *the set* $J_{\mathscr{L}}$ *is a closed two sided ideal of* $\operatorname{Alg}\mathscr{L}$, *contained in* $\operatorname{Rad}\operatorname{Alg}\mathscr{L}$.

We say that $\operatorname{Alg}\mathscr{L}$ satisfies the radical condition if $J_{\mathscr{L}} = \operatorname{Rad}\operatorname{Alg}\mathscr{L}$. Ringrose [16] has shown that nest algebras satisfy the radical condition and Hopenwasser [8] proves the conjecture in a variety of cases, including finite C.S.L.'s.

PROPOSITION 4.2. *Let* $\mathscr{L}$ *be a* C.S.L. *and* $A \in \operatorname{Alg}\mathscr{L}$. *Then the following are equivalent:*

a) $A \in J_{\mathscr{L}}$,

b) *A belongs to the norm closure of* $\bigcup\{\operatorname{Rad}\operatorname{Alg}\mathscr{F} ; \mathscr{F}$ *is a finite sublattice of* $\mathscr{L}\}$,

c) $\mathscr{D}(A)$ *exists and equals zero.*

Proof. a) $\Rightarrow$ c) Let $\varepsilon > 0$. By Theorem 4.1 there exists a finite partition $\mathfrak{S}_0 = \{E_i, F_i ; i \in I\}$ of $\mathscr{L}$, such that, $\|(E_i - F_i)A(E_i - F_i)\| < \varepsilon$ for all $i \in I$.

Let $\mathscr{F}_0$ be the finite sublattice of $\mathscr{L}$, generated by $\mathfrak{S}_0$. If $\mathscr{F} \supseteq \mathscr{F}_0$ is a finite sublattice, Lemma 3.2 implies that for each $P \in \mathscr{F}$ there is a $j \in I$ such that $\Delta_{\mathscr{F}}(P) \leqslant E_j - F_j$, and hence

$$\|\Delta_{\mathscr{F}}(P)A\Delta_{\mathscr{F}}(P)\| = \|\Delta_{\mathscr{F}}(P)(E_j - F_j)A(E_j - F_j)\Delta_{\mathscr{F}}(P)\| < \varepsilon.$$

Observing that the $\Delta_{\mathscr{F}}(P)$'s are mutually orthogonal we have

$$\|\mathscr{D}_{\mathscr{F}}(A)\| = \sup_{P \in \mathscr{F}} \|\Delta_{\mathscr{F}}(P)A\Delta_{\mathscr{F}}(P)\| < \varepsilon.$$

c) $\Rightarrow$ a) Obvious by definition of $J_{\mathscr{L}}$.

b) $\Rightarrow$ c) Given $\varepsilon > 0$, (b) implies that there exist a finite sublattice $\mathscr{F}$ of $\mathscr{L}$ and a $B \in \operatorname{Rad}\operatorname{Alg}\mathscr{F}$ such that $\|A - B\| < \varepsilon$. Thus $\|\Delta_{\mathscr{F}}(P)(A - B)\Delta_{\mathscr{F}}(P)\| < \varepsilon$ for all $P \in \mathscr{F}$. But since finite C.S.L's satisfy the radical condition:

$$\Delta_{\mathscr{F}}(P)B\Delta_{\mathscr{F}}(P) = 0 \quad \text{for all } P \in \mathscr{F}.$$

Therefore

$$\|\Delta_{\mathcal{F}}(P)A\Delta_{\mathcal{F}}(P)\| < \varepsilon$$

hence $\mathcal{D}(A)$ exists and equals 0.

c $\Rightarrow$ b) It can be proved by similar arguments.

REMARKS. a) The proof of b) $\Rightarrow$ c) does not use the fact that A belongs to $\mathrm{Alg}\,\mathcal{L}$. In fact this condition is essentially used in the proof of b) $\Rightarrow$ a).

b) The Hopenwasser's conjecture is equivalent to the following.

"For $A \in \mathrm{Alg}\,\mathcal{L}$, $A \in \mathrm{Rad}\,\mathrm{Alg}\,\mathcal{L} \Rightarrow \mathcal{D}(A)$ exists and equals 0".

Using Lemma 3.1, one can prove the following.

PROPOSITION 4.3. *Let $\mathcal{L}$ be a* C.S.L. *and $A \in B(\mathcal{H})$. If $\mathcal{L}(A)$ exists, then $\mathcal{L}(A)$ belongs to $J_{\mathcal{L}}$.*

NOTE. Similarly, one can prove that for every $A \in B(\mathcal{H})$, if $\mathcal{U}(A)$ exists then $\mathcal{U}(A) \in \mathrm{Alg}\,\mathcal{L}$ and that if $\mathcal{D}(A)$ exists then $\mathcal{D}(A) \in \mathcal{L}'$.

DEFINITION. An *atom* Δ for a C.S.L. $\mathcal{L}$ is a non zero projection of the form

$$\Delta = \Delta(P) \equiv P - \sup\{Q \in \mathcal{L}\ ; Q < P\}, \quad \text{for some } P \text{ in } \mathcal{L}.$$

It can be shown that Δ is an atom for $\mathcal{L}$ if and only if, for any $Q \in \mathcal{L}$, either Q contains Δ or is orthogonal to it. Hopenwasser in [9] gives the following criterion.

THEOREM 4.4. *Let $\mathcal{L}$ be a* C.S.L. *and $K \in \mathrm{Alg}\,\mathcal{L}$, compact. Then $K \in J_{\mathcal{L}}$ if and only if $\Delta(P)K\Delta(P) = 0$ for every P in $\mathcal{L}$.*

It is also important that a reformulation of the proof of the previous theorem shows that for every C.S.L. $\mathcal{L}$, the ideal $J_{\mathcal{L}}$ and the Rad Alg $\mathcal{L}$ contain the same compact operators.

5. THE DIAGONAL INTEGRAL

The main results of this section are that the diagonal integral exists for every compact operator and that this coincides with the "diagonal part" of that operator.

For $1 \leqslant p \leqslant +\infty$, C_p will denote the von Neumann-Schatten class of all compact operators A on $\mathcal{H}$, such that the eigenvalues of $(A^*A)^{1/2}$, repeated according to multiplicity, are p-summable. It is well known that, for $1 \leqslant p \leqslant +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the formula $\langle A, B\rangle = \mathrm{tr}(BA)$, $A \in C_q$, $B \in C_p$ establishes a duality between C_p and C_q. In particular, the Hilbert-Schmidt class C_2 is a Hilbert space.

PROPOSITION 5.1. *Let $1 < p < +\infty$. Then the net $\{\mathscr{D}_{\mathscr{F}}\}$ indexed by the finite sublattices $\mathscr{F}$ of $\mathscr{L}$ is a monotonically decreasing net of idempotents on the Banach space C_p. Furthermore for each such $\mathscr{F}$, $\mathscr{D}_{\mathscr{F}} : C_q \to C_q$ is the dual mapping of $\mathscr{D}_{\mathscr{F}} : C_p \to C_p$ $(1/p + 1/q = 1)$. In particular, $\mathscr{D}_{\mathscr{F}}$ is a selfadjoint projection on C_2.*

Proof. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ to be finite sublattices of $\mathscr{L}$, such that $\mathscr{F}_1 \subseteq \mathscr{F}_2$. Using Corollary 3.3, one can prove that

$$(\mathscr{D}_{\mathscr{F}_1}\mathscr{D}_{\mathscr{F}_2})(A) = (\mathscr{D}_{\mathscr{F}_2}\mathscr{D}_{\mathscr{F}_1})(A) = \mathscr{D}_{\mathscr{F}_2}(A)$$

and in particular

$$\mathscr{D}^2_{\mathscr{F}}(A) = \mathscr{D}_{\mathscr{F}}(A).$$

This shows that $\{\mathscr{D}_{\mathscr{F}}\}$ is a decreasing net of idempotents.

The final statement follows from the calculation

$$\langle \mathscr{D}_{\mathscr{F}}(A),\ B\rangle = \operatorname{tr}(B\mathscr{D}_{\mathscr{F}}(A)) = \operatorname{tr}\big(B \sum_{P\in\mathscr{F}} \Delta_{\mathscr{F}}(P)A\Delta_{\mathscr{F}}(P)\big) =$$

$$= \sum_{P\in\mathscr{F}} \operatorname{tr}(\Delta_{\mathscr{F}}(P)B\Delta_{\mathscr{F}}(P)A) = \operatorname{tr} A\big(\sum_{P\in\mathscr{F}} \Delta_{\mathscr{F}}(P)B\Delta_{\mathscr{F}}(P)\big) = \langle A,\ \mathscr{D}_{\mathscr{F}}(B)\rangle$$

where $A \in C_q$, $B \in C_p$.

REMARK. Using similar arguments as in the previous proof one can prove that $\{\mathscr{U}_{\mathscr{F}}\}_{\mathscr{F}}$ is a decreasing net of idempotents on the Banach space C_p, where $1 < p < +\infty$. In particular these idempotents are selfadjoint on C_2.

THEOREM 5.2. *If $1 < p < +\infty$ then for each finite sublattice $\mathscr{F}$ of $\mathscr{L}$, $\mathscr{D}_{\mathscr{F}} : C_p \to C_p$ is a contractive projection and the net $\{\mathscr{D}_{\mathscr{F}}\}_{\mathscr{F}}$ converges in the strong operator topology to a contractive projection*

$$\mathscr{D} : C_p \to C_p.$$

Proof. (i) We claim that $|||\mathscr{D}_{\mathscr{F}}|||_{2p} \leqslant |||\mathscr{D}_{\mathscr{F}}|||_p$, where $|||\cdot|||_p$ denotes the norm of an operator on C_p.

Indeed, if $A \in C_{2p}$ and $\|A\|_{2p} = 1$, then

$$\|\mathscr{D}_{\mathscr{F}}(A)\|^2_{2p} = \|\mathscr{D}_{\mathscr{F}}(A)^*\mathscr{D}_{\mathscr{F}}(A)\|_p = \|\mathscr{D}_{\mathscr{F}}(A^*\mathscr{D}_{\mathscr{F}}(A))\|_p \leqslant$$

$$\leqslant |||\mathscr{D}_{\mathscr{F}}|||_p\|A^*\mathscr{D}_{\mathscr{F}}(A)\|_p \leqslant |||\mathscr{D}_{\mathscr{F}}|||_p\|A^*\|_{2p}\|\mathscr{D}_{\mathscr{F}}(A)\|_{2p}$$

so that dividing:

$$\|\mathscr{D}_{\mathscr{F}}(A)\|_{2p} \leqslant |||\mathscr{D}_{\mathscr{F}}|||_p \Rightarrow |||\mathscr{D}_{\mathscr{F}}|||_{2p} \leqslant |||\mathscr{D}_{\mathscr{F}}|||_p.$$

(ii) Inductively, for an arbitrary sublattice, we obtain:

$$|||\mathcal{D}_{\mathcal{F}}|||_{2^n} \leqslant |||\mathcal{D}_{\mathcal{F}}|||_2 \leqslant 1 \quad \text{for all } n \in \mathbf{N}.$$

An interpolation theorem (see [4], Theorem 2.4) now shows that

$$|||\mathcal{D}_{\mathcal{F}}|||_p \leqslant 1 \quad \text{for } 2 \leqslant p < +\infty.$$

By duality (Proposition 5.1) the result follows for all $1 < p < +\infty$.

(iii) Thus the net $\{\mathcal{D}_{\mathcal{F}}\}_{\mathcal{F}}$ is a monotonically decreasing net of contractive projections on the reflexive Banach space C_p $(1 < p < +\infty)$. A theorem of Lorch (see [4], Theorem 2.1) now shows that there exists a contraction $\mathcal{D}: C_p \to C_p$ such that, for all $A \in C_p$,

$$\lim_{\mathcal{F}} \|\mathcal{D}_{\mathcal{F}}(A) - \mathcal{D}(A)\|_p = 0.$$

But if $\mathcal{F}_0$ is any finite sublattice of $\mathcal{L}$ and $A \in C_p$ then:

$$\mathcal{D}(\mathcal{D}_{\mathcal{F}_0}(A)) = \lim_{\mathcal{F}} \mathcal{D}_{\mathcal{F}}(\mathcal{D}_{\mathcal{F}_0}(A)) = \lim_{\mathcal{F} \supseteq \mathcal{F}_0} \mathcal{D}_{\mathcal{F}}(A) = \mathcal{D}(A)$$

and hence

$$\mathcal{D}(\mathcal{D}(A)) = \mathcal{D}(\lim_{\mathcal{F}} \mathcal{D}_{\mathcal{F}}(A)) = \lim_{\mathcal{F}} \mathcal{D}(\mathcal{D}_{\mathcal{F}}(A)) = \mathcal{D}(A).$$

So $\mathcal{D}$ is a projection.

PROPOSITION 5.3. *The diagonal integral $\mathcal{D}$ is a contractive projection on the Banach space of compact operators.*

Proof. It is clear that for any finite sublattice $\mathcal{F} \subseteq \mathcal{L}$ and any $A \in B(\mathcal{H})$

$$\|\mathcal{D}_{\mathcal{F}}(A)\| \leqslant \|A\|.$$

Let K be a compact operator. Given $\varepsilon > 0$, choose a finite rank operator K_ε, such that $\|K - K_\varepsilon\| < \varepsilon$. By Theorem 5.2 $\{\mathcal{D}_{\mathcal{F}}(K_\varepsilon)\}$ converges in the Hilbert--Schmidt norm $\|\cdot\|_2$. Hence there exists a sublattice $\mathcal{F}_0$ such that whenever $\mathcal{F}_1, \mathcal{F}_2 \supseteq \mathcal{F}_0$

$$\|\mathcal{D}_{\mathcal{F}_1}(K_\varepsilon) - \mathcal{D}_{\mathcal{F}_2}(K_\varepsilon)\|_2 < \varepsilon/3.$$

We then have:

$$\|\mathcal{D}_{\mathcal{F}_1}(K) - \mathcal{D}_{\mathcal{F}_2}(K)\| \leqslant \|\mathcal{D}_{\mathcal{F}_1}(K - K_\varepsilon)\| + \|\mathcal{D}_{\mathcal{F}_1}(K_\varepsilon) - \mathcal{D}_{\mathcal{F}_2}(K_\varepsilon)\| + \|\mathcal{D}_{\mathcal{F}_2}(K - K_\varepsilon)\| \leqslant$$

$$\leqslant 2\|K - K_\varepsilon\| + \|\mathcal{D}_{\mathcal{F}_1}(K_\varepsilon) - \mathcal{D}_{\mathcal{F}_2}(K_\varepsilon)\|_2 < 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

The proposition follows from the fact that the set of compact operators is a Banach space in the operator norm.

REMARK. The crucial element in the precedings proofs is that the net $\{\mathscr{D}_{\mathscr{F}}\}_{\mathscr{F}}$ is monotonic, a consequence of our results on refinement (Corollary 3.3). Given this fact, the proofs are analogous to the ones given in [4], for the nest case.

By contrast, our proof of the following theorem is quite different from the one in [5], which relies on compactness of a nest, in the strong operator topology. It is not the case that an arbitrary C.S.L. is compact. (See Wagner [17].)

For any $A \in B(\mathscr{H})$, the sum $\delta(A) = \sum_{P \in \mathscr{L}} \Delta(P) A \Delta(P)$ exists as a strong operator limit. This "diagonal part" of the operator A might be thought to be the "natural" value for the diagonal integral $\mathscr{D}(A)$. However, easy examples show that this need not to be the case, even in nests: the diagonal integral may not exist, or it may exist and not equal $\delta(A)$. The situation is better for compact operators:

THEOREM 5.4. *If $K \in B(\mathscr{H})$ is a compact operator then*

$$\mathscr{D}(K) = \sum_{P \in \mathscr{L}} \Delta(P) K \Delta(P).$$

Proof. It is easy to see that the sum $\sum_{P \in \mathscr{L}} \Delta(P) K \Delta(P)$ converges in the operator norm, since K is compact.

Observe that since the $\Delta(P)$'s are atoms of $\mathscr{L}$, $\Delta(P)Q = 0$ or $\Delta(P)Q = \Delta(P)$ for every $Q \in \mathscr{L}$. Let $\{P_i, i \in I_0\}$ be the set of all elements of $\mathscr{L}$ which give non-zero atoms. Then:

$$\delta(K)Q = \sum_{i \in I_0} \Delta(P_i) K \Delta(P_i) Q = \sum_{i \in I_0} Q \Delta(P_i) K \Delta(P_i) = Q\delta(K).$$

Hence $\delta(K) \in \mathscr{L}'$.

(i) Assume first that $K \in \operatorname{Alg} \mathscr{L}$. Now

$$\Delta(Q)(K - \delta(K))\Delta(Q) = 0 \quad \text{for all } Q \in \mathscr{L}$$

and therefore $K - \delta(K)$ belongs to $J_{\mathscr{L}}$ by Theorem 4.4 and so

$$\mathscr{D}(K - \delta(K)) = 0 \Rightarrow \mathscr{D}(K) = \mathscr{D}(\delta(K)) = \delta(K)$$

since $\delta(K) \in \mathscr{L}'$.

(ii) Now let K be an arbitrary compact operator. Since $\mathscr{D}(K)$ exists and belongs to $\mathscr{L}'$, part (i) gives:

$$\mathscr{D}(K) = \mathscr{D}(\mathscr{D}(K)) = \delta(\mathscr{D}(K)).$$

Given $\varepsilon > 0$, there exists a finite set $I_\varepsilon \subseteq I_0$ such that:

$$\left\|\mathscr{D}(K) - \sum_{i \in I} \Delta(P_i)\mathscr{D}(K)\Delta(P_i)\right\| < \varepsilon/2$$

for all finite subsets $I \subseteq I_0$ such that $I \supseteq I_\varepsilon$.

Let $\mathscr{F}_0 \subseteq \mathscr{L}$ be a finite sublattice, such that

$$\|\mathscr{D}(K) - \mathscr{D}_{\mathscr{F}}(K)\| < \varepsilon/2 \tag{1}$$

whenever $\mathscr{F} \supseteq \mathscr{F}_0$ is a finite sublattice of $\mathscr{L}$. Then

$$\left\|\sum_{i \in I} \Delta(P_i)(\mathscr{D}(K) - \mathscr{D}_{\mathscr{F}}(K))\Delta(P_i)\right\| < \varepsilon/2, \quad I \supseteq I_\varepsilon. \tag{2}$$

Fix an $I \subseteq I_0$ finite such that $I \supseteq I_\varepsilon$. Then (2) is valid in particular for the finite sublattice $\mathscr{F}_1$, generated by $\mathscr{F}_0$ and $\{P_i, E(P_i); i \in I\}$. Observe that for $i \in I$, $\Delta(P_i) = \Delta_{\mathscr{F}_1}(P_i)$, since $\mathscr{F}_1$ contains the immediate predecessor $E(P_i)$ of P_i, in $\mathscr{L}$. So

$$\sum_{i \in I} \Delta(P_i)\mathscr{D}_{\mathscr{F}_1}(K)\Delta(P_i) = \sum_{i \in I} \Delta_{\mathscr{F}_1}(P_i)\mathscr{D}_{\mathscr{F}_1}(K)\Delta_{\mathscr{F}_1}(P_i) =$$

$$= \sum_{i \in I} \Delta_{\mathscr{F}_1}(P_i)\Big(\sum_{Q \in \mathscr{F}_1} \Delta_{\mathscr{F}_1}(Q)K\Delta_{\mathscr{F}_1}(Q)\Big)\Delta_{\mathscr{F}_1}(P_i) = \sum_{i \in I} \Delta_{\mathscr{F}_1}(P_i)K\Delta_{\mathscr{F}_1}(P_i) = \tag{3}$$

$$= \sum_{i \in I} \Delta(P_i)K\Delta(P_i).$$

Therefore:

$$\left\|\mathscr{D}(K) - \sum_{i \in I} \Delta(P_i)K\Delta(P_i)\right\| \leqslant \left\|\mathscr{D}(K) - \sum_{i \in I} \Delta(P_i)\mathscr{D}(K)\Delta(P_i)\right\| +$$

$$+ \left\|\sum_{i \in I} \Delta(P_i)(\mathscr{D}(K) - \mathscr{D}_{\mathscr{F}_1}(K))\Delta(P_i)\right\| + \left\|\sum_{i \in I} \Delta(P_i)(\mathscr{D}_{\mathscr{F}_1}(K) - K)\Delta(P_i)\right\| <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0$$

by (1), (2) and (3).

Since the above inequality holds for an arbitrary $I \supseteq I_\varepsilon$, the theorem has been proved.

6. THE DOMAIN OF THE THREE INTEGRALS

Erdos and Longstaff in [5] characterized the domain of the three integrals in the case of a nest. Many of their results carry over to a general C.S.L., but there are significant differences.

Our first result is a decomposition of an operator with respect to a C.S.L. algebra. In the nest-algebra case this generalizes to the decomposition of a matrix into strictly upper triangular, a diagonal and a strictly lower triangular part.

DEFINITION. Let $\mathscr{F}$ be a C.S.L. We define as $\mathscr{R}(\mathscr{F})$, the set of all operators $A \in B(\mathscr{H})$ such that

$$AQ = Q_-^{\mathscr{F}} AQ \quad \text{for all } Q \in \mathscr{F}.$$

Using a well known proposition of Longstaff (see [3], Theorem 23.3), one can prove that $\mathscr{R}(\mathscr{F})$ is exactly the annihilator of the rank one operators, that belong to Alg $\mathscr{F}$ (if we see them as a subset of the trace class operators).

THEOREM 6.1. *Let $\mathscr{L}$ be a* C.S.L. *and $A \in B(\mathscr{H})$. If any two of $\mathscr{L}(A)$, $\mathscr{U}(A)$, $\mathscr{D}(A)$ exist then so does the third and*

$$\mathscr{U}(A) = \mathscr{L}(A) + \mathscr{D}(A).$$

In this case A decomposes as a sum $A = A_1 + A_2 + A_3$, where

$$A_1 = \mathscr{L}(A) \quad \textit{belongs to } J_{\mathscr{L}}$$

$$A_2 = \mathscr{D}(A) \quad \textit{belongs to } \mathscr{L}'$$

and

$$A_3^* = (A - \mathscr{U}(A))^*$$

belongs to the norm closure of the $\bigcup\{\mathscr{R}(\mathscr{F})\,;\, \mathscr{F}$ finite sublattice of $\mathscr{L}\}$.

Proof. The first assertion is immediate since for every finite sublattice $\mathscr{F}$ of $\mathscr{L}$:

$$\mathscr{U}_{\mathscr{F}}(A) = \mathscr{L}_{\mathscr{F}}(A) + \mathscr{D}_{\mathscr{F}}(A).$$

If all integrals exist then:

$$A = \mathscr{L}(A) + \mathscr{D}(A) + (A - \mathscr{U}(A)).$$

We only need to prove that $A_3 \equiv A - \mathscr{U}(A)$ has the desired property. Given $\varepsilon > 0$, let $\mathscr{F} \subseteq \mathscr{L}$ be a finite sublattice, such that,

$$\Big\|A_3 - \sum_{P \in \mathscr{F}} P^{\perp} A \Delta_{\mathscr{F}}(P)\Big\| < \varepsilon.$$

For $Q \in \mathscr{F}$, we have that, if $P^{\perp}Q \neq 0$ then $Q \nleq P$ and so $P \leqslant Q_-^{\mathscr{F}}$. Hence

$$(A - \mathscr{U}_{\mathscr{F}}(A))^* Q = \Big(\sum_{P \in \mathscr{F}} \Delta_{\mathscr{F}}(P) A^* P^{\perp}\Big) Q = \sum_{P \in \mathscr{F}} Q_-^{\mathscr{F}} \Delta_{\mathscr{F}}(P) A^* P^{\perp} Q = Q_-^{\mathscr{F}} (A - \mathscr{U}_{\mathscr{F}}(A))^* Q$$

which shows that $A - \mathscr{U}_{\mathscr{F}}(A)$ belongs to $\mathscr{R}(\mathscr{F})$. Thus, the theorem has been proved.

REMARKS. (i) One might hope that, as in the nest algebra case, A_3^* would belong to $J_{\mathscr{L}}$ (see [5]). This is certainly true if A^* belongs to Alg $\mathscr{L}$, but not in general:

EXAMPLE. Let $\mathscr{L} = \{0, P, Q, 1\}$ where $PQ = 0$ and $P + Q = 1$. Here $E(P) = E(Q) = 0$, so

$$\Delta(P) = P \quad P_- = Q$$

$$\Delta(Q) = Q \quad Q_- = P.$$

Thus, if

$$A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}$$

we have that $\mathscr{L}(A) = 0$, $\mathscr{D}(A) = PAP + QAQ$ and $\mathscr{U}(A) = \mathscr{D}(A)$

$$(A - \mathscr{U}(A))^* = \begin{bmatrix} 0 & A_3^* \\ A_3^* & 0 \end{bmatrix}$$

which, certainly, does not belong to Alg $\mathscr{L}$.

(ii) It is easy to see that the norm closure of $\bigcup\{\mathscr{R}(\mathscr{F}) : \mathscr{F}$ finite sublattice of $\mathscr{L}\}$ is contained in $\mathscr{R}(\mathscr{L})$, but equality fails, even in nests.

EXAMPLE. Let $\{e_k ; k \in \mathbf{N}\}$ be an orthonormal basis for a separable Hilbert space $\mathscr{H}$ and let us denote by X the back-ward shift ($Xe_1 = 0$, and $Xe_n = e_{n-1}$, $n \in \mathbf{N}$). If $\mathscr{N} = \{P_n, n \in \mathbf{N}\}$, where $P_n \equiv [e_k ; k \leqslant n]$, then $X(P_n) \leqslant P_{n-1}$, so $X \in \mathscr{R}(\mathscr{N})$.

On the other hand, the norm closure of $\bigcup\{\mathscr{R}(\mathscr{F}) ; \mathscr{F} \subseteq \mathscr{N}$ finite subnest$\}$ coincides with Rad Alg $\mathscr{N}$, in the nest algebra case. Since X is not quasinilpotent, we have the desired counterexample.

The following theorem is a partial converse of Theorem 6.1. Note that the full converse holds in the nest case (see [5]).

THEOREM 6.2. *Let* $A \in B(\mathscr{H})$ *and* $A = A_1 + A_2 + A_3$ *such that:*

$$A_1 \in J_{\mathscr{L}}, \quad A_2 \in \mathscr{L}'$$

and

$$A_3^* \in \bigcup\{\mathscr{R}(\mathscr{F}) ; \mathscr{F} \subseteq \mathscr{L} \textit{ finite sublattice}\}.$$

Then the triangular integrals of A *all exist.*

Proof. Let $A = A_1 + A_2 + A_3$ as above and $A_3 \in \mathscr{R}(\mathscr{F}_0)$ where $\mathscr{F}_0 \subseteq \mathscr{L}$ finite sublattice. It is easily seen that $A_3 \in \mathscr{R}(\mathscr{F})$ where $\mathscr{F}$ is any finite sublattice of $\mathscr{L}$, containing $\mathscr{F}_0$. So if $\mathscr{F} \supseteq \mathscr{F}_0$

$$(\mathscr{L}_{\mathscr{F}}(A_3))^* = \sum \Delta_{\mathscr{F}}(P)A_3^*E_{\mathscr{F}}(P) =$$

$$= \sum \Delta_{\mathscr{F}}(P)(E_{\mathscr{F}}(P))_-^{\mathscr{F}}A_3^*E_{\mathscr{F}}(P) = \sum \Delta_{\mathscr{F}}(P)P(E_{\mathscr{F}}(P))_-^{\mathscr{F}}A_3^*E_{\mathscr{F}}(P).$$

We claim that $P(E_{\mathscr{F}}(P))_-^{\mathscr{F}} \leqslant E_{\mathscr{F}}(P)$. Indeed, let $Q \in \mathscr{F}$, $Q \not\geqslant P$. Then $PQ < P$ so that $PQ \leqslant E_{\mathscr{F}}(P)$. Thus:

$$P(E_{\mathscr{F}}(P))_-^{\mathscr{F}} = P\cdot\sup\{Q\,;\, Q \in \mathscr{F},\ Q \not\geqslant P\} = \sup\{PQ\,;\, Q \not\geqslant P\} \leqslant E_{\mathscr{F}}(P).$$

Therefore $\Delta_{\mathscr{F}}(P)P(E_{\mathscr{F}}(P))_-^{\mathscr{F}} = 0$ for all $P \in \mathscr{F}$ and hence $\mathscr{L}_{\mathscr{F}}(A_3)^* = 0 = \mathscr{L}_{\mathscr{F}}(A_3)$. So

$$\mathscr{L}_{\mathscr{F}}(A) = \mathscr{L}_{\mathscr{F}}(A_1) \quad \text{for all } \mathscr{F} \supseteq \mathscr{F}_0.$$

But since $A_1 \in J_{\mathscr{L}}$, given $\varepsilon > 0$ we can find a finite sublattice $\mathscr{F}_\varepsilon \supseteq \mathscr{F}_0$ such that:

$$\|\mathscr{D}_{\mathscr{F}}(A_1)\| < \varepsilon \quad \text{for all } \mathscr{F} \supseteq \mathscr{F}_\varepsilon.$$

So:

$$\|A_1 - \mathscr{L}_{\mathscr{F}}(A)\| = \|A_1 - \mathscr{L}_{\mathscr{F}}(A_1)\| = \|\mathscr{D}_{\mathscr{F}}(A_1)\| < \varepsilon.$$

This proves that $\mathscr{L}(A)$ exists and equals A_1. Using similar arguments, one can prove that $\mathscr{D}(A)$ exists and equals A_2. This completes the proof.

REMARK. It is obvious that the intersection of the domain of the three integrals with $\mathrm{Alg}\,\mathscr{L}$ is fully characterized by the above theorem. In fact it is the set $J_{\mathscr{L}} + \mathscr{L}'$.

The following proposition guarantees that the domain of the three integrals contains a remarkable class of compact operators.

PROPOSITION 6.3. *Let $\mathscr{L}$ be a* C.S.L. *Then the triangular integrals of every Hilbert-Schmidt operator exist in the Hilbert-Schmidt norm.*

Proof. Obvious by Proposition 5.1 and the remarks following it.

The next theorem characterizes the domain of the diagonal integral.

THEOREM 6.4. *Let $\mathscr{L}$ be a* C.S.L. *and $A \in B(\mathscr{H})$. Then the following are equivalent:*

a) *The diagonal integral of A exists.*

b) *Given $\varepsilon > 0$, there is a $B = B_1 + B_2 + B_3$, such that:*

$$\|B - A\| < \varepsilon$$

where :

$$B_1 \in \bigcup\{\operatorname{Rad}\operatorname{Alg}\mathscr{F}\ ;\ \mathscr{F} \subseteq \mathscr{L} \textit{ finite sublattice}\},$$

$$B_2^* \in \bigcup\{\mathscr{R}(\mathscr{F})\ ;\ \mathscr{F} \subseteq \mathscr{L}, \textit{ finite sublattice}\},$$

$$B_3 \in \mathscr{L}'.$$

Proof. a) $\Rightarrow$ b). Let us suppose that the diagonal integral of $A \in B(\mathscr{H})$ exists.

Given $\varepsilon > 0$, there is a finite sublattice $\mathscr{F}$ of $\mathscr{L}$ such that $\|\mathscr{D}(A) - \mathscr{D}_{\mathscr{F}}(A)\| < \varepsilon$.

Let $B = B_1 + B_2 + B_3$, where

$$B_1 = \mathscr{L}_{\mathscr{F}}(A), \quad B_2 = A - \mathscr{U}_{\mathscr{F}}(A) \quad \text{and} \quad B_3 = \mathscr{D}(A).$$

Then simple calculations show that:

$$\|A - B\| = \|\mathscr{D}_{\mathscr{F}}(A) - \mathscr{D}(A)\| < \varepsilon.$$

But:

$$\mathscr{L}_{\mathscr{F}}(A) \in \operatorname{Rad}\operatorname{Alg}\mathscr{F}$$

$(A - \mathscr{U}_{\mathscr{F}}(A))^* \in \mathscr{R}(\mathscr{F})$ (it can be proved as in Theorem 6.1) and $\mathscr{D}(A) \in \mathscr{L}'$.

b) $\Rightarrow$ a) Let $B = B_1 + B_2 + B_3$ where

$$B_1 \in \bigcup\{\operatorname{Rad}\operatorname{Alg}\mathscr{F}\ ;\ \mathscr{F} \subseteq \mathscr{L} \text{ finite sublattice}\},$$

$$B_2^* \in \bigcup\{R(\mathscr{F})\ ;\ \mathscr{F} \subseteq \mathscr{L}, \text{ finite sublattice}\},$$

and

$$B_3 \in \mathscr{L}'.$$

We will show that the diagonal integral of B exists.

Indeed, using Theorem 4.1 and the remarks following it, one can find a finite sublattice $\mathscr{F}_1$ of $\mathscr{L}$ such that:

$$\mathscr{D}_{\mathscr{F}}(B_1) = 0, \quad \text{for all } \mathscr{F} \supseteq \mathscr{F}_1 \text{ finite sublattices.}$$

Also, there is a finite sublattice $\mathscr{F}_2$, such that,

$$\mathscr{D}_{\mathscr{F}}(B_2) = 0, \quad \text{for every } \mathscr{F} \supseteq \mathscr{F}_2 \text{ finite sublattice}$$

(see the proof of Theorem 6.2).

So, if $\mathscr{F}$ is an arbitrary sublattice containing the finite sublattice $\mathscr{F}_0$ generated by $\mathscr{F}_1$ and $\mathscr{F}_2$, we have:

$$\mathscr{D}_{\mathscr{F}}(B) - B_3 = \mathscr{D}_{\mathscr{F}}(B_1) + \mathscr{D}_{\mathscr{F}}(B_2) = 0.$$

So the diagonal integral of B exists and equals B_3.

The proof now follows, using similar arguments as in the proof of Proposition 5.3.

7. SOME APPLICATIONS

A well known theorem of Ringrose [15] states that any compact operator K has a maximal nest $\mathscr{N}$ of invariant subspaces and, moreover, the "diagonal elements" $\Delta(N)K\Delta(N)$ $(N \in \mathscr{N})$ of K are one dimensional operators, corresponding precisely to the non-zero eigenvalues of K. This was generalized by Erdos-Longstaff [5] for arbitrary nests. We generalize it to arbitrary CSL's. Our proof, even in the nest case, is more direct than that of [5].

DEFINITION (see [11]). A collection $\mathscr{S}$ of operators acting on a Hilbert space $\mathscr{H}$ is called *simultaneously triangulatizable* (abbreviated S.T.) if there is a maximal nest, each of whose members is invariant under all the operators in $\mathscr{S}$; such a nest will be said *triangularizing* for $\mathscr{S}$.

THEOREM 7.1. ([11]). *If A and B are compact then $\{A, B\}$ is* S.T. *if and only if $q(A, B)\cdot(AB - BA)$ is quasinilpotent for all polynomials q.*

THEOREM 7.2. *Let $\mathscr{L}$ be a* C.S.L. *and K a compact operator in* $\operatorname{Alg}\mathscr{L}$. *Then K decomposes uniquely as a sum*

$$K = K_1 + K_2$$

where $K_1 \in \mathscr{L}'$ and $K_2 \in \operatorname{Rad}\operatorname{Alg}\mathscr{L}$. Moreover $\sigma(K_1) = \sigma(K)$.

Proof. Since K is compact, $K_1 = \mathscr{D}(K)$ exists. Also, $\mathscr{U}(K)$ exists and equals K, since $K \in \operatorname{Alg}\mathscr{L}$. So by Theorem 6.1, $K_2 \equiv \mathscr{L}(K)$ exists and

$$K = K_1 + K_2.$$

Also $\sigma(K_2) = 0$ since $K_2 \in J_{\mathscr{L}} \subseteq \operatorname{Rad}\operatorname{Alg}\mathscr{L}$.

The uniqueness of the decomposition follows from the fact that if $A \in B(\mathscr{H})$ is in $\mathscr{L}'$ and in $\operatorname{Rad}\operatorname{Alg}\mathscr{L}$, then $A^* \in \operatorname{Alg}\mathscr{L}$, so $A^*A \in \operatorname{Rad}\operatorname{Alg}\mathscr{L}$, hence $A^*A = 0$.

It remains to show that $\sigma(K_1) = \sigma(K)$. Observe first that for each $A \in B(\mathscr{H})$ which belongs to the algebra $\mathscr{A}$ generated by K and K_1, we have

$$A(K_1K - KK_1) = A(KK_2 - K_2K)$$

and also, K_2 belongs to $\mathscr{A}$. But, since $K_2 \in \operatorname{Rad}\operatorname{Alg}\mathscr{L}$ and $K, A \in \operatorname{Alg}\mathscr{L}$ it follows that $A(KK_2 - K_2K)$ is quasinilpotent. The previous theorem now shows that there is a maximal nest $\mathscr{N}$ such that $K, K_1, K_2 \in \operatorname{Alg}\mathscr{N}$.

By Ringrose's theorem, the non-zero diagonal coefficients $\alpha_{\mathscr{N}}(K)$, $\alpha_{\mathscr{N}}(K_1)$ and $\alpha_{\mathscr{N}}(K_2)$ are precisely the non zero eigenvalues of K, K_1 and K_2. But

$$K = K_1 + K_2$$

so $\alpha_{\mathscr{N}}(K) = \alpha_{\mathscr{N}}(K_1) + \alpha_{\mathscr{N}}(K_2) = \alpha_{\mathscr{N}}(K_1)$ since $\sigma(K_2) = \{0\}$. This proves the theorem.

The following generalizes a theorem of Brodskii (see [7], Theorem I.5.1).

THEOREM 7.3. *Let $\mathscr{L}$ be a* CSL *and K a compact operator in* $\operatorname{Rad}\operatorname{Alg}\mathscr{L}$. *Then:*

$$K = 2\mathscr{U}(\operatorname{Re} K) = 2i\,\mathscr{U}(\operatorname{Im} K).$$

Proof. By Theorem 7.1

$$K = \mathscr{U}(K) = \mathscr{L}(K) + \mathscr{D}(K) = \mathscr{L}(K)$$

since K is in $\operatorname{Rad}\operatorname{Alg}\mathscr{L}$.

A short calculation shows that $\mathscr{L}_{\mathscr{F}}(K^*) = 0$ for any finite sublattice $\mathscr{F}$ of $\mathscr{L}$. Thus $\mathscr{L}(K^*)$ exists and is zero. Also $\mathscr{D}(K^*) = \mathscr{D}(K)^* = 0$. Thus:

$$0 = \mathscr{L}(K^*) = \mathscr{U}(K^*) - \mathscr{D}(K^*) = \mathscr{U}(K^*)$$

and hence:

$$K = \mathscr{U}(K \pm K^*) = \mathscr{L}(K \pm K^*).$$

So the integrals of both $2\operatorname{Re} K = K + K^*$ and $2i\operatorname{Im} K = K - K^*$ exist and equals K.

REMARK. Note that the converse of Brodskii's theorem ([7], Theorem I.6.1) fails for a general C.S.L., even in finite dimensions.

EXAMPLE. Consider the C.S.L. in $\mathbf{C}^3$

$$\mathscr{L} = \{0, P_1, P_2, P_3, 1\}$$

where P_1 is the projection onto the subspace spanned by the first unit vector e_1, and P_2, P_3 are the projections on the subspaces spanned by $\{e_1, e_2\}$ and $\{e_1, e_3\}$ respectively. If K is the selfadjoint operator with matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

it can easily be seen that $\mathcal{U}(K)$ exists and is zero, and hence K is not the imaginary part of $2i\mathcal{U}(K)$.

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