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MINIMAL SIGNATURE IN LIFTING OF OPERATORS. I

TIBERIU CONSTANTINESCU and AURELIAN GHEONDEA

The approach, developed in [14], of extrapolation problems in the theory of functions by means of lifting of commutants makes natural to consider problems of lifting of operators with control over the (negative) signatures, in connection with various generalizations of classical extrapolation problems as formulated in different papers [1], [12], [6] and many others. In [3] it was formulated a problem of lifting of operators with one-sided control of the negative signature and it was proved that, in the case of minimal signature and the underlying spaces of Pontryagin type, this problem has a solution similar with that in the positive definite case. In the well-known interplay between lifting of operators and lifting of commutants, this made the possibility to prove a theorem of lifting of commutants for contractions on Pontryagin spaces [9].

The aim of this paper is to obtain a variant of the commutants' lifting theorem which can be applied to problems of extrapolation of meromorphic functions as those formulated in [1], [12], [6]. In the first part of the paper, we state a framework in which such a result can be formulated, but, as Example 3.4 shows, in this case the problem of lifting of commutants has not always a solution. In Theorem 3.5 there is considered a situation when there exist solutions, this containing the results in [9], [11].

In the second part of the paper, which will be published elsewhere, we will state necessary and sufficient conditions for the existence of lifting of commutants in indefinite setting.

We have specified in Section 1 the notation and terminology concerning linear operators on Krein spaces. As general references for these we recommend [5] or [7]. Also, we have stated some auxiliary results concerning indefinite factorizations, link operators, etc. (cf. [3]) and unitary dilations (cf. [9]; a different approach was pointed out in [5], see also [4], [10]).

1. NOTATION AND PRELIMINARY RESULTS

1.1. KREIN SPACES. A *Krein space* $\mathscr{K}$ is a complex vector space endowed with an indefinite inner product $[\cdot\,,\cdot]$ such that $\mathscr{K}$ admits a decomposition $\mathscr{K} = \mathscr{K}^+ \dotplus \mathscr{K}^-$, called a *fundamental decomposition*, with the properties: $\mathscr{K}^+$ and $\mathscr{K}^-$ are subspaces of $\mathscr{K}$, orthogonal with respect to $[\cdot\,,\cdot]$ and $(\mathscr{K}^+, [\cdot,\cdot])$ and $(\mathscr{K}^-, -[\cdot\,,\cdot])$ are Hilbert spaces. With respect to such a decomposition one associates a *fundamental symmetry* J on $\mathscr{K}$,

$$J(x^+ + x^-) = x^+ - x^-, \quad x^\pm \in \mathscr{K}^\pm. \tag{1.1}$$

A fundamental symmetry, f.s. for short, J is determined by the following properties: it is a linear operator on $\mathscr{K}$ satisfying $J^2 = I$ and such that the relation

$$(x, y)_J = [Jx, y] \quad x, y \in \mathscr{K} \tag{1.2}$$

determines a positive inner product $(\cdot\,,\cdot)_J$ which turns $\mathscr{K}$ into a Hilbert space. The norm associated to a f.s. is called a *unitary norm*. All unitary norms of the Krein space $\mathscr{K}$ are equivalent, in particular one can speak about the strong topology and about linear bounded operators on Krein spaces. The *signatures*

$$\varkappa^\pm(\mathscr{K}) = \dim \mathscr{K}^\pm \tag{1.3}$$

are independent of the chosen fundamental decomposition. If $\varkappa(\mathscr{K}) = \min\{\varkappa^+(\mathscr{K}), \varkappa^-(\mathscr{K})\}$ is finite (without restricting the generality we can assume $\varkappa^-(\mathscr{K}) < \infty$) then $\mathscr{K}$ is called a *Pontryagin space*.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be Krein spaces. The *direct sum Krein space* $\mathscr{K}_1[+]\mathscr{K}_2$ is defined as follows: take J_i f.s. on $\mathscr{K}_i$, $i = 1, 2$, and consider the Hilbert space $\mathscr{K}_1 \oplus \mathscr{K}_2$. The symmetry $J = J_1 \oplus J_2$ turns this Hilbert space into the desired Krein space $\mathscr{K}_1[+]\mathscr{K}_2$. The construction does not depend on the chosen f.s. J_i.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be Krein spaces and $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. Then, its *adjoint* is defined by

$$[Tx, y] = [x, T^\# y], \quad x \in \mathscr{K}_1, y \in \mathscr{K}_2. \tag{1.4}$$

If J_1 and J_2 are f.s. on $\mathscr{K}_1$ and respectively $\mathscr{K}_2$, and T^* is the adjoint of T with respect to the Hilbert spaces $(\mathscr{K}_i, (\cdot\,,\cdot)_{J_i})$, $i = 1, 2$, then

$$T^\# = J_1 T^* J_2. \tag{1.5}$$

A (possible unbounded) operator T with domain $\mathscr{D}(T) \subset \mathscr{K}_1$ and range $\mathscr{R}(T) \subset \mathscr{K}_2$ is called *isometric* if

$$[Tx, Ty] = [x, y], \quad x, y \in \mathscr{D}(T). \tag{1.6}$$

A surjective isometry $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ is called *unitary*. A necessary and sufficient condition to exist unitary operators between $\mathscr{K}_1$ and $\mathscr{K}_2$ is that $\varkappa^{\pm}(\mathscr{K}_1) = \varkappa^{\pm}(\mathscr{K}_2)$ hold. An operator $V \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ is called a *partial isometry* if $\ker V$ is a Krein subspace of $\mathscr{K}_1$ and with respect to the decomposition $\mathscr{K}_1 = \ker V[+]\mathscr{L}_1$, $V \mid \mathscr{L}_1$ is an isometry. The operator $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ is called *contraction* if

$$[Tx, \ Tx] \leqslant [x, \ x], \quad x \in \mathscr{K}_1, \tag{1.7}$$

equivalently

$$[(I - T^{\#}T)x, \ x] \geqslant 0, \quad x \in \mathscr{K}_1, \tag{1.8}$$

i.e. $I - T^{\#}T$ is *positive*. T is called *doubly contractive* if both T and $T^{\#}$ are contractions.

1.2. INDEFINITE FACTORIZATIONS. Let $\mathscr{H}$ be a Hilbert space and $A \in \mathscr{L}(\mathscr{H})$, $A = A^{*}$. If sgn denotes the function signum and we put $S_A = \mathrm{sgn}(A)$ then S_A is the selfadjoint partial isometry appearing in the polar decomposition of A,

$$A = S_A|A|, \quad \ker(S_A) = \ker A, \quad S_A\mathscr{H} = \overline{\mathscr{R}(A)}. \tag{1.9}$$

The *signature numbers* of A are defined as follows:

$$\varkappa^{\pm}(A) = \dim\ker(I \mp S_A), \quad \varkappa^{0}(A) = \dim\ker(S_A). \tag{1.10}$$

Considering the quadratic form (Ax, x), $x \in \mathscr{H}$, we notice that $\varkappa^{-}(A)$ ($\varkappa^{+}(A)$) represents also the number of negative (positive) squares associated with it, equivalently the dimension of any subspace which is maximal negative (maximal positive) with respect to this form.

Denote by $\mathscr{H}_A$ the Krein space obtained from $\overline{\mathscr{R}(A)}$ with the inner product

$$[x, y] = (S_A x, y), \quad x, y \in \mathscr{H}_A \tag{1.11}$$

(here $(\cdot\,,\cdot)$ denotes the inner product of the Hilbert space $\mathscr{H}$). In particular $S_A|\mathscr{H}_A$ (which will be denoted simply by S_A) is a f.s. of $\mathscr{H}_A$; the corresponding fundamental decomposition is

$$\mathscr{H}_A = \mathscr{H}_A^{+}[+]\mathscr{H}_A^{-}, \tag{1.12}$$

where $\mathscr{H}_A^{+} = \ker(I - S_A)$, $\mathscr{H}_A^{-} = \ker(I + S_A)$ and

$$\varkappa^{\pm}(\mathscr{H}_A) = \varkappa^{\pm}(A) = \varkappa^{\pm}(S_A). \tag{1.13}$$

The next result can be found in [3]. Its first part was earlier obtained in [8] (for the matrix case see [13]).

1.1. PROPOSITION. *Let* $\mathcal{H}$ *be a Hilbert space,* $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$, *and let* $\mathcal{K}$ *be a Krein space with* J *a f.s. Then:*

(i) *There exists* $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *such that*

$$A = B^* J B \tag{1.14}$$

if and only if

$$\varkappa^+(\mathcal{K}) \geqslant \varkappa^+(A), \quad \varkappa^-(\mathcal{K}) \geqslant \varkappa^-(A). \tag{1.15}$$

(ii) *Suppose* (1.15) *is fulfilled; then* $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *verifies* (1.14) *if and only if*

$$B = [C|A|^{1/2}|\mathcal{H}_A \quad X] \tag{1.16}$$

(with respect to $\mathcal{H} = \mathcal{H}_A \oplus \ker A$*), where* $C: \mathcal{R}(|A|^{1/2})$ $(\subset \mathcal{H}_A) \to \mathcal{K}$ *is isometric such that* $C|A|^{1/2} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *and* $X \in \mathcal{L}(\ker A, \mathcal{K})$ *satisfies* $X^\# X = 0$ *and* $\mathcal{R}(X) \subset \subset \mathcal{R}(C)^\perp$.

In the next corollaries the notation is as in Proposition 1.1; the following one is proved in [3].

1.2. COROLLARY. *Assume that either* $\varkappa^+(A) = \varkappa^+(\mathcal{K}) < \infty$ *or* $\varkappa^-(A) = \varkappa^-(\mathcal{K}) < < \infty$. *Then the formula*

$$B = C|A|^{1/2} \tag{1.17}$$

establishes a bijective correspondence between all operators $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *satisfying* (1.14) *and all isometric operators* $C \in \mathcal{L}(\mathcal{H}_A, \mathcal{K})$. ▨

During the second section we will make intensive use of the following.

1.3. COROLLARY. *Assume that either* $\varkappa^-(A)$ *or* $\varkappa^+(A)$ *are finite and there exists* $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *with dense range and satisfying* (1.14).

Then B *is of the form* (1.17), *where* $C \in \mathcal{L}(\mathcal{H}_A, \mathcal{K})$ *is unitary.*

Proof. Assume that $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ satisfies (1.14). Then (1.15) holds and B is of the form (1.16). It is easy to see that $\mathcal{R}(X)$ is the isotropic part of $\mathcal{R}(B)$, hence if B has dense range then $X = 0$.

In particular, the representation (1.17) holds. Now, it is easy to see that, without restricting the generality, we can assume $\ker A = 0$. Further, if, say, $\varkappa^-(A) < \infty$ then we claim that

$$\varkappa^-(A) = \varkappa^-(\mathcal{K}). \tag{1.18}$$

Indeed, if this is not true then from (1.15) it follows that $\varkappa^-(\mathscr{K}) > \varkappa^-(A)$ would hold. An argument of Pontryagin Lemma type ([7, Theorem IX.1.4]) shows that there exists a finite dimensional subspace $\mathscr{L} \subset \mathscr{R}(B)$ such that $\dim \mathscr{L} > \varkappa^-(A)$ and $\mathscr{L}$ is negative in $\mathscr{K}$, i.e. $[y, y] < 0$, $y \in \mathscr{L} \setminus \{0\}$. Then, we consider the subspace $\mathscr{L}' = \{x \in \mathscr{H}_A \mid Bx \in \mathscr{L}\}$. It is easy to see that B is injective on $\mathscr{H}_A$ and that $\mathscr{L}'$ is a negative subspace in $\mathscr{H}_A$ (all these follow from (1.14)). Since $\dim(\mathscr{L}') = \dim(\mathscr{L}) > \varkappa^-(A)$, this is a contradiction. Finally, taking account of (1.18) and Corollary 1.2 it follows that $C \in \mathscr{L}(\mathscr{H}_A, \mathscr{K})$ and is isometric. Moreover if B has dense range the same holds for C; hence C is actually unitary. ▨

1.3. Some duality relations. Let $\mathscr{K}$ be a Krein space and $A \in \mathscr{L}(\mathscr{K})$, $A = A^*$. If J is an arbitrary f.s. of $\mathscr{K}$ then $(JAx, y)_J = [Ax, y]$, $x, y \in \mathscr{K}$, hence the following signature numbers are well-defined:

$$\varkappa^{\pm}[A] = \varkappa^{\pm}(JA), \quad \varkappa^0[A] = \varkappa^0(A).$$

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be Krein spaces and $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. Then, by [3, Proposition 3.1] the following identities hold

$$\varkappa^{\pm}[I - T^*T] + \varkappa^{\pm}(\mathscr{K}_2) = \varkappa^{\pm}[I - TT^*] + \varkappa^{\pm}(\mathscr{K}_1) \tag{1.19}$$

$$\varkappa^0[I - T^*T] = \varkappa^0[I - TT^*]. \tag{1.20}$$

1.4. Elementary rotations. Let $\mathscr{K}_1, \mathscr{K}_2$ be Krein spaces and $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. We fix on the Krein spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ the f.s. J_1 and, respectively, J_2 and define the following operators:

$$J_T = \operatorname{sgn}(J_1 - T^*J_2T), \quad J_{T^*} = \operatorname{sgn}(J_2 - TJ_1T^*); \tag{1.21}$$

$$D_T = |J_1 - T^*J_2T|^{1/2}, \quad D_{T^*} = |J_2 - TJ_1T^*|^{1/2}. \tag{1.22}$$

Then, the following relations hold

$$J_TD_T = D_TJ_T, \quad J_{T^*}D_{T^*} = D_{T^*}J_{T^*}; \tag{1.23}$$

$$J_TD_T^2 = J_1 - T^*J_2T, \quad J_{T^*}D_{T^*}^2 = J_2 - TJ_1T^*; \tag{1.24}$$

$$TJ_1(J_1 - T^*J_2T) = (J_2 - TJ_1T^*)J_2T; \tag{1.25}$$

$$T^*J_2(J_2 - TJ_1T^*) = (J_1 - T^*J_2T)J_1T^*. \tag{1.26}$$

The operators D_T and D_{T^*} are the *defect operators* associated to T. With respect to these, the *defect spaces* can be introduced: they are the Krein spaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ constructed from $\overline{\mathscr{R}(D_T)}$ and $\overline{\mathscr{R}(D_{T^*})}$ with respect to the f.s. J_T and, respectively, J_{T^*}.

Then, the following identities hold

$$\varkappa^{\pm}(\mathscr{D}_T) = \varkappa^{\pm}[I - T^{\#}T], \quad \varkappa^{\pm}(\mathscr{D}_{T^*}) = \varkappa^{\pm}[I - TT^{\#}]. \tag{1.27}$$

The following result is proved in [3].

1.4. PROPOSITION. *With previous notation, there exist uniquely determined operators* $L_T \in \mathscr{L}(\mathscr{D}_T, \mathscr{D}_{T^*})$ *and* $L_{T^*} \in \mathscr{L}(\mathscr{D}_{T^*}, \mathscr{D}_T)$ *such that*

$$D_{T^*}L_T = TJ_1D_T|\mathscr{D}_T, \quad D_TL_{T^*} = T^*J_2D_{T^*}|\mathscr{D}_{T^*}. \tag{1.28}$$

Moreover, the following identities hold

$$L_{T^*} = J_TL_T^*J_{T^*}|\mathscr{D}_{T^*} \tag{1.29}$$

$$(J_T - D_TJ_1D_T)|\mathscr{D}_T = L_T^*J_{T^*}L_T \tag{1.30}$$

$$(J_{T^*} - D_{T^*}J_2D_{T^*})|\mathscr{D}_{T^*} = L_{T^*}^*J_TL_{T^*}. \tag{1.31}$$

In particular, the following operator

$$R(T): \mathscr{K}_1[+]\mathscr{D}_{T^*} \to \mathscr{K}_2[+]\mathscr{D}_T$$

(1.32)

$$R(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -J_TL_{T^*} \end{bmatrix}$$

is unitary, being called the elementary rotation of T.

1.5. UNITARY AND ISOMETRIC DILATIONS. Let $\mathscr{H}$ be a Krein space and $T \in \mathscr{L}(\mathscr{H})$. An *isometric* (*unitary*) *dilation* of T is a pair $(U, \widetilde{\mathscr{H}})$ where $\widetilde{\mathscr{H}}$ is a Krein space extension of $\mathscr{H}$ and $U \in \mathscr{L}(\widetilde{\mathscr{H}})$ is an isometry (unitary operator) satisfying $T^n = P_{\mathscr{H}}^{\widetilde{\mathscr{H}}}U^n|\mathscr{H}$, $n \geqslant 0$ ($P_{\mathscr{H}}^{\widetilde{\mathscr{H}}}$ denotes the selfadjoint projection of $\widetilde{\mathscr{H}}$ onto $\mathscr{H}$). The isometric (unitary) dilation is called *minimal* if $\widetilde{\mathscr{H}} = \bigvee_{n\geqslant 0} U^n\mathscr{H}$ ($\widetilde{\mathscr{H}} = \bigvee_{n\in\mathbf{Z}} U^n\mathscr{H}$).

Two isometric (unitary) dilations $(U', \tilde{\mathscr{H}}')$ and $(U'', \tilde{\mathscr{H}}'')$ of T are *unitary equivalent* if there exists a unitary operator $W \in \mathscr{L}(\tilde{\mathscr{H}}', \tilde{\mathscr{H}}'')$ such that W acts as the identity on $\mathscr{H}$ and $WU' = U''W$.

Let us fix on $\mathscr{H}$ a f.s. J with respect to which we consider the objects D_T, J_T, $\mathscr{D}_T, \ldots$ introduced in subsection 1.4. Define the Hilbert space

$$\tilde{\mathscr{H}} = \ldots \oplus \mathscr{D}_{T^*} \oplus \mathscr{D}_{T^*} \oplus \boxed{\mathscr{H}} \oplus \mathscr{D}_T \oplus \mathscr{D}_T \oplus \ldots \tag{1.33}$$

and consider the operator $\tilde{J} \in \mathscr{L}(\tilde{\mathscr{H}})$

$$\tilde{J} = \ldots \oplus J_{T^*} \oplus J_{T^*} \oplus \boxed{J} \oplus J_T \oplus \ldots . \tag{1.34}$$

$\tilde{J}$ is a symmetry on the Hilbert space $\tilde{\mathscr{H}}$, hence, considering the inner product $[x, y] = (\tilde{J}x, y)$, $x, y \in \tilde{\mathscr{H}}$, $(\tilde{\mathscr{H}}, [\cdot, \cdot])$ becomes a Krein space and $\tilde{J}$ is a f.s. on $\tilde{\mathscr{H}}$. Defining the linear operator

$$U : \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$$

$$U = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \ldots & I & 0 & 0 & 0 & \ldots \\ \ldots & 0 & D_{T^*} & \boxed{T} & 0 & \ldots \\ \ldots & 0 & -J_T L_{T^*} & D_T & 0 & \ldots \\ \ldots & 0 & 0 & 0 & I & \ldots \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{1.35}$$

it follows from Proposition 1.4 that $(U, \tilde{\mathscr{H}})$ is a minimal unitary dilation of T.

Let us assume now that T is doubly contractive, hence $J_T | \mathscr{D}_T = I | \mathscr{D}_T$ and, $J_{T^*} | \mathscr{D}_{T^*} = I | \mathscr{D}_{T^*}$, in particular

$$\tilde{\mathscr{H}} = \mathscr{H} [+] \mathscr{P} \tag{1.36}$$

where $(\mathscr{P}, [\cdot, \cdot])$ is a Hilbert space. Consider now another minimal unitary dilation $(U', \tilde{\mathscr{H}}')$ of T. Then the linear manifolds $\mathscr{D} = \text{lin}\{U^n x \mid x \in \mathscr{H},\ n \in \mathbf{Z}\}$ and $\mathscr{D}' = \text{lin}\{U'^n y \mid y \in \mathscr{H},\ n \in \mathbf{Z}\}$ are dense in $\tilde{\mathscr{H}}$ and respectively, in $\tilde{\mathscr{H}}'$, hence nondegenerate, and the linear mapping W defined by

$$\mathscr{D} \ni \sum_{k=-N_1}^{N_2} U^k x_k \overset{W}{\mapsto} \sum_{k=-N_1}^{N_2} U'^k x_k \in \mathscr{D}' \tag{1.37}$$

is correctly defined, isometric and injective. The boundedness of W is now a consequence of (1.36) (indeed, W acts like the identity on $\mathscr{H}$ and on $\mathscr{P}$ is bounded because it is isometric). It follows that the unitary dilations $(U, \widetilde{\mathscr{H}})$ and $(U', \widetilde{\mathscr{H}}')$ are unitarily equivalent. Summing up the above considerations we have ([5] and [10])

1.4. THEOREM. (a) *Any operator* $T \in \mathscr{L}(\mathscr{H})$ *has a minimal unitary dilation.*

(b) *If T is doubly contractive then the minimal unitary dilation* $(U, \widetilde{\mathscr{H}})$ *of T is uniquely determined up to unitary equivalence and* $\varkappa^-(\widetilde{\mathscr{H}}) = \varkappa^-(\mathscr{H})$.

Let now $(U, \widetilde{\mathscr{H}})$ be the minimal unitary dilation constructed in (1.33)—(1.35). Consider the subspace $\mathscr{K}$ of $\widetilde{\mathscr{H}}$

$$\mathscr{K} = \ldots \oplus 0 \oplus 0 \oplus \boxed{\mathscr{H}} \oplus \mathscr{D}_T \oplus \mathscr{D}_T \oplus \ldots ; \tag{1.38}$$

it follows that $\mathscr{K}$ is a Krein subspace of $\widetilde{\mathscr{H}}$ (it is invariant for the f.s. $\widetilde{J}$). $\mathscr{K}$ is also invariant for U and denoting $V = U \mid \mathscr{K}$ it follows that $(V, \mathscr{K})$ is a minimal isometric dilation of T. If, in addition, T is contractive then $J_T \mid \mathscr{D}_T = I \mid \mathscr{D}_T$ hence

$$\mathscr{K} = \mathscr{H}[+]\mathscr{R} \tag{1.39}$$

where $(\mathscr{R}, [\cdot\,,\cdot])$ is a Hilbert space. The uniqueness of the minimal isometric dilation of T follows now as above (see also [11]). Thus we have ([5] and [9])

1.5. COROLLARY. (a) *Any operator* $T \in \mathscr{L}(\mathscr{H})$ *has a minimal isometric dilation.*

(b) *If T is contractive then the minimal isometric dilation* $(V, \mathscr{K})$ *is unique up to unitary equivalence and* $\varkappa^-(\mathscr{K}) = \varkappa^-(\mathscr{H})$.

In the following we describe a construction which will be used in the third section. Let $T \in \mathscr{L}(\mathscr{H})$ be arbitrary and, for any integer $n \geqslant 0$, define the Krein spaces

$$\mathscr{K}^{(n)} = \begin{cases} \mathscr{H}, & n = 0, \\ \mathscr{H}[+]\underbrace{\mathscr{D}_T[+]\ldots[+]\mathscr{D}_T}_{n \text{ copies}}, & n \geqslant 1. \end{cases} \tag{1.40}$$

$\mathscr{K}^{(n)}$ is naturally embedded into $\mathscr{K}$, the Krein space defined by (1.38), and this yields also the natural embedding of $\mathscr{K}^{(n)}$ into $\mathscr{K}^{(n+1)}$ for all integers $n \geqslant 0$. $\{\mathscr{K}^{(n)}\}_{n\geqslant 0}$

is a non-decreasing chain of subspaces such that

$$\mathscr{K} = \bigvee_{n\geqslant 0} \mathscr{K}^{(n)}. \tag{1.41}$$

Now, with respect to this chain of subspaces, the minimal isometric dilation $V = U|\mathscr{K}$, where U is defined in (1.35), produces a sequence $\{T^{(n)}\}_{n\geqslant 0}$, $T^{(n)} \in \mathscr{L}(\mathscr{K}^{(n)})$ of dilations of T,

$$T^{(n)} = P^{\mathscr{K}}_{\mathscr{K}^{(n)}} V | \mathscr{K}^{(n)}, \quad n \geqslant 0. \tag{1.42}$$

If we identify $T^{(n)}$ with its trivial extension to an operator in $\mathscr{L}(\mathscr{K})$ then

$$V = \text{so-}\lim_{n\to\infty} T^{(n)}. \tag{1.43}$$

For any integer $n \geqslant 0$, $J^{(n)} = \tilde{J} | \mathscr{K}^{(n)}$, where $\tilde{J}$ is given at (1.34), is a f.s. on $\mathscr{K}^{(n)}$, with respect to which we will consider defect operators, defect subspaces etc. on $\mathscr{K}^{(n)}$.

1.6. LEMMA. (a) *For any integer $n \geqslant 1$, $T^{(n)}$ is a partial isometry and $\mathscr{D}_{T^{(n)}}$ is naturally identified with $\mathscr{D}_T$ as Krein spaces.*

(b) *For any integer $n \geqslant 1$, $\mathscr{D}_{T^{(n)*}}$ can be embedded into $\mathscr{D}_{T^*}$ as Krein spaces. If $\mathscr{D}_{T^*}$ is separable (respectively, Pontryagin space) then it can be (naturally) identified with $\mathscr{D}_{T^{(n)*}}$.*

Proof. (a) is obvious. For (b), we compute:

$$J^{(1)} - T^{(1)}J^{(1)}T^{(1)*} = \begin{bmatrix} D_{T^*} \\ -L_T^* J_{T^*} \end{bmatrix} J_{T^*}[D_{T^*} \quad -J_{T^*}L_T] \tag{1.44}$$

where we used Proposition 1.4. From here, via Proposition 1.2, it follows that $\mathscr{D}_{T^{(1)*}}$ can be embedded into $\mathscr{D}_{T^*}$. Let us observe that the row operator

$$[\mathscr{D}_{T^*} \quad -J_{T^*}L_T] \in \mathscr{L}(\mathscr{H}[+]\mathscr{D}_T, \mathscr{D}_{T^*}) \tag{1.45}$$

has dense range. Assuming that $\mathscr{D}_{T^*}$ is separable, an argument of Pontryagin Lemma type (as in the proof of Corollary 1.3) shows that the ranks of negativity and positivity of $\mathscr{D}_{T^*}$ and $\mathscr{D}_{T^{(1)*}}$ are the same, hence they are unitary equivalent. On the other hand, if $\mathscr{D}_{T^*}$ is a Pontryagin space, then application of Corollary 1.3 to the factorization (1.44) yields the natural identification of $\mathscr{D}_{T^*}$ and $\mathscr{D}_{T_{1;*}}$. The case $n \geqslant 2$ reduces to $n = 1$. ▨

2. A LIFTING THEOREM

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be Krein spaces and $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. We assume that the signature numbers $\varkappa_1 = \varkappa^-[I - T^* T]$ and $\varkappa_2 = \varkappa^-[I - TT^*]$ are both finite. From (1.19) we have the following identity

$$\varkappa_1 + \varkappa^-(\mathscr{K}_2) = \varkappa_2 + \varkappa^-(\mathscr{K}_1), \tag{2.1}$$

in particular, $\varkappa^-(\mathscr{K}_1)$ and $\varkappa^-(\mathscr{K}_2)$ are simultaneously finite or not. Let also $\mathscr{K}_1'$ and $\mathscr{K}_2'$ be Krein spaces and denote $\widetilde{\mathscr{K}}_1 = \mathscr{K}_1[+]\mathscr{K}_1'$ and $\widetilde{\mathscr{K}}_2 = \mathscr{K}_2[+]\mathscr{K}_2'$. Given two natural numbers $\tilde{\varkappa}_1$ and $\tilde{\varkappa}_2$, the lifting problem that we consider in this section is the following:

$$(*)\qquad \begin{cases} \textit{Determine all operators } \widetilde{T} \in \mathscr{L}(\widetilde{\mathscr{K}}_1, \widetilde{\mathscr{K}}_2) \textit{ such that} \\ P^{\widetilde{\mathscr{K}}_2}_{\mathscr{K}_2}\widetilde{T}\,|\,\mathscr{K}_1 = T,\ \varkappa^-[I - \widetilde{T}^*\widetilde{T}] = \tilde{\varkappa}_1 \textit{ and } \varkappa^-[I - \widetilde{T}\widetilde{T}^*] = \tilde{\varkappa}_2. \end{cases}$$

The approach adopted here requires to consider first the particular problems corresponding to rows (i.e. $\mathscr{K}_2' = 0$):

$$(*)_{\mathrm{r}}\qquad \begin{cases} \textit{Determine all operators } T_{\mathrm{r}} \in \mathscr{L}(\widetilde{\mathscr{K}}_1, \mathscr{K}_2) \textit{ such that} \\ T_{\mathrm{r}}\,|\,\mathscr{K}_1 = T,\ \varkappa^-[I - T_{\mathrm{r}}^{\#} T_{\mathrm{r}}] = \tilde{\varkappa}_1 \textit{ and } \varkappa^-[I - T_{\mathrm{r}}T_{\mathrm{r}}^{\#}] = \tilde{\varkappa}_2 \end{cases}$$

and columns (i.e. $\mathscr{K}_1' = 0$):

$$(*)_{\mathrm{c}}\qquad \begin{cases} \textit{Determine all operators } T_{\mathrm{c}} \in \mathscr{L}(\mathscr{K}_1, \widetilde{\mathscr{K}}_2) \textit{ such that} \\ P^{\widetilde{\mathscr{K}}_2}_{\mathscr{K}_2} T_{\mathrm{c}} = T,\ \varkappa^-[I - T_{\mathrm{c}}^{\#} T_{\mathrm{c}}] = \tilde{\varkappa}_1 \textit{ and } \varkappa^-[I - T_{\mathrm{c}}T_{\mathrm{c}}^{\#}] = \tilde{\varkappa}_2. \end{cases}$$

In the following we fix f.s. J_i, J_i' on $\mathscr{K}_i$ and, respectively, $\mathscr{K}_i'$, $i = 1, 2$. With respect to these we will consider defect operators, defect spaces, etc.

2.1. LEMMA. (i) *If problem* $(*)_{\mathrm{r}}$ *has at least one solution then the following relations hold:*

$$\tilde{\varkappa}_1 + \varkappa^-(\mathscr{K}_2) = \tilde{\varkappa}_2 + \varkappa^-(\mathscr{K}_1) + \varkappa^-(\mathscr{K}_1'), \tag{2.2}$$
$$\tilde{\varkappa}_1 \geqslant \varkappa_1, \quad \tilde{\varkappa}_2 + \varkappa^-(\mathscr{K}_1') \geqslant \varkappa_2.$$

(ii) *Assume* $\varkappa^-(\mathscr{K}_1') < \infty$ *and* $\tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}_1') \geqslant 0$. *Then the problem* $(*)$ *has solutions if and only if* $\tilde{\varkappa}_1 = \varkappa_1$. *In this case, the formula*

$$T_r = [T \quad D_{T^*}\Gamma] \tag{2.3}$$

establishes a bijective correspondence between the set of solutions of problem $(*)_r$ *and the set of contractions* $\Gamma \in \mathscr{L}(\mathscr{K}_1', \mathscr{D}_{T^*})$.

Moreover, in this situation, the operators $U(T_r)$ *and* $U_*(T_r)$ *defined by*

$$\begin{cases} U(T_r): \mathscr{D}_{T_r} \to \mathscr{D}_T[+]\mathscr{D}_\Gamma \\ U(T_r)D_{T_r} = \begin{bmatrix} D_T & -J_T L_{T^*}\Gamma \\ 0 & D_\Gamma \end{bmatrix} \end{cases} \tag{2.4}$$

and

$$\begin{cases} U_*(T_r): \mathscr{D}_{T_r^*} \to \mathscr{D}_{\Gamma^*} \\ U_*(T_r)D_{T_r^*} = D_{\Gamma^*}D_{T^*} \end{cases} \tag{2.5}$$

are unitary.

Proof. In the following we denote $\tilde{J}_1 = J_1 \oplus J_1'$ the f.s. which is fixed on $\tilde{\mathscr{K}}_1$.

(i) Let us assume that the problem $(*)_r$ has solutions.

Then the identity from (2.2) is a direct consequence of (1.19) written for T_r. Also, if we denote

$$T_r = [T \quad A] \tag{2.6}$$

then

$$J_1 - T^*J_2T = P^{\tilde{\mathscr{K}}_1}_{\mathscr{K}_1}(\tilde{J}_1 - T_r^*J_2T_r)\,|\,\mathscr{K}_1 \tag{2.7}$$

hence the first inequality in (2.2) follows from Proposition 1.1(i).

On the other hand, the identity

$$J_2 - T_r\tilde{J}_1T_r^* = J_2 - TJ_1T^* - AJ_1'A^* \tag{2.8}$$

can be written in the form

$$J_2 - TJ_1T^* = [A \quad D_{T_r^*}]\,[J_1' \oplus J_{T_r^*}]\,[A \quad D_{T_r^*}]^* \tag{2.9}$$

whence, also using Proposition 1.1(i), the second inequality in (2.2) follows.

(ii) Assuming $\varkappa^-(\mathscr{K}_1') < \infty$ and $\tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}_1') \geqslant 0$, let T_r be a solution to problem $(*)_r$. Then (2.9) holds and we can apply Corollary 2.6 in [3] and get a uniquely determined isometric operator $\Lambda \in \mathscr{L}(\mathscr{D}_{T^*}, \mathscr{K}_1'[+]\mathscr{D}_{T_r^*})$ such that

$$\begin{bmatrix} A^* \\ D_{T_r^*} \end{bmatrix} = \Lambda D_{T^*} \tag{2.10}$$

hence

$$\Lambda = D_{T^*}\Gamma \tag{2.11}$$

for a certain $\Gamma \in \mathscr{L}(\mathscr{K}_1', \mathscr{D}_{T^*})$. Now, we observe that the following factorizations hold:

$$D_{T_r^*}J_{T_r^*}D_{T_r^*} = J_2 - T_r\tilde{J}_1T_r^* = D_{T^*}(J_{T^*} - \Gamma J_1'\Gamma^*)D_{T^*} = D_{T^*}D_{\Gamma^*}J_{\Gamma^*}D_{\Gamma^*}D_{T^*} \tag{2.12}$$

and

$$D_{T_r}J_{T_r}D_{T_r} = \begin{bmatrix} D_T & 0 \\ -\Gamma^*L_{T^*}^*J_T & D_\Gamma \end{bmatrix}\begin{bmatrix} J_T & 0 \\ 0 & J_\Gamma \end{bmatrix}\begin{bmatrix} D_T & -J_TL_{T^*}\Gamma \\ 0 & D_\Gamma \end{bmatrix}. \tag{2.13}$$

Since the operators $D_{\Gamma^*}D_{T^*} : \mathscr{K}_2 \to \mathscr{D}_{\Gamma^*}$ and

$$\begin{bmatrix} D_T & -J_TL_{T^*}\Gamma \\ 0 & D_\Gamma \end{bmatrix} : \begin{matrix} \mathscr{K}_1 \\ [+] \\ \mathscr{K}_1' \end{matrix} \longrightarrow \begin{matrix} \mathscr{D}_T \\ [+] \\ \mathscr{D}_\Gamma \end{matrix}$$

have dense ranges, application of Corollary 1.3 to these factorizations is possible and one gets the unitary operators $U(T_r)$ and $U_*(T_r)$ defined by (2.4) and (2.5). In particular, the following identities hold:

$$\tilde{\varkappa}_1 = \varkappa_1 + \varkappa^-(J_1' - \Gamma^*J_{T^*}\Gamma) \tag{2.14}$$

and

$$\tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}_1') = \varkappa^-(J_{T^*} - \Gamma J_1'\Gamma^*). \tag{2.15}$$

Writing the duality relation (1.19) for Γ we have

$$\varkappa^-(J_1' - \Gamma^*J_{T^*}\Gamma) + \varkappa_2 = \varkappa^-(J_{T^*} - \Gamma J_1'\Gamma^*) + \varkappa^-(\mathscr{K}_1'), \tag{2.16}$$

and then using (2.15) in (2.16) we obtain $\varkappa^-(J_1' - \Gamma^*J_{T^*}\Gamma) = 0$, i.e. Γ is contractive, and now (2.14) gives $\tilde{\varkappa}_1 = \varkappa_1$.

Conversely, assuming $\tilde{\varkappa}_1 = \varkappa_1$ it is easy to see that there exist contractions $\Gamma \in \mathscr{L}(\mathscr{K}_1', \mathscr{D}_{T^*})$. Taking T_r as in (2.6) with A given by (2.11) for some contraction $\Gamma \in \mathscr{L}(\mathscr{K}_1', \mathscr{D}_{T^*})$ the factorizations (2.12) and (2.13) still hold. Since Γ being contraction implies (by (2.16)) $\varkappa^-(J_{T^*} - \Gamma J_1' \Gamma^*) = \varkappa_2 - \varkappa^-(\mathscr{K}_1')$, the application of Corollary 1.3 to (2.12) and (2.13) gives $\varkappa^-(J_2 - T_r \tilde{J}_1 T_r^*) = \tilde{\varkappa}_2$ and $\tilde{\varkappa}^-(\tilde{J}_1 - T_r^* J_2 T_r) = \tilde{\varkappa}_1$, hence T_r is a solution for problem $(*)_r$.

The fact that the correspondence given in (2.3) is a bijection is clear. ▨

The corresponding result for the problem $(*)_c$ can be obtained by duality from Lemma 2.1. This is contained in the following.

2.2. LEMMA. (i) *If problem* $(*)_c$ *has at least one solution then the following relations hold:*

$$\tilde{\varkappa}_1 + \varkappa^-(\mathscr{K}_2) + \varkappa^-(\mathscr{K}_2') = \tilde{\varkappa}_2 + \varkappa^-(\mathscr{K}_1),$$

(2.17)

$$\tilde{\varkappa}_1 + \varkappa^-(\mathscr{K}_2') \geqslant \varkappa_1, \quad \tilde{\varkappa}_2 \geqslant \varkappa_2.$$

(ii) *Assume* $\varkappa^-(\mathscr{K}_2') < \infty$ *and* $\tilde{\varkappa}_1 = \varkappa_1 - \varkappa^-(\mathscr{K}_2') \geqslant 0$. *Then the problem* $(*)_c$ *has solutions if and only if* $\tilde{\varkappa}_2 = \varkappa_2$. *In this case, the formula*

$$T_c = \begin{bmatrix} T \\ \Gamma D_T \end{bmatrix} \tag{2.18}$$

establishes a bijective correspondence between the set of solutions of problem $(*)_c$ *and the set of operators* $\Gamma \in \mathscr{L}(\mathscr{D}_T, \mathscr{K}_2')$ *such that* $\Gamma^\#$ *is contractive. Moreover, in this situation, the operators* $U(T_c)$ *and* $U_*(T_c)$ *defined by*

$$\begin{cases} U(T_c) : \mathscr{D}_{T_c} \to \mathscr{D}_\Gamma \\ U(T_c) D_{T_c} = D_\Gamma D_T \end{cases} \tag{2.19}$$

and

$$\begin{cases} U_*(T_c) : \mathscr{D}_{T_c^*} \to \mathscr{D}_{T^*}[+]\mathscr{D}_{\Gamma^*} \\ U_*(T_c) D_{T_c^*} = \begin{bmatrix} D_{T^*} & -J_T L_T \Gamma^* \\ 0 & D_{\Gamma^*} \end{bmatrix} \end{cases} \tag{2.20}$$

are unitary.

We can now prove a result concerning problem (*).

2.3. THEOREM. (i) *If problem (*) has solutions then the following relations hold*

$$(2.21)\qquad \begin{cases} \tilde{\varkappa}_1 + \varkappa^-(\mathscr{K}_2) + \varkappa^-(\mathscr{K}_2') = \tilde{\varkappa}_2 + \varkappa^-(\mathscr{K}_1) + \varkappa^-(\mathscr{K}_1') \\ \tilde{\varkappa}_1 + \varkappa^-(\mathscr{K}_2') \geqslant \varkappa_1, \quad \tilde{\varkappa}_2 + \varkappa^-(\mathscr{K}_1') \geqslant \varkappa_2. \end{cases}$$

(ii) *Assume* $\varkappa^-(\mathscr{K}_1'), \varkappa^-(\mathscr{K}_2') < \infty$, $\tilde{\varkappa}_1 = \varkappa_1 - \varkappa^-(\mathscr{K}_2') \geqslant 0$ *and* $\tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}_1') \geqslant 0$. *Then the formula*

$$(2.22)\qquad \tilde{T} = \begin{bmatrix} T & D_{T^*}\Gamma_1 \\ \Gamma_2 D_T & -\Gamma_2 L_T^* J_{T^*}\Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1} \end{bmatrix}$$

establishes a bijective correspondence between the set of solutions $\tilde{T}$ *of problem* (*) *and the set of triplets* $\{\Gamma_1, \Gamma_2, \Gamma\}$ *where* $\Gamma_1 \in \mathscr{L}(\mathscr{K}_1', \mathscr{D}_{T^*})$, $\Gamma_2^\# \in \mathscr{L}(\mathscr{K}_2', \mathscr{D}_T)$ *and* $\Gamma \in \mathscr{L}(\mathscr{D}_{\Gamma_1}, \mathscr{D}_{\Gamma_2^*})$ *are contractions. Moreover, in this situation the operators* $U(\tilde{T})$ *and* $U_*(\tilde{T})$ *defined by*

$$(2.23)\qquad \begin{cases} U(\tilde{T}) : \mathscr{D}_{\tilde{T}} \to \mathscr{D}_{\Gamma_2}[+]\mathscr{D}_\Gamma \\ U(\tilde{T})D_{\tilde{T}} = \begin{bmatrix} D_{\Gamma_2}D_T & -(D_{\Gamma_2}L_T^* J_{T^*}\Gamma_1 + J_{\Gamma_2}L_{\Gamma_2}^*\Gamma D_{\Gamma_1}) \\ 0 & D_\Gamma D_{\Gamma_1} \end{bmatrix} \end{cases}$$

and

$$(2.24)\qquad \begin{cases} U_*(\tilde{T}) : \mathscr{D}_{\tilde{T}^*} \longrightarrow \mathscr{D}_{\Gamma_1^*}[+]\mathscr{D}_{\Gamma^*} \\ U_*(\tilde{T}) = U(\tilde{T}^*) \end{cases}$$

are unitary operators.

Proof. (i) Let $\tilde{T}$ be a solution to problem (*). The identity in (2.21) follows from (1.19) written for $\tilde{T}$ instead of T. Regarding $\tilde{T}$ as a row extension of its left-hand column, the inequalities from (2.21) can be immediately obtained by the corresponding inequalities from Lemma 2.1 and Lemma 2.2.

(ii) Let us assume now that $\varkappa^-(\mathscr{K}_1'), \varkappa^-(\mathscr{K}_2') < \infty$ and

$$(2.25)\qquad \tilde{\varkappa}_1 = \varkappa_1 - \varkappa^-(\mathscr{K}_2') \geqslant 0, \quad \tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}_1') \geqslant 0$$

hold. Also, assuming that problem (*) has a solution, let $\tilde{T}$ be one of them. We represent $\tilde{T}$ with respect to $\tilde{\mathscr{K}}_i = \mathscr{K}_i[+]\mathscr{K}'_i$ by

$$\tilde{T} = \begin{bmatrix} T & A \\ B & X \end{bmatrix} \tag{2.26}$$

and let T_r and T_c denote its upper row and respectively left-hand column. Regarding $\tilde{T}$ as a row extension of T_c, from the second identity in (2.25) and Lemma 2.1 we obtain

$$\varkappa^-[I - T_c^\# T_c] = \tilde{\varkappa}_1 = \varkappa_1 - \varkappa^-(\mathscr{K}'_2) \tag{2.27}$$

and

$$\tilde{T} = [T_c \quad D_{T_c^*}\Delta] \tag{2.28}$$

for a certain contraction $\Delta \in \mathscr{L}(\mathscr{K}'_1, \mathscr{D}_{T^*})$. Further, from (2.27) and Lemma 2.2 we obtain $\varkappa^-[I - T_c T_c^\#] = \varkappa_2$ and

$$T_c = \begin{bmatrix} T \\ \Gamma_2 D_T \end{bmatrix} \tag{2.29}$$

for a certain contraction $\Gamma_2^\# : \mathscr{K}'_2 \to \mathscr{D}_T$.

Similarly, regarding $\tilde{T}$ as a column extension of T_r, from the first identity in (2.25) and Lemma 2.2 it follows

$$\varkappa^-[I - T_r T_r^\#] = \tilde{\varkappa}_2 = \varkappa_2 - \varkappa^-(\mathscr{K}'_1) \tag{2.30}$$

and then, by Lemma 2.1, we obtain $\varkappa^-[I - T_r^\# T_r] = \varkappa_1$ and

$$T_r = [T \quad D_{T^*}\Gamma_1] \tag{2.31}$$

for a certain contraction $\Gamma_1 \in \mathscr{L}(\mathscr{K}'_1, \mathscr{D}_{T^*})$.

The rest of the proof follows as in [3, Theorem 5.3]. ▨

2.4. Remarks. (a) If $\mathscr{K}_1$ and $\mathscr{K}_2$ are Pontryagin spaces the above Theorem 2.3 coincides with [3, Theorem 5.3]. Also, if $\varkappa^-(\mathscr{K}'_1) = \varkappa^-(\mathscr{K}'_2) = \varkappa_1 = \tilde{\varkappa}_1 = \varkappa_2 = \tilde{\varkappa}_2 = 0$, the fact that formula (2.22) gives all the solutions of problem (*) is proved in [11].

(b) Assuming that the hypothesis in Theorem 2.3 (ii) hold, we define

$$\hat{\Gamma}_1 = I \oplus \Gamma_1, \quad \hat{\Gamma}_2 = I \oplus \Gamma_2, \quad \hat{\Gamma} = 0 \oplus \Gamma \tag{2.32}$$

and the formula (2.22) can be written as follows

$$\tilde{T} = \hat{\Gamma}_2 R(T)\hat{\Gamma}_1 + D_{\hat{\Gamma}_2^*}\hat{\Gamma} D_{\hat{\Gamma}_1}. \tag{2.33}$$

(c) Using the above Theorem 2.3, one can say, in some cases, when a given operator T can be lifted to operators in certain classes (e.g. contractive, doubly contractive, isometric, unitary etc.) and to describe (some of) them in terms of the parameters Γ_1, Γ_2 and Γ.

2.5. COROLLARY. *Assume* $\varkappa^-(\mathscr{K}_1') = \varkappa^-(\mathscr{K}_2') = \varkappa_1 = \tilde{\varkappa}_1 = \varkappa_2 = \tilde{\varkappa}_2 = 0$. *Then the set of solutions of problem* (∗) *is uniformly bounded, more precisely for any solution* $\tilde{T}$ *of problem* (∗) *it holds*

$$\|\tilde{T}\| \leqslant \|R(T)\| + 1.$$

Proof. We consider the unitary norms $\|\cdot\|$ associated to f.s. $\tilde{J}_i = J_i \oplus I$ on $\tilde{\mathscr{K}}_i$, $i = 1, 2$. Then, observe that in this case, the operators $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ and $\hat{\Gamma}$ from (2.32) are Hilbert space contractions hence the representation (2.33) yields immediately the estimation for $\|\tilde{T}\|$. ∎

3. STRICTLY INTERTWINING DILATIONS

Let $\mathscr{H}_i$ be Krein spaces and let $T_i \in \mathscr{L}(\mathscr{H}_i)$ be contractions, $i = 1, 2$.

We fix a f.s. J_i on $\mathscr{H}_i$ with respect to which will be considered defect spaces, defect operators etc. Denote by $(V_i, \mathscr{K}_i)$ the minimal isometric dilation of T_i in the representation given in (1.38) and consider the chain of subspaces $\{\mathscr{K}_i^{(n)}\}_{n\geqslant 0}$ introduced in (1.40) and the sequence of partial isometries $\{T_i^{(n)}\}_{n\geqslant 1}$ defined in (1.42).

Let $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be an operator which intertwines T_1 and T_2, i.e.

$$AT_1 = T_2A. \tag{3.1}$$

We define the set of *strictly intertwining dilations of order* $n \geqslant 0$, by

$$\mathfrak{S}^{(n)}(A;\ T_1,\ T_2) = \{A_n \in \mathscr{L}(\mathscr{K}_1^{(n)},\ \mathscr{K}_2^{(n)}) \mid P_{\mathscr{H}_2}^{\mathscr{K}_2^{(n)}} A_n = AP_{\mathscr{H}_1}^{\mathscr{K}_1^{(n)}},$$

$$A_nT_1^{(n)} = T_2^{(n)}A_n,\ \varkappa^-[I - A_n^{\#}A_n] = \varkappa^-[I - A^{\#}A],\ \varkappa^-[I - A_nA_n^{\#}] = \varkappa^-[I - A\,A^{\#}]\}.$$

It is convenient to remark that an operator $A_n \in \mathscr{L}(\mathscr{K}_1^{(n)}, \mathscr{K}_2^{(n)})$ satisfies the first two conditions in the definition of $\mathfrak{S}^{(n)}(A\,;\,T_1, T_2)$ if and only if, with respect to the decompositions (1.40) of $\mathscr{K}_i^{(n)}$, it has a lower triangular matrix $A_n = (X_{ij})$, where $X_{00} = A$, $X_{ij} \in \mathscr{L}(\mathscr{D}_{T_1}, \mathscr{D}_{T_2})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant i+1$, and satisfy the relations

$$X_{11}T_1 + X_{12}D_{T_1} = D_{T_2}A, \tag{3.2}$$

$$X_{k1}T_1 + X_{k2}D_{T_1} = X_{k-1,1}, \quad 2 \leqslant k \leqslant n, \tag{3.3}$$

$$X_{kp} = X_{k-1,p-1}, \quad 2 \leqslant k \leqslant n,\ 3 \leqslant p \leqslant k+1. \tag{3.4}$$

Let $m \geqslant n \geqslant 0$ be given. For any $A_m \in \mathfrak{S}^{(m)}(A\,;T_1, T_2)$ denote

$$\varphi_{mn}(A_m) = P^{\mathscr{K}_2^{(m)}}_{\mathscr{K}_2^{(n)}} A_m \mid \mathscr{K}_1^{(n)}. \tag{3.5}$$

A direct application of the above remark shows that $\varphi_{mn}(A_m)$ satisfies the first two conditions in the definition of $\mathfrak{S}^{(n)}(A\,;T_1, T_2)$. Moreover, since $\mathscr{D}_{T_1}$ and $\mathscr{D}_{T_2}$ are Hilbert spaces, from Theorem 2.3(i) we have

$$\varkappa^-[I - A^\# A] \leqslant \varkappa^-[I - \varphi_{mn}(A_m)^\# \varphi_{mn}(A_m)] \leqslant \varkappa^-[I - A_m^\# A_m] \tag{3.6}$$

$$\varkappa^-[I - AA^\#] \leqslant \varkappa^-[I - \varphi_{mn}(A_m)\varphi_{mn}(A_m)^\#] \leqslant \varkappa^-[I - A_m A_m^\#] \tag{3.7}$$

hence the mapping φ_{mn} takes values in $\mathfrak{S}^{(n)}(A\,;T_1, T_2)$. It is now easy to see that $\{\mathfrak{S}^{(n)}(A\,;T_1, T_2)\}_{n\geqslant 0}$ with the canonical mappings $\{\varphi_{mn}\}_{m\geqslant n\geqslant 0}$ verifies the axioms of a projective system of sets, and let us consider its *projective limit*

$$\varprojlim_{n\geqslant 0} \mathfrak{S}^{(n)}(A\,;T_1, T_2) = \{(A_n)_{n\geqslant 0} \mid A_n \in \mathfrak{S}^{(n)}(A\,;T_1, T_2),\ A_n = \\ = \varphi_{n+1,n}(A_{n+1})\}. \tag{3.8}$$

Let us observe that up to now, nothing enables us to say that this projective limit is non-void. This is our first problem. The second one is to find a labelling of the objects of the projective limit from (3.8) in case when it is non-void.

In order to relate these problems with other results we introduce the set of *strictly intertwining dilations* of A, by

$$\mathfrak{S}(A\,;T_1, T_2) = \{A_\infty \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2) \mid P^{\mathscr{K}_2}_{\mathscr{H}_2} A_\infty = AP^{\mathscr{K}_1}_{\mathscr{H}_1},\ A_\infty V_1 = V_2 A_\infty,$$

$$\varkappa^-[I - A_\infty^\# A_\infty] = \varkappa^-[I - A^\# A],$$

$$\varkappa^-[I - A_\infty A_\infty^\#] = \varkappa^-[I - AA^\#]\}.$$

3.1. LEMMA. *The set* $\Xi(A\,;\,T_1,\,T_2)$ *is naturally embedded into* $\varprojlim_{n\geq 0}\Xi^{(n)}(A;\,T_1,\,T_2)$, *more precisely, each operator* $A_\infty\in\Xi(A;\,T_1,\,T_2)$ *can be identified with the sequence* $(A_n)_{n\geq 0}$, *where*

$$A_n = P^{\mathscr{K}_2}_{\mathscr{K}_2^{(n)}}A_\infty\,|\,\mathscr{K}_1^{(n)},\quad n\geq 0. \tag{3.9}$$

With respect to this embedding, $\Xi(A\,;\,T_1,\,T_2)$ *coincides with the set of those sequences* $(A_n)_{n\geq 0}\in\varprojlim_{n\geq 0}\Xi^{(n)}(A\,;\,T_1,\,T_2)$ *which are uniformly bounded, and in this case*

$$A_\infty = \operatorname{so-lim}_{n\to\infty} A_n \tag{3.10}$$

where for each $n\geq 0$, A_n *is identified with its trivial extension to an operator in* $\mathscr{L}(\mathscr{K}_1,\,\mathscr{K}_2)$.

The proof of this lemma is straightforward and we omit it.

3.2. LEMMA. *If the operator* A *is doubly contractive, then*

$$\Xi(A\,;\,T_1,\,T_2) = \varprojlim_{n\geq 0}\Xi^{(n)}(A;\,T_1,\,T_2). \tag{3.11}$$

Proof. In view of Lemma 3.1, we have to prove that any sequence $(A_n)_{n\geq 0}\in\varprojlim_{n\geq 0}\Xi^{(n)}(A\,;\,T_1,\,T_2)$ is uniformly bounded, which is a direct consequence of Corollary 2.5. ∎

We present now a criterion which ensures that $\varprojlim_{n\geq 0}\mathfrak{S}^{(n)}(A\,;\,T_1,\,T_2)$ is non-void. Let us denote $C = AT_1\ (=T_2A)$. Then

$$J_1 - A^* J_2 C = [D_{T_1}\ \ T_1^* D_A][I\oplus J_A][D_{T_1}\ \ T_1^* D_A]^*$$

hence by Proposition 2.1 in [3], it follows that

$$\varkappa^-[I - C^\# C]\leq \varkappa^-[I - A^\# A]. \tag{3.12}$$

Similarly,

$$J_2 - CJ_1C^* = [D_{A^*}\ \ AD_{T_1^*}][J_{A^*}\oplus J_{T_1^*}][D_{A^*}\ \ AD_{T_1^*}]^*$$

and this yields

$$\varkappa^-[I - CC^\#]\leq\varkappa^-[I - AA^\#] + \varkappa^-[I - T_1T_1^\#]. \tag{3.13}$$

In the following we denote $\varphi_n = \varphi_{n+1,n}$ for any integer $n \geqslant 0$.

3.3. LEMMA. *Assume that* $\varkappa^-[I - A^\# A]$, $\varkappa^-[-AA^\#]$ *and* $\varkappa^-[I - T_1 T_1^\#]$ *are all finite and also that* (3.12) *and* (3.13) *both hold with equality. Then the canonical mapping* φ_0 $(=\varphi_{10})$ *is surjective, equivalently* $\mathfrak{S}^{(1)}(A\,;\,T_1, T_2)$ *is non-void.*

Proof. We search for an operator $A_1 \in \mathscr{L}(\mathscr{K}_1^{(1)}, \mathscr{K}_2^{(1)})$ of the form

$$A_1 = \begin{bmatrix} A & 0 \\ X_{11} & X_{12} \end{bmatrix} \tag{3.14}$$

such that (3.2) holds. Considering the operator $B_1 = A_1 R(T_1) \in \mathscr{L}(\mathscr{H}_1[+]\mathscr{D}_{T_1^*}, \mathscr{H}_2[+]\mathscr{D}_{T_2})$ we get

$$B_1 = \begin{bmatrix} AT_1 & AD_{T_1^*} \\ D_{T_2}A & S_1 \end{bmatrix} \tag{3.15}$$

where $S_1 = X_{11}D_{T_1^*} - X_{12}J_{T_1}L_{T_1^*}$. Since the elementary rotation $R(T_1)$ is unitary, we have also

$$\varkappa^-[I - B_1^\# B_1] = \varkappa^-[I - A_1^\# A_1], \quad \varkappa^-[I - B_1 B_1^\#] = \varkappa^-[I - A_1 A_1^\#]$$

hence A_1 would be in $\mathfrak{S}^{(1)}(A\,;\,T_1, T_2)$ if we can show the existence of an operator B_1 as in (3.15) such that $\varkappa^-[I - B_1^\# B_1] = \varkappa^-[I - A^\# A]$, $\varkappa^-[I - B_1 B_1^\#] = \varkappa^-[I - AA^\#]$. But,

$$J_2 - [AT_1 \;\; AD_{T_1^*}]\,[J_1 \oplus J_{T_1^*}]\,[AT_1 \;\; AD_{T_1^*}]^* = J_2 - AJ_1A^*$$

and taking account that, by assumption, in (3.13) equality holds, application of Lemma 2.1 yields a unique contraction $X_1 \in \mathscr{L}(\mathscr{D}_{T_1^*}, \mathscr{D}_{C^*})$ such that $AD_{T_1^*} = D_{C^*}X_1$. Similarly, since

$$J_1 - [A^*T_2^* \;\; A^*D_{T_2}]\,[J_2 \oplus J_{T_2}]\,[A^*T_2^* \;\; A^*D_{T_2}]^* = J_1 - A^*J_2A$$

and taking account that, by assumption, in (3.12) equality holds, application of Lemma 2.2 yields a unique operator $X_2 \in \mathscr{L}(\mathscr{D}_C, \mathscr{D}_{T_2})$ such that $X_2^\#$ is contractive and $D_{T_2}A = X_2D_C$. Finally, application of Theorem 2.3 shows that we can choose

$$S_1 = -X_2L_C^*J_{C^*}X_1 + D_{X_2^*}\Gamma D_{X_1}$$

with $\Gamma \in \mathscr{L}(\mathscr{D}_{X_1}, \mathscr{D}_{X_2^*})$ contraction. ▨

The next example shows that in order that $\mathfrak{S}^{(1)}(A\,;\,T_1,\,T_2)$ be non-void, the assumption in Lemma 3.3 concerning equality in (3.12) and (3.13) is not of technical nature.

3.4. EXAMPLE. We consider the Hilbert space $\mathbf{C}^2$ and the operators

$$T_1 = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & \sqrt{17}/8 \\ 4/\sqrt{17} & 0 \end{bmatrix} \quad A = \begin{bmatrix} \sqrt{17}/4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then T_1 and T_2 are contractions, $\varkappa^-(I - A^*A) = 1 = \varkappa^-(I - AA^*)$ and $C = \begin{bmatrix} 0 & \sqrt{17}/8 \\ 1 & 0 \end{bmatrix}$, in particular C is also a contraction hence in (3.12) strict inequality holds. On the other hand, in view of the proof of Lemma 3.3, $\mathfrak{S}^{(1)}(A\,;\,T_1,\,T_2)$ is non-void if and only if there exist complex numbers α and β such that (see (3.15)) for

$$B_1 = \begin{bmatrix} 0 & \sqrt{17}/8 & \sqrt{51}/8 \\ 1 & 0 & 0 \\ 1/4 & 0 & \alpha \\ 0 & \sqrt{47}/8 & \beta \end{bmatrix}$$

the matrix $I - B_1^*B_1$ has only one negative eigenvalue. Performing elementary operations in this matrix it follows that for any choice of complex numbers α and β it always has two negative eigenvalues, hence $\mathfrak{S}^{(1)}(A\,;\,T_1,\,T_2) = \emptyset$ in this case. □

3.5. THEOREM. *Assume that* $\varkappa^-[I - A^\# A]$, $\varkappa^-[I - AA^\#]$ *and* $\varkappa^-[I - T_1T_1^\#]$ *are all finite, and in addition* $\varkappa^-[I - T_1T_1^\#] = \varkappa^-[I - T_2T_2^\#]$.

If (3.12) *and* (3.13) *both hold with equality then for any* $m \geqslant n \geqslant 0$ *the canonical mappings* φ_{mn} *are surjective, in particular the set* $\varprojlim_{n\geqslant 0} \mathfrak{S}^{(n)}(A\,;\,T_1,\,T_2)$ *is non-void.*

Proof. Since $\{\varphi_{mn}\}_{m\geqslant n\geqslant 0}$ is a projective system it is sufficient to prove that for any $n \geqslant 0$ the mappings $\varphi_n = \varphi_{n+1,n}$ are surjective.

The case $n = 0$ is contained in Lemma 3.3 so that assume $n \geqslant 1$ and take $A_n \in \mathfrak{S}^{(n)}(A\,;\,T_1,\,T_2)$. A_n has a lower triangular matrix (X_{ij}), $X_{00} = A$ and such that (3.2) and (3.3) hold. Consider the operator $C_n = A_nT_1^{(n)} = T_2^{(n)}A_n \in \mathscr{L}(\mathscr{K}_1^{(n)}, \mathscr{K}_2^{(n)})$ and we will show that

$$\varkappa^-[I - C_n^\# C_n] = \varkappa^-[I - A_n^\# A_n] \tag{3.16}$$

and

$$\varkappa^-[I - C_n C_n^*] = \varkappa^-[I - A_n A_n^*] + \varkappa^-[I - T_1^{(n)} T_1^{(n)*}]. \tag{3.17}$$

To this end we compute the matrix of C_n and observe that after multiplication on the left with the unitary operator

$$\begin{bmatrix} J_2 T_2^* J_2 & J_2 D_{T_2} \\ D_{T_2^*} J_2 & -J_{T_2^*} L_{T_2} \end{bmatrix} \oplus I$$

we obtain the matrix

$$\begin{bmatrix} A & 0 & \dots & & & 0 \\ 0 & 0 & \dots & & & 0 \\ X_{11} & X_{12} & 0 & \dots & & 0 \\ \vdots & \vdots & & & & \\ X_{n-1,1} & X_{n-1,2} & \dots & \dots & X_{n-1,n} & 0 \end{bmatrix}$$

which, at its turn, after a rearrangement of rows, i.e. a multiplication on the left with a unitary operator, becomes $A_{n-1} \oplus 0$, where $A_{n-1} = \varphi_{n-1}(A_n) \in \mathfrak{S}^{(n-1)}(A\,;\,T_1, T_2)$. From here, and taking account of the identifications of the defect spaces which follow from Lemma 1.6, the identities (3.16) and (3.17) can be immediately obtained.

Now, these identities enable us to apply Lemma 3.3 to the operators A_n, $T_1^{(n)}$ and $T_2^{(n)}$, instead of A, T_1 and respectively T_2, and obtain an operator $A_{n+1} \in \mathfrak{S}^{(1)}(A_n\,;\,T_1^{(n)}, T_2^{(n)})$. But, with the identifications of defect subspaces which follow from Lemma 1.6, we can prove the inclusion $\mathfrak{S}^{(1)}(A_n\,;\,T_1^{(n)}, T_2^{(n)}) \subset \mathfrak{S}^{(n+1)}(A\,;\,T_1, T_2)$ and finally remark that $\varphi_n(A_{n+1}) = A_n$. ▨

3.6. Remark. In view of Lemma 3.2, Theorem 3.5 contains the following particular cases:

(a) If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and A is contractive, this is the classical theorem of lifting of commutants [15], [16].

(b) If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Pontryagin spaces and A is contractive, this is Theorem 6.1 in [10].

(c) If A, T_1 and T_2 are doubly contractive then this is the result obtained in [11].

Finally, let us note a variant of Theorem 3.5 in the form of dilation of commuting doubly contractive operators (i.e. generalizing the dilation theorem in [2]).

3.7. COROLLARY. *Let $\mathscr{H}$ be a Krein space and $T_1, T_2 \in \mathscr{L}(\mathscr{H})$ be two commuting double contractions. Then, there exist a Krein space $\mathscr{K}$ containing $\mathscr{H}$, with $\varkappa^-(\mathscr{K}) = \varkappa^-(\mathscr{H})$, and two commuting isometric operators $V_1, V_2 \in \mathscr{L}(\mathscr{K})$ such that*

(a) *$V_1^\#$ and $V_2^\#$ are also contractive.*

(b) *$T_1^m T_2^n = P_{\mathscr{H}}^{\mathscr{K}} V_1^m V_2^n \mid \mathscr{H}$ for any $m, n \geqslant 0$.*

(c) *$\bigvee_{m,n \geqslant 0} V_1^m V_2^n \mathscr{H} = \mathscr{K}$.*

Proof. First, we remark that if $TA = AT$, where T is a double contraction, A is isometric and $A^\#$ is contraction, then there exists $A_\infty \in \Xi(A\,;\, T, T)$ which is also an isometry. Indeed, in this case $\begin{bmatrix} AT \\ D_T A \end{bmatrix}$ is also an isometry and by Lemma 2.2, $D_T A = X D_{AT}$, with an isometric operator X. Now, we can follow the proof of Lemma 3.3 and use Theorem 2.3 in order to find an operator B_1 (with the notation in the proof of Lemma 3.3) which is an isometry. Our claim now follows by Corollary 2.5.

Returning to the proof of the Corollary, we use Theorem 3.5 in order to obtain a double contraction $S_2 \in \mathscr{L}(\mathscr{K}_1)$ such that $P_{\mathscr{H}}^{\mathscr{K}_1} S_2 \mid \mathscr{H} = T_2$ and $S_2 U_1 = U_1 S_2$, where $(U_1, \mathscr{K}_1)$ is the minimal isometric dilation of T_1. Using the above claim, we get an isometry $V_1 \in \mathscr{L}(\mathscr{K})$ such that $P_{\mathscr{K}_1}^{\mathscr{K}} V_1 \mid \mathscr{K}_1 = U_1$ and $V_1 V_2 = V_2 V_1$, where $(V_2, \mathscr{K})$ is the minimal isometric dilation of S_2. The rest is plain. ∎

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T. CONSTANTINESCU and A. GHEONDEA
Department of Mathematics, INCREST,
Bdul Păcii 220, 79622 Bucharest,
Romania.

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