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THE CYCLIC COHOMOLOGY OF COMPACT LIE GROUPS AND THE DIRECT SUM FORMULA

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1. PROLOGUE

The group C^*-algebra $C^*(G)$ of a locally compact group G is often a useful "dual object" of G. For instance, it contains all information about unitary representations of G. When G is abelian, by Fourier transform, $C^*(G)$ is identified with the C^*-algebra $C_0(\hat{G})$ of all complex-valued continuous functions vanishing at infinity on the spectrum $\hat{G}$ of G. When G is compact, each irreducible representation is finite dimensional, and we have

$$C^*(G) \cong \hat{\bigoplus_{\pi \in \hat{G}}} \mathrm{End}(V_\pi),$$

where V_π is the representation space of π in $\hat{G}$, and $\hat{\oplus}$ denotes the C^*-algebraic direct sum. This leads to the well known results $K_1(C^*(G)) = 0$ and $K_0(C^*(G)) \cong \cong \bigoplus_{\pi \in \hat{G}} \mathbf{Z} \cong R(G)$. Moreover, for each π in $\hat{G}$, the corresponding generator is given by a projection obtained from a normalized matrix coefficient of π.

Recently, as a sort to dual of K-theory, cyclic cohomology theory has been developed. In the study of cyclic cohomology C^*-algebras are sometimes rather uninteresting objects. For instance, there is no obvious way to construct nontrivial cyclic cocycles for a given C^*-algebra. A possible candidate is a tracial state. Even in this case, a tracial state is quite often defined only on a dense subalgebra.

On the other hand, the main point of noncommutative differential geometry ([1]) is that one uses an auxiliary smooth structure given by a subalgebra. It is well known that for G compact, the character of every irreducible representation gives rise to a central distribution on G, in other words a trace on the convolution algebra $C^\infty(G)$. In this note we study the cyclic cohomology of the Fréchet algebra $C^\infty(G)$. It is natural to expect that irreducible representations play a central role, and actually they do so (Theorem 1).

Throughout the paper, the tensor product means completed topological tensor product.

A. Wassermann has announced some computations of the cyclic cohomology for reductive linear Lie groups, which carries a ceritan overlap with our result. Since our result was obtained independently, we present it here.

2. COMPACT LIE GROUPS

In this section we briefly review the basic facts about the structure of $C^\infty(G)$ (for details, see [2]).

Let $\hat{G}$ be the spectrum of a compact Lie group G, and let $\mathrm{d}g$ be the bi-invariant measure with $\int_G \mathrm{d}g = 1$. Denote by $F(\hat{G})$ the algebra $\prod_{\pi \in \hat{G}} \mathrm{End}(V_\pi)$. Consider the space $L^2(\hat{G})$ of all families $\varphi = (\varphi_\pi) \in F(\hat{G})$ such that

$$\sum_{\pi \in \hat{G}} \dim(V_\pi)\mathrm{Tr}(\varphi_\pi^* \varphi_\pi) < \infty.$$

Then $L^2(\hat{G})$ is a Hilbert space where the inner product is given by

$$\langle \varphi, \psi \rangle = \sum_{\pi \in \hat{G}} \dim(V_\pi)\mathrm{Tr}(\varphi_\pi^* \psi_\pi).$$

Let f be an integrable function on G. For each $\pi \in \hat{G}$,

$$\tilde{f}_\pi = \int_G f(g)\pi(g)\mathrm{d}g \in \mathrm{End}(V_\pi),$$

and we put $\tilde{f} = (\tilde{f}_\pi) \in F(\hat{G})$. The map $f \to \tilde{f}$ defines an isomorphism of the Hilbert spaces $L^2(G, \mathrm{d}g)$ and $L^2(\hat{G})$. The function $\tilde{f}$ is called the Fourier transform of f.

Let $\Gamma \in U(G)$ be a Casimir operator. For every π in $\hat{G}$, the operator $\pi(\Gamma)$ on V_π is a scalar multiple of the identity $\lambda_\pi 1$. Thus we get a complex-valued function $\pi \mapsto \lambda_\pi$ on $\hat{G}$. It is known that λ_π is in fact real-valued, and positive unless π is the trivial representation. Recall that for some integer $k > 0$, we have $\sum_\pi (1 + \lambda_\pi)^{-k} < \infty$.

Denote by $\mathscr{S}(\hat{G})$ the space of all φ in $F(\hat{G})$ such that for each integer $n > 0$,

$$\sup_{\pi \in \hat{G}} (1 + \lambda_\pi)^n \|\varphi_\pi\| < \infty.$$

Then $\mathscr{S}(\hat{G})$ is well defined, i.e. it does not depend on the choice of a Casimir operator. Through Fourier transform $C^\infty(G)$ is topologically isomorphic to $\mathscr{S}(\hat{G})$, where the latter is topologized by the family of seminorms:

$$\|\varphi\|_n = \sup_{\pi\in\hat{G}}(1 + \lambda_\pi)^n\|\varphi_\pi\| \quad (n = 0, 1, \ldots).$$

A complex-valued function φ on $\hat{G}$ is called slowly increasing if there exist an integer $n > 0$ such that

$$\sup_{\pi\in\hat{G}}(1 + \lambda_\pi)^{-n}\|\varphi_\pi\| < \infty.$$

The vector space of all slowly increasing functions is denoted by $\mathcal{O}(\hat{G})$.

For $\varphi \in \mathcal{O}(\hat{G})$, consider the functional τ_φ on $C^\infty(G)$ defined by

$$\tau_\varphi(f) = \sum_{\pi\in\hat{G}} \frac{1}{\dim(V_\pi)} \operatorname{Tr}(\tilde{f}_\pi)\varphi_\pi .$$

Then τ_π is a trace on $C^\infty(G)$, and the map which assigns τ_φ to φ is an injection of $\mathcal{O}(\hat{G})$ into $H^0_\lambda(G)$.

Our main result is as follows:

THEOREM 1. (I) *If m is odd, then* $SH^m_\lambda(C^\infty(G))$ *is zero in* $H^{m+2}_\lambda(C^\infty(G))$. *In particular,* $H^{\mathrm{odd}}(C^\infty(G)) = \{0\}$.

(II) *If $m = 2n$, then* $SH^m_\lambda(C^\infty(G)) = S^{n+1}\mathcal{O}(\hat{G})$ *in* $H^{2n+2}_\lambda(C^\infty(G))$. *Moreover, the canonical map* $\mathcal{O}(\hat{G}) \to H^{\mathrm{ev}}(C^\infty(G))$ *is an isomorphism.*

Proof. Denote by $L(V_\pi, V_{\pi'})$ the space of all linear maps from $V_{\pi'}$ to V_π. Assume that $f = (f_{\pi,\pi'}) \in \prod_{(\pi,\pi')\in\hat{G}^2} L(V_{\pi'}, V_\pi)$ satisfies the following condition: for any integer $n > 0$,

$$\sup_{(\pi,\pi')\in\hat{G}\times\hat{G}} (1 + \lambda_\pi + \lambda_{\pi'})^n\|f_{\pi,\pi'}\| < \infty.$$

The totality $\mathscr{S}(\hat{G}\times\hat{G})$ of such functions is an algebra, where the product is given by

$$(fg)_{\pi,\pi'} = \sum_{\pi''} f_{\pi,\pi''}g_{\pi'',\pi'} .$$

Notice that this product is well defined, because $\sum (1 + \lambda_\pi)^{-k} < \infty$ for some $k > 0$.

Each $f \in \mathcal{S}(\hat{G}\times\hat{G})$ defines an operator T_f on $L^2(\hat{G})$ by the formula:

$$(T_f\xi)_\pi = \sum_{\pi'\in\hat{G}} f_{\pi,\pi'}\xi_{\pi'}.$$

By a straightforward computation we see that T_f is a trace class operator. We cal T_f an integral operator with rapidly decreasing kernel function f. The totality $\mathcal{K}$ of those T_f is a Fréchet algebra with seminorms:

$$\|T_f\|_n = \sup_{(\pi,\pi')}(1 + \lambda_\pi + \lambda_{\pi'})^n\|f_{\pi,\pi'}\| \quad (n = 0, 1, \ldots).$$

Moreover $\mathcal{K}$ is nuclear as topological vector space.

The algebra $\mathcal{K}$ is a discrete analogue of the algebra of smooth compact operators studied in [3].

For each π in $\hat{G}$, choose a minimal projection e_π in $\mathrm{End}(V_\pi)$. Denote by e_0 the minimal projection corresponding to the trivial one-dimensional representation. Let us consider the two embeddings α and β of $\mathcal{S}(\hat{G})$ into $\mathcal{S}(\hat{G}) \otimes \mathcal{K}$ defined by

$$\alpha(a)_\pi = a_\pi \otimes e_0,$$

$$\beta(a)_\pi = e_\pi \otimes a_\pi,$$

respectively. Notice that e_π, $a_\pi \in \mathrm{End}(V_\pi) \subset \mathcal{S}(\hat{G}) \subset \mathcal{K}$. Since $\hat{G}$ is discrete, the latter inclusion is meaningful. As we have pointed out above, $\mathcal{K}$ is contained in the algebra of all trace class operators.

Hence we have a trace, denoted by Tr, on $\mathcal{K}$. For a given φ in $H^n_\lambda(\mathcal{S}(\hat{G}))$, we see that

$$\alpha^*(\varphi \# \mathrm{Tr}) = \varphi.$$

The character χ_π of $\pi \in \hat{G}$ is a trace on $\mathcal{S}(\hat{G})$.

LEMMA 2. (1) *If n is odd, then* $\beta^*(\varphi \# \mathrm{Tr}) = 0$.

(2) *If $n = 2k$, then*

$$\beta^*(\varphi \# \mathrm{Tr}) = (2\pi i)^{-k}(k!)^{-1}\sum_\pi \varphi(e_\pi, \ldots, e_\pi)S^k\chi_\pi,$$

where the convergence of the infinite sum on the right is with respect to the topology of simple convergence.

Proof. By a straightforward computation, for $a^0, \ldots, a^n \in \mathcal{S}(\hat{G})$, we have

$$\beta^*(\varphi \# \mathrm{Tr})(a^0, \ldots, a^n) = \sum_{\pi_0,\ldots,\pi_n} \varphi(e_{\pi_0}, \ldots, e_{\pi_n})\mathrm{Tr}(a^0_{\pi_0} \ldots a^n_{\pi_n}).$$

Notice that $a^0_{\pi_0} \ldots a^n_{\pi_n}$ is nonzero only if $\pi_0 = \ldots = \pi_n$. Thus

$$\beta^*(\varphi \# \mathrm{Tr})(a^0, \ldots, a^n) = \sum_\pi \varphi(e_\pi, \ldots, e_\pi)\mathrm{Tr}(a^0_\pi \ldots a^n_\pi).$$

As φ is cyclic, then $\varphi(e_\pi, \ldots, e_\pi) = 0$ if n is odd. Thus we get the assertion (1).

When $n = 2k$, the value $\mathrm{Tr}(a^0_\pi \ldots a^n_\pi)$ is nothing but

$$(2\pi \mathrm{i})^{-k}(k!)^{-1}(\mathrm{S}^k\chi_\pi)(a^0, \ldots, a^n).$$

So we get the assertion (2). ▨

LEMMA 3. *There exists a path of homomorphisms* $\rho_t : \mathscr{S}(\hat{G}) \to \mathscr{S}(\hat{G}) \otimes \mathscr{K}$ *such that for all* $a \in \mathscr{S}(\hat{G})$, *the map* $t \in [0, 1] \to \rho_t(a) \in \mathscr{S}(\hat{G}) \otimes \mathscr{K}$ *is differentiable, and such that* $\rho_0 = \alpha$, $\rho_1 = \beta$.

Proof. Let $\varepsilon_{\pi,0}$ (resp. $\varepsilon_{0,\pi}$) denote the partial isometry from e_0 (resp. e_π) to e_π (resp. e_0). Denote by 1_π the identity of $\mathrm{End}(V_\pi)$, and by 1 the identity of $B(L^2(\hat{G}) \otimes \otimes L^2(\hat{G}))$.

Put

$$U = \sum_\pi 1_\pi \otimes (\varepsilon_{\pi,0} + \varepsilon_{0,\pi}) + (1 - \sum_\pi 1_\pi \otimes (e_0 + e_\pi)).$$

Then $U^* = U$ and $U^*U = UU^* = 1$, where $*$-operation is taken in the multiplier algebra of $\mathscr{S}(\hat{G}) \otimes \mathscr{K}$.

Consider now the subalgebra

$$(1_\pi \mathscr{S}(\hat{G}) 1_\pi) \otimes (1_\pi \mathscr{K} 1_\pi) \cong \mathrm{End}(V_\pi \otimes V_\pi).$$

Let U_π be the unitary in $\mathrm{End}(V_\pi \otimes V_\pi)$ such that $\mathrm{Ad}(U_\pi)$ is the "flip" on $\mathrm{End}(V_\pi) \otimes \otimes \mathrm{End}(V_\pi)$. Put

$$V = \sum_\pi U_\pi + (1 - \sum_\pi 1_\pi \otimes 1_\pi)$$

to get an element of $B(L^2(\hat{G}) \otimes L^2(\hat{G}))$. Then V is a unitary, and belongs to the multiplier algebra of $\mathscr{S}(\hat{G}) \otimes \mathscr{K}$. By construction,

$$\mathrm{Ad}(VU)\alpha = \beta.$$

It is also evident that $\mathrm{Ad}(VU)$ is connected to the identity by a C^∞-path in $\mathrm{Aut}(\mathscr{S}(\hat{G}) \otimes \mathscr{K})$. ▨

By Lemma 3 together with the homotopy invariance of cyclic cohomology [1] (see also [3]), we see that

$$\alpha^*([\varphi \# \mathrm{Tr}]) - \beta^*([\varphi \# \mathrm{Tr}]) \in \mathrm{Ker}\,\mathrm{S}.$$

This means that when $n = 2k$,

$$S((2\pi i)^{-k}(k!)^{-1} \sum_{\pi} \varphi(e_\pi, \ldots, e_\pi) S^k \chi_\pi) = S\varphi$$

in cohomology, and that, if n is odd, $S\varphi = 0$ in cohomology.

It remains to show that the function ψ defined by $\psi_\pi = \dim V_\pi \varphi(e_\pi, \ldots, e_\pi)$ belongs to $\mathscr{C}(\hat{G})$. This follows from the continuity of φ and the fact that $\pi \mapsto \dim V_\pi$ is slowly increasing ([2]).

Finally, for each $\pi \in \hat{G}$, choose a projection ρ_π, so that $\{\rho_\pi\}$ is a set of generators for $K_0(C^*(G)) = \bigoplus_\pi \mathbf{Z}$. Then,

$$\chi_\pi(\rho_{\pi'}) = \begin{cases} 1 & \text{if } \pi = \pi' \\ 0 & \text{otherwise.} \end{cases}$$

This easily implies that the map $\mathscr{C}(\hat{G}) \mapsto H^{ev}(C^\infty(G))$ is an isomorphism.

This completes the proof of Theorem 1.

3. EPILOGUE

In the preceding section we have considered a certain completion of a direct sum of Banach algebras. The technique developed there enables us to deal with direct sums of Fréchet algebras. Assume that each A_i is topologized by a family of seminorms $\|\cdot\|_{i,n}$ $(n = 0, 1, \ldots)$. Let λ_i $(i = 0, 1, 2, \ldots)$ be a sequence of positive numbers such that for some integer $k > 0$,

$$\sum \lambda_i^{-k} < \infty.$$

On the algebraic direct sum $\oplus A_i$ of (A_i), define seminorms $\|\cdot\|_n$ by

$$\|a\|_n = \sup_i \lambda_i^n \|a_i\|_{i,n},$$

where a_i is the i-component of $a \in \oplus A_i$. The completion of $\oplus A_i$ with respect to the family $\|\cdot\|_n$ $(n = 0, 1, \ldots)$ is denoted by $\hat{\oplus} A_i$. The space $\hat{\oplus} A_i$ turns out to be a nuclear Fréchet algebra.

Denote by $\bar{C}_\lambda^n(\hat{\oplus} A_i)$ the space of sequences $(\varphi_i) \in \prod_i C_\lambda^n(A_i)$ such that $\sum \rho_i^* \varphi_i$ exists with respect to simple convergence, where ρ_i is the canonical projection from $\hat{\oplus} A_i$ onto A_i. Put $b(\varphi_i) = (b\varphi_i)$ to get a cochain complex $\{\bar{C}_\lambda^*(\hat{\oplus} A_i), b\}$. Its cohomology is denoted by $\hat{\bigoplus}_i H_\lambda^*(A_i) = \bigoplus_n \hat{\bigoplus}_i H_\lambda^n(A_i)$. The complex $\bar{C}_\lambda^*(\hat{\oplus} A_i)$ can be regarded as a sort of completion of the direct sum $\bigoplus_i \rho_i^* C_\lambda^*(A_i)$. Hence the notation $\hat{\bigoplus}_i H_\lambda^*(A_i)$ is justified.

Let j be the canonical embedding of $\bar{C}_\lambda^*(\hat{\oplus} A_i)$ into $C_\lambda^*(\hat{\oplus} A_i)$. Then j induces an embedding of $\hat{\oplus} H^*(A_i)$ into $H_\lambda^*(\hat{\oplus} A_i)$. The following theorem say that $\bar{C}_\lambda^*(\hat{\oplus} A_i)$ is a retract of $C_\lambda^*(\hat{\oplus} A_i)$.

THEOREM 4. *For each integer $n \neq 0$, we have that*

$$\mathrm{S}(H_\lambda^n(\hat{\oplus} A_i)) = \mathrm{S}(\hat{\oplus} H_\lambda^n(A_i))$$

in $H_\lambda^{n+2}(\hat{\oplus} A_i)$.

Sketch of proof. Let $\mathscr{S}(\mathrm{N}^2, \Lambda)$ be the space of rapidly decreasing sequences on N^2 with respect to the weight $\Lambda = (\lambda_i)$. In an obvious way, $\mathscr{S}(\mathrm{N}^2, \Lambda)$ acts on $\ell^2(\mathrm{N})$. The assumption on (λ_i) enables us to furnish the totality $\mathscr{K}$ of bounded operators on $\ell^2(\mathrm{N})$ arising from $\mathscr{S}(\mathrm{N}^2, \Lambda)$ with a structure of a nuclear Fréchet algebra. Let e_i be the projection of $\ell^2(\mathrm{N})$ onto its i-th component. Obviously $e_i \in \mathscr{K}$. Define two embeddings α and β of $\hat{\oplus} A_i$ into $(\hat{\oplus} A_i) \otimes \mathscr{K}$ by $\alpha(a)_i = a_i \otimes e_0$, and $\beta(a)_i = a_i \otimes e_i$. Then as in the proof of Theorem 1, we see that α and β are connected by a differentiable path of embeddings. Therefore, again by the homotopy invariance, for $[\varphi] \in H_\lambda^n(\hat{\oplus} A_i)$, we have

$$\alpha^*([\varphi \# \mathrm{Tr}]) - \beta([\varphi \# \mathrm{Tr}]) \in \mathrm{Ker}\, \mathrm{S}.$$

By straightforward computations, we get the desired result. ▨

COROLLARY TO THE PROOF OF THEOREM 3. *For two algebras A, B, we have that*

$$\mathrm{S}H_\lambda^n(A \oplus B) \cong \mathrm{S}H_\lambda^n(A) \oplus \mathrm{S}H_\lambda^n(B)$$

in $H_\lambda^{n+2}(a \oplus B)$. In particular, $H^*(A \oplus B) \cong H^*(A) \oplus H^*(B)$. Moreover, if both A and B are unital, then

$$H_\lambda^n(A \oplus B) \cong H_\lambda^n(A) \oplus H_\lambda^n(B).$$

Outline of proof. In this case, instead of $\mathscr{K}$ in the proof of Theorem 4, we use $M_2(\mathrm{C})$, and all the arguments go through in the multiplier algebra of $(A \oplus B) \otimes M_2(\mathrm{C})$.

In particular, if A and B are unital, two embeddings of $(A \oplus B)$ into $(A \oplus B) \otimes \otimes M_2(\mathrm{C})$ are intertwined by a unitary of $(A \oplus B) \otimes M_2(\mathrm{C})$. Since inner automorphisms act trivially on cyclic cohomology ([1]), the second statement follows. ▨

REMARK 5. When A and B are unital, the second statement of Corollary to the proof of Theorem 4 follows also from the direct sum formula for Hochschild cohomology and the Five Lemma, using a long exact sequence involving cyclic cohomology and Hochschild cohomology ([1]).

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