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FURUTA'S INEQUALITY AND ITS MEAN THEORETIC APPROACH

Dedicated to the semicentennial presence of
Professor Masahiro Nakamura as a mathematician

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1. INTRODUCTION

Let A and B be (bounded linear) operators acting on a Hilbert space. Motivated by the conjecture due to Chan and Kwong [2],

$$(BA^2B)^{1/2} \geqslant B^2 \quad \text{if } A \geqslant B \geqslant 0, \tag{1}$$

Furuta recently proved the following excellent inequality [4]:

THEOREM A. (Furuta's inequality). *If* $A \geqslant B \geqslant 0$, *then*

$$(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q} \tag{2}$$

holds for all $r, p \geqslant 0$ *and* $q \geqslant 1$ *such that*

$$(1 + 2r)q \geqslant p + 2r. \tag{3}$$

It may be one of the essential developments in operator inequalities, whose proof and the condition (3) are a crystallization of his inspiration based on computations for many appropiate examples.

Very recently, Kamei [8] gave a curious proof to (1) in the light of the (operator) geometric mean g, definition below. As a matter of fact, (1) is rephrased as

$$1\, \mathrm{g}\, BA^2B \geqslant B^2,$$

or equivalently (if B is invertible)

$$B^{-2}\, \mathrm{g}\, A^2 \geqslant 1.$$

His proof in itself is very interesting, yet the importance of his work might be to suggest the possibility of a mean theoretic proof for Furuta's inequality.

Following Kubo and Ando [9], also [1], a binary operation m among positive operators is said to be a mean if m satisfies

(I) $A \leqslant C$ and $B \leqslant D$ imply $A \,\mathrm{m}\, B \leqslant C \,\mathrm{m}\, D$,

(II) $T^*(A \,\mathrm{m}\, B)T \leqslant T^*AT \,\mathrm{m}\, T^*BT$ for any T,

and

(III) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \,\mathrm{m}\, B_n \downarrow A \,\mathrm{m}\, B$.

The condition (II) is called the transformer inequality. However, if T is invertible, then equality holds in (II), that is,

(II′) $T^*(A \,\mathrm{m}\, B)T = T^*AT \,\mathrm{m}\, T^*BT$ for any invertible T.

Here we call it the transformer equality.

The geometric mean g is defined by

$$A \,\mathrm{g}\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Then, since $1 \,\mathrm{g}\, C = C^{1/2}$, we have

$$A \,\mathrm{g}\, B = A^{1/2}(1 \,\mathrm{g}\, A^{-1/2}BA^{-1/2})A^{1/2},$$

which follows from the transformer equality for g. In the same manner, given a non-negative operator monotone function f, one can define a mean m by

(4) $$A \,\mathrm{m}\, B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

The principal result in the Kubo-Ando theory is that the correspondence (4) between the means and the non-negative operator monotone functions on $[0, \infty)$ is an affine isomorphism.

In this note, we will be able to enjoy a beautiful aspect of Furuta's inequality by virtue of the Kubo-Ando theory. This short play is in three parts; namely, the bottom lemma, the basic inequality and the reduction lemma. Moreover, Furuta [6] gave the following extension of the bottom lemma with an elementary and self-contained proof:

THEOREM B. *If $A \geqslant B \geqslant 0$, then for each r with $0 \leqslant r \leqslant 1/2$*

(5) $$B^rA^pB^r \geqslant (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)}$$

holds for each p and s such that $p \geqslant s \geqslant 0$ and $p + 2r \geqslant 2s$.

So we also discuss the above theorem; precisely, we give it a mean theoreic interpretation.

2. FURUTA'S INEQUALITY

Throughout this note, we may assume that positive operators A and B are invertible. First of all, we state the known fact, e.g., [10]:

THEOREM C. *For* $0 \leqslant s \leqslant 1$, *the function* $f(x) = x^s$ *for* $x \in [0, \infty)$ *is operator monotone, that is,*

$$C^s \leqslant D^s \quad \text{if } 0 \leqslant C \leqslant D.$$

Next we pose the bottom lemma, whose name will be clarified in the proof of Futura's inequality. We give it a mean theoretic proof though it appeared in [4], [6].

THE BOTTOM LEMMA. *If* $0 \leqslant r \leqslant 1/2$, *then Furuta's inequality holds.*

Proof. Let m be the mean corresponding to the operator monotone function $f(x) = x^{1/q}$, that is, $f(x) = 1 \, \mathrm{m} \, x$. Then (2) is rephrased

$$(2') \qquad B^{-2r} \mathrm{m} \, A^p \geqslant B^{-2r} \mathrm{m} \, B^p$$

via the transformer equality. Since $0 \leqslant 2r \leqslant 1$, we have $B^{2r} \leqslant A^{2r}$ by Theorem C, and so $B^{-2r} \geqslant A^{-2r}$. Thus it follows that

$$B^{-2r} \mathrm{m} \, A^p \geqslant A^{-2r} \mathrm{m} \, A^p = A^{-2r}(1 \, \mathrm{m} \, A^{p+2r}) = A^{(p+2r)/q-2r}.$$

The assumption (3) ensures that the last term in the above dominates

$$B^{(p+2r)/q-2r} = B^{-2r} \mathrm{m} \, B^p$$

by Theorem C, and (2') has been proved.

The following is the special case $r = 1/2$, $2q = p + 1$ in the bottom lemma, for which the above proof works without Theorem C.

THE BASIC INEQUALITY. *If* $A \geqslant B \geqslant 0$, *then*

$$A_1 = (B^{1/2} A^p B^{1/2})^{2/(p+1)} \geqslant B_1 = B^2$$

for all $p \geqslant 1$.

Next, suppose that $p \geqslant 1$ and $c > 1/2$. Let m be the mean corresponding to $f(x) = x^{1/q}$, where $q = (p + 2c)/(1 + 2c)$. Then we have

THE REDUCTION LEMMA. *Let p, c, q and* m *be as above,* $b = (c - 1/2)/2$. *If the inequality*

$$B^{-2b} \,\mathrm{m}\, A^{(p+1)/2} \geqslant B^{-2b} \,\mathrm{m}\, B^{(p+1)/2} \quad \text{if } A \geqslant B \geqslant 0$$

is known, then we can infer

$$B^{-2c} \,\mathrm{m}\, A^{p} \geqslant B^{-2c} \,\mathrm{m}\, B^{p} \quad \textit{if } A \geqslant B \geqslant 0.$$

Proof. If $A \geqslant B \geqslant 0$, then $A_1 \geqslant B_1 \geqslant 0$ by the basic inequality and so

$$B_1^{-2b} \,\mathrm{m}\, A_1^{(p+1)/2} \geqslant B_1^{-2b} \,\mathrm{m}\, B_1^{(p+1)/2}.$$

That is,

$$B^{-4b} \,\mathrm{m}\, B^{1/2} A^{p} B^{1/2} \geqslant B^{-4b} \,\mathrm{m}\, B^{p+1}.$$

By the transformer equality, we have

$$B^{-4b-1} \,\mathrm{m}\, A^{p} \geqslant B^{-4b-1} \,\mathrm{m}\, B^{p}.$$

That is,

$$B^{-2c} \,\mathrm{m}\, A^{p} \geqslant B^{-2} \,\mathrm{m}\, B^{p}.$$

Consequently one can complete the proof of Furuta's inequality:

Proof of Theorem A. Let p, q, r satisfy the requirements in the Theorem. (By Theorem C, we can assume equality in (3).) Now the reduction lemma deduces (2′) with a value $r = c$ from (2′) with a lower value $r = b < c/2$. (Equality continues to hold in (3). The value of p changes, but that is without effect.) Thus by using the reduction lemma a suitable number of times we can drop into the range where the bottom lemma applies. The proof is complete.

REMARK. In [5], Futura gave a simplified proof to $(BA^2B)^{3/4} \geqslant B^3$ for $A \geqslant B \geqslant 0$. His plan is as follows: To show

(i) $$(B^{1/4} A^{3/2} B^{1/4})^{3/4} \geqslant B^{3/2},$$

(ii) $$A_1 = (B^{1/2} A^2 B^{1/2})^{2/3} \geqslant B_1 = B^2.$$

And apply (i) to A_1 and B_1.

This is just the case where $c = 1$, $p = 2$, $q = (p + 2c)/(1 + 2c) = 4/3$ in the reduction lemma and the basic inequality.

3. EXTENDED FURUTA'S INEQUALITY

In this section, we will make a mean theoretic consideration on Theorem B.

LEMMA. *If f is a non-negative operator monotone function on* $[0, \infty)$, *then* $g(x) = xf(x)$ *satisfies*

$$g(C^*AC) \leqslant C^*g(A)C \tag{6}$$

for any $A \geqslant 0$ *and contraction* C.

Proof. Let $C^* = UH$ be the polar decomposition of C^*. Then, since g is analytic and $g(0) = 0$, we have

$$g(C^*AC) = g(UHAHU^*) = Ug(HAH)U^*.$$

Therefore we may assume that $C \geqslant 0$, and moreover C and A are invertible.

Now, if m is the corresponding mean for f, then

$$g(CAC) = CAC(1\,\mathrm{m}\,CAC) = CAC\,\mathrm{m}\,(CAC)^2 \leqslant$$

$$\leqslant CAC\,\mathrm{m}\,CA^2C = C(A\,\mathrm{m}\,A^2)C = Cg(A)C.$$

Applying Theorem C as in Furuta's discussion, we have

COROLLARY. ([3], [7]). *If $g(x) = x^s$ for $1 \leqslant s \leqslant 2$, then g satisfies* (6).

Proof of Theorem B. Put $t = (p + 2r)/(p - s + 2r)$. We note that if $p \geqslant s \geqslant \geqslant 0$, then $1 \leqslant t \leqslant 2$ is the same as the condition $p + 2r \geqslant 2s$. Since $0 \leqslant r \leqslant 1/2$, $A^{-r}B^r$ is a contraction. Hence it follows from the above corollary that

$$(B^rA^{p-s}B^r)^t = (B^rA^{-r}A^{p-s+2r}A^{-r}B^r)^t \leqslant B^rA^{-r}A^{(p-s+2r)t}A^{-r}B^r =$$

$$= B^rA^{-r}A^{p+2r}A^{-r}B^r = B^rA^pB^r.$$

REMARK. If we put $s = p - 1$ in Theorem B, then we obtain the bottom lemma. As a matter of fact, it follows from Theorem B that

$$B^rA^pB^r \geqslant (B^rAB^r)^{(p+2r)/(1+2r)}.$$

Since $p \geqslant 1$, we have $0 < (1 + 2r)/(p + 2r) \leqslant 1$ and so

$$(B^rA^pB^r)^{(1+2r)/(p+2r)} \geqslant B^rAB^r \geqslant B^rBB^r = B^{1+2r}$$

by the assumption $A \geqslant B$, which is nothing but the bottom lemma.

Added in proof. After the preparation, we are informed that Furuta also gives a mean theoretic discussion to his inequality in [11].

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