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A DIFFEOMORPHISM OF AN IRRATIONAL ROTATION C^*-ALGEBRA BY A NON-GENERIC ROTATION

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1. INTRODUCTION

Recently Elliott [2] proved the following. Any diffeomorphism of irrational rotation C^*-algebras by generic rotations is composed of three diffeomorphisms induced by a smooth unitary element, an element in the integral special group of degree 2 and an element in the two dimensional torus.

In the present paper we will show that there are an irrational rotation C^*-algebra by a non-generic rotation and a diffeomorphism of it which does not satisfy the result of Elliott [2].

2. MAIN RESULT

Let A_θ be an irrational rotation C^*-algebra by θ and let u and v be unitary elements in A_θ with $uv = e^{2\pi i\theta}vu$. Then they generate A_θ. Let A_θ^∞ be a dense $*$-subalgebra of all smooth elements with respect to the canonical action of the two dimensional torus.

DEFINITION. Let α be an automorphism of A_θ. We say that it is a *diffeomorphism of* A_θ if $\alpha(A_\theta^\infty) = A_\theta^\infty$.

For any $s, t \in \mathbf{R}$ let $\alpha_{(s,t)}$ be the diffeomorphism of A_θ defined by $\alpha_{(s,t)}(u) = e^{2\pi i t}u$ and $\alpha_{(s,t)}(v) = e^{2\pi i t}v$. Let $SL(2, \mathbf{Z})$ be the group of all 2×2 matrices over $\mathbf{Z}$ with determinant 1. For any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{Z})$, let α_g be the diffeomorphism of A_θ defined by $\alpha_g(u) = u^a v^c$ and $\alpha_g(v) = u^b v^d$.

DEFINITION. Let θ be an irrational number. θ is *generic* if there are $r > 1$ and $C > 0$ such that $|e^{2\pi i n\theta} - 1| \geqslant \frac{C}{|n|^r}$ for any integer $n \neq 0$ that is, not a *Liouville number*.

Let τ be the unique tracial state on A_θ and for any automorphism α of A_θ let $\tilde{\tau}$ be a tracial state on the crossed product $A_\theta \times \mathbf{Z}$ and $\tilde{\tau}_*$ be the homomorphism of $\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})$ into $\mathbf{R}$ induced by $\tilde{\tau}$.

LEMMA. 1. *With the above notations let* $\alpha = \mathrm{Ad}(w)\circ\alpha_g\circ\alpha_{(s,t)}$. *Then* α *is inner if and only if* $\alpha_* = \mathrm{id}$ *on* $\mathrm{K}_1(A_\theta)$ *and* $\tilde{\tau}_*(\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$.

Proof. It is trivial that $\alpha_* = \mathrm{id}$ on $\mathrm{K}_1(A_\theta)$ and $\tilde{\tau}_*(\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$ if α is inner. We suppose that $\alpha_* = \mathrm{id}$ on $\mathrm{K}_1(A_\theta)$ and $\tilde{\tau}_*(\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$. Then since $\alpha_* = \mathrm{id}$ on $\mathrm{K}_1(A_\theta)$, $g = I_2$. Thus $\alpha = \mathrm{Ad}(w)\circ\alpha_{(s,t)}$. Hence we can see by Pimsner [6, Theorem 3] that

$$\tilde{\tau}_*(\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta + \mathbf{Z}s + \mathbf{Z}t.$$

Since $\tilde{\tau}_*(\mathrm{K}_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$, we obtain that $s, t \in \mathbf{Z} + \mathbf{Z}\theta$. Thus there are some k, l, m and $n \in \mathbf{Z}$ such that $s = k + l\theta$ and $t = m + n\theta$. Let $z = u^n v^{-l}$. Then

$$zuz^* = u^n v^{-l} u v^l u^{-n} =$$

$$= e^{2\pi i l\theta} u = e^{2\pi i(k+l\theta)} u = e^{2\pi i s} u$$

and

$$zvz^* = u^n v^{-l} v v^l u^{-n} = e^{2\pi i n\theta} v = e^{2\pi i(m+n\theta)} v = e^{2\pi i t} v.$$

Hence $\alpha_{(s,t)} = \mathrm{Ad}(z)$. Therefore we can see that α is inner. Q.E.D.

Let $\mathbf{T}$ be the one dimensional torus and $C(\mathbf{T})$ be the abelian C^*-algebra of all continuous functions on $\mathbf{T}$. We identify it with the set of all continuous functions on $\mathbf{R}$ with period 1. Let α be an automorphism of A_θ defined by $\alpha(u) = f(v)u$ and $\alpha(v) = v$ where f is a unitary element in $C(\mathbf{T})$. Let δ_1 and δ_2 be the canonical derivations induced by the canonical action of $\mathbf{T}^2$. We note that $A_\theta^\infty = \bigcap_{m,n\geqslant 0} D(\delta_1^m\circ\delta_2^n)$ where $D(\delta_1^m\circ\delta_2^n)$ is the domain of $\delta_1^m\circ\delta_2^n$ for any $m, n \geqslant 0$. Furthermore it is well known that $A_\theta^\infty = \{ \sum_{m,n\in\mathbf{Z}} c_{mn} u^m v^n \mid \{c_{mn}\} \in S(\mathbf{Z}^2)\}$ where $S(\mathbf{Z}^2)$ is the set of all rapidly decreasing functions on $\mathbf{Z}^2$.

LEMMA 2. *Let* α *be an automorphism of* A_θ *defined by* $\alpha(u) = f(v)u$ *and* $\alpha(v) = v$ *where* f *is a unitary element in* $C(\mathbf{T})$. *If* $f \in C^\infty(\mathbf{T})$, α *is a diffeomorphism of* A_θ.

Proof. It is sufficient to show that $\alpha(x) \in A_\theta^\infty$ for any $x \in A_\theta^\infty$. Let $x = \sum_{m,n\in\mathbf{Z}} c_{mn} u^m v^n$ where $\{c_{mn}\} \in S(\mathbf{Z}^2)$. Then

$$\alpha(x) = \sum c_{mn}\alpha(u)^m \alpha(v)^n = \sum c_{mn} f(v) f(e^{2\pi i\theta} v) \dots f(e^{2\pi i(m-1)\theta} v) u^m v^n.$$

Let $y_k = \sum_{|m|,|n| \leq k} c_{mn} f(v) f(e^{2\pi i\theta}v) \dots f(e^{2\pi i(m-1)\theta}v) u^m v^n$ where k is a positive integer. Clearly $y_k \in D(\delta_j)$ for $j = 1, 2$ and $\|y_k - \alpha(x)\| \to 0$ as $k \to \infty$. Since $\{c_{mn}\} \in S(\mathbf{Z}^2)$ and $|f(t)| = 1$ for any $t \in \mathbf{T}$, $\{\delta_j(y_k)\}_{k=1}^{\infty}$ is a Cauchy sequence for $j = 1, 2$. Hence since δ_j is closed, $\alpha(x) \in D(\delta_j)$ for $j = 1, 2$. Similarly we can see that $\alpha(x) \in D(\delta_1^m \circ \delta_2^n)$ for any $m, n \in \mathbf{Z}$. Therefore $\alpha(x) \in \bigcap_{m,n \geq 0} D(\delta_1^m \circ \delta_2^n)$. Q.E.D.

For any automorphism α of A_θ we denote by $\Gamma(\alpha)$ its Connes spectrum.

LEMMA 3. *Let α be as in Lemma 2. Then $\Gamma(\alpha) = \mathbf{T}$ if and only if there is no unitary element R in $C(\mathbf{T})$ satisfying the functional equation*

$$[f(t)]^n = R(t)R(t+\theta)^{-1}$$

for any integer $n \neq 0$.

Proof. We will show that $\Gamma(\alpha) \subsetneqq \mathbf{T}$ if and only if there are a non zero integer n and a unitary element $R \in C(\mathbf{T})$ such that $[f(t)]^n = R(t)R(t+\theta)^{-1}$ for any $t \in \mathbf{T}$. It is clear that $\Gamma(\alpha) \subsetneqq \mathbf{T}$ if and only if there are a non zero integer n and a unitary element $w \in A_\theta$ such that $\alpha^n = \mathrm{Ad}(w)$. Then since $\alpha^n(u) = f(v)^n u$ and $\alpha^n(v) = v$, $wuw^* = f(v)^n$ and $wvw^* = v$. Hence there is a unitary element $R \in C(\mathbf{T})$ such that $w = R(v)$ since $wv = vw$. And since $wuw^* = f(v)^n u$, $R(v)uR(v)^* = f(v)^n u$. Then $R(v)R(e^{2\pi i\theta}v)^* = f(v)^n$. Therefore we obtain that

$$[f(t)]^n = R(t)R(t+\theta)^{-1}$$

for any $t \in \mathbf{T}$. Conversely suppose there are a non zero integer n and a unitary element $R \in C(\mathbf{T})$ such that $[f(t)]^n = R(t)R(t+\theta)^{-1}$. Then let $w = R(v)$. By trivial computation $\alpha^n = \mathrm{Ad}(w)$. Q.E.D.

LEMMA 4. *Let α be the automorphism of A_θ defined by* $\alpha(u) = e^{2\pi i g(v)}u$ and $\alpha(v) = v$ *where g is a selfadjoint element in $C(\mathbf{T})$. If $\int_0^1 g(t)dt = 0$ and there is no selfadjoint element $k \in C(\mathbf{T})$ satisfying*

$$g(t) = k(t) - k(t - \theta)$$

for any $t \in \mathbf{T}$, then $\Gamma(\alpha) = \mathbf{T}$.

Proof. We will show that if $\Gamma(\alpha) \subsetneqq \mathbf{T}$, there is a selfadjoint element k in $C(\mathbf{T})$ such that $g(t) = k(t) - k(t+\theta)$ for any $t \in \mathbf{T}$. By Lemma 3 we may suppose that there are a non zero integer n and a unitary element $R \in C(\mathbf{T})$ such that

$$e^{2\pi i n g(t)} = R(t)R(t+\theta)^{-1}.$$

Now we consider g as a real valued continuous function on $\mathbf{R}$ with period 1 and R as a complex valued continuous function on $\mathbf{R}$ with period 1 and $|R(t)| = 1$ for any $t \in \mathbf{R}$. Then there is a real valued continuous function k on $\mathbf{R}$ such that $R(t) = e^{2\pi i k(t)}$ for any $t \in \mathbf{R}$. Thus

$$e^{2\pi i n g(t)} = e^{2\pi i (k(t) - k(t+\theta))}$$

for any $t \in \mathbf{R}$. Hence

$$ng(t) = k(t) - k(t+\theta) + m(t)$$

for any $t \in \mathbf{R}$ where m is a $\mathbf{Z}$-valued function on $\mathbf{R}$. Since g and k are continuous, so is m. Thus it is a constant integer. And since $R(t+1) = R(t)$, $e^{2\pi i k(t+1)} = e^{2\pi i k(t)}$. Thus $k(t+1) - k(t) = l(t)$ for any $t \in \mathbf{R}$ where l is a $\mathbf{Z}$-valued function on $\mathbf{R}$. Since k is continuous, so is l. Hence it is a constant integer. Let $\tilde{k}(t) = k(t) - lt$ for any $t \in \mathbf{R}$. Then for any $t \in \mathbf{R}$

$$\tilde{k}(t+1) = k(t+1) - l(t+1) = k(t) + l - lt - l = k(t) - lt = \tilde{k}(t).$$

Therefore $\tilde{k}$ is a continuous function on $\mathbf{R}$ with period 1. Furthermore for any $t \in \mathbf{R}$

$$\tilde{k}(t) - \tilde{k}(t+\theta) = k(t) - lt - (k(t+\theta) - l(t+\theta)) =$$

$$= k(t) - k(t+\theta) + l\theta = ng(t) - m + l\theta.$$

Hence $ng(t) = \tilde{k}(t) - \tilde{k}(t+\theta) + m - l\theta$ for any $t \in \mathbf{R}$. Since $\int_0^1 g(t)\mathrm{d}t = 0$, $\int_0^1 (\tilde{k}(t) - \tilde{k}(t+\theta))\mathrm{d}t + m - l\theta = 0$. Hence we obtain that $m - l\theta = 0$ since $\tilde{k}$ has period 1. Thus $m = l = 0$. Therefore we see that k is a real valued continuous function on $\mathbf{R}$ with period 1 and satisfies that

$$ng(t) = k(t) - k(t+\theta)$$

for any $t \in \mathbf{R}$. Q.E.D.

LEMMA 5. *There are an irrational number θ and an analytic function $g : \mathbf{R} \to \mathbf{R}$ with period 1 and $\int_0^1 g(t)\mathrm{d}t = 0$ such that there is no continuous function $k : \mathbf{R} \to \mathbf{R}$ with period 1 satisfying*

$$g(t) = k(t) - k(t+\theta)$$

for any $t \in \mathbf{R}$.

Proof. Let $\{\nu_j\}_{j=1}^{\infty}$ be the sequence of integers defined by $\nu_1 = 1$ and $\nu_{j+1} = 2^{\nu_j} + \nu_j + 1$. Let $\theta = \sum_{j=1}^{\infty} 2^{-\nu_j}$ and $n_j = 2^{\nu_j}$. Then

$$\frac{1}{2} \cdot 2^{-n_j} = \frac{2^{\nu_j}}{2^{\nu_{j+1}}} \leqslant |n_j\theta - [n_j\theta]| \leqslant \frac{2 \cdot 2^{\nu_j}}{2^{\nu_{j+1}}} = 2^{-n_j}$$

where $[t]$ denotes the largest integer not exceeding t. Hence we obtain that

$$2 \cdot 2^{-n_j} \leqslant |e^{2\pi i n_j \theta} - 1| \leqslant 2\pi \cdot 2^{-n_j}.$$

By the above left hand inequality θ is irrational. We define g by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t}$$

where

$$a_n = \begin{cases} \dfrac{1}{j} 2^{-n_j} \dfrac{1 - e^{2\pi i n_j \theta}}{|1 - e^{2\pi i n_j \theta}|} & \text{if } n = n_j \\[2ex] \dfrac{1}{j} 2^{-n_j} \dfrac{1 - e^{-2\pi i n_j \theta}}{|1 - e^{-2\pi i n_j \theta}|} & \text{if } n = -n_j \\[2ex] 0 & \text{elsewhere.} \end{cases}$$

Clearly g is a real valued continuous function and $\int_0^1 g(t)\mathrm{d}t = 0$. Since $\limsup \frac{1}{n} \log|a_n| = -\log 2$, we can see that g is analytic. We suppose that there is a continuous functions $k : \mathbf{R} \to \mathbf{R}$ with period 1 satisfying

$$g(t) = k(t) - k(t + \theta)$$

for any $t \in \mathbf{R}$. Then the Fourier series of k is equal to $\sum_n \frac{a_n}{1 - e^{2\pi i n \theta}} e^{2\pi i n t} + c$ where c is a constant number. Since k is continuous, it is Cesàro summable at $t = 0$. On the other hand

$$\sum_n \frac{a_n}{1 - e^{2\pi i n \theta}} = 2 \sum_{j=1}^{\infty} \frac{1}{j} \frac{2^{-n_j}}{|1 - e^{2\pi i n_j \theta}|}.$$

By the inequality $|e^{2\pi i n_j \theta} - 1| \leqslant 2\pi \cdot 2^{-n_j}$, we have

$$\frac{2^{-n_j}}{|1 - e^{2\pi i n_j \theta}|} \geqslant \frac{1}{2\pi}.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j}$ is not Cesàro summable, neither is $\sum_n \frac{a_n}{1 - e^{2\pi i n \theta}}$. Thus we get a contradiction. Q.E.D.

THEOREM 6. *Let θ and g be as above. Let A_θ be the irrational rotation C^*-algebra by θ. Let α be the automorphism of A_θ defined by $\alpha(u) = e^{2\pi i g(v)} u$ and $\alpha(v) = v$. Then α is a diffeomorphism of A_θ such that*

$$\alpha \neq \mathrm{Ad}(w) \circ \alpha_h \circ \alpha_{(s,t)}$$

or any unitary element $w \in A_\theta^\infty$, any $h \in \mathrm{SL}(2, \mathbf{Z})$ and any $s, t \in \mathbf{R}$.

Proof. By Lemma 2 α is a diffeomorphism. We suppose that there are a unitary element $w \in A_\theta^\infty$, some $h \in \mathrm{SL}(2, \mathbf{Z})$ and $s, t \in \mathbf{R}$ such that

$$\alpha = \mathrm{Ad}(w) \circ \alpha_h \circ \alpha_{(s,t)}.$$

By the definition of α it is clear that $\alpha_* = \mathrm{id}$ on $K_1(A_\theta)$. Thus $\mathrm{Ker}(\mathrm{id} - \alpha_*) = \mathbf{Z}[u] \oplus \mathbf{Z}[v]$. We can see that $\alpha(v)v^* = 1$ and $\alpha(u)u^* = e^{2\pi i g(v)}$. Let ξ be the continuously differentiable path from 1 to $e^{2\pi i g(v)}$ on $[0, 1]$ defined by $\xi(r) = e^{2\pi i r g(v)}$ for $r \in [0, 1]$. Then

$$\frac{1}{2\pi i} \int_0^1 \tau\left(\xi(r)^* \frac{d}{dr} \xi(r) \right) dr = \int_0^1 \tau(g(v)) dr = \tau(g(v)) = \int_0^1 g(r) dr = 0.$$

Therefore we obtain by Pimsner [6, Theorem 3] that $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$. Thus by Lemma 1 α is inner. On the other hand by Lemma 4 $\Gamma(\alpha) = \mathbf{T}$. This is a contradiction. Therefore we obtain the conclusion. Q.E.D.

REMARK. By Elliott [2] θ in Theorem 6 is non-generic.

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