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TRIANGULAR AF ALGEBRAS

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A Banach algebra $\mathscr{T}$ is said to be triangular AF, or TAF, if there is a unital AF C^*-algebra $\mathfrak{A}$ such that $\mathscr{T}$ is a norm-closed subalgebra of $\mathfrak{A}$ and the diagonal $\mathfrak{D} = \mathscr{T} \cap \mathscr{T}^*$ is a maximal abelian self-adjoint subalgebra (maşa) in $\mathfrak{A}$. In addi-s tion, we require that $\mathfrak{D}$ be a masa of the type studied by Strătilă and Voiculescu [15]; namely, there exists a nested sequence of finite dimensional unital subalgebras $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ such that $\mathfrak{A} = \overline{\bigcup_{k=1}^{\infty} \mathfrak{A}_k}$ and $\mathfrak{D}_k = \mathfrak{D} \cap \mathfrak{A}_k$ is a masa in $\mathfrak{A}_k$. If $\mathscr{T}$ is a TAF algebra, it follows that $\mathscr{T} = \overline{\bigcup_k \mathscr{T}_k}$, where $\mathscr{T}_k = \mathscr{T} \cap \mathfrak{A}_k$ (Corollary 2.3). Conversely, if $\mathscr{T}_k \subseteq \mathfrak{A}_k$, $k = 1,2,\ldots$, is an increasing sequence of triangular algebras with diagonal $\mathfrak{D}_k$, then $\mathscr{T} = \overline{\bigcup_k \mathscr{T}_k}$ is TAF (Theorem 2.6).

Let M_n denote the $n \times n$ complex matrices and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers, strictly increasing, such that n_k divides n_{k+1}. Let $\mathfrak{A}_k = \mathrm{M}_{n_k}$, and $i_k, j_k : \mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ each be unital embeddings which map $\mathscr{T}_k$ into $\mathscr{T}_{k+1}$, where $\mathscr{T}_k$ is the set of upper triangular matrices in $\mathfrak{A}_k$. From Glimm's theorem [5], we know of course that the UHF algebras $\mathfrak{A} = \varinjlim (\mathfrak{A}_k, i_k)$ and $\mathscr{B} = \varinjlim (\mathfrak{A}_k, j_k)$ are isomorphic, for as Glimm showed, the isomorphism class depends only on the dimensions of the finite dimensional factors, and not on the particular form of the embeddings. However, if $\mathscr{S} = \varinjlim (\mathscr{T}_k, i_k)$ and $\mathscr{T} = \varinjlim (\mathscr{T}_k, j_k)$ are the respective Banach algebra inductive limits, then $\mathscr{S}$ and $\mathscr{T}$ will not in general be isomorphic (i.e., isometrically isomorphic as Banach algebras), and they may in fact exhibit startlingly different qualities. It can happen, for suitable i_k, j_k, that $\mathscr{S}$ possesses a nest of invariant projections which generates the diagonal, whereas $\mathscr{T}$ may have only the trivial lattice of invariant projections (Example 1.1). On the other hand, with another choice of embeddings, $\mathscr{S}$ and $\mathscr{T}$ can fail to be isomorphic yet still have the same lattice of invariant projections. Thus we introduce another invariant, which is an ordering on projections in the diagonal.

Let $\mathscr{W}_{\mathfrak{D}}$ be the set of partial isometries w in $\mathfrak{A}$ which satisfy $w^*\mathfrak{D}w \subseteq \mathfrak{D}$ and $w\mathfrak{D}w^* \subseteq \mathfrak{D}$. Then for any TAF algebra $\mathscr{T} \subseteq \mathfrak{A}$ with $\mathscr{T} \cap \mathscr{T}^* = \mathfrak{D}$, or more generally for any norm-closed $\mathfrak{D}$-bimodule $\mathcal{T}$, we have that $\mathcal{T}$ is the closed linear span of $\mathcal{T} \cap \mathcal{W}_{\mathfrak{D}}$. If $\mathcal{T}$ is TAF, the set $\mathcal{T} \cap \mathcal{W}_{\mathfrak{D}}$ can be used to define a partial ordering on $\mathcal{P}(\mathfrak{D})$, the projections in $\mathfrak{D}$, as follows: let $e, f \in \mathcal{P}(\mathfrak{D})$; write $e \prec_{\mathcal{T}} f$ in case there is partial isometry $v \in \mathcal{T} \cap \mathcal{W}_{\mathfrak{D}}$ such that $v^*v = f$ and $vv^* = e$. The ordering thus defined, which is not the usual ordering on $\mathcal{P}(\mathfrak{D})$, is reflexive, antisymmetric. and transitive. It turns out that this ordering on $\mathcal{P}(\mathfrak{D})$ is an isomorphism invariant (Proposition 3.20). Returning to the algebras $\mathcal{S}$ and $\mathcal{T}$ above, it is possible to choose the embeddings i_k, j_k in such a way that $\mathcal{S}$ and $\mathcal{T}$ have the same lattice of invariant projections (e.g., the trivial lattice), and yet the diagonal orderings defined by $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic – and hence $\mathcal{S}$ and $\mathcal{T}$ are a fortiori not isomorphic TAF algebras (Example 3.27).

There are still other ways to choose the embeddings i_n, j_n so that the resulting TAF algebras fail to be isomorphic in a rather subtle way, which is not completely understood. It can happen that $\mathcal{S}$ and $\mathcal{T}$ are both nest algebras, that is, the lattice of invariant projections is linearly ordered and generates the diagonal, and the diagonal ordering defined by $\mathcal{S}$ is isomorphic with the diagonal ordering defined by $\mathcal{T}$, and yet $\mathcal{S}$ and $\mathcal{T}$ can still fail to be isomorphic (Example 4.4).

There are several notions of maximality for TAF algebras; we mention two such here. Let $\mathfrak{A}$, $\mathfrak{A}_k$, $\mathfrak{D}$, $\mathfrak{D}_k$ be as in the first paragraph. If $\mathcal{T}_k$ is triangular in $\mathfrak{A}_k$ with diagonal $\mathfrak{D}_k$, and $\mathcal{T} = \overline{\bigcup_k \mathcal{T}_k}$, then $\mathcal{T}$ is said to be strongly maximal if $\mathcal{T}_k$ is maximal in $\mathfrak{A}_k$, $k \geqslant 1$. Theorem 2.2 implies that $\mathcal{T}$ is maximal TAF; i.e., $\mathcal{T}$ is a maximal element, in the sense of set inclusion, among all TAF subalgebras of $\mathfrak{A}$ with diagonal $\mathfrak{D}$. However, there are maximal TAF algebras which are not strongly maximal (Example 3.25).

Section 1 sets down notation and also shows that certain TAF algebras arise "naturally" as subalgebras of semicrossed products. Section 2 is concerned with the notion inductivity of $\mathfrak{D}$-bimodules and triangular algebras. The main tool developed in Section 3 is the set $\mathcal{W}_{\mathfrak{D}}$ of partial isometries associated with a masa $\mathfrak{D}$. One use of the important technical result $\mathcal{W}_{\mathfrak{D}} = \bigcup_{k=1}^{\infty} \mathcal{U}(\mathfrak{D})(\mathcal{W}_{\mathfrak{D}} \cap \mathfrak{A}_k)$ (Theorem 3.6) is to show that an arbitrary isomorphism of TAF algebras can be replaced by an isomorphism that has a particularly tractable form (Corollary 4.2). Another consequence of the isomorphism results of this section is a generalization of the principal result of R. Baker [1, Theorem 1] (Theorem 3.26). Finally, Section 4 gives some necessary conditions for certain strongly maximal TAF subalgebras of UHF algebras to be isomorphic, and exhibits an uncountable collection of strongy maximal. pairwise nonisomorphic TAF algebras inside the 2^{∞} UHF algebra.

While there is no direct connection between our setting and weakly-closed triangular subalgebras of von Neumann algebras (except in finite dimensions), the authors looked for analogues of some of the results of [7] for TAF algebras. The work of Power [11] was directly relevant, though it is restricted to the nest case.

The classification problem for TAF algebras is far from complete. In fact, even restricting to strongly maximal TAF algebras inside a given UHF algebra, the classification problem is unsolved. Nor do we have a classification of the diagonal order types.

1. PRELIMINARIES

A C^*-algebra $\mathfrak{A}$ is *almost finite dimensional* (AF) if $\mathfrak{A}$ contains a sequence $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ of finite dimensional subalgebras such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots$ and $\mathfrak{A} = \overline{\bigcup_{k=1}^{\infty} \mathfrak{A}_k}$. All AF algebras in this paper will be unital, and in this case we require that $\mathfrak{A}_1$ contains the unit 1 of $\mathfrak{A}$. In the special case that each $\mathfrak{A}_n$ is a full matrix algebra, then $\mathfrak{A}$ is called a UHF *algebra*. The reader is referred to [5] and [3], [4] for more details on UHF and AF algebras, respectively. If $\mathscr{S} \subseteq \mathfrak{A}$, then $\mathscr{S}^c$ will denote $\{x \in \mathfrak{A} : xs = sx \text{ for all } s \in \mathscr{S}\}$, the commutant of $\mathscr{S}$ in $\mathfrak{A}$.

The term *masa* will be used in the sense of [15]. Specifically, if $\mathfrak{A}$ is a finite dimensional C^*-algebra, then masa has the usual meaning, i.e., a maximal abelian selfadjoint subalgebra of $\mathfrak{A}$. On the other hand, if $\mathfrak{A}$ is an infinite dimensional AF algebra, we say that a subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ is a *masa in* $\mathfrak{A}$ if there are increasing sequences $\{\mathfrak{A}_k\}$ and $\{\mathfrak{D}_k\}$ of finite dimensional C^*-algebras such that $\mathfrak{D}_k$ is a masa in $\mathfrak{A}_k$ for every k, $\mathfrak{A} = \overline{\bigcup_k \mathfrak{A}_k}$, and $\mathfrak{D} = \overline{\bigcup_k \mathfrak{A}_k}$. It follows that $\mathfrak{D}_k = \mathfrak{D} \cap \mathfrak{A}_k$. Given any increasing sequence $\{\mathfrak{A}_k\}$ with $\mathfrak{A} = \overline{\bigcup_k \mathfrak{A}_k}$, a masa $\mathfrak{D}$ can be constructed as follows: let $\mathfrak{D}_1$ be a masa in $\mathfrak{A}_1$, and define $\{\mathfrak{D}_k\}$ inductively by $\mathfrak{D}_{k+1} = C^*(\mathfrak{D}_k, \mathfrak{E}_{k+1})$, where $\mathfrak{E}_{k+1}$ is an arbitrary masa in $\mathfrak{A}_k^c \cap \mathfrak{A}_{k+1}$. It then follows that $\mathfrak{E}_{k+1} = \mathfrak{A}_k^c \cap \mathfrak{D}_{k+1}$. We remark that these masas are referred to as *canonical masas* in [12], and are sometimes called *approximately finite Cartan subalgebras*.

Given a masa $\mathfrak{D}$ in $\mathfrak{A}$, it is very useful to define systems of matrix units for $\mathfrak{A}$ which respect the relationship between $\mathfrak{D}$ and $\mathfrak{A}$ (see [15]). First of all, suppose $\mathscr{F}$ is an n^2-dimensional factor with a masa $\mathfrak{D}$. By a *set of matrix units* $\{e_{ij} : 1 \leqslant i, j \leqslant n\}$ *for* $\mathscr{F}$ *with respect to* $\mathfrak{D}$ we mean a set of matrix units in the usual sense with the additional property that $\{e_{ii} : 1 \leqslant i \leqslant n\} \subseteq \mathfrak{D}$. Obviously, such a system of matrix units is not unique (if $\dim(\mathscr{F}) > 1$), for if ω is any complex number with modulus one, then $\{f_{ij} = \omega^{j-i} e_{ij}\}$ is another set of matrix units for $\mathscr{F}$ with respect to $\mathfrak{D}$. If $\mathfrak{A}$ is a finite dimensional C^*-algebra with minimal central projections $\{e^{(l)} : 1 \leqslant l \leqslant m\}$, and if $\mathfrak{D}$ is a given masa in $\mathfrak{A}$, then a set $\{e_{ij}^{(l)} : 1 \leqslant i, j \leqslant n_l, 1 \leqslant l \leqslant m\}$ is called a *set of matrix units for* $\mathfrak{A}$ *with respect to* $\mathfrak{D}$ if for each l, $1 \leqslant l \leqslant m$, $\{e_{ij}^{(l)} : 1 \leqslant i, j \leqslant n_l\}$ is a set of matrix units for the factor $e^{(l)}\mathfrak{A}$ with respect to $e^{(l)}\mathfrak{D}$.

Now let $i: \mathfrak{A} \hookrightarrow \mathfrak{B}$ be a unital embedding, with $\mathfrak{B}$ finite dimensional. Let $\mathfrak{D} \subseteq \mathfrak{A}$, $\mathfrak{E} \subseteq \mathfrak{B}$ be masas such that $\mathfrak{E} = C^*(i(\mathfrak{D}), \mathfrak{E}_0)$ for some masa $\mathfrak{E}_0 \subseteq i(\mathfrak{A})^c$.

Then given a set of matrix units $\{e_{ij}^{(l)}\}$ for $\mathfrak{A}$ with respect to $\mathfrak{D}$, the set of all nonzero products of minimal projections of $\mathfrak{E}_0$ with $\{i(e_{ij}^{(l)})\}$ form a set of matrix units for $C^*(i(\mathfrak{A}), \mathfrak{E}_0)$. Furthermore, one can check directly or apply standard results to see that this set of matrix units can be extended to a full set of matrix units for $\mathfrak{B}$ with respect to $\mathfrak{E}$. Thus, $i(e_{ij}^{(l)})$ can be expressed as a sum of matrix units for $\mathfrak{B}$ with respect to $\mathfrak{E}$.

Finally, given an AF algebra $\mathfrak{A}$ with masa $\mathfrak{D}$, then sets of matrix units $\{e_{ij}^{(kl)} : 1 \leqslant l \leqslant m_k,\ 1 \leqslant i, j \leqslant n_{kl}\}$ for $\mathfrak{A}_k$ with respect to $\mathfrak{D}_k$ can be constructed inductively as above. We then say that $\{e_{ij}^{(kl)} : 1 \leqslant k < \infty,\ 1 \leqslant l \leqslant m_k,\ 1 \leqslant i, j \leqslant n_{kl}\}$ is a *set of matrix units for* $\mathfrak{A}$ *with respect to* $\mathfrak{D}$. Whenever we use matrix units in $\mathfrak{A}$, we will always assume that they are contained in a set of matrix units for $\mathfrak{A}$ with respect to $\mathfrak{D}$.

If $\mathscr{S} \subseteq \mathfrak{A}$, let $\mathscr{P}(\mathscr{S})$ denote the set of (selfadjoint) projections in $\mathscr{S}$. If $\mathfrak{D}$ is a masa in $\mathfrak{A}$, then $\mathscr{P}(\mathfrak{D}) = \bigcup_{n=1}^{\infty} \mathscr{P}(\mathfrak{D}_n)$, for if $p \in \mathscr{P}(\mathfrak{D})$, then there is some $q \in \mathfrak{D}_n$ with $q = q^*$ and $\|p - q\| < 1/2$, and by using the functional calculus we can assume that q is a projection. But then $p = q$ since they commute. It follows that the spectrum of $\mathfrak{D}$ is totally disconnected, and therefore zero dimensional since it is compact Hausdorff.

Given an AF algebra $\mathfrak{A}$ with masa $\mathfrak{D}$, the term $\mathfrak{D}$-*module* will always mean a norm-closed unital $\mathfrak{D}$-bimodule, unless otherwise indicated. Subalgebras of $\mathfrak{A}$ will be assumed to also be $\mathfrak{D}$-modules (i.e., closed and containing $\mathfrak{D}$) for some masa $\mathfrak{D}$, usually clear from the context. A $\mathfrak{D}$-module $\mathscr{S}$ is *triangular* if $\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$, and $\mathfrak{D}$ is the *diagonal* of $\mathscr{S}$. $\mathscr{T} \subseteq \mathfrak{A}$ is a *triangular AF* (TAF) *algebra* if $\mathscr{T}$ is a closed triangular subalgebra of $\mathfrak{A}$.

Suppose $\mathscr{S}$ is a subset of $\mathfrak{A}$ with $\mathfrak{D} \subseteq \mathscr{S}$. A projection $e \in \mathfrak{A}$ is *invariant* for $\mathscr{S}$ if $se = ese$ for all $s \in \mathscr{S}$. Then e is also invariant for $\mathfrak{D}$, so $e \in \mathfrak{D}^c$ since $\mathfrak{D}$ is selfadjoint. But $\mathfrak{D}$ is a masa, so $\mathfrak{D}^c = \mathfrak{D}$. Thus, the set of invariant projections of $\mathscr{S}$ lies in $\mathfrak{D}$, and therefore is commutative. This set is also a lattice with $e \vee f = e + f - ef$ and $e \wedge f = ef$, and is denoted Lat $\mathscr{S}$.

By choosing matrix units for $\{\mathfrak{A}_k\}$ as indicated above, we can think of $\mathfrak{D}_k$ as the usual diagonal of $\mathfrak{A}_k$, a direct sum of matrix algebras. However, there are still many ways in which $\mathfrak{A}_k$ can be embedded into $\mathfrak{A}_{k+1}$ with $\mathfrak{D}_k$ embedded into $\mathfrak{D}_{k+1}$. We will define two such embeddings now for UHF algebras, and discuss others in Section 4. We will use the notation $\mathbf{M}_n$ throughout this paper to indicate a fixed representation of the n^2-dimensional factor as $n \times n$ matrices, and then use $\{g_{ij}^{(n)} : 1 \leqslant i, j \leqslant n\}$ to denote the usual matrix units for $\mathbf{M}_n$.

Now suppose $\{p_k\}$ is a sequence of positive integers such that p_k divides p_{k+1}, with $q_k = p_{k+1}/p_k$. Let $e_{ij}^{(k)} = g_{ij}^{(p_k)}$. The embedding $\sigma_k : \mathbf{M}_{p_k} \hookrightarrow \mathbf{M}_{p_{k+1}}$, denoted by $\sigma_k(x) = I_{q_k} \otimes x$, is defined by

$$\sigma_k(e_{ij}^{(k)}) = \sum_{t=0}^{q_k - 1} e_{i+tp_k,\, j+tp_k}^{(k+1)}$$

and will be called the *standard embedding* of $\mathbf{M}_{p_k}$ into $\mathbf{M}_{p_{k+1}}$. Note that $\varinjlim(\mathbf{M}_{p_k}, \sigma_k)$ is the UHF algebra of type $(p_1 p_2 \ldots)$, and $\sigma_k(\mathfrak{D}_k) \subseteq \mathfrak{D}_{k+1}$, where $\mathfrak{D}_k$ is the diagonal of $\mathbf{M}_{p_k}$.

Alternatively, let p_k, q_k, and $e_{ij}^{(k)}$ be the same as above and define $\nu_k : \mathbf{M}_{p_k} \hookrightarrow \mathbf{M}_{p_{k+1}}$ by

$$\nu_k(e_{ij}^{(k)}) = \sum_{t=1}^{q_k} e_{(i-1)q_k+t,(j-1)q_k+t}^{(k+1)}.$$

This embedding is denoted by $\nu_k(x) = x \otimes I_{q_k}$, and will be called the *nest embedding* for reasons explained below. $\varinjlim(\mathbf{M}_{p_k}, \nu_k)$ is once again the UHF algebra of type $(p_1 p_2 \ldots)$, since the isomorphism class of the inductive limit is determined by dimensions of the finite dimensional factors, and not the form of the embedding. Again, note that ν_k embeds the diagonal of $\mathbf{M}_{p_k}$ into the diagonal of $M_{p_{k+1}}$.

EXAMPLE 1.1. Let $\mathfrak{U} = \varinjlim(\mathbf{M}_{2^k}, \sigma_k)$ be the 2^∞ UHF algebra with masa $\mathfrak{D}$ via the standard embeddings, let $\mathfrak{B} = \varinjlim(\mathbf{M}_{2^k}, \nu_k)$ be the 2^∞ UHF algebra with masa $\mathfrak{C}$ via the nest embeddings, and let $\{e_{ij}^{(k)}\}$ be the set of matrix units for $\mathbf{M}_{2^k}$ as described, above. Let $\mathscr{S}_k$ and $\mathscr{T}_k$ both represent the upper triangular subalgebra of $\mathbf{M}_{2^k}$, and observe that $\sigma_k(\mathscr{S}_k) \subseteq \mathscr{S}_{k+1}$ and $\nu_k(\mathscr{T}_k) \subseteq \mathscr{T}_{k+1}$. Finally, let $\mathscr{S} = \varinjlim(\mathscr{S}_k, \sigma_k)$ and $\mathscr{T} = \varinjlim(\mathscr{T}_k, \nu_k)$. We will show in Section 2 that $\mathscr{S}$ and $\mathscr{T}$ are TAF algebras, but at this point we want to identify Lat $\mathscr{S}$ and Lat $\mathscr{T}$. The only invariant projections in $\mathscr{S}$ are 0 and 1. For suppose $0 \neq e$ is an invariant projection. Then $e \in \mathfrak{D}_k$ for some k by remarks given above. In particular, e must be invariant for $\mathscr{S}_k$, he upper triangular subalgebra of $\mathbf{M}_{2^k}$. This forces

$$e = e_{11}^{(k)} + e_{22}^{(k)} + \ldots + e_{jj}^{(k)}$$

for some j, $1 \leqslant j \leqslant 2^k$. But the image of e in $\mathscr{S}_{k+1}$, namely

$$\sigma_k(e) = e_{11}^{(k+1)} + \ldots + e_{jj}^{(k+1)} + e_{2^k+1,2^k+1}^{(k+1)} + \ldots + e_{2^k+j,2^k+j}^{(k+1)},$$

is not invariant for $\mathscr{S}_{k+1}$ unless $j = 2^k$; that is, unless $e = 1$. We refer to algebras whose only invariant projections are trivial as *transitive*.

On the other hand, let f be an invariant projection of $\mathscr{T}$. Then for some j, and k, $f = e_{11}^{(k)} + \ldots + e_{jj}^{(k)}$ is an invariant projection of $\mathscr{T}_k$, and

$$\nu_k(f) = e_{11}^{(k+1)} + \ldots + e_{2j,2j}^{(k+1)}$$

is invariant in $\mathcal{T}_{k+1}$. Similarly, $v_{m-1} \circ \ldots \circ v_k(f)$ is invariant in $\mathcal{T}_m$ for every $m > k$, so (identifying $\mathcal{T}_k$ as a subalgebra of $\mathcal{T}_m$) f is invariant in $\bigcup_{m=k}^{\infty} \mathcal{T}_m$, and hence for the closure $\mathcal{T}$; that is, $tf = ftf$ for all $t \in \mathcal{T}$. In fact, the invariant projections generate the diagonal $\mathfrak{C} = \mathcal{T} \cap \mathcal{T}^*$, and are totally ordered. In analogy with the terminology used for weakly closed algebras of operators in Hilbert space, we call such triangular algebras *nest* TAF *algebras*, and this justifies the term *nest embedding*. As noted above, $\mathfrak{A}$ and $\mathfrak{B}$ are two realizations of the 2^∞ UHF algebra. However, there is no C^*-isomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ which maps $\mathcal{S}$ onto $\mathcal{T}$, for such a map would necessarily be a bijective map of Lat $\mathcal{S}$ onto Lat $\mathcal{T}$, which is impossible as $\mathcal{S}$ has no nontrivial invariant projections and T has many. In Section 3 we will prove an even stronger statemtent: viewing $\mathcal{S}$ and $\mathcal{T}$ as Banach algebras, we will show there is no isometric Banach algebra isomorphism from $\mathcal{S}$ onto $\mathcal{T}$. Finally, in Section 4 we will use other embeddings to show that there is an uncountable family $\mathcal{T}_{(\alpha)}$, $0 \leqslant \alpha \leqslant 1$, of nonisomorphic triangular subalgebras of the 2^∞ UHF algebra such that $\mathcal{T}_{(\alpha)} \cap \mathbf{M}_{2^k}$ is the full upper triangular matrix algebra.

EXAMPLE 1.2. We give an example of a TAF algebra which arises as a non-selfadjoint subalgebra of a crossed product. Let $\{e_{ij}^{(k)}\}$, $\mathfrak{A}$, and $\mathcal{S}$ be as in the last example. Consider the "binary odometer" dynamical system : $X = \prod_{k=1}^{\infty} \{0,1\}$, and $T : X \to X$ the homeomorphism $T\vec{x} = \vec{y}$, where

$$y_1 = x_1 + 1 \pmod 2, \quad y_2 = \begin{cases} x_2 & \text{if } x_1 = 0 \\ x_2 + 1 \pmod 2 & \text{if } x_1 = 1, \end{cases}$$

$$y_3 = \begin{cases} x_3 & \text{if } x_1 = 0 \text{ or } x_2 = 0 \\ x_3 + 1 \pmod 2 & \text{if } x_1 = x_2 = 1, \end{cases} \qquad \text{etc.}$$

Let $[i_1, \ldots, i_k]$ be the cylinder set $\{x \in X : x_1 = i_1, \ x_2 = i_2, \ldots, \ x_k = i_k\}$. The cylinder sets are clopen sets which generate the topology of X. Set $\vec{x}_0 = (0,0,0,\ldots) \in X$, and let $\mathfrak{C}$ be the C^*-subalgebra of the crossed product $\mathbf{Z} \times_T C(X)$ generated by $C(X)$ and $\{Uf : f \in C(X), f(\vec{x}_0) = 0\}$. Here U is the unitary satisfying $Uf = (f \circ T)U$, $f \in C(X)$. Let X_k be the cylinder set $[0,0, \ldots, 0]$ (k zeroes), and denote by $\mathfrak{C}_k$ the linear span of $\{U^{i-j}\chi_{X_k} \circ T^{2^k-j} : 1 \leqslant i, j \leqslant 2^k\}$. One checks that $\mathfrak{C}_k$ is in fact a $*$-algebra, and that $\mathfrak{C}_k \subseteq \mathfrak{C}$. Furthermore, $\bigcup_{k=1}^{\infty} \mathfrak{C}_k$ is dense in $\mathfrak{C}$. Observe that T maps cylinder sets to cylinder sets, and if $\chi_{X_k} \circ T^{2^k-j} = \chi_{[i_1,\dots,i_k]}$, then

$$U^{j-i}\chi_{X_k} \circ T^{2^k-j} = U^{j-i}\chi_{[i_1,\dots,i_k]} =$$

$$= U^{j-i}\chi_{[i_1,\dots,i_k,0]} + U^{(j+2^k)-(i+2^k)}\chi_{[i_1,\dots,i_k,1]} =$$

$$= U^{j-i}\chi_{X_{k+1}} \circ T^{2^{k+1}-j} + U^{(j+2^k)-(i+2^k)}\chi_{X_{k+1}} \circ T^{2^{k+1}-(j+2^k)}.$$

Thus, the map $\psi_k : \mathfrak{C}_k \to \mathbf{M}_{2^k} \subseteq \mathfrak{A}$ defined by

$$\psi_k(U^{j-i}\chi_{X_k} \circ T^{2^k-j}) = e_{ij}^{(k)},$$

$1 \leqslant i, j \leqslant 2^k$, is a $*$-isomorphism. If $\mathfrak{C}_k^+$ is the nonselfadjoint subalgebra of $\mathfrak{C}_k$ generated by $\{U^{j-i}\chi_{X_k} \circ T^{2^k-j} : 1 \leqslant i \leqslant j \leqslant 2^k\}$, then $\psi_k(\mathfrak{C}_k^+) = \mathscr{S}_k$. It follows that if $\mathfrak{C}^+$ is the nonselfadjoint subalgebra of $\mathbf{Z} \times_T C(X)$ generated by $C(X)$ and $\{Uf : f(\vec{x}_0) = 0, f \in C(X)\}$, then the C^*-isomorphism $\psi : \mathfrak{C} \to \varinjlim(\mathbf{M}_{2^k}, \sigma_k) = \mathfrak{A}$ defined by the sequence (ψ_k) satisfies $\psi(\mathfrak{C}^+) = \mathscr{S}$. Thus, the crossed product $\mathbf{Z} \times_T C(X)$ contains a copy of the triangular algebra $\mathscr{S}$.

$\mathfrak{C}^+$ can also be characterized as the intersection of $\mathfrak{C}$ with the semicrossed product $\mathbf{Z}^+ \times_T C(X)$.

EXAMPLE 1.3. TAF subalgebras of semi-crossed products. Let T be a homeomorphism of a compact zero dimensional space X. For a closed subset $Y \subseteq X$, let $\mathfrak{A}_Y$ denote the C^*-subalgebra of the crossed product $\mathbf{Z} \times_T C(X)$ generated by $C(X)$ and $UC(X \setminus Y)$. Here U is a unitary satisfying $Uf = (f \circ T)U, f \in C(X)$, and such that U and $C(X)$ together generate $\mathbf{Z} \times_T C(X)$. Let $\mathscr{D}(X, T)$ be the collection of all closed subsets Y of X having the following property: for any clopen set $W \supseteq Y$, $\bigcup_{n \in Z} T^n(W) = X$. Let $\mathscr{T}_Y = \mathfrak{A}_Y \cap (\mathbf{Z}^+ \times_T C(X))$ (the semi-crossed product $\mathbf{Z}^+ \times_T C(X)$ generated by $C(X)$ and the non-negative powers of U [9]). Then $\mathscr{T}_Y$ is TAF iff $Y \in \mathscr{D}(X, T)$. For if $Y \notin \mathscr{D}(X, T)$, then $\mathfrak{A}_Y = C^*(\mathscr{T}_Y)$ is not AF by [10, Theorem 2.2]. On the other hand, if $Y \in \mathscr{D}(X, T)$ then $\mathfrak{A}_Y$ is AF, and by uniqueness of the Fourier series expansion of elements of $\mathbf{Z}^+ \times_T C(X)$, one verifies that $\mathscr{T}_Y \cap \mathscr{T}_Y^* = C(X)$. Thus $\mathscr{T}_Y$ is TAF.

In particular, $\mathscr{T}_Y = \mathbf{Z}^+ \times_T C(X)$ if $Y = \emptyset$, so it follows from the above that the semi-crossed product is not TAF if X is zero dimensional. One can also observe that any TAF algebra has topological stable rank (tsr) one, whereas $\operatorname{tsr}(\mathbf{Z}^+ \times_T C(X)) = 2$ by [8]. In fact, since $\operatorname{tsr}(\mathbf{Z}^+ \times_T C(X)) \geqslant \operatorname{tsr}(C(X))$ by [8], the semi crossed product is no TAF for any space X.

2. INDUCTIVE $\mathfrak{D}$-MODULES AND TRIANGULARITY

Let $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$ be an AF algebra with masa $\mathfrak{D} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{D}_n}$. In this section we will prove two important results. Theorem 2.2 and 2.6, which will be needed later. The first result allows one to reduce many questions about a $\mathfrak{D}$-module $\mathscr{S}$ to its finite dimensional components $\mathscr{S} \cap \mathfrak{A}_n$, and will be used often in the sequel. In addition, these two results taken together characterize TAF algebras as inductive limits of finite dimensional triangular algebras. We will also use them to show that every TAF algebra is contained in a maximal one.

DEFINITION. We will call a subset $\mathscr{S}$ of $\mathfrak{A}$ *inductive* if $\mathscr{S} \subseteq \overline{\bigcup_{n=1}^{\infty} (\mathscr{S} \cap \mathfrak{A}_n)}$. Note that if $\mathscr{S}$ is closed, then the right side is contained in the left, so equality holds. This is a generalization of the term used in [11], in which $\mathscr{S}$ was always assumed to be closed.

We first introduce some notation which will be used throughout this section. With $\mathfrak{A}$, $\mathfrak{A}_n$, $\mathfrak{D}$, $\mathfrak{D}_n$ as above, express $\mathfrak{A}_n = \bigoplus_{k=1}^{k_n} \mathbf{M}_{(nk)}$, $\mathbf{M}_{(nk)}$ a factor, and $\mathfrak{D}_n = \bigoplus_{k=1}^{k_n} \mathfrak{D}_{(nk)}$, $\mathfrak{D}_{(nk)}$ a masa in $\mathbf{M}_{(nk)}$. Let $e^{(nk)}$be the central projection in $\mathfrak{A}_n$ onto the factor $\mathbf{M}_{(nk)}$. Set $\mathfrak{D}_m^{(nk)} = e^{(nk)}\mathfrak{D}_m \cap \mathbf{M}_{(nk)}^c$, $m > n$, and $\mathfrak{D}^{(nk)} = e^{(nk)}\mathfrak{D} \cap \mathbf{M}_{(nk)}^c$. Then $\mathfrak{D}^{(nk)} = \overline{\bigcup_{m>n} \mathfrak{D}_m^{(nk)}}$. Let $\mathfrak{B}_n$ be the norm closed algebra generated by $\mathfrak{A}_n$ and $\mathfrak{D}$. Then

$$(*) \qquad \mathfrak{B}_n = \bigoplus_{k=1}^{k_n} \mathbf{M}_{(nk)} \otimes \mathfrak{D}^{(nk)}.$$

The proofs of these facts are elementary and will be omitted.

LEMMA 2.1 [11]. *Let $1 \leqslant n < r$ and let $e_1, \ldots, e_p$ be the minimal projections of $\mathfrak{A}_n^c \cap \mathfrak{D}_r$, and define*

$$P_{n,r}(x) = \sum_{i=1}^{p} e_i x e_i, \quad x \in \mathfrak{A}.$$

Then $P_n(x) = \lim_r P_{n,r}(x)$ exists and may be written as $P_n(x) = \sum_{i=1}^{l} v_i d_i$ where v_i is a matrix unit in $\mathfrak{A}_n$ and d_i is in the closed span of $\mathfrak{D}_{n+1}, \mathfrak{D}_{n+2}, \ldots$. Thus, $P_n(x) \in \mathfrak{B}_n$ for all x, and if $x \in \mathfrak{B}_n$, then $x = P_n(x) = P_{n,r}(x)$, $r > n$.

Note that if $x \in \mathfrak{A}_m$ with $m > n$, then $P_n(x) = \lim_r P_{n,r}(x) = \lim_r P_{m,r}(P_{n,m}(x)) = \lim_r P_{n,m}(x) = P_{n,m}(x)$. Also, if $x \in \mathfrak{A}$ and $\varepsilon > 0$, then there is some $y \in \mathfrak{A}_N$, some N, such that $\|x - y\| < \varepsilon$. It follows that if $n > N$, then $\|x - P_n(x)\| \leqslant$

$\leqslant \|x - y\| + \|y - P_n(x)\| < 2\varepsilon$ since $y = P_n(y)$ and $\|P_n\| = 1$, so $P_n(x) \to x$. This lemma was proved in [11, Lemma 1.2]. The proof of the lemma does not depend on any particular choice of embeddings. The following theorem was also proved in [11, Lemma 1.3] for (closed) $\mathfrak{D}$-modules. We give a different proof here, and also prove the result for a slightly more general case. We say that a $\mathfrak{D}$-bimodule $\mathscr{S}$ is *locally closed* if for each $x \in \mathscr{S}$, the closed $\mathfrak{D}$-bimodule generated by x is contained in $\mathscr{S}$. Note that $\mathscr{S}$ need not be unital.

THEOREM 2.2. *Every locally closed $\mathfrak{D}$-module in $\mathfrak{A}$ is inductive.*

Proof. Let $\mathscr{S} \subseteq \mathfrak{A}$ be a locally closed $\mathfrak{D}$-module. Since $P_n(S) \to S$ and $P_n(S) \in S \cap \mathfrak{B}_n$ for all n, it follows by Lemma 2.1 that $\mathscr{S} \subseteq \overline{\bigcup_n (\mathscr{S} \cap \mathfrak{B}_n)}$. Thus, it suffices to showthat

$$(**) \qquad \mathscr{S} \cap \mathfrak{B}_n \subseteq \overline{\bigcup_{m>n} (\mathscr{S} \cap \mathfrak{B}_n \cap \mathfrak{A}_m)}$$

for each n.

Fix n, and choose a system of matrix units $\{e_{ij}^{(nk)}\}$ for $\mathfrak{A}_n$ so that for each k, $\{e_{ij}^{(nk)}\}$ spans the factor $\mathbf{M}_{(nk)} \subseteq \mathfrak{A}_n$ and $\{e_{ii}^{(nk)}\}$ spans $\mathbf{M}_{(nk)} \cap \mathfrak{D}_n$. If $S \in \mathscr{S} \cap \mathfrak{B}_n$, then $e_{ii}^{(nk)} S e_{jj}^{(nk)} = e_{ij}^{(nk)} \otimes D$ for some $D \in \mathfrak{D}^{(nk)} \subseteq \mathfrak{D}$ by (*). Thus, the result will follow if we can show that $e_{ij}^{(nk)} \otimes D$ belongs to the right side of (**). Now the spectrum X of $\mathfrak{D}$ is zero dimensional. Therefore, given $\varepsilon > 0$, and viewing $\mathfrak{D}$ as $C(X)$, ther is a collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_l\}$ of clopen sets in X such that $\{x : |D(x)| \geqslant 2\varepsilon\} \subseteq \bigcup_{s=1}^{l} \mathcal{O}_s \subseteq$ $\subseteq \{x : |D(x)| > \varepsilon\}$ and such that $|D(x) - D(y)| < \varepsilon/2$ for all $x, y \in \mathcal{O}_s$. Pick $x_s \in \mathcal{O}_s$ arbitrarily, and let $D_0(x) = \sum_s D(x_s)\chi_{\mathcal{O}_s}$. Then $\|D - D_0\| < 2\varepsilon$, and $D_0 \in \mathfrak{D}_m$ for m sufficiently large. If

$$D'(x) = \begin{cases} D(x)^{-1}, & x \in \bigcup_{i=1}^{l} O_i \\ 0, & \text{otherwise} \end{cases}$$

then $D' \in C(X) \subseteq \mathfrak{D}$. Thus $e_{ij}^{(nk)} \otimes D_0 = (e_{ij}^{(nk)} \otimes D)(1 \otimes D')(1 \otimes D_0) \in (\mathscr{S} \cap \mathfrak{B}_n)\mathfrak{D}\mathfrak{D} \subseteq$ $\subseteq \mathscr{S} \cap \mathfrak{B}_n$. Since $e_{ij}^{(nk)} \otimes D_0 \in \mathscr{S} \cap \mathfrak{A}_m$ and $\|e_{ij}^{(nk)} \otimes D_0 - e_{ij}^{(nk)} \otimes D\| = \|D_0 - D\| <$ $< 2\varepsilon$, the proof is complete. ▨

COROLLARY 2.3. *If $\mathscr{T} \subseteq \mathfrak{A}$ is a* TAF *algebra with diagonal $\mathfrak{D}$, then* $\mathscr{T} =$ $= \overline{\bigcup_{n=1}^{\infty} (\mathscr{T} \cap \mathfrak{A}_n)}$.

Conversely, we will show that given an increasing sequence of triangular algebras $\mathscr{T}_n \subseteq \mathfrak{A}_n$ with diagonal $\mathfrak{D}_n$, then $\overline{\bigcup_{n=1}^{\infty} \mathscr{T}_n}$ is a TAF algebra in $\mathfrak{A}$. Actually, this is true in the more general context of modules. We first need a couple of preliminary results. Suppose $\{\mathcal{S}_n : 1 \leqslant n < \infty, \mathcal{S}_n \subseteq \mathfrak{A}_n\}$ is a nested sequence of $\mathfrak{D}_n$-modules, and let $\mathcal{S} = \overline{\bigcup_n \mathcal{S}_n}$. Then it is not true in general that $\mathcal{S}_n = \mathcal{S} \cap \mathfrak{A}_n$. However, we will show in Proposition 2.5 below that the sequence $\{\mathcal{S}_n\}$ can be replaced with another which has this property.

LEMMA 2.4. *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$, $\mathcal{S}_n \subseteq \mathfrak{A}_n$, be a nested sequence of $\mathfrak{D}_n$-modules. Then there exists a nested sequence $\{\mathcal{T}_n\}_{n=1}^{\infty}$, $\mathcal{T}_n \subseteq \mathfrak{A}_n$, of $\mathfrak{D}_n$-modules satisfying*

(i) *$\mathcal{S}_n \subseteq \mathcal{T}_n$ for all n;*

(ii) *$\bigcup_{n=1}^{\infty} \mathcal{S}_n = \bigcup_{n=1}^{\infty} \mathcal{T}_n$;*

(iii) *$\mathcal{T}_n = \mathcal{T}_{n+1} \cap \mathfrak{A}_n$ for all n,*

and hence $\mathcal{T}_n = \mathcal{T}_{n+k} \cap \mathfrak{A}_n$ for all n, $k \geqslant 1$. Moreover, if each $\mathcal{S}_n$ is triangular ($\mathcal{S}_n \cap \mathcal{S}_n^* = \mathfrak{D}_n$), then so is each $\mathcal{T}_n$.

Proof. Set $\mathcal{T}_n = \left(\bigcup_{k=1}^{\infty} \mathcal{S}_k\right) \cap \mathfrak{A}_n$, $n = 1,2, \ldots$. Then properties (i), (ii) and (iii) are obvious. Suppose each $\mathcal{S}_n$ is triangular. Clearly $\mathcal{T}_n \cap \mathcal{T}_n^* \supseteq \mathcal{S}_n \cap \mathcal{S}_n^* = \mathfrak{D}_n$. On the other hand, if $T \in \mathcal{T}_n \cap \mathcal{T}_n^*$, then $T \in \mathcal{S}_{n_1}$ and $T \in \mathcal{S}_{n_2}^*$ for some n_1, $n_2 \geqslant n$. Hence $T \in \mathcal{S}_m \cap \mathcal{S}_m^*$ for $m = \max\{n_1, n_2\}$, so $T \in \mathfrak{D}_m \cap \mathfrak{A}_n = \mathfrak{D}_n$. We conclude that $\mathcal{T}_n \cap \mathcal{T}_n^* = \mathfrak{D}_n$. ▨

DEFINITION. We say that a sequence $\{\mathcal{S}_n\}$ is in *canonical form* if it satisfies $\mathcal{S}_n = \mathcal{S}_{n+1} \cap \mathfrak{A}_n$ for all n. Thus, the sequence $\{\mathcal{T}_n\}$ in the above lemma is in canonical form.

PROPOSITION 2.5. *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be a nested sequence of $\mathfrak{D}_n$-modules, $\mathcal{S}_n \subseteq \mathfrak{A}_n$, and let $\mathcal{S} = \overline{\bigcup_{n=1}^{\infty} \mathcal{S}_n}$. If $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is in canonical form, then $\mathcal{S}_n = \mathcal{S} \cap \mathfrak{A}_n$.*

Proof. Let $\{e_{ij}^{(nk)}\}$ be a system of matrix units for $\mathfrak{A}_n$ so that for fixed k, $\{e_{ij}^{(nk)}\}$ spans the factor $\mathbf{M}_{(nk)} \subseteq \mathfrak{A}_n$ and $\{e_{ii}^{(nk)}\}$ spans $\mathbf{M}_{(nk)} \cap \mathfrak{D}_n$. Since $\mathcal{S} \cap \mathfrak{A}_n \supseteq \mathfrak{D}_n$, $\mathcal{S} \cap \mathfrak{A}_n$ is generated, as a $\mathfrak{D}_n$-module, by the matrix units it contains. Fix i, j, k such that $e_{ij}^{(nk)} \in \mathcal{S} \cap \mathfrak{A}_n$, and let $S = e_{ij}^{(nk)}$. Write $S = \lim_m S_m$, $S_m \in \mathcal{S}_m$, so $S = P_n(S) = \lim_m P_n(S_m) = \lim_m P_{n,m}(S_m)$. For $m > n$, set

$$R_m = e_{ii}^{(nk)} P_{n,m}(S_m) e_{jj}^{(nk)} \in \mathcal{S}_m \cap (\mathbf{M}_{(nk)} \otimes \mathfrak{D}_m^{(nk)}) \subseteq \mathcal{S}_m \cap \mathfrak{B}_n.$$

Then $\{R_m\}$ converges to $e_{ii}^{(nk)} S e_{jj}^{(nk)} = S$, and $R_m = e_{ij}^{(nk)} \otimes D_m$, $D_m \in \mathfrak{D}_m^{(nk)}$. Now

$$\|R_m - S\| = \|e_{ij}^{(nk)} \otimes D_m - e_{ij}^{(nk)} \otimes e^{(nk)}\| = \|D_m - e^{(nk)}$$

so it follows that for m sufficiently large $D_m \in \mathfrak{D}_m^{(nk)}$ is invertible in $\mathfrak{D}_m^{(nk)}$, and hence $S = e_{ij}^{(nk)} = R_m(I_{(nk)} \otimes D_m^{-1}) \in \mathscr{S}_m \cap \mathfrak{A}_n = \mathscr{S}_n$. Therefore, $\mathscr{S} \cap \mathfrak{A}_n \subseteq \mathscr{S}_n$. Of course, the reverse inequality is obvious. ▨

THEOREM 2.6. *Let $\{\mathscr{S}_n\}_{n=1}^{\infty}$ be a nested sequence of $\mathfrak{D}_n$-modules with $\mathscr{S}_n$ triangular in $\mathfrak{A}_n$, $\mathscr{S}_n \cap \mathscr{S}_n^* = \mathfrak{D}_n$, $n = 1,2,\ldots$. Then $\mathscr{S} = \overline{\bigcup_{n=1}^{\infty} \mathscr{S}_n}$ is a triangular $\mathfrak{D}$-module in $\mathfrak{A}$ with diagonal $\mathfrak{D}$.*

Proof. By Lemma 2.4, we can assume the sequence $\{\mathscr{S}_n\}$ is in canonical form. Let S be self-adjoint in $\mathscr{S}$. Since $P_n(S) \to S$ and $\mathfrak{D}$ is closed, it suffices to show that $P_n(S) \in \mathfrak{D}$ for all n. Lemma 2.1 implies that $P_n(S) = \sum_{i=1}^{r} w_i d_i$, where the w_i's are distinct matrix units in $\mathfrak{A}_n$ and $d_i \in \mathfrak{D}$ for all i. Let e_i and f_i be the final and initial projections of w_i, respectively. Then $e_i, f_i \in \mathfrak{D}_n$ and $e_i P_n(S) f_i = w_i d_i$ for all i.

Suppose $w_j d_j \neq 0$ and $w_j \notin \mathfrak{D}_n$ for some j, $1 \leqslant j \leqslant r$. Then $f_j e_j = 0$ and $f_j d_j \neq 0$. Viewing $\mathfrak{D}$ as $C(X)$, there is some $\varepsilon > 0$ such that $U = \{x : (f_j d_j)(x) > \varepsilon\} \neq \emptyset$. Let V be a clopen subset of U, and let

$$d(x) = \begin{cases} ((f_j d_j)(x))^{-1}, & x \in V \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_j d_j d = \chi_V$ is a projection in $\mathfrak{D}$, so $f_j d_j d \in \mathscr{P}(\mathfrak{D}_m)$ for some $m \geqslant n$, and $v = w_j f_j d_j d \neq 0$. $v \in \mathfrak{A}_m$ and $v = e_j P_n(S) f_j d \in \mathscr{S} \cap \mathscr{S}^*$ since $\mathscr{S} \cap \mathscr{S}^*$ is a $\mathfrak{D}$-module and $P_n(S) = P_n(S)^*$. Thus, $v \in \mathfrak{A}_m \cap \mathscr{S} \cap \mathscr{S}^* = \mathscr{S}_m \cap \mathscr{S}_m^* = \mathfrak{D}_m \subseteq \mathfrak{D}$ by Proposition 2.5. But then $v = vf_j = f_j v = f_j e_j v = 0$, a contradiction. It follows that $P_n(S) \in \mathfrak{D}$ for all n. ▨

DEFINITION. A not necessarily closed triangular subalgeba $\mathscr{T}$ of $\mathfrak{A}$ is *maximal triangular* if $\mathscr{T}$ is not contained in any larger triangular subalgebra. Simple Zorn's lemma arguments show that every triangular subalgebra is contained in a maximal one, and every locally closed triangular subalgebra is contained in a triangular subalgebra wihch is maximal in the class of locally closed triangular subalgebras. However it requires a little more work to obtain the corresponding result for TAF algebras.

COROLLARY 2.7. *Every* TAF *algebra in $\mathfrak{A}$ is contained in a maximal* TAF *algebra in $\mathfrak{A}$.*

Proof. By Zorn's Lemma, it is enough to show that if $\{\mathscr{T}^{(\alpha)} : \alpha \in A\}$ is an increasing chain of TAF subalgebras of $\mathfrak{A}$, then $\mathscr{T} = \overline{\bigcup_{\alpha \in A} \mathscr{T}^{(\alpha)}}$ is also triangular. First, choose a set M of matrix units for $\mathfrak{A}$. Since M is countable, and each

$\mathcal{T}^{(\alpha)}$ is generated by the matrix units it containes (by Theorem 2.2), it follows that $\{\mathcal{T}^{(\alpha)}\}$ has only a countable number of distinct elements. Therefore, $\{\mathcal{T}^{(\alpha)}\}$ contains an increasing sequence $\{\mathcal{T}^{(m)}:1 \leqslant m < \infty\}$ such that $\mathcal{T} = \overline{\bigcap_{m=1}^{\infty} \mathcal{T}^{(m)}}$.

Now let $T \in \mathcal{T}$ and choose $\varepsilon > 0$. Then there is an R in some $\mathcal{T}^{(M)}$ such that $\|T - R\| < \varepsilon/2$. By Theorem 2.2, $\mathcal{T}^{(M)} = \overline{\bigcup_{n=1}^{\infty} (\mathcal{T}^{(M)} \cap \mathfrak{A}_n)}$, so there is an S in some $\mathcal{T}^{(M)} \cap \mathfrak{A}_N$ such that $\|R - S\| < \varepsilon/2$. Therefore $S \in \mathcal{T}^{(L)} \cap \mathfrak{A}_L$, where $L = \max\{M, N\}$, and $\|T - S\| < \varepsilon$, so it follows that $\mathcal{T} = \overline{\bigcup_{n=1}^{\infty} (\mathcal{T}^{(n)} \cap \mathfrak{A}_n)}$. Theorem 2.6 then implies that $\mathcal{T}$ is triangular. ∎

Suppose $\mathcal{L}$ is a subset of $\mathcal{P}(\mathfrak{D})$. As in the theory of commutative subspace lattices in Hilbert space, we define $\mathrm{Alg}\,\mathcal{L} = \{a \in \mathfrak{A} : ap = pap \text{ for all } p \in \mathcal{L}\}$. $\mathrm{Alg}\,\mathcal{L}$ is a norm-closed subalgebra of $\mathfrak{A}$ which contains $\mathfrak{D}$. $\mathrm{Alg}\,\mathcal{L} \cap (\mathrm{Alg}\,\mathcal{L})^* = \mathcal{L}^c \supseteq \mathfrak{D}$, of $\mathrm{Alg}\,\mathcal{L}$ is triangular iff $\mathcal{L}^c = \mathfrak{D}$ iff $(\mathcal{L}^c)^c = \mathfrak{D}$. Note that $\mathfrak{D} \supseteq (\mathcal{L}^c)^c \supseteq C^*(\mathcal{L})$. We record these facts in the following proposition. Also, if $\mathrm{Alg}\,\mathcal{L}$ is triangular and $\mathcal{L}$ is linearly ordered, then we say that $\mathrm{Alg}\,\mathcal{L}$ is a *nest* TAF *algebra.*

PROPOSITION 2.8. *If* $\mathcal{L} \subseteq \mathcal{P}(\mathfrak{D})$, *then* $\mathrm{Alg}\,\mathcal{L}$ *is a triangular algebra iff* $(\mathcal{L}^c)^c = \mathfrak{D}$. *Thus, if* $\mathcal{T}$ *is a maximal triangular subalgebra of* $\mathfrak{A}$ *such that* $C^*(\mathrm{Lat}\,\mathcal{T}) = \mathfrak{D}$, *then* $\mathcal{T} = \mathrm{Alg}(\mathrm{Lat}\,\mathcal{T})$.

The next proposition shows that if in addition $\mathfrak{A}$ is a UHF algebra, then $\mathrm{Lat}\,\mathcal{T}$ is a nest, so $\mathcal{T}$ is a nest TAF algebra.

PROPOSITION 2.9. *If* $\mathcal{T}$ *is a maximal* TAF *subalgebra of a* UHF *algebra* $\mathfrak{A}$, *then* $\mathrm{Lat}\,\mathcal{T}$ *is lineary ordered.*

Proof. The proof is a variation of the proofs of [6, Lemmas 2.3.2 and 2.3.3]. In our situation, any two projections in $\mathfrak{D}$ lie in some factor, and this reduces to the setting in [6, Lemma 2.3.3]. ∎

3. PARTIAL ISOMETRIES ASSOCIATED WITH A MASA

Just as in [5] and [3], the partial isometries of an AF algebra will prove to be very useful in our study of TAF algebras. Because of the relationship between a TAF algebra and its diagonal, we will need to restrict our attention to certain partial isometries. We will actually be working in the more general context of $\mathfrak{D}$-modules, and we will then use these partial isometries to introduce the notion of a diagonal ordering induced by a $\mathfrak{D}$-module. In this section, $\mathfrak{A}$ will be a fixed AF algebra and $\mathfrak{D}$ a fixed masa in $\mathfrak{A}$. Thus, $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_n}$ and $\mathfrak{D} = \overline{\bigcup \mathfrak{D}_n}$, where the $\mathfrak{A}_n$'s are finite-dimensional C^*-algebras and each $\mathfrak{D}_n$ is a masa in $\mathfrak{A}_n$. We will also assume that a system of matrix units for $\mathfrak{A}$ with respect to $\mathfrak{D}$ have been chosen as described in Section 1. Although this system depends on the sequences $\{\mathfrak{A}_n\}$ and $\{\mathfrak{D}_n\}$, we stress that the results of this section will actually be independent of these sequences. Finally, $\mathfrak{D}_{sa}$ and $\mathcal{U}(\mathfrak{D})$ will denote the self-adjoint and unitary elements of $\mathfrak{D}$, respectively.

DEFINITION. $\mathcal{W}_{\mathfrak{D}}(\mathfrak{A}) = \{$partial isometries $w \in \mathfrak{A} : w^*\mathfrak{D}w \subseteq \mathfrak{D}$ and $w\mathfrak{D}w^* \subseteq \mathfrak{D}\}$. If the AF algebra $\mathfrak{A}$ and the masa $\mathfrak{D}$ are understood from the context we will simply write $\mathcal{W}$ instead $\mathcal{W}_{\mathfrak{D}}(\mathfrak{A})$. If $\mathcal{S} \subseteq \mathfrak{A}$, then $\mathcal{W}(\mathcal{S}) = \mathcal{W} \cap \mathcal{S}$. If $v,\ w \in \mathcal{W}$, then $v \perp w$ means that the initial projections of v and w are orthogonal and the final projections of v and w are orthogonal.

Note that $\mathcal{P}(\mathfrak{D}) \subseteq \mathcal{W}$ and $\mathcal{W} = \mathcal{W}^*$, and if $w \in \mathcal{W}$ then $w^*w,\ ww^* \in \mathcal{P}(\mathfrak{D})$ (ince $1 \in \mathfrak{D}$.

LEMMA 3.1. (a) *If* $w \in \mathcal{W}$ *and* $e,\ f \in \mathcal{P}(\mathfrak{D})$, *then* $ewf \in \mathcal{W}$.

(b) $\mathcal{W}$ *is multiplicative.*

(c) $\mathcal{U}(\mathfrak{D})\mathcal{W}\mathcal{U}(\mathfrak{D}) \subseteq \mathcal{W}$.

Proof: (a) Let $p = ww^*$ and $q = w^*w$, so $pwq = w$. Then ew is a partial isometry since $(ew)(ew)^*(ew) = eww^*ew = epew = epw = ew$. Also, if $d \in \mathfrak{D}$, then $(ew)^* \cdot d(ew) = w^*(ede)w \in \mathfrak{D}$ and $(ew)d(ew)^* = e(wdw^*)e \in e\mathfrak{D}e \subseteq \mathfrak{D}$. Therefore, $ew \in \mathcal{W}$. It now follows that $w^*e \in \mathcal{W}$, so $f(w^*e) \in \mathcal{W}$ by the same argument, and this implies that $ewf \in \mathcal{W}$, thus proving (a).

For (b), suppose $v,\ w \in \mathcal{W}$, and let $e = ww^*$, $f = v^*v$. Then $vww^*v^*vw = vefw = vfew = vv^*vww^*w = vw$, so vw is a partial isometry. Also, if $d \in \mathfrak{D}$, then $w^*v^*dvw \subseteq w^*\mathfrak{D}w \subseteq \mathfrak{D}$, and similarly $vwdw^*v^* \subseteq \mathfrak{D}$. The proof of (c) is similar. ▨

It will be convenient later to use the following alternate characterization of $\mathcal{W}$.

LEMMA 3.2. $\mathcal{W} = \{$*partial isometries* w *in* $\mathfrak{A} : w\mathfrak{D}_{sa} = \mathfrak{D}_{sa}w\}$.

Proof. Suppose that w is a partial isometry in $\mathfrak{A}$ such that $w\mathfrak{D}_{sa} = \mathfrak{D}_{sa}w$, and suppose $s \in \mathfrak{D}_{sa}$. Then $sw = wt$ and $ws = t'w$ for some $t,\ t' \in \mathfrak{D}_{sa}$, and therefore $w^*ws = w^*t'w = sw^*w$. It follows that $w^*w \in \mathfrak{D}^c = \mathfrak{D}$, so $w^*sw = w^*wt \in \mathfrak{D}$. Similarly, $wsw^* \in \mathfrak{D}$, and thus $w \in \mathcal{W}$.

Conversely, if $w \in \mathcal{W}$ and $e \in \mathcal{P}(\mathfrak{D})$, then $we \in \mathcal{W}$ by Lemma 3.1. Now $wew^* = f$ for some self-adjoint $f \in \mathfrak{D}$. Thus, $we = ww^*we = wew^*w = fw$, and it follows that $w\mathfrak{D}_{sa} \subseteq \mathfrak{D}_{sa}w$ since $\mathfrak{D}_{sa}$ is generated by $\mathcal{P}(\mathfrak{D})$. Similarly, $\mathfrak{D}_{sa}w \subseteq w\mathfrak{D}_{sa}$. ▨

LEMMA 3.3. *If* w *is a matrix unit of* $\mathfrak{A}_n$, *then* $w \in \mathcal{W}$.

Proof. Let $e \in \mathcal{P}(\mathfrak{D})$. Then $e \in \mathfrak{D}_m$ for some m. If $m \leqslant n$, then $ew = w$ or 0, which implies that $w^*ew \in \mathfrak{D}$, and $we = w$ or 0, so $wew^* \in \mathfrak{D}$. If $m > n$, then we can write $w = w_1 + \ldots + w_k$, where the w_i's are distinct matrix units of $\mathfrak{A}_m$. Because of the way that $\mathfrak{A}_n$ embeds into $\mathfrak{A}_m$, the initial projections $\{f_i\}$ of $\{w_i\}$ are distinct and thus orthogonal) minimal projections in $\mathfrak{D}_m$, and the same is true of the final projections $\{e_i\}$. For instance, suppose $f_1 = f_j$ for some $j \neq 1$. Let $E = \{i : f_i = f_1\}$, and note that $e_i \neq e_1$ for all $i \in E$ since the w_i's are distinct. Then

$$1 \geq \|(\sum_{i\in E} e_i)wf_1\|^2 = \|\sum_{i\in E} f_1 w^* e_i w f_1\| =$$

$$= \|\sum_{i\in E} w_i^* w_i\| = \text{card}(E)\|f_1\| = \text{card}(E),$$

a contradiction. The other cases are similar. Now $w^*ew = \sum_{i=1}^{k} \alpha_i f_i \in \mathfrak{D}$, where $\alpha_i = 0$ or 1, and $wew^* = \sum_{i=1}^{k} \alpha_i e_i \in \mathfrak{D}$. The result follows since $\mathfrak{D}$ is generated by $\mathscr{P}(\mathfrak{D})$. ▨

LEMMA 3.4. *Suppose $w \in \mathscr{W}$ and $u \in \mathfrak{A}_n$ such that $\|w - u\| < 1/5$. Let $\{e_i\}_{i=1}^m$ be the minimal projections in $\mathfrak{D}_n$. Then*

(a) *if $e_i g \neq 0 = e_j g$ for some central projection $g \in \mathfrak{A}_n$, then $e_i w e_j = e_j w e_i = 0$,*

(b) *for each i, j, either $e_i w e_j = 0$, or else $e_j w^* e_i w e_j = e_j$ and $e_i w e_j w^* e_i = e_i$ (i.e., $e_i w e_j$ is a partial isometry with initial projection e_j and final projection e_i)*

(c) *for each j, $e_i w e_j \neq 0$ for at most one i,*

(d) *for each i, $e_i w e_j \neq 0$ for at most one j.*

Proof. By Lemma 3.1, $\|e_i w e_j\| = 1$ or 0. If $e_i g = e_i$ and $e_j g = 0$, then $e_i u e_j = e_i g u e_j = e_i u e_j g = 0$. Therefore, $1/5 > \|e_i w e_j - e_i u e_j\| = \|e_i w e_j\|$, which implie that $e_i w e_j = 0$. The same argument shows that $e_i w^* e_j = 0$, and (a) is proved.

We can now assume that $e_i g = e_i$ and $e_j g = e_j$ for some minimal centrals projection $g \in \mathfrak{A}_n$. Let e_{ij} be a matrix unit of $\mathfrak{A}_n g$ such that $e_i e_{ij} e_j = e_{ij}$, and suppose $e_i w e_j \neq 0$. Then

$$\frac{1}{5} > \|e_i w e_j - e_i u e_j\| = \|e_i w e_j - \mu_{ij} e_{ij}\|, \quad \text{some } \mu_{ij} \in \mathbf{C},$$

which implies that $|1 - |\mu_{ij}|| < 1/5$. Also,

$$\|e_j w^* e_i w e_j - e_j u^* e_i u e_j\| \leq \|w^* - u^*\|\,\|w\| + \|u^*\|\,\|w - u\| \leq$$

$$\leq \frac{1}{5} + \left(1 + \frac{1}{5}\right)\frac{1}{5} < \frac{1}{2},$$

and thus $\|f - |\mu_{ij}|^2 e_j\| < 1/2$, where $f = e_j w^* e_i w e_j$. Now $\|f - e_j\| < 1$ since $|1 - |\mu_{ij}|^2| < 1/2$, so $f = e_j$ because f and e_j are commuting projections. Similarly, $e_i w e_j w^* e_i = e_i$, and (b) is proved.

Now suppose $e_i w e_j \neq 0$ and $e_k w e_j \neq 0$, $i \neq k$. Then $(e_i + e_k) w e_j \in \mathscr{W}$ by Lemma 3.1, so $\|(e_i + e_k) w e_j\| = 1$ or 0. But then

$$\|(e_i + e_k) w e_j\|^2 = \|e_i w e_j + e_k w e_j\|^2 =$$

$$= \|(e_j w^* e_i + e_j w^* e_k)(e_i w e_j + e_k w e_j)\| =$$

$$= \|e_j w^* e_i w e_j + e_j w^* e_k w e_j\| = 2\|e_j\| = 2$$

by part (b), a contradiction. Therefore, (c) is proved, and (d) follows by symmetry. ▨

LEMMA 3.5. *Suppose that $v, w \in \mathscr{W}$ with $\|v - w\| < 1$. Then*
(a) *for each $e, f \in \mathscr{P}(\mathfrak{D})$, the initial projections of evf and ewf are the same,*
(b) *for each $e, f \in \mathscr{P}(\mathfrak{D})$, the final projections of evf and ewf are the same,*
(c) *$w = vu = u'v$ for some $u, u' \in \mathscr{U}(\mathfrak{D})$.*

Proof. Let $e, f \in \mathscr{P}(\mathfrak{D})$, $g = fv^*evf$, and $h = fw^*ewf$. $g, h \in \mathscr{P}(\mathfrak{D})$ by Lemma 3.1. If $g \neq h$, then either $g - gh \neq 0$ or $h - gh \neq 0$. In the first case, let $g' = g - gh$. Then $g'g = g'$ and $g'h = 0$. Now

$$1 > \|evf - ewf\|^2 \geq \|evfg' - ewfg'\|^2 = \|evfgg' - ewfhg'\|^2 =$$

$$= \|evfgg'\|^2 = \|g'gfv^*evfgg'\| = \|g'gg'\| = \|g'\| = 1,$$

a contradiction. The second case is similar. Therefore, $g = h$, and (a) is proved. (b) follows by taking adjoints.

Now let $f \in \mathscr{P}(\mathfrak{D})$, and let $e = wfw^*$. Then $vfv^* = e$ by (b), and simple calculations show that $vf = ev$ and $wf = ew$. Then $v^*wf = v^*ew = fv^*w$, so v^*w commutes with f. Since f is arbitrary, it follows that $v^*w \in \mathfrak{D}^c = \mathfrak{D}$. Define $u = v^*w + (1 - w^*w)$. (a) implies that $w^*w = v^*v$, so $v(1 - w^*w) = v - vv^*v = 0$, and easy calculations now yield that u is a unitary in $\mathfrak{D}$ and $w = vu$. The other equatility follows by considering w^* and v^*. ▨

THEOREM 3.6. $\mathscr{W} = \bigcup_{n=1}^{\infty} (\mathscr{W} \cap \mathfrak{A}_n)\mathscr{U}(\mathfrak{D}) = \bigcup_{n=1}^{\infty} \mathscr{U}(\mathfrak{D})(\mathscr{W} \cap \mathfrak{A}_n)$.

Proof. Let $w \in \mathscr{W}$, $p = ww^*$, and $q = w^*w$. Choose n large enough so that $p, q \in \mathfrak{D}_n$ and so that there is some $u \in \mathfrak{A}_n$ such that $\|w - u\| < 1/5$. By replacing u with puq, we can assume that $puq = u$. By Lemma 3.4, $p = \sum_{j=1}^{m} e_j$ and $q = \sum_{j=1}^{m} f_j$, where the e_j's and f_j's are minimal projections in $\mathfrak{D}_n$, and there is a permutation π of $\{1, \ldots, m\}$ such that $w = \sum_{j=1}^{m} e_{\pi(j)} w f_j$ with each $e_{\pi(j)} w f_j \neq 0$. Also, $e_{\pi(j)} w f_j w^* e_{\pi(j)} = e_{\pi(j)}$ and $f_j w^* e_{\pi(j)} w f_j = f_j$ for all j.

Since $e_{\pi(j)}wf_j \neq 0$, it follows by Lemma 3.4 that $e_{\pi(j)}g = e_{\pi(j)}$ and $f_jg = f_j$ for some minimal central projection g in $\mathfrak{A}_n$. Therefore, $e_{\pi(j)}uf_j = \mu_j x_j$, where $\mu_j \in \mathbf{C}$ and x_j is a matrix unit of $\mathfrak{A}_n g$ such that $x_j^* x_j = f_j$ and $x_j x_j^* = e_{\pi(j)}$. Then

$$\frac{1}{5} > \|e_{\pi(j)}wf_j - e_{\pi(j)}uf_j\| = \|e_{\pi(j)}wf_j - \mu_j x_j\|$$

implies that $|1 - |\mu_j|| < 1/5$, and thus

$$\left\| e_{\pi(j)}wf_j - \frac{\mu_j}{|\mu_j|} x_j \right\| \leqslant \|e_{\pi(j)}wf_j - \mu_j x_j\| + \left\| \mu_j x_j - \frac{\mu_j}{|\mu_j|} x_j \right\| <$$

$$< \frac{1}{5} + \left| \mu_j - \frac{\mu_j}{|\mu_j|} \right| = \frac{1}{5} + ||\mu_j| - 1| < \frac{2}{5}.$$

Let $v_j = \frac{\mu_j}{|\mu_j|} x_j$. Then $v_j^* v_j = f_j$ and $v_j v_j^* = e_{\pi(j)}$, $\|e_{\pi(j)}wf_j - v_j\| < 2/5$, and $v_j \in \mathscr{W}$ since $x_j \in \mathscr{W}$ (by Lemma 3.3).

By Lemma 3.5, for each j there is some unitary u_j in $\mathfrak{D}$ such that $e_{\pi(j)}wf_j = v_j u_j$. Let $\tilde{u} = 1 - q + \sum_{j=1}^{m} f_j u_j$. Simple calculations show that $\tilde{u} \in \mathscr{U}(\mathfrak{D})$ and that $w = \sum e_{\pi(j)}wf_j = \sum v_j u_j = (\sum v_j)\tilde{u}$. Finally, $\sum v_j = w\tilde{u}^* \in \mathscr{W}\mathscr{U}(\mathfrak{D}) \subseteq \mathscr{W}$ and therefore $\mathscr{W} = \bigcup_{n=1}^{\infty} (\mathscr{W} \cap \mathfrak{A}_n)\mathscr{U}(\mathfrak{D})$. The other equality follows by taking adjoints. ∎

A related result appears in [12, Lemma 6.3].

Corollary 3.7. *If $\mathscr{S} \subseteq \mathfrak{A}$ such that $\mathscr{S}\mathscr{U}(\mathfrak{D}) \subseteq \mathscr{S}$, then $\mathscr{W} \cap \mathscr{S} = \bigcup_{n=1}^{\infty} (\mathscr{W} \cap \mathscr{S} \cap \mathfrak{A}_n)\mathscr{U}(\mathfrak{D})$. If $\mathscr{U}(\mathfrak{D})\mathscr{S} \subseteq \mathscr{S}$, then $\mathscr{W} \cap \mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{U}(\mathfrak{D})(\mathscr{W} \cap \mathscr{S} \cap \mathfrak{A}_n)$.*

Proof. Suppose $w \in \mathscr{W} \cap \mathscr{S}$. Then $w = vu$ for some $u \in \mathscr{U}(\mathfrak{D})$ and $v \in \mathscr{W} \cap \mathfrak{A}_n$. Therefore, $wu^* = v$. But $wu^* \in \mathscr{S}$, so $v \in \mathscr{S}$. The other statement is proved similarly. ∎

We will now use $\mathscr{W}$ to make a more detailed study of $\mathfrak{D}$-modules. If $\mathscr{V} \subseteq \mathscr{W}$, let $\mathscr{M}(\mathscr{V})$ denote the (closed) $\mathfrak{D}$-module generated by $\mathscr{V}$. We first note that if $\mathscr{S}$ is a $\mathfrak{D}$-module, then $\mathscr{S}$ is generated by $\mathscr{W}(\mathscr{S}) = \mathscr{W} \cap \mathscr{S}$.

Proposition 3.8. *Let $\mathscr{S}$ be a $\mathfrak{D}$-module in $\mathfrak{A}$. Then $\mathscr{M}(\mathscr{W}(\mathscr{S})) = \mathscr{S}$.*

Proof. $\mathscr{W}(\mathscr{S}) \subseteq \mathscr{S}$ implies that $\mathscr{M}(\mathscr{W}(\mathscr{S})) \subseteq \mathscr{S}$. For the opposite inclusion, suppose $s \in \mathscr{S}$. By Theorem 2.2, there is a sequence $\{s_n\} \in \bigcup_m (\mathscr{S} \cap \mathfrak{A}_m)$ such that $s_n \to s$. Now $s_n = \sum \alpha_i e_i$, where $\alpha \in \mathbf{C}$ and the e_i's are matrix units of $\mathfrak{A}_m$, some m. If $\alpha_i \neq 0$, then $e_i \in \mathscr{W}(\mathscr{S})$ by Lemma 3.3 and the fact that $\mathscr{S}$ is a $\mathfrak{D}$-module. It follows that $s_n \in \mathscr{M}(\mathscr{W}(\mathscr{S}))$, and therefore $s \in \mathscr{M}(\mathscr{W}(\mathscr{S}))$. ▨

Next, note that $\mathscr{W}(\mathscr{S})$ has the following properties:

(†)
(i) $\mathscr{P}(\mathfrak{D}) \subseteq \mathscr{W}(\mathscr{S})$,
(ii) if $w \in \mathscr{W}(\mathscr{S})$ and $e, f \in \mathscr{P}(\mathfrak{D})$, then $ewf \in \mathscr{W}(\mathscr{S})$,
(iii) $\mathscr{U}(\mathfrak{D})\mathscr{W}(\mathscr{S})\mathscr{U}(\mathfrak{D}) \subseteq \mathscr{W}(\mathscr{S})$,
(iv) if $v, w \in \mathscr{W}(\mathscr{S})$ and $v \perp w$, then $v + w \in \mathscr{W}(\mathscr{S})$.

(i) and (iii) follow directly from the definition of $\mathscr{W}(\mathscr{S})$, and (ii) follows from Lemma 3.1. (iv) follows from the definition and the fact that the initial and final projections of v and w are in $\mathscr{P}(\mathfrak{D})$. As the following proposition shows, properties (†) completely characterize those subsets of $\mathscr{W}$ which are of the form $\mathscr{W}(\mathscr{S})$ for some $\mathfrak{D}$-module $\mathscr{S}$.

PROPOSITION 3.9. *Suppose $\mathscr{V} \subseteq \mathscr{W}$ has properties* (†). *Then $\mathscr{W}(\mathscr{M}(\mathscr{V})) = \mathscr{V}$*

Proof. We must show that $\mathscr{W}(\mathscr{M}(\mathscr{V})) \subseteq \mathscr{V}$. Let $\mathscr{M}_n$ be the $\mathfrak{D}_n$-module generated by $\mathscr{V} \cap \mathfrak{A}_n$. By properties (†), this is the same as the linear span of $\mathscr{V} \cap \mathfrak{A}_n$. By Corollary 3.7, $\mathscr{M}(\mathscr{V}) = \overline{\bigcup_n \mathscr{M}_n}$. Also, once we know that the sequence $\{\mathscr{M}_n\}$ is in canonical form, it will follow by Proposition 2.5 that $\mathscr{M}_n = \mathscr{M}(\mathscr{V}) \cap \mathfrak{A}_n$.

To show that $\{\mathscr{M}_n\}$ is in canonical form, it suffices to show that if x is a matrix unit in $\left(\bigcup_{m=1}^{\infty} \mathscr{M}_m\right) \cap \mathfrak{A}_n$, then $x \in \mathscr{M}_n$. Now $x \in \mathscr{M}_m$ for some m. The result is trivial if $m \leqslant n$, so we can assume $m > n$. Thus, $x = \sum_{j=1}^{k} \alpha_k v_k$ for some $\alpha_k \in \mathbf{C}$ and $v_k \in \mathscr{V} \cap \mathfrak{A}_m$. Using property (ii) of (†), x can be rewritten as $x = \sum_{i=1}^{l} \beta_i x_i$, where $\beta_i \in \mathbf{C}$, $\beta_i \neq 0$, and the x_i's are matrix units in $\mathscr{V} \cap \mathfrak{A}_m$. In this decomposition, $\beta_i x_i = exf$ for some minimal projections $e, f \in \mathfrak{D}_m$. Therefore, since $x \in \mathscr{W}$ by Lemma 3.3, it follows from Lemma 3.4 that $\beta_i x_i \perp \beta_j x_j$ for all $i \neq j$. Now apply property (ii) again to get that each $\beta_i x_i \in \mathscr{V}$, so $x \in \mathscr{V}$ by proerty (iv). Thus, $x \in \mathscr{V} \cap \mathfrak{A}_n \subseteq \mathscr{M}_n$.

Let $w \in \mathscr{W}(\mathscr{M}(\mathscr{V}))$. Then $w = uv$ for some $u \in \mathscr{U}(\mathfrak{D})$ and $v \in \mathscr{W} \cap \mathscr{M}(\mathscr{V}) \cap \mathfrak{A}_n$, some n, by Corollary 3.7. $\mathscr{M}(\mathscr{V}) \cap \mathfrak{A}_n = \mathscr{M}_n$ by the above, so $v \in \mathscr{W} \cap \mathscr{M}_n$. Now we can replace x with v and repeat the argument in the last paragraph to get that $v \in \mathscr{V}$. It follows that $w = uv \in \mathscr{V}$ by property (iii) of (†). ▨

Thus, we have a correspondence between $\mathfrak{D}$-modules in $\mathfrak{A}$ and subsets of $\mathscr{W}$ which satisfy properties (†). The previous two theorems can be thought of as reflexivity results for this correspondence, and they show that the correspondence is bijective.

DEFINITION. Let $\mathcal{S}$ be a $\mathfrak{D}$-module in $\mathfrak{A}$. Define a relation $\prec_{\mathcal{S}}$ on $\mathcal{P}(\mathfrak{D})$ by $e \prec_{\mathcal{S}} f$ if there is some $w \in \mathcal{W}(\mathcal{S})$ such that $ww^* = e$ and $w^*w = f$. We say that $\prec_{\mathcal{S}}$ is the *diagonal ordering induced by* $\mathcal{S}$. In general, this relation is not necessarily a partial ordering, i.e., reflexive, anti-symmetric, and transitive. However, we are primarily interested in the case in which $\mathcal{S}$ is a triangular algebra, and we will show later (Theorem 3.13) that in this case the relation is a partial ordering. Given a diagonal ordering $\prec_{\mathcal{S}}$, define $\mathcal{W}_{\max}(\mathcal{S}) = \{w \in \mathcal{W} : wfw^* \prec_{\mathcal{S}} fw^*wf$ for all $f \in \mathcal{P}(\mathfrak{D})\}$.

LEMMA 3.10. *$\mathcal{W}_{\max}(\mathcal{S})$ satisfies properties* (†).

Proof. (i) and (iii) of (†) clearly hold since $\mathcal{P}(\mathfrak{D}) \subseteq \mathcal{W}(\mathcal{S})$. Now let $w \in \mathcal{W}_{\max}(\mathcal{S})$ and $e, f \in \mathcal{P}(\mathfrak{D})$. Let e' be the initial projection of ew. Then $ew = we'$, so $ewf = we'f \in \mathcal{W}_{\max}(\mathcal{S})$ by definition. Therefore property (ii) holds.

To prove that (iv) holds, let $w_1, w_2 \in \mathcal{W}_{\max}(\mathcal{S})$ with $w_1 \perp w_2$, and let $f \in \mathcal{P}(\mathfrak{D})$. Then there are $v_1, v_2 \in \mathcal{W}(\mathcal{S})$ such that $fw_i^*w_if = v_i^*v_i$ and $w_ifw_i^* = v_iv_i^*$ for $i = 1, 2$. Note that $v_1 \perp v_2$ since $w_1f \perp w_2f$, so $v_1 + v_2 \in \mathcal{W}(\mathcal{S})$. It follows that $w_1 + w_2 \in \mathcal{W}_{\max}(\mathcal{S})$ because $f(w_1^* + w_2^*)(w_1 + w_2)f = (v_1^* + v_2^*)(v_1 + v_2)$ and $(w_1 + w_2)f(w_1^* + w_2^*) = (v_1 + v_2)(v_1^* + v^*)$. ∎

THEOREM 3.11. *Given a diagonal ordering $\prec_{\mathcal{S}}$ induced by $\mathcal{S}$, let Ω be the collection of all $\mathfrak{D}$-modules $\mathcal{T}$ such that $\prec_{\mathcal{T}} = \prec_{\mathcal{S}}$*, i.e., *$e \prec_{\mathcal{T}} f$ iff $e \prec_{\mathcal{S}} f$. Then Ω has a largest element, namely $\mathcal{M}(\mathcal{W}_{\max}(\mathcal{S}))$. $\mathcal{M}(\mathcal{W}_{\max}(\mathcal{S}))$ is called the unique maximal module induced by $\mathcal{S}$, and will also be denoted $\mathcal{M}_{\max}(\mathcal{S})$.*

Proof. $\mathcal{M}(\mathcal{W}_{\max}(\mathcal{S})) \in \Omega$ since $\mathcal{W}(\mathcal{M}(\mathcal{W}_{\max}(\mathcal{S}))) = \mathcal{W}_{\max}(\mathcal{S})$ by Theorem 3.9. Now suppose $\mathcal{T} \in \Omega$ and $\mathcal{T} \nsubseteq \mathcal{M}(\mathcal{W}_{\max}(\mathcal{S}))$. Since $\mathcal{T}$ is generated by $\mathcal{T} \cap \mathcal{W}$, there is some $w \in \mathcal{T} \cap \mathcal{W}$ with $w \notin \mathcal{M}(\mathcal{W}_{\max}(\mathcal{S})) \cap \mathcal{W} = \mathcal{W}_{\max}(\mathcal{S})$. Therefore, there is some $f \in \mathcal{P}(\mathfrak{D})$ such that $wfw^* \not\prec_{\mathcal{S}} fw^*wf$. But this contradicts the fact that $\prec_{\mathcal{T}} = \prec_{\mathcal{S}}$, since $wf \in \mathcal{T} \cap \mathcal{W}$ implies that $wfw^* \prec_{\mathcal{T}} fw^*wf$. ∎

$\mathcal{W}_{\max}(\mathcal{S})$ can also be characterized by the following proposition. The proof is straight forward and is omitted.

PROPOSITION 3.12. *$\mathcal{W}_{\max}(\mathcal{S}) = \bigcup \{\mathcal{V} \subseteq \mathcal{W} : \mathcal{V}$ satisfies properties (†), and $vv^* \prec_{\mathcal{S}} v^*v$ for all $v \in \mathcal{V}\}$.*

In general, as Theorem 3.11 suggests, a diagonal ordering may be induced by more than one $\mathfrak{D}$-module. We will show in Example 3.17 that this can indeed occur. However, the theorem does associate a unique object with each diagonal ordering. Since this is a maximal object, it is natural to ask whether two different maximal TAF algebras can induce the same diagonal ordering, i.e., are these two notions of maximality the same for triangular algebras? We can answer this question by investigating the relationship between the properties of the diagonal ordering $\prec_{\mathscr{S}}$ and the algebraic properties of $\mathscr{S}$.

THEOREM 3.13. *Let $\mathscr{S}$ be a $\mathfrak{D}$-module in $\mathfrak{A}$, and let $\prec_{\mathscr{S}}$ be the diagonal ordering on $\mathscr{P}(\mathfrak{D})$ induced by $\mathscr{S}$.*

(a) *$\prec_{\mathscr{S}}$ is reflexive.*

(b) *$\mathscr{S}$ is an algebra iff $\mathscr{W}(\mathscr{S})$ is multiplicative.*

(c) *If $\mathscr{S}$ is an algebra, then $\prec_{\mathscr{S}}$ is transitive.*

(d) *$\prec_{\mathscr{S}}$ is transitive iff $\mathscr{M}_{\max}(\mathscr{S})$ is an algebra.*

(e) *$\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$ iff $\mathscr{W}(\mathscr{S}) \cap \mathscr{W}(\mathscr{S})^* \subseteq \mathfrak{D}$ iff $\mathscr{W}(\mathscr{S}) \cap \mathscr{W}(\mathscr{S})^* = \mathscr{P}(\mathfrak{D})\mathscr{U}(\mathfrak{D})$.*

(f) *If $\prec_{\mathscr{S}}$ is antisymmetric, then $\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$.*

(g) *If $\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$ and $\mathscr{S}$ is an algebra, then $\prec_{\mathscr{S}}$ is antisymmetric.*

Proof. (a) is clear since $\mathscr{P}(\mathfrak{D}) \subseteq \mathscr{W}(\mathscr{S})$.

(b) If $\mathscr{S}$ is an algebra, then $\mathscr{W}(\mathscr{S})$ is multiplicative by Lemma 3.1. Conversely, suppose $\mathscr{W}(\mathscr{S})$ is multiplicative and $s, t \in \mathscr{S}$. By Theorem 2.2, there are sequences $\{s_n\}, \{t_n\} \in \bigcup_m (\mathscr{S} \cap \mathfrak{A}_m)$ such that $s_n \to s$ and $t_n \to t$. Now for fixed n, there is some m such that $s_n, t_n \in \mathfrak{A}_m$. Then $s_n = \sum_{i=1}^{k} \alpha_i e_i$ and $t_n = \sum_{i=1}^{k} \beta_i e_i$ where $\alpha_i, \beta_i \in \mathbf{C}$ and the e_i's are matrix units in $\mathfrak{A}_m$. Suppose $\alpha_i \neq 0$. Let $e, f \in \mathscr{P}(\mathfrak{D})$ be the final and initial projections, respectively, of e_i. Then $e_i = \alpha_i^{-1} e s_n f$, so $e_i \in \mathscr{W}(\mathscr{S})$ by Lemma 3.3 and the fact that $\mathscr{S}$ is a $\mathfrak{D}$-module. Similarly, for each i with $\beta_i \neq 0$ we have $e_i \in \mathscr{W}(\mathscr{S})$. Finally, $s_n t_n = \sum_{i,j=1}^{k} \alpha_i \beta_j e_i e_j$, and each nonzero term is in $\mathscr{S}$ because $\mathscr{W}(\mathscr{S})$ is multiplicative. Therefore, $s_n t_n \in \mathscr{S}$, and it follows that $st \in \mathscr{S}$. Thus, $\mathscr{S}$ is an algebra.

(c) Let $e, f, g \in \mathscr{P}(\mathfrak{D})$ such that $e \prec_{\mathscr{S}} f \prec_{\mathscr{S}} g$. By definition there are $v, w \in \mathscr{W}(\mathscr{S})$ such that $vv^* = e$, $v^*v = f = ww^*$, and $w^*w = g$. Then $vw \in \mathscr{W}(\mathscr{S})$ by (b), and $(vw)^*(vw) = w^*v^*vw = w^*fw = w^*ww^*w = w^*w = g$ and $(vw)(vw)^* = vww^*v^* = vfv^* = vv^*vv^* = vv^* = e$, so $e \prec_{\mathscr{S}} g$.

(d) The 'if' implication follows from (c). Now suppose that $\prec_{\mathscr{S}}$ is transitive. From (b), it is enough to show that $\mathscr{W}(\mathscr{M}_{\max}(\mathscr{S}))$ is multiplicative, and note that $\mathscr{W}(\mathscr{M}_{\max}(\mathscr{S})) = \mathscr{W}(\mathscr{M}(\mathscr{W}_{\max}(\mathscr{S}))) = \mathscr{W}_{\max}(\mathscr{S})$ by Theorem 3.9. Let $v, w \in \mathscr{W}_{\max}(\mathscr{S})$. To see that $vw \in \mathscr{W}_{\max}(\mathscr{S})$, we must show that $vwew^*v^* \prec_{\mathscr{S}} ewv^*v^*we$ for all $e \in \mathscr{P}(\mathfrak{D})$. Let $p = v^*v$ and $f = pwew^*p$. $f \prec_{\mathscr{S}} ew^*pwe$ since $pwe \in \mathscr{W}_{\max}(\mathscr{S})$. Also, $fv^*vf = fpf = pff = f$ and $vf \in \mathscr{W}_{\max}(\mathscr{S})$ imply that $vfv^* \prec_{\mathscr{S}} f$. Therefore, $vfv^* \prec_{\mathscr{S}} \prec_{\mathscr{S}} ew^*pwe$ by transitivity, and it follows that

$$vwew^*v^* = vfv^*vwew^*v^*vv^* =$$

$$= vpwew^*pv^* = vfv^* \prec_{\mathscr{S}} ew^*pwe = ew^*v^*vwe.$$

(e) If $\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$, then $\mathscr{W}(\mathscr{S}) \cap \mathscr{W}(\mathscr{S})^* = \mathscr{W} \cap \mathscr{S} \cap \mathscr{S}^* = \mathscr{W} \cap \mathfrak{D} \subseteq \mathfrak{D}$. Conversely, if $\mathscr{W}(\mathscr{S}) \cap \mathscr{W}(\mathscr{S})^* \subseteq \mathfrak{D}$, then $\mathscr{S} \cap \mathscr{S}^* \cap \mathscr{W} \subseteq \mathfrak{D}$ and therefore, $\mathscr{S} \cap \mathscr{S}^* \supseteq \mathfrak{D} = \mathscr{M}(\mathfrak{D}) \supseteq \mathscr{M}(\mathscr{S} \cap \mathscr{S}^* \cap \mathscr{W}) = \mathscr{S} \cap \mathscr{S}^*$ by Proposition 3.8. Finally, suppose $w \in \mathcal{W}(\mathcal{S}) \cap \mathcal{W}(\mathcal{S})^* \subseteq \mathfrak{D}$. Then $w^*w = ww^*$, so $u = (1 - w^*w) + w$ is a unitary in $\mathfrak{D}$ and $w = (w^*w)u$.

(f) Suppose $\prec_{\mathcal{S}}$ is antisymmetric and $w \in \mathcal{W}(\mathcal{S}) \cap \mathcal{W}(\mathcal{S})^*$. Let $e = ww^*$ and $f = w^*w$. Then $e \prec_{\mathcal{S}} f$ and $f \prec_{\mathcal{S}} e$, so $e = f$ and w and w^* commute. Now if $d \in \mathcal{P}(\mathfrak{D})$, then wd, $dw^* \in \mathcal{W}(\mathcal{S})$ and thus $wd \in \mathcal{W}(\mathcal{S}) \cap \mathcal{W}(\mathcal{S})^*$. Therefore, the above argument applied to wd implies that $dw = dew = dfw = dfdw = dw^*wdw = wddw^*w = wdf = wfd = wd$. Hence, $w \in \mathcal{P}(\mathfrak{D})^c = \mathfrak{D}^c = \mathfrak{D}$, and the result follows from (e).

(g) Let $p, q \in \mathcal{P}(\mathfrak{D})$, such that $p \prec_{\mathcal{S}} q$ and $q \prec_{\mathcal{S}} p$. Then there are $v, w \in \mathcal{W}(\mathcal{S})$ such that $v^*v = ww^* = q$ and $vv^* = w^*w = p$. By Corollary 3.7 there is an n such that $v = uv_1$ and $w = u'w_1$ for some $u, u' \in \mathcal{U}(\mathfrak{D})$ and $v_1, w_1 \in \mathcal{W}(\mathcal{S}) \cap \mathfrak{A}_n$. Now $v_1v_1^* = u^*vv^*u = u^*pu = pu^*u = p$ and $v_1^*v_1 = v^*uu^*v = v^*v = q$, and similarly $w_1w_1^* = q$ and $w_1^*w_1 = p$. Therefore, by replacing v and w with w_1 and w_1, respectively, we can assume that $v, w \in \mathcal{W}(\mathcal{S}) \cap \mathfrak{A}_n$ and $p, q \in \mathfrak{D}_n$.

Write $p = \sum_{i=1}^{k} e_i$ and $q = \sum_{i=1}^{m} f_i$, where the e_i's and f_i's are minimal projections in $\mathfrak{D}_n$. These representations define a fixed ordering on $\{e_i\}$ and $\{f_i\}$. Also, for each i and j, let $x_{j,i}$ be a matrix unit of $\mathfrak{A}_n$ with initial projection f_i and final projection e_j. Now Lemmas 3.1 and 3.4 show that $k = m$ and that there is a permutation π of $\{1, \ldots, k\}$ defined as follows: $\pi(i)$ is the unique integer such that $vf_i = e_{\pi(i)}vf_i = e_{\pi(i)}v \neq 0$. Likewise, there is a permutation τ of $\{1, \ldots, k\}$ defined by $we_j = f_{\tau(j)}we_j = f_{\tau(j)}w \neq 0$. Therefore, $\tau \circ \pi$ is a permutation of $\{1, \ldots, k\}$ such that $0 \neq f_{\tau(\pi(i))}wvf_i$, and similarly $0 \neq f_{(\tau\circ\pi)^t(i)}(wv)^tf_i$ for every positive integer t. Let s be the smallest positive integer such that $(\tau \circ \pi)^s = \mathrm{id}$. Then $0 \neq f_{(\tau\circ\pi)^s(i)}(wv)^sf_i = f_i(wv)^{s-1}we_{\pi(i)}vf_i$. Therefore, $f_i(wv)^{s-1}we_{\pi(i)} = \alpha_i x_{i,\pi(i)}$ and $e_{\pi(i)}vf_i = \beta_i x_{\pi(i),i}$ for some nonzero $\alpha_i, \beta_i \in \mathbf{C}$. Since $\mathcal{S}$ is an algebra, it follows that $x_{\pi(i),i}$ and $x_{i,\pi(i)}$ are both in $\mathcal{S}$, and thus $x_{\pi(i),i} \in \mathcal{S} \cap \mathcal{S}^*$. If $e_{\pi(i)} \neq f_i$ for some i, this contradicts the fact that $\mathcal{S} \cap \mathcal{S}^* = \mathfrak{D}$. Therefore $e_{\pi(i)} = f_i$ for all i, and it follows that $p = q$. ▨

We will give an example shortly of a $\mathfrak{D}$-module $\mathcal{S}$ which is triangular but $\prec_{\mathcal{S}}$ is not antisymmetric, and also an example in which $\prec_{\mathcal{S}}$ is transitive but $\mathcal{S}$ is not an algebra.

COROLLARY 3.14. *If $\mathcal{S}$ is a* TAF *algebra, then so is $\mathcal{M}_{\max}(\mathcal{S})$. Thus, if $\mathcal{S}$ is a maximal* TAF *algebra, then $\mathcal{S} = \mathcal{M}_{\max}(\mathcal{S})$.*

Proof. Theorem 3.13 implies that $\prec_{\mathcal{S}}$ is transitive and antisymmetric, and therefore $\mathcal{M}_{\max}(\mathcal{S})$ is a closed algebra which is triangular since $\prec_{\mathcal{M}_{\max}(\mathcal{S})} = \prec_{\mathcal{S}}$ by Theorem 3.11. ▨

COROLLARY 3.15. *$\mathcal{S}$ is a maximal* TAF *algebra if and only if $\mathcal{W}(\mathcal{S})$ is a maximal subset of $\mathcal{W}$ with respect to satisfying properties (†) and $\prec_{\mathcal{S}}$ being reflexive, transitive, and antisymmetric.*

REMARK. Parts (f) and (g) of Theorem 3.13 can be used to give a simple proof of the special case of Theorem 2.6 in which each $\mathscr{T}_n$ is a triangular algebra in $\mathfrak{A}_n$ (note that Theorem 2.6 played no role in the proof of Theorem 3.13 or any of the results used in the proof). First, if $\mathscr{T}$ is not triangular, then $\prec_{\mathscr{T}}$ is not antisymmetric by 3.13 (f). Therefore, there are $p, q \in \mathscr{P}(\mathfrak{D})$, $p \neq q$, and $v, w \in \mathscr{W}(\mathscr{T})$ such that $vv^* = p = w^*w$ and $v^*v = q = ww^*$. As in the first part of the proof of 3.13 (g), we can assume that $p, d, v, w \in \mathfrak{A}_n$ for some n. It follows that $\prec_{\mathscr{T}_n}$ is not antisymmetric. Since $\mathscr{T}_n$ is an algebra, 3.13 (g) implies that $\mathscr{T}_n \cap \mathscr{T}_n^* \neq \mathfrak{D}$, a contradiction. *Thus, $\mathscr{T}$ is triangular.*

EXAMPLE 3.16. As indicated in Section 1, we let $\{g_{ij}^{(n)}\}$ denote the usual matrix units for $\mathbf{M}_n$. Let $\mathfrak{A}_1 = \mathbf{M}_2$, $\mathfrak{A}_2 = \mathbf{M}_4$, and let $\mathfrak{A} = \bigcup_{k=1}^{2} \mathfrak{A}_k = \mathfrak{A}_2$, using the standard embedding $\sigma_1 : \mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2$ (see Section 1). Define $\mathscr{S} = \bigcup_{k=1}^{2} \mathscr{S}_k = \mathscr{S}_2$, where $\mathscr{S}_1 = \operatorname{span}\{g_{ij}^{(2)} : i \leqslant j\} \subseteq \mathfrak{A}_1$ and $\mathscr{S}_2 = \operatorname{span}(\{g_{ij}^{(4)} : j - 1 \leqslant i \leqslant j\} \cap \{g_{41}^{(4)}\}) \subseteq \mathfrak{A}_2$. Then $\mathscr{S} \cap \mathscr{S}^* = \mathfrak{D}$, the diagonal of $\mathbf{M}_4$, but $\prec_{\mathscr{S}}$ is not antisymmetric since $g_{11}^{(2)} \prec_{\mathscr{S}}$ $\prec_{\mathscr{S}} g_{22}^{(2)}$ via $g_{12}^{(2)}$ and $g_{22}^{(2)} \prec_{\mathscr{S}} g_{11}^{(2)}$ via $g_{23}^{(4)} + g_{41}^{(4)}$. Thus, the converse of 3.13 (f) does not hold.

EXAMPLE. 3.17. Let $\{e_{ij}^{(k)} = g_{ij}^{(3\cdot 2^k)}\}$ be matrix units for $\mathfrak{A}_k = \mathbf{M}_{3\cdot 2^k}$, $k = 0,1,2,\ldots$, and let $\mathfrak{A} = \overline{\bigcup_k \mathfrak{A}_k}$ via the standard embeddings $\sigma_k : \mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$. Let $\mathscr{S} = \overline{\bigcup_k \mathscr{S}_k}$, where $\mathscr{S}_0 = \operatorname{span}\{e_{ij}^{(0)} : j - 1 \leqslant i \leqslant j\} \subseteq \mathfrak{A}_0$ and for $k \geqslant 0$, $\mathscr{S}_{k+1}$ is the $\mathfrak{D}_{k+1}$-module generated by $\sigma(\mathscr{S}_k) \cup \{e^{(k+1)}_{3\cdot 2^k+3(j-1)+1,3j},\ e^{(k+1)}_{3(j-1)+1,3\cdot 2^k+3j}\}_{j=1}^{2^k} \subseteq \mathfrak{A}_{k+1}$. Then $\prec_{\mathscr{S}}$ is transitive, but $e_{12}^{(0)} e_{23}^{(0)} = e_{13}^{(0)} \notin \mathscr{S}$, so $\mathscr{S}$ is not an algebra. It follows that the converse of 3.13(c) is not true. To see that $e_{13}^{(0)} \notin \mathscr{S}$, notice that otherwise there would be some $u \in \mathscr{S}_k$ such that $\|e_{13}^{(0)} - u\| < 1/2$, and therefore $\|e_{11}^{(k)} u e_{33}^{(k)}\| > 1/2$. But this contradicts the fact that $e_{11}^{(k)} u e_{33}^{(k)} = 0$ for all k and all $u \in \mathscr{S}_k$.

Note that this is also a concrete example of a module $\mathscr{S}$ for which $\mathscr{S} \neq \mathscr{M}_{\max}(\mathscr{S})$, since $\prec_{\mathscr{S}} = \prec_{\mathscr{M}_{\max}(\mathscr{S})}$ implies that $\mathscr{M}_{\max}(\mathscr{S})$ is an algebra by Theorem 3.13(d).

We now turn to the study of isomorphisms of modules, and especially triangular algebras. For the remainder of this section, $\mathfrak{A}$ and $\mathfrak{B}$ will be AF algebras with masas $\mathfrak{D}$ and $\mathfrak{E}$, respectively, $\mathscr{S}$ will be a $\mathfrak{D}$-module in $\mathfrak{A}$, and $\mathscr{T}$ will be a $\mathfrak{E}$-module in $\mathfrak{B}$. As usual, all other modules and subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$ mentionedwill be assumed to be closed. We first prove the following key lemma.

LEMMA 3.18. *Suppose $\varphi : \mathscr{S} \to \mathscr{T}$ is an isometric module isomorphism (so in particular $\varphi(\mathfrak{D}) = \mathfrak{E}$ and φ is multiplicative on $\mathfrak{D}$) and $v \in \mathscr{W}_{\mathfrak{D}}(\mathscr{S})$. Then $\varphi(v)^*\varphi(v) = \varphi(v^*v)$ and $\varphi(v)\varphi(v)^* = \varphi(vv^*)$.*

Proof. $\varphi \mid \mathfrak{D} : \mathfrak{D} \to \mathfrak{E}$ is an isomorphism of abelian C^*-algebras, so it is a C^*-isomorphism. Therefore, $\varphi(\mathscr{P}(\mathfrak{D})) = \mathscr{P}(\mathfrak{E})$. Now since $v^*v \in \mathscr{P}(\mathfrak{D})$, $\varphi(v^*v)$ is defined and represents a projection in $\mathfrak{E}$. To show that $\varphi(v)^*\varphi(v) \in \mathfrak{E}$, we show it commutes with $\mathscr{P}(\mathfrak{E})$. Let $f \in \mathscr{P}(\mathfrak{E}), f = \varphi(e)$, $e \in \mathscr{P}(\mathfrak{D})$. Then

$$f\varphi(v)^*\varphi(v) = \varphi(e)\varphi(v)^*\varphi(v) = \varphi(e)^*\varphi(v)^*\varphi(v) =$$

$$= [\varphi(v)\varphi(e)]^*\varphi(v) = \varphi(ve)^*\varphi(v).$$

By Lemma 3.1, $ve \in \mathscr{W}_{\mathfrak{D}}(\mathscr{S})$. Letting $e' \in \mathscr{P}(\mathfrak{D})$ be the final projection of ve, it follows that $ve = e'v$, so

$$f\varphi(v)^*\varphi(v) = \varphi(e'v)^*\varphi(v) = [\varphi(e')\varphi(v)]^*\varphi(v) =$$

$$= \varphi(v)^*\varphi(e')\varphi(v) = \varphi(v)^*\varphi(e'v) =$$

$$= \varphi(v)^*\varphi(ve) = \varphi(v)^*\varphi(v)f.$$

Next, let $s = \varphi(v)^*\varphi(v) \in \mathfrak{E}$. Viewing s as a function on X, the spectrum of $\mathfrak{E}$, we have $0 \leqslant s \leqslant 1$. X is zero dimensional, so if s assumes some value strictly between 0 and 1, then there is a projection (i.e., characteristic function of a clopen set) $h \in \mathscr{P}(\mathfrak{E})$ such that $0 \neq hs$ and $\|hs\| < 1$. Setting $h = \varphi(g)$, $hs = hsh = [\varphi(v)\varphi(g)]^*[\varphi(v)\varphi(g)] = \varphi(vg)^*\varphi(vg)$. Now either $vg = 0$, or else vg is a partial isometry with initial projection v^*vg. Since $vg = 0$ implies $hs = 0$, vg must be a partial isometry. Hence $\varphi(vg)$ has norm one, so $1 = \|\varphi(vg)^*\varphi(vg)\|$, contradicting $\|hs\| < 1$.

To finish the proof of the claim, let $p = v^*v$ and $q = \varphi(p)$. Then $\varphi(v)^*\varphi(v)q = \varphi(v)^*\varphi(vv^*v) = \varphi(v)^*\varphi(v)$, so $\varphi(v)^*\varphi(v) \leqslant q$. Suppose $q' \leqslant q$ is such that $\varphi(v)^*\varphi(v)q' = 0$. Set $q' = \varphi(p')$ and note that $\varphi(v)^*\varphi(v)q' = q'\varphi(v)^*\varphi(v)q' = \varphi(vp')^*\varphi(vp')$, so that $vp' = 0$. But $p' \leqslant p$, the initial projection of v. Hence $vp' = 0$ implies $p' = 0$, and so $q' = 0$. We conclude that $\varphi(v^*v) = \varphi(v)^*\varphi(v)$. Similarly, $\varphi(vv^*) = \varphi(v)\varphi(v^*)$. ▨

It now follows that the diagonal ordering induced by a module is an isomorphism invariant.

DEFINITION. Two diagonal orderings $(\mathfrak{D}, \prec_{\mathscr{S}})$ and $(\mathfrak{E}, \prec_{\mathscr{T}})$ are *order isomorphic* if there is a C^*-isomorphism ψ of $\mathfrak{D}$ onto $\mathfrak{E}$ such that $e \prec_{\mathscr{S}} f$ iff $\psi(e) \prec_{\mathscr{T}} \psi(f)$. In this case we say that ψ is an *order isomorphism* of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$.

THEOREM 3.19. *If $\varphi : \mathscr{S} \to \mathscr{T}$ is an isometric module isomorphism, then φ is an order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$. Moreover, $\varphi(\mathscr{W}_{\mathfrak{D}}(\mathscr{S})) = \mathscr{W}_{\mathfrak{E}}(\mathscr{T})$ and if $e \prec_{\mathscr{S}} f$ via $v \in \mathscr{W}_{\mathfrak{D}}(\mathscr{S})$, then $\varphi(e) \prec_{\mathscr{S}} \varphi(f)$ via $\varphi(v)$.*

Proof. $\varphi \mid \mathfrak{D} : \mathfrak{D} \to \mathfrak{E}$ is a C^*-isomorphism onto $\mathfrak{E}$. Now if $e \prec_{\mathscr{S}} f$, then there is some $v \in \mathscr{W}_{\mathfrak{D}}(\mathscr{S})$ such that $vv^* = e$ and $v^*v = f$. By Lemma 3.2, $v\mathfrak{D}_{sa} = \mathfrak{D}_{sa}v$, so $\varphi(v)\mathfrak{E}_{sa} = \mathfrak{E}_{sa}\varphi(v)$. Lemma 3.18 shows that $\varphi(v)$ is a partial isometry, and thus $\varphi(v) \in \mathscr{W}_{\mathfrak{E}}(\mathscr{T})$. Lemma 3.18 also implies that $\varphi(e) = \varphi(vv^*) = \varphi(v)\varphi(v^*)$ and $\varphi(f) = \varphi(v^*v) = \varphi(v^*)\varphi(v)$, so $\varphi(e) \prec_{\mathscr{T}} \varphi(f)$. ▨

If $\mathscr{S}$ and $\mathscr{T}$ are triangular algebras, then we only need to assume that φ is an algebra isomorphism.

PROPOSITION 3.20. *Suppose that $\mathscr{S}$ and $\mathscr{T}$ are triangular algebras. If $\varphi : \mathscr{S} \to \mathscr{T}$ is an isometric algebra isomorphism, then $\varphi \mid \mathfrak{D}$ is a C^*-isomorphism onto $\mathfrak{E}$. Thus, φ is an order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$.*

Proof. First we observe that $\varphi \mid \mathfrak{D}$ maps $\mathfrak{D}$ into $\mathfrak{E}$. Let $d = d^* \in \mathfrak{D}$. Since $\exp(\sqrt{-1}td)$ is unitary ($t \in \mathbf{R}$) and φ is isometric, $\|\varphi(\exp(\sqrt{-1}td))\| = 1$. But $\varphi(\exp(\sqrt{-1}td)) = \exp(\sqrt{-1}t\varphi(d)), t \in \mathbf{R}$, so $\|\exp(\sqrt{-1}t\varphi(d))\| = 1$ and it follows that $\varphi(d)$ is hermitian in the sense of [2, Corollary 13, p. 55]. As a hermitian element of a C^*-algebra is self-adjoint [2, Proposition 20, p. 67], $\varphi(d) = \varphi(d)^*$. It now follows that $\varphi(\mathfrak{D}) \subseteq \mathfrak{E}$ and $\varphi \mid \mathfrak{D}$ is a C^*-isomorphism. The same argument, applied to $\varphi^{-1} \mid \mathfrak{E}$, shows that $\varphi(\mathfrak{D}) = \mathfrak{E}$, and the result follows by Theorem 3.19. ▨

REMARK. Another proof can be based on the fact that if ρ is a state on $C^*(\mathscr{T})$, then $\rho \circ \varphi$ is a linear functional on $\mathfrak{D}$ of norm 1 such that $(\rho \circ \varphi)(1) = 1$, and hence $\rho \circ \varphi$ is a state on $\mathfrak{D}$. It follows that $\varphi(h)$ is self-adjoint if $h \in \mathfrak{D}$ is self-adjoint. Note also that the proof of the proposition does not use the fact that $\mathfrak{A}$, $\mathfrak{B}$ are AF algebras, and the only reason for the assumption is that triangular algebras have only been defined in the context of AF algebras in this paper.

We now consider the question of whether the diagonal ordering is a complete isomorphism invariant. This seems unlikely, since we know by Example 3.17 that two different modules can have the same diagonal ordering. However, there is a unique maximal module with a particular ordering. More generally, if $(\mathfrak{D}, \prec_{\mathscr{S}})$ and $(\mathfrak{E}, \prec_{\mathscr{T}})$ are order isomorphic, are $\mathscr{M}_{\max}(\mathscr{S})$ and $\mathscr{M}_{\max}(\mathscr{T})$ isomorphic?

PROPOSITION 3.21. *Let $\mathscr{S}$ be a $\mathfrak{D}$-module in $\mathfrak{A}$ and $\mathscr{T}$ an $\mathfrak{E}$-module in $\mathfrak{B}$. Suppose $\varphi : \mathfrak{D} \to \mathfrak{E}$ is an order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$ which extends to a C^*-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Then $\varphi(\mathscr{M}_{\max}(\mathscr{S})) = \mathscr{M}_{\max}(\mathscr{T})$.*

Proof. It is clear that $\varphi(\mathscr{W}_{\mathfrak{D}}(\mathfrak{A})) = \mathscr{W}_{\mathfrak{E}}(\mathfrak{B})$. Now if $v \in \mathscr{W}_{\mathfrak{D},\max}(\mathscr{S}) = \mathscr{W}_{\mathfrak{D}}(\mathscr{M}_{\max}(\mathscr{S}))$, then $vev^* \prec_{\mathscr{S}} ev^*ve$ for each $e \in \mathscr{P}(\mathfrak{D})$. Thus $\varphi(v)f\varphi(v)^* \prec_{\mathscr{T}} f\varphi(v)^*\varphi(v)f$ for each $f \in \mathscr{P}(\mathfrak{E})$, since $f = \varphi(e)$ for some $e \in \mathscr{P}(\mathfrak{D})$. Therefore, $\varphi(v) \in \mathscr{W}_{\mathfrak{E}}(\mathscr{M}_{\max}(\mathscr{T}))$ by definition, and Proposition 3.8 implies that $\varphi(\mathscr{M}_{\max}(\mathscr{S})) \subseteq \mathscr{M}_{\max}(\mathscr{T})$. The result follows by considering φ^{-1}. ▨

However, we will later show (Example 4.4) that not all order isomorphisms of diagonal orderings extend, and they may not even extend to algebra isomorphisms in the case that the maximal modules are algebras. In fact this is true even if we impose a rather strong condition, that the modules be strongly maximal TAF algebras (defined below). However, it is not known if order isomorphisms extend to module isomorphisms.

The lattice of invariant projections is also an isomorphism invariant, since if $\varphi : \mathscr{S} \to \mathscr{T}$ is an isometric module isomorphism and $e \in \operatorname{Lat}\mathscr{S}$, then $\varphi(e) \in \mathscr{P}(\mathfrak{E})$ by Theorem 3.19 and, for $s \in \mathscr{S}$, $se = ese$ iff $\varphi(s)\varphi(e) = \varphi(e)\varphi(s)\varphi(e)$. Therefore, $\varphi(\operatorname{Lat}\mathscr{S}) = \operatorname{Lat}\mathscr{T}$. Of course the same is true for an isometric algebra isomorphism of triangular algebras by Proposition 3.20. Actually, though, the diagonal ordering $\prec_{\mathscr{S}}$ alone is enough to determine $\operatorname{Lat}\mathscr{S}$.

LEMMA 3.22. *A projection $p \in \mathfrak{D}$ is invariant for $\mathscr{S}$ if and only if for any two projections $e, f \in \mathscr{S}$ such that $e \prec_{\mathscr{S}} f \leqslant p$, we have $e \leqslant p$.*

Proof. Suppose p is an invariant projection and $v \in \mathscr{W}(\mathscr{S})$ such that $e = vv^*$ and $v^*v = f \leqslant p$. Then

$$(1-p)vp = 0 \Rightarrow (1-p)vf = 0 \Rightarrow (1-p)v = 0 \Rightarrow$$
$$\Rightarrow (1-p)vv^* = 0 \Rightarrow (1-p)e = 0 \Rightarrow e = pe.$$

Conversely, suppose p is a projection satisfying the stated condition. Then for every $v \in \mathscr{W}(\mathscr{S})$, we have

$$[(1-p)vp][(1-p)vp]^* \prec_{\mathscr{S}} pv^*(1-p)vp \leqslant p.$$

Hence,

$$[(1-p)vp][(1-p)vp]^* = p[(1-p)vp][(1-p)vp]^* = 0,$$

so $(1-p)vp = 0$. Since linear combinations of elements of $\mathscr{W}(\mathscr{S})$ are dense in $\mathscr{S}$, we conclude that $(1-p)sp = 0$ for all $s \in \mathscr{S}$, and thus p is an invariant projection. ▨

COROLLARY 3.23. *If φ is an order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$, then $\varphi(\operatorname{Lat}\mathscr{S}) = \operatorname{Lat}\mathscr{T}$.*

DEFINITION. We will say that two lattices $\operatorname{Lat}\mathscr{S}$ and $\operatorname{Lat}\mathscr{T}$ are *isomorphic* if there is a C^*-isomorphism φ of $\mathfrak{D}$ onto $\mathfrak{E}$ such that $\varphi(\operatorname{Lat}\mathscr{S}) = \operatorname{Lat}\mathscr{T}$, and we then say that φ is a *lattice isomorphism*. Thus, isometric module isomorphisms are order isomorphisms, which in turn are lattice isomorphisms. However, we will show in Example 3.27 that the converse of Corollary 3.23 is false, i.e., there may be no order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ and $(\mathfrak{E}, \prec_{\mathscr{T}})$ even though $\operatorname{Lat}\mathscr{S}$ and $\operatorname{Lat}\mathscr{T}$ are isomorphic.

In the context of a TAF algebra $\mathscr{T}$, we have several notions of maximality:

(1) $\mathscr{T}$ is maximal with respect to $\prec_{\mathscr{T}} = \prec_{\mathscr{S}}$ for a given reflexive, transitive, and antisymmetric partial ordering $\prec_{\mathscr{S}}$, i.e., $\mathscr{T} = \mathscr{M}_{\max}(\mathscr{T})$.

(2) $\mathscr{W}(\mathscr{T})$ is a maximal subset of $\mathscr{W}$ with respect to satisfying properties (†) and $\prec_{\mathscr{T}}$ being reflexive, transitive, and antisymmetric.

(3) $\mathscr{T}$ is a maximal TAF algebra.

Corollary 3.15 implies that (2) and (3) are equivalent, and Corollary 3.14 shows that (3) implies (1). However, (1) is strictly weaker than (3) since the masa $\mathfrak{D}$ satisfies (1) but not (3). We will now introduce a fourth type of maximality.

DEFINITION. A TAF algebra $\mathscr{T} \subseteq \mathfrak{A}$ is *strongly maximal* if there is a sequence $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of finite dimensional C^*-algebras such that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathscr{T} \cap \mathfrak{A}_n$ is a maximal triangular algebra in $\mathfrak{A}_n$.

PROPOSITION 3.24. *A strongly maximal* TAF *algebra is a maximal* TAF *algebra.*

Proof. If $\mathscr{S}$ is a TAF algebra which strictly contains $\mathscr{T}$, then $\mathscr{S} \cap \mathfrak{A}_n$ is a triangular algebra in $\mathfrak{A}_n$, and $(\mathscr{S} \cap \mathfrak{A}_n) \supsetneq (\mathscr{T} \cap \mathfrak{A}_n)$ for some n by Theorem 2.2. But this contradicts the maximality of $\mathscr{T} \cap \mathfrak{A}_n$. ▨

EXAMPLE 3.25. We will show that maximal TAF algebras need not be strongly maximal. It suffices to find a maximal TAF algebra $\mathscr{T}$ such that $\overline{\mathscr{T} + \mathscr{T}^*} \neq \mathfrak{A}$, for equality clearly holds if $\mathscr{T}$ is strongly maximal. This example will in addition have the property $\overline{\mathscr{T} + \mathscr{T}^*} \neq C^*(\mathscr{T}) \neq \mathfrak{A}$.

Let $\mathfrak{A}_0 = \mathbf{M}_2$, $\mathfrak{A}_k = \mathbf{M}_{2\cdot 3^k}$, $k \geqslant 1$, and $\mathfrak{A} = \overline{\bigcup_{k=0}^{\infty} \mathfrak{A}_k}$ via the standard embeddings σ_k. Let $\{g_{ij}^{(n)}\}$ be the usual matrix units for $\mathbf{M}_n$, $e_{ij}^{(0)} = g_{ij}^{(2)}$, $1 \leqslant i, j \leqslant 2$, and $e_{ij}^{(k)} = g_{ij}^{(2\cdot 3^k)}$, $1 \leqslant i, j \leqslant 2\cdot 3^k$, $k \geqslant 1$. If $\mathscr{R} \subseteq \mathfrak{A}_k$, let $\mathscr{M}_k(\mathscr{R})$ denote the $\mathfrak{D}_k$-module generated by $\mathscr{R}$. Now define $\mathscr{T}_0 = \mathfrak{D}_0$, the diagonal of $\mathfrak{A}_0$, $\mathscr{T}_1 = \mathscr{M}_1(\sigma_0(\mathscr{T}_0), g_{12}^{(3)} \otimes \mathfrak{A}_0, g_{13}^{(3)} \otimes \mathfrak{A}_0, e_{34}^{(1)}, e_{65}^{(1)}) \subseteq \mathfrak{A}_1$, $\mathscr{T}_2 = \mathscr{M}_2(\sigma_1(\mathscr{T}_1), g_{12}^{(3)} \otimes \mathfrak{A}_1, g_{13}^{(3)} \otimes \mathfrak{A}_1, g_{56}^{(9)} \otimes \mathfrak{A}_0, g_{98}^{(9)} \otimes \mathfrak{A}_0, e_{78}^{(2)}, e_{14,13}^{(2)}) \subseteq \mathfrak{A}_2$, and then define $\mathscr{T}_k \subseteq \mathfrak{A}_k$ inductively by $\mathscr{T}_{k+1} = \mathscr{M}_{k+1}(\sigma_k(\mathscr{T}_k), g_{12}^{(3)} \otimes \mathfrak{A}_k, g_{13}^{(3)} \otimes \mathfrak{A}_k, g_{56}^{(9)} \otimes \mathfrak{A}_{k-1}, g_{98}^{(9)} \otimes \mathfrak{A}_{k-1}, g_{11,12}^{(27)} \otimes \mathfrak{A}_{k-2}, g_{21,20}^{(27)} \otimes \mathfrak{A}_{k-2}, e_{2\cdot 3^k+1, 2\cdot 3^k+2}^{(k+1)}, e_{2\cdot 3^{k+1}+2, 2\cdot 3^{k+1}+1}^{(k+1)})$. Observe that each $\mathscr{T}_k$ is a triangular algebra in $\mathfrak{A}_k$. For example, the embedding $\sigma_1 : \mathscr{T}_1 \hookrightarrow \mathscr{T}_2$ looks like

$$
\left[\begin{array}{c|c|c}
\begin{matrix} a & 0 \\ 0 & b \end{matrix} & A & B \\ \hline
0 & \begin{matrix} c & d \\ 0 & e \end{matrix} & 0 \\ \hline
0 & 0 & \begin{matrix} f & 0 \\ g & h \end{matrix}
\end{array}\right]
\longrightarrow
\left[\begin{array}{c|c|c}
\begin{array}{c|c|c} \begin{matrix} a & 0 \\ 0 & b \end{matrix} & A & B \\ \hline 0 & \begin{matrix} c & d \\ 0 & e \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} f & 0 \\ g & h \end{matrix} \end{array} & * & * \\ \hline
0 & \begin{array}{c|c|c} \begin{matrix} a & * \\ 0 & b \end{matrix} & A & B \\ \hline 0 & \begin{matrix} c & d \\ 0 & e \end{matrix} & * \\ \hline 0 & 0 & \begin{matrix} f & 0 \\ g & h \end{matrix} \end{array} & 0 \\ \hline
0 & 0 & \begin{array}{c|c|c} \begin{matrix} a & 0 \\ * & b \end{matrix} & A & B \\ \hline 0 & \begin{matrix} c & d \\ 0 & e \end{matrix} & 0 \\ \hline 0 & * & \begin{matrix} f & 0 \\ g & h \end{matrix} \end{array}
\end{array}\right].
$$

Subsequent embeddings map nine blocks into twenty-seven blocks. Finally, let $\mathscr{T} = \overline{\bigcup_k \mathscr{T}_k}$.

First of all, note that $\mathscr{T}$ is maximal TAF. For if there were some TAF algebra $\mathscr{S} \supsetneq \mathscr{T}$, then $\mathscr{S} \cap \mathfrak{A}_k \supsetneq \mathscr{T} \cap \mathfrak{A}_k$ for some k by Theorem 2.2. But then $\sigma_k(\mathscr{S} \cap \mathfrak{A}_k)$ would not be triangular in $\mathfrak{A}_{k+1}$ because of the way we have defined the sequence $\{\mathscr{T}_k\}$.

Now $C^*(\mathscr{T}_k) = C^*(\mathscr{T}_{k+1}) \cap \mathfrak{A}_k$ for each k, so the sequence $\{C^*(\mathscr{T}_k)\}$ of $\mathfrak{D}_k$-modules is in canonical form. Thus $C^*(\mathscr{T}_k) = C^*(\mathscr{T}) \cap \mathfrak{A}_k$ by Proposition 2.5 (note that $C^*(\mathscr{T}) = \overline{\bigcup_k C^*(\mathscr{T}_k)}$). But $C^*(\mathscr{T}_k) \neq \mathfrak{A}_k$ since $e_{12}^{(k)} \notin C^*(\mathscr{T}_k)$, so $C^*(\mathscr{T}) \neq \mathfrak{A}$. Therefore, $\mathscr{T}$ is not strongly maximal.

Finally, to see that $\overline{\mathscr{T} + \mathscr{T}^*} \neq C^*(\mathscr{T})$, observe that the matrix unit $e_{35}^{(1)}$ lies in $C^*(\mathscr{T}_1)$. However, $e_{35}^{(1)} \notin \overline{\mathscr{T} + \mathscr{T}^*}$, forotherwise there would be a sequence $\{t_j\} \subseteq \bigcup_k (\mathscr{T}_k + \mathscr{T}_k^*)$ such that $t_j \to e_{35}^{(1)}$. By passing to a subsequence of $\{\mathscr{T}_k + \mathscr{T}_k^*\}$, we can assume that $t_j \in \mathscr{T}_j + \mathscr{T}_j^*$. Thus, there is some t_j such that $\|e_{35}^{(1)} - t_j\| < 1$. But then $0 = \|e_{33}^{(j)}(e_{35}^{(1)} - t_j)e_{55}^{(j)}\| < 1$, a contradiction since) $\|e_{33}^{(j)}t_j e_{55}^{(j)}\| = 0$.

We close this section with a stronger result for isomorphisms of strongly maximal TAF algebras, and as a consequence obtain a generalization of [1, Theorem I].

THEOREM 3.26. *Let $\mathscr{S} \subseteq \mathfrak{A}$ and $\mathscr{T} \subseteq \mathfrak{B}$ be strongly maximal* TAF *algebras with diagonals $\mathfrak{D}$ and $\mathfrak{E}$, respectively. If $\Phi : \mathscr{S} \to \mathscr{T}$ is an isometric algebra isomorphism, then Φ extends to a C^*-isomorphism $\tilde{\Phi}: \mathfrak{A} \to \mathfrak{B}$.*

Proof. Let $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_n}$ and $\mathfrak{B} = \overline{\bigcup \mathfrak{B}_n}$ such that $\mathscr{S} \cap \mathfrak{A}_n$, $\mathscr{T} \cap \mathfrak{B}_n$ are maximal triangular subalgebras of $\mathfrak{A}_n$ and $\mathfrak{B}_n$, respectively. By Proposition 3.20, $\Phi|\mathfrak{D}$ is a C^*-isomorphism onto $\mathfrak{E}$. If $\mathscr{F} \subseteq \mathfrak{A}_n$ (n fixed) is a factor, then a set of matrix units $\{e_{ij}\}_{1 \leqslant i,j \leqslant N}$ can be chosen for $\mathscr{F}$ such that $\mathfrak{D} \cap \mathscr{F} = \operatorname{span}\{e_{ii} : 1 \leqslant i \leqslant N\}$ and $\mathscr{S} \cap \mathscr{F} = \operatorname{span}\{e_{ij} : 1 \leqslant i \leqslant j \leqslant N\}$. Define $\psi: \mathscr{F} \to \mathfrak{B}$ by $\psi(e_{ij}) = \Phi(e_{ij})$ if $i \leqslant j$, and $\psi(e_{ij}) = \Phi(e_{ji})^*$ if $i > j$, and extend by linearity to $\mathscr{F}$. We claim that $\{f_{ij} = \psi(e_{ij}) : 1 \leqslant i, j \leqslant N\}$ forms a system of matrix units for $\psi(\mathscr{F})$, and hence $\psi(\mathscr{F})$ is a finite dimensional factor in $\mathfrak{B}$. We need to show

$$(*) \qquad f_{ij}f_{kl} = \delta_{jk}f_{il}$$

for all $i, j, k, l \in \{1, \ldots, N\}$, and that $f_{ij}^* = f_{ji}$. The second fact follows immediately from the definition of ψ and the fact that the f_{ii}'s are projections. By Lemma 3.18, the f_{ij}'s are partial isometries with initial projections $f_{ij}^*f_{ij} = f_{jj}$ and final projections $f_{ij}f_{ij}^* = f_{ii}$. (*) will be proved by cases:

(1) $i \leqslant j$ and $k \leqslant l$. (*) holds since Φ is multiplicative.

(2) $i > j$ and $k > l$. Take adjoints of both sides and use (1).

(3) $i > j$ and $k \leqslant l$ If $j \neq k, f_{ij}f_{kl} = 0$ since the initial projection f_{jj} of f_{ij} is orthogonal to the final projection f_{kk} of f_{kl}. So we suppose $j = k$. If $i \leqslant l$, then $f_{jl} = f_{ji}f_{il}$, and thus $f_{ij}f_{jl} = f_{ij}f_{ji}f_{il} = f_{ii}f_{il} = f_{il}$. If $i > l$, then $f_{ji} = f_{jl}f_{li}$. Multiplying on the left by f_{lj}, we have $f_{lj}f_{ji} = f_{lj}f_{jl}f_{li} = f_{ll}f_{li} = f_{li}$. By taking adjoints, $f_{ij}f_{jl} = f_{il}$.

(4) $i \leqslant j$ and $k > l$. This proof is similar to (3).

Thus, $\psi(\mathscr{F}) \subseteq \mathfrak{B}$ is a finite dimensional factor and $\psi: \mathscr{F} \to \psi(\mathscr{F})$ is a C^*-isomorphism. By construction, ψ is the unique C^*-morphism of $\mathscr{F}$ which extends $\Phi|(\mathscr{F} \cap \mathscr{S})$. Since $\mathscr{F} \subseteq \mathfrak{A}_n$ was an arbitrary factor, there is a C^*-isomorphism $\psi_n : \mathfrak{A}_n \to \psi_n(\mathfrak{A}_n) \subseteq \mathfrak{B}$ which extends $\Phi|(\mathfrak{A}_n \cap \mathscr{S})$, and ψ_n is unique with these properties. Furthermore, by uniqueness, ψ_{n+1} must extend ψ_n. Thus the map $\tilde{\Phi} : \mathfrak{A} \to \mathfrak{B}$ defined by $\tilde{\Phi}|\mathfrak{A}_n = \psi_n$ is a C^*-isomorphism which extends Φ. Since $\tilde{\Phi}(\mathfrak{A}) = C^*(\mathscr{T}) = \mathfrak{B}$, $\tilde{\Phi}$ is onto, and this finishes the proof. ▨

REMARK. Thus, strongly maximal TAF subalgebras of nonisomorphic AF algebras can never be isomorphic. Theorem 3.26 generalizes [1, Theorem I] in two directions: it is valid for all AF algebras, and for any embeddings which generate strongly maximal TAF algebras. Note also that this theorem uses very little of the theory presented so far. The only results used are Lemma 3.18 (which in turn depended only on Lemma 3.1) and Proposition 3.20.

In view of Theorem 3.26, it makes sense now to investigate the case of two strongly maximal triangular subalgebras of the same AF algebra, or even the simpler case of a UHF algebra. This will be the subject of the next section. However, we can immediately show that even in this setting the converse of Corollary 3.23 is false.

EXAMPLE 3.27. We will exhibit two transitive triangular subalgebras of the 2^∞ UHF algebra $\mathfrak{A}$ whose diagonal orderings are not order isomorphic (and, a fortiori, the algebras themselves are not isomorphic).

Let $\mathfrak{A}_n = M_{2^n}$ with matrix units $\{e_{ij}^{(n)}\}_{1 \leqslant i,j \leqslant 2^n}$, $\sigma_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ the standard embedding, and $j_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ defined by

$$j_n(x) = \begin{cases} I_2 \otimes x, & n \text{ even} \\ x \otimes I_2, & n \text{ odd}. \end{cases}$$

Thus, j_n coincides with the standard embedding if n is even, and with the nest embedding if n is odd. Let $\mathscr{T}_n$ be the upper triangular algebra in $\mathfrak{A}_n$, and set $\mathscr{S} = \varinjlim(\mathscr{T}_n, \sigma_n)$ and $\mathscr{T} = \varinjlim(\mathscr{T}_n, j_n)$. Further, let $\mathfrak{D}_n = \operatorname{span}\{e_{ii}^{(n)} : 1 \leqslant i \leqslant n\}$, $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \sigma_n)$, and $\mathfrak{E} = \varinjlim(\mathscr{D}_n, j_n)$. It was shown in Example 1.1 that $\mathscr{S}$ is transitive, and the same argument in fact shows that $\mathscr{T}$ is transitive as well.

Suppose that $\varphi : (\mathfrak{D}, \prec_{\mathscr{S}}) \to (\mathfrak{E}, \prec_{\mathscr{T}})$ is an order isomorphism. Then there exist integers $1 < l < l + 1 < m < n$ such that the diagram

$$\begin{array}{ccc} \mathfrak{D}_1 & \xrightarrow{\sigma_{n-1}\circ\ldots\circ\sigma_1} & \mathfrak{D}_n \\ {\scriptstyle \varphi|\mathfrak{D}_1}\downarrow & & \uparrow{\scriptstyle \psi|\mathfrak{D}_m} \\ \mathfrak{D}_l & \xrightarrow[j_{m-1}\circ\ldots\circ j_l]{} & \mathfrak{D}_m \end{array}$$

commutes, where $\psi = \varphi^{-1}$. Note that the orderings $\prec_{\mathscr{S}}$ and $\prec_{\mathscr{T}}$ on the minimal projections of $\mathfrak{D}_k$ coincide; indeed, $e_{11}^{(k)} \prec e_{22}^{(k)} \prec \ldots \prec e_{2^k,2^k}^{(k)}, k = 1, 2, \ldots$. We arrive at a contradiction through the following observations:

(i) $e_{22}^{(n)} \cdot \sigma_{n-1} \circ \ldots \circ \sigma_1(e_{22}^{(1)}) \neq 0$.

(ii) $\varphi(e_{22}^{(1)})e_{11}^{(l)} = 0$. This is an immediate consequence of the fact φ is an order isomorphism.

(iii) $j_{m-1} \circ \ldots \circ j_l(e_{jj}^{(l)})e_{ii}^{(m)} = 0,\ i = 1,\ 2,\ 2 \leqslant j \leqslant 2^l$. This follows from the fact that $m - l \geqslant 2$, so that at least one of the embeddings $j_l, \ldots, j_{m-1}$ is the nest embedding.

(iv) $\psi \circ j_{m-1} \circ \ldots \circ j_l(e_{jj}^{(l)})e_{ii}^{(n)} = 0,\ i = 1, 2,\ 2 \leqslant j \leqslant 2^l$. This follows from (iii) and the facts that $e_{11}^{(n)} + e_{22}^{(n)} \leqslant \psi(e_{11}^{(m)} + e_{22}^{(m)})$ and ψ is an order isomorphism.

(v) $\psi \circ j_{m-1} \circ \ldots \circ j_l \circ \varphi(e_{22}^{(1)})e_{ii}^{(n)} = 0,\ i = 1, 2$. From (ii), $\varphi(e_{22}^{(1)})$ is a sum of $\{e_{jj}^{(l)}\}$ for j in some subset of $\{j : 2 \leqslant j \leqslant 2^l\}$. Thus, (v) follows from (iv) and (ii). But (i) contradicts (v), so we conclude that $(\mathfrak{D}, \prec_{\mathscr{S}})$ and $(\mathfrak{E}, \prec_{\mathscr{T}})$ are not order isomorphic.

Note that the "identity" map $\imath : \mathscr{P}(\mathfrak{D}) \to \mathscr{P}(\mathfrak{E})$ defined by $\imath(e_{ii}^{(k)}) = e_{ii}^{(k)}$ is a bijection which is order-preserving: $e \prec_{\mathscr{S}} f$ iff $\imath(e) \prec_{\mathscr{T}} \imath(f)$. Of course, $\imath$ is not a C*-isomorphism, so this example also shows that our definition of an order isomorphism is not equivalent to this weaker notion. In fact, it can be shown in general that if τ is an order-preserving bijection from $\mathscr{P}(\mathfrak{D})$ onto $\mathscr{P}(\mathfrak{E})$ which also preserves order in the usual sense ($e \leqslant f$ iff $\tau(e) \leqslant \tau(f)$), then τ can be extended to an order isomorphism of $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$. Thus, an order isomorphism is determined by the diagonal ordering and the usual ordering.

4. ISOMORPHISMS OF TRIANGULAR UHF ALGEBRAS

It is well known that a unital embedding of one finite dimensional factor into another is the composition of an ampliation with a spatial isomorphism. Recall that $\mathbf{M}_n$ denotes a fixed representation of the n^2-dimensional factor as $n \times n$ complex matrices, with the usual matrix units $\{g_{ij}^{(n)}\}_{1 \leqslant i,j \leqslant n}$. Thus if $\mathfrak{A} = \mathbf{M}_n$ and $\mathfrak{B} = \mathbf{M}_n \otimes \mathbf{M}_m$, any unital embedding $\varphi : \mathfrak{A} \to \mathfrak{B}$ has the form $\varphi(x) = (\mathrm{Ad}\, W)(x \otimes$

$\otimes I_m$) for some unitary W in $\mathfrak{B}$. φ is said to be *orthographic* if W is also a permutation matrix.

Let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers, let $n_1 \in \mathbf{Z}^+$, and define $\{n_k\}_{k=1}^{\infty}$ inductively by $n_{k+1} = m_k n_k$. Define $\mathfrak{A}_1 = \mathbf{M}_{n_1}$ and $\mathfrak{A}_{k+1} = \mathfrak{A}_k \otimes \mathbf{M}_{m_k}$, $k \geqslant 1$. Also, define $e_{ij}^{(k)} = g_{ij}^{(n_k)}$. Let $\nu_k : \mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ be the nest embedding $\nu_k(x) = x \otimes I_{m_k}$, and set $\nu_{k,l} = \nu_{l-1} \circ \nu_{l-2} \circ \ldots \circ \nu_k$ for $l > k$.

A UHF algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_k, \varphi_k)$ is said to be given in *orthographic form* if each embedding $\varphi_k : \mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ is orthographic, i.e., $\varphi_k = \operatorname{Ad} P_{k+1} \circ \nu_k$ for some permutation matrix P_{k+1} in $\mathfrak{A}_{k+1}$. Since each $\mathfrak{A}_k$ is canonically assciated with a subalgebra of $\mathfrak{A}$ [13], we will also say that the sequence of subalgebras $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ is in orthographic form if the inclusion $\mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ is orthographic.

With no loss of generality we may assume that every UHF algebra $\mathfrak{A}$ is given in orthographic form; indeed as Glimm showed [5], the isomorphism class of $\mathfrak{A}$ depends only on the generalized integer $\prod_{k=1}^{\infty} n_k$ and not on the particular form of the embeddings. However, as we have already seen, the isomorphism class of a triangular algebra given by $\mathscr{T} = \varinjlim(\mathscr{T}_k, \varphi_k)$ does depend on the form of the embeddings φ_k.

With notation as above, suppose $\mathfrak{A} = \overline{\bigcup_k \mathfrak{A}_k}$ is in orthographic form, and let $\mathfrak{D}_k$ be the masa in $\mathfrak{A}_k$ generated by $\{e_{ii}^{(k)} : 1 \leqslant i \leqslant n_k\}$. If $\mathscr{S}_k$ is a maximal triangular subalgebra of $\mathfrak{A}_k$ with diagonal $\mathfrak{D}_k$, $k \geqslant 1$, then there is a permutation matrix Q_k such that $Q_k \mathscr{S}_k Q_k^* = \mathscr{T}_k$, the upper triangular matrices in $\mathfrak{A}_k$. For each k, suppose that the inclusion $\mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$, given by $\operatorname{Ad} P_{k+1} \circ \nu_k$, also maps $\mathscr{S}_k$ into $\mathscr{S}_{k+1}$. Then the following diagram commutes:

$$\begin{array}{ccccccc} \longrightarrow & \mathfrak{A}_k & \xrightarrow{\operatorname{Ad} P_{k+1} \circ \nu_k} & \mathfrak{A}_{k+1} & \longrightarrow \\ & \downarrow{\scriptstyle \operatorname{Ad} Q_k} & & \downarrow{\scriptstyle \operatorname{Ad} Q_{k+1}} & \\ \longrightarrow & \mathfrak{A} & \xrightarrow[\operatorname{Ad} R_{k+1} \circ \nu_k]{} & \mathfrak{A}_{k+1} & \longrightarrow \end{array}$$

where $R_{k+1} = Q_{k+1} P_{k+1} \nu_k(Q_k^*)$. Thus the triangular algebra $\overline{\bigcup \mathscr{S}_k}$ is isomorphic to $\overline{\bigcup \mathscr{T}_k}$, where the inclusion $\mathscr{T}_k \hookrightarrow \mathscr{T}_{k+1}$ given by $\operatorname{Ad} R_{k+1} \circ \nu_k$ is in orthographic form. Hence we will assume that all strongly maximal TAF subalgebras of UHF algebras in this section are given in this form, i.e., as the closure of the union of upper triangular matrix algebras with orthographic embeddings.

LEMMA 4.1. *Let $\varphi_k : \mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$, $k = 1,2$, be orthographic embeddings, with $\varphi_k = \operatorname{Ad} P_{k+1} \circ \nu_k$, and suppose $\varphi_k(\mathscr{T}_k) \subseteq \mathscr{T}_{k+1}$, where $\mathscr{T}_k$ is the algebra of upper triangular matrices in $\mathfrak{A}_k$.*

(a) *If $\varphi_2 \circ \varphi_1 \mid \mathfrak{D}_1 = \nu_2 \circ \nu_1 \mid \mathfrak{D}_1$, then $\varphi_1 \mid \mathfrak{D}_1 = \nu_1 \mid \mathfrak{D}_1$.*

(b) *If* $\varphi_2 \circ \varphi_1 \mid \mathscr{T}_1 = \nu_2 \circ \nu_1 \mid \mathscr{T}_1$, *then* $\varphi_1 = \nu_1$.

Proof. Let $\{e_{ij}^{(k)} : 1 \leqslant i, j \leqslant n_k\}$ be the matrix units of $\mathfrak{A}_k$, as above, and let $1 \leqslant j < j' \leqslant n_1$. φ_k is order-preserving because $\varphi_k(\mathscr{T}_k) \subseteq \mathscr{T}_{k+1}$. Therefore, since $\varphi_2 \circ \varphi_1 = \nu_2 \circ \nu_1$ is the nest embedding, there exist no nonzero subprojections $p \leqslant \varphi_1(e_{jj}^{(1)})$, $q \leqslant \varphi_1(e_{j'j'}^{(1)})$, for which $q \prec_{\mathscr{T}_2} p$. Hence $\varphi_1(e_{jj}^{(1)}) = \sum_{l=1}^{m_1} e_{(j-1)m_1+l,(j-1)m_1+l}^{(2)} = \nu_1(e_{jj}^{(1)})$ where $m_1 = n_2/n_1$. This proves (a), and also implies that for every $1 \leqslant j < j' \leqslant n_1$, there exists a permutation π of $\{1, \ldots, m_1\}$ such that $\varphi_1(e_{jj'}^{(1)}) = \sum_{l=1}^{m_1} e_{(j-1)m_1+l,(j'-1)m_1+\pi(l)}^{(2)}$. Now given the hypothesis of (b), it suffices to prove that π is the identity permutation.

For simplicity of notation, we may assume that $n_1 = 2$ and therefore $j = 1$ and $j' = 2$. The general case follows from part (a) by restricting to $(e_{jj}^{(1)} + e_{j'j'}^{(1)})\mathfrak{A}_1(e_{jj}^{(1)} + e_{j'j'}^{(1)})$. For each $1 \leqslant i \leqslant n_2$, let $\varphi_2(e_{ii}^{(2)}) = \sum_{l \in A(i)} e_{ll}^{(3)}$, where $A(i)$ is a subset of $\{1, \ldots, n_3\}$. Define $\lambda_i = \max\{l : l \in A(i)\}$. Then the following are equivalent:

(i) $1 \leqslant i \leqslant j \leqslant n_2$,

(ii) $\sum_{l \in A(i)} e_{ll}^{(3)} \prec_{\mathscr{T}_3} \sum_{l \in A(j)} e_{ll}^{(3)}$,

(iii) $\lambda_i \leqslant \lambda_j$.

Let $1 \leqslant i \leqslant m_1$. $\lambda_{m_1} = n_3/2$ by part (a), and (i)–(iii) then implies that $\lambda_i \leqslant n_3/2$. Observe that $e_{m_1+\pi(i),\, m_1+\pi(i)}^{(2)} = [e_{ii}^{(2)}\varphi_1(e_{12}^{(1)})]^*[e_{ii}^{(2)}\varphi_1(e_{12}^{(1)})]$, from which

$$\varphi_2(e_{m_1+\pi(i),\, m_1+\pi(i)}^{(2)}) = \varphi_2(e_{ii}^{(2)}\varphi_1(e_{12}^{(1)}))^*\varphi_2(e_{ii}^{(2)}\varphi_1(e_{12}^{(1)})) =$$

$$= \varphi_2 \circ \varphi_1(e_{12}^{(1)})^*\varphi_2(e_{ii}^{(2)})\varphi_2 \circ \varphi_1(e_{12}^{(1)}) =$$

$$= \left(\sum_{l=1}^{n_3/2} e_{l,l+n_3/2}^{(3)}\right)^* \left(\sum_{l \in A(i)} e_{ll}^{(3)}\right) \left(\sum_{l=1}^{n_3/2} e_{l,l+n_3/2}^{(3)}\right) = \sum_{l \in A(i)} e_{l+n_3/2,l+n_3/2}^{(3)}.$$

Thus $\lambda_{m_1+\pi(i)} = (n_3/2) + \lambda_i$, and therefore $\pi(i) = i$, $1 \leqslant i \leqslant m_1$. ∎

DEFINITION. Let $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_k}$ and $\mathfrak{B} = \overline{\bigcup \mathfrak{B}_k}$ be UHF algebras given in orthographic form, with masas $\mathfrak{D}$ and $\mathfrak{E}$, respectively. A C^*-homomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ is said to be *orthographic* if for every k there exists some $l(k)$ such that $\Phi(\mathfrak{A}_k) \subseteq \mathfrak{B}_{l(k)}$ and $\Phi \mid \mathfrak{A}_k$ is orthographic. If $\Phi, \Psi : \mathfrak{A} \to \mathfrak{B}$ are C^*-isomorphisms such that $\Phi(\mathfrak{D}) = \Psi(\mathfrak{D}) = \mathfrak{E}$, we will say that Φ and Ψ are *orthographically equivalent* if $\Psi(e_{ij}^{(k)}) \in \mathscr{U}(\mathfrak{E})\Phi(e_{ij}^{(k)})$ for each of the matrix units $e_{ij}^{(k)}$ of $\mathfrak{A}_k$.

Now suppose $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a C^*-isomorphism which satisfies $\Phi(\mathfrak{D}) = \mathfrak{E}$. Hence $\Phi(\mathscr{W}_{\mathfrak{D}}) = \mathscr{W}_{\mathfrak{E}}$. For each k, let $\{e_{ij}^{(k)} : 1 \leqslant i, j \leqslant n_k\}$ and $\{f_{ij}^{(k)} : 1 \leqslant i, j \leqslant m_k\}$

be the fixed matrix units of $\mathfrak{A}_k$ and $\mathfrak{B}_k$, respectively. Since $\Phi(e_{ij}^{(k)}) \in \mathscr{W}_{\mathfrak{E}}$ by Theorem 3.3, then Theorem 3.6 implies that there is some l such that $\Phi(e_{ij}^{(k)}) = u_{ij}^{(k)} \sum_{(i',j')\in\Lambda} f_{i'j'}^{(l)}$, where $u_{ij}^{(k)} \in \mathscr{U}(\mathfrak{E})$ and $\Lambda = \Lambda(i,j,k) \subseteq \{(i',j') : 1 \leqslant i', j' \leqslant m_l\}$. Thus, we can define $\widetilde{\Phi}(e_{ij}^{(k)}) = \sum_{(i',j')\in\Lambda(l)} f_{i'j'}^{(l)}$. The proof of Theorem 3.6 shows that Λ is uniquely determined, and since the sequence $\{\mathfrak{B}_k\}$ is given in orthographic form, the definition of $\widetilde{\Phi}(e_{ij}^{(k)})$does not depend on l. One then checks directly that $\widetilde{\Phi}$ extends to an orthographic C^*-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. We thus have the following result.

COROLLARY 4.2. *Any C^*-isomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ with $\Phi(\mathfrak{D}) = \mathfrak{E}$ is orthographically equivalent to a C^*-isomorphism $\widetilde{\Phi} : \mathfrak{A} \to \mathfrak{B}$ such that $\widetilde{\Phi}(\mathfrak{D}) = \mathfrak{E}$ and $\widetilde{\Phi}$ is orthographic. Moreover, if $\mathscr{S}_k$ and $\mathscr{T}_k$ are the upper triangular matrices of $\mathfrak{A}_k$ and $\mathfrak{B}_k$ (with diagonals $\mathfrak{D}_k$ and $\mathfrak{E}_k$), respectively, and if $\Phi(\overline{\bigcup \mathscr{S}_k}) = \overline{\bigcup \mathscr{T}_k}$, then $\widetilde{\Phi}(\overline{\bigcup \mathscr{S}_k}) = \overline{\bigcup \mathscr{T}_k}$.*

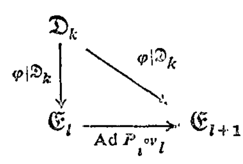
Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_k, \nu_k)$ and $\mathfrak{B} = \varinjlim(\mathfrak{A}_k, j_k)$ be UHF algebras in orthographic form, where ν_k is the nest embedding and $j_k = \operatorname{Ad} P_k \circ \nu_k$ for some permutation matrix P_k. Let $\mathscr{S}_k = \mathscr{T}_k$ be the upper triangular matrix algebra in $\mathfrak{A}_k$, $\mathscr{S} = \varinjlim(\mathscr{S}_k, \nu_k)$, and $\mathscr{T} = \varinjlim(\mathscr{T}_k, j_k)$. Let $\mathfrak{D}$ and $\mathfrak{E}$ be the diagonals of $\mathscr{S}$ and $\mathscr{T}$, respectively, and note that $\mathfrak{D}_k = \mathfrak{D} \cap \mathfrak{A}_k$ and $\mathfrak{E}_k = \mathfrak{E} \cap \mathfrak{A}_k$ coincide with the diagonal of $\mathfrak{A}_k$ (with respect to the fixed set of matrix units of $\mathfrak{A}_k$). Of course, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic UHF algebras. Howevever, the following theorem shows that $\mathscr{S}$ and $\mathscr{T}$ are not in general isomorphic, and in fact their diagonal orderings may not even be order isomorphic.

THEOREM 4.3. *With notation as above,*

(i) *If $(\mathfrak{D}, \prec_{\mathscr{S}})$ and $(\mathfrak{E}, \prec_{\mathscr{T}})$ are order isomorphic, then for every k there exists an $l(k)$ such that for $l > l(k)$, $P_l = \bigoplus_{i=1}^{n_k} P(l, i)$, where $P(l, i)$ is a permutation matrix of size n_{l+1}/n_k.*

(ii) *If $\mathscr{S}$ is isomorphic to $\mathscr{T}$, then, in addition to the conclusion in* (i), *$P(l, i) = P(l, i')$ for each l, $1 \leqslant i, i' \leqslant n_k$.*

Proof. Let φ be an order isomorphism from $(\mathfrak{D}, \prec_{\mathscr{S}})$ onto $(\mathfrak{E}, \prec_{\mathscr{T}})$. For each k, there exists an $l(k) > k$ such that $\varphi(\mathfrak{D}_k) \subseteq \mathfrak{E}_{l(k)}$. Then for each $l \geqslant l(k)$, there is some $m > l$ such that $\varphi^{-1}(\mathfrak{E}_l) \subseteq \mathfrak{D}_m$. Since φ and φ^{-1} are order isomorphisms, the proof of Lemma 4.1(a) applies in this case as well to show that $\varphi \mid \mathfrak{D}_k : \mathfrak{D}_k \to \mathfrak{E}_l$ equals $\nu_{k,l} \mid \mathfrak{D}_k$. Since the diagram



commutes, it follows that $\nu_{k,l+1} = \operatorname{Ad} P_l \circ \nu_{k,l+1}$. This forces $P_l = \bigoplus_{i=1}^{n_k} P(l, i)$, where $P(l, i)$ is a permutation matrix of size n_{l+1}/n_k (recall that $\dim \mathfrak{A}_j = n_j^2$).

To prove (ii), let Φ be an isomorphism from $\mathcal{S}$ onto $\mathcal{T}$. By Theorem 3.26, Φ is the restriction of a C^*-automorphism of $\mathfrak{A}$. By Corollary 4.2, we may assume Φ is orthographic. Therefore, for each k there exists some $l(k)$ such that $\Phi(\mathcal{S}_k) \subseteq \mathcal{T}_{l(k)}$. Viewing $\Phi : \mathcal{S}_k \to \mathcal{T}_l$ $(l \geq l(k))$, we then have from Lemma 4.1 (b) that Φ is given by $\nu_{k,l}$ and hence $\nu_{k,l+1} = \operatorname{Ad} P_l \circ \nu_{k,l+1}$. Therefore $P_l \nu_{k,l+1}(e_{ii'}^{(k)}) = \nu_{k,l+1}(e_{ii'}^{(k)}) P_l$, and this implies that $P(l, i) = P(l, i')$. ▨

EXAMPLE 4.4. Let $\mathcal{S}_k = \mathcal{T}_k$ be the full upper triangular matrix algebra in $\mathfrak{A}_k$, where $n_k = 2^k$. Set $P_l = \bigoplus_{i=1}^{2^l} Q(l, i)$, where $Q(l, i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $1 \leq i < 2^l$, and $Q(l, 2^l) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. As in the last theorem, $\mathcal{S} = \overline{\bigcup \mathcal{S}_k}$, $\mathcal{T} = \overline{\bigcup \mathcal{T}_k}$, $\mathfrak{D} = \mathcal{S} \cap \mathcal{S}^*$, and $\mathfrak{E} = \mathcal{T} \cap \mathcal{T}^*$. One verifies directly that the "identity" map ι on the diagonal $\mathfrak{D}_k$ of $\mathcal{T}_k$ yields an order isomorphism between $(\mathfrak{D}, \prec_{\mathcal{S}})$ and $(\mathfrak{E}, \prec_{\mathcal{T}})$. On the other hand, $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic by part (ii) of the above theorem. Thus, the diagonal ordering is not a complete isomorphism invariant for strongly maximal TAF algebras, even in the UHF case. Also, $\mathcal{S} = \mathcal{M}_{\max}(\mathcal{S})$ and $\mathcal{T} = \mathcal{M}_{\max}(\mathcal{T})$ by Prdposition 3.24 and Corollary 3.14, so this example also shows that the diagonal ordering is not a complete isomorphism invariant for maximal modules, at least in regard to algebra isomorphisms.

Now $\iota(\operatorname{Lat} \mathcal{S}) = \operatorname{Lat} \mathcal{T}$ by Corollary 3.23, and $C^*(\operatorname{Lat} \mathcal{S}) = \mathfrak{D}$, so $C^*(\operatorname{Lat} \mathcal{T}) = \mathfrak{E}$ and it follows by Propositions 2.8 and 2.9 that $\mathcal{S}$ and $\mathcal{T}$ are both nest TAF algebras. Thus, nest TAF subalgebras of a UHF algebra need not be isomorphic.

Let $\mathfrak{A}$ be a UHF algebra and $\mathcal{T} \subseteq \mathfrak{A}$ a strongly maximal TAF algebra (of the type considered in this section) with diagonal $\mathfrak{D}$. Set $\tau(\mathcal{T}) = \{\operatorname{tr} p : p \in \operatorname{Lat} \mathcal{T}\}$ and $\rho(\mathcal{T}) = \sup\{t \in \tau(\mathcal{T}) : t \neq 1\}$, where tr is the unique normalized trace on $\mathfrak{A}$. Then by Proposition 3.20 and Corollary 3.23, $\tau(\mathcal{T})$ and $\rho(\mathcal{T})$ are isomorphism invariants of $(\mathfrak{D}, \prec_{\mathcal{T}})$ and $\mathcal{T}$. Example 3.27 shows that neither of these invariants is complete for either $(\mathfrak{D}, \prec_{\mathcal{T}})$ or $\mathcal{T}$, but they are still quite useful. We will use $\rho(\mathcal{T})$ to show that there is an uncountable family of nonisomorphic lattices; in fact, for each $\alpha \in [0, 1]$ we will associate a strongly maximal TAF algebra $\mathcal{T}_{(\alpha)}$ with $\rho(\mathcal{T}_{(\alpha)}) = \alpha$. This will imply, of course, that the corresponding diagonal orderings and the triangular algebras themselves are pairwise nonisomorphic.

THEOREM 4.5. *Let $\mathfrak{A}$ be a* UHF *algebra and $\alpha \in [0, 1]$. Then there exists a strongly maximal triangular subalgebra $\mathcal{T}_{(\alpha)} \subseteq \mathfrak{A}$ such that $\rho(\mathcal{T}_{(\alpha)}) = \alpha$.*

Proof. We will carry out the construction for the 2^∞ UHF algebra; the construction is similar in the general case. Let matrix units $\{e_{ij}^{(n)}\}_{1 \leq i,j \leq 2^n}$ for $\mathfrak{A}_n = \mathbf{M}_{2^n}$ be given. For every positive integer N, let $Q(N)$ be a permutation matrix Q in $\mathbf{M}_{2N}$ such that

$$Q\operatorname{diag}(a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_1^{(N)}, a_2^{(N)})\, Q^{\mathrm{T}} =$$
$$= \operatorname{diag}(a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(N)}, a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(N)}),$$

where $\operatorname{diag}(b_1, \ldots, b_l)$ denotes the diagonal matrix in $\mathbf{M}_l$ with diagonal entries $b_1, \ldots, b_l$. For each n and $1 \leqslant m < 2^n$, let $R(n, m) = I_{2m} \oplus Q(2^n - m)$. Observe that $\operatorname{Ad} R(n, m) \circ \nu_n(\mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$, where $\mathcal{T}_n \subseteq \mathfrak{A}_n$ is the upper triangular algebra, and $\nu_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ is the nest embedding, $\nu_n(x) = x \otimes I_2$.

Define $\mathcal{T}_{(1)}$ to be the nest algebra $\mathcal{T}_{(1)} = \varinjlim(\mathcal{T}_n, \nu_n)$, and $\mathcal{T}_{(0)} = \varinjlim(\mathcal{T}_n, \sigma_n)$, where σ_n is the standard embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$, $\sigma_n(x) = I_2 \otimes x$. It follows immediately from Example 1.1 that $\rho(\mathcal{T}_{(0)}) = 0$ and $\rho(\mathcal{T}_{(1)}) = 1$,

For $0 < \alpha < 1$, choose a sequence $\{k_n\}_{n=1}^{\infty}$, $0 \leqslant k_n < 2^n$, such that $\alpha = \sum_{n=1}^{\infty} k_n/2^n$. If $\alpha = m/2^n$ for some positive integers m and n, choose $k_i = 0$ for al $i > n$. Let $M_n = \sum_{i=1}^{n} 2^{n-i}k_i$, and define $\mathcal{T}_{(\alpha)} = \overline{\bigcup_n \mathcal{T}_n}$, where the embedding $\mathcal{T}_n \hookrightarrow \mathcal{T}_{n+1}$ is given by $R(n, M_n)\circ\nu_n$. For each n, let $p_n = \sum_{i=1}^{M_n} e_{ii}^{(n)}$. Then the following hold:

(i) $p_n \leqslant p_{n+1} \leqslant \ldots$ because $p_n = \sum_{i=1}^{2M_n} e_{ii}^{(n+1)} \leqslant \sum_{i=1}^{M_{n+1}} e_{ii}^{(n+1)} = p_{n+1}$.

(ii) Every p_n is an invariant projection of $\mathcal{T}_{(\alpha)}$. Indeed, since the embedding p_n into $\mathcal{T}_{n+k}$ is via the nest embedding, then $p_n = \sum_{i=1}^{2^k M_n} e_{ii}^{(n+k)}$, which is an invariant projection of $\mathcal{T}_{n+k}$. Hence, p_n is an invariant projection of $\overline{\bigcup_{k>0} \mathcal{T}_{n+k}} = \mathcal{T}_{(\alpha)}$.

(iii) Let q be a projection in $\mathcal{T}_n$ with $p_n < q < 1$. We will show that $\mathcal{T}_{n+1}$ is not invarant under q. Write $q = \sum_{i=1}^{M} e_{ii}^{(n)}$ for some $M_n < M < 2^n$. From the way the embedding $\mathcal{T}_n \hookrightarrow \mathcal{T}_{n+1}$ was defined, q embeds into $\mathcal{T}_{n+1}$ as

$$q = \sum_{i=1}^{M_n+M} e_{ii}^{(n+1)} + \sum_{i=2^n+M_n+1}^{2^n+M} e_{ii}^{(n+1)}.$$

Thus,

$$(1 - q)(e_{M_n+M+1,2^n+M}^{(n+1)})q = e_{M_n+M+1,2^n+M}^{(n+1)} \neq 0.$$

Therefore, q is not an invariant projection for $\mathcal{T}_{n+1}$.

Finally, let $\rho_n = \frac{M_n}{2^n} = \sum_{i=1}^{n} \frac{k_i}{2^n}$ and $s = \rho(T_{(\alpha)})$. Since p_n is an invariant projection for $\mathscr{T}_{(\alpha)}$, and $\mathrm{tr}\, p_n = \rho_n$, clearly $s \geqslant \sup \rho_n = \alpha$. However if $s > \alpha$, there must be some n and an invariant projection $q \in \mathscr{T}_n$ for $\mathscr{T}_{(\alpha)}$ such that $p_n < q < 1$. But we just showed that no such q can exist. It follows that $s = \alpha$, and $\mathscr{T}_{(\alpha)}$ has the required property. ▨

COROLLARY 4.6. *Every* UHF *algebra* $\mathfrak{A}$ *contains an uncountable number of pairwise nonisomorphic strongly maximal triangular subalgebras whose diagonal orderings are also not order isomorphic.*

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