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SHAPE THEORY AND (CONNECTIVE) K-THEORY

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1. INTRODUCTION AND PRELIMINARIES

1.1. INTRODUCTION

E. G. Effros [16] posed the problem of finding suitable invariants for studying inductive limits of the form

$$C(X_1) \otimes A_1 \to C(X_2) \otimes A_2 \to \dots$$

where the X_i are CW-complexes and A_i are finite dimensional C^*-algebras. In the present paper we study this problem from the viewpoint of homotopy theory and shape theory. Our algebraic models for shape invariants are based on ordered K-theory.

The material is organized as follows:

1. Introduction and preliminaries.
2. Ordered K-theory and large denominators.
3. Connective KK-theory for spaces.
4. Homotopy computations for large homomorphisms.
5. Shape theory.
6. A connectivity result.

We consider the category $\mathscr{C}(n)$ whose objects are C^*-algebras of the form

$$\bigoplus_k C(X_k) \otimes M_{m_k} \text{ (finite sums)}$$

where the X_k are finite connected CW-complexes of dimension $\leqslant n$. The set $\mathrm{Hom}(A, B)$ of morphisms in $\mathscr{C}(n)$ from A to B consists of all $*$-homomorphisms of C^*-algebras (including the nonunital ones).

Our main object of study will be the inductive limits of C^*-algebras from $\mathscr{C}(n)$. This requires a satisfactory classification of the morphisms of $\mathscr{C}(n)$. Now it is clear that the space $\mathrm{Hom}(A, B)$ with $A, B \in \mathscr{C}(n)$ is too big. Therefore, as in commutative topology, it is natural to consider first the question of homotopy classification of such maps. This is a difficult problem whose approach requires certain technical considerations of homotopy theory occuping many pages of our paper. The starting point of our calculations of homotopy classes of $*$-homomorphisms is a deep result of G. Segal [39] concerning the realization of the connective BO-spectrum as a sequence of spaces of $*$-homomorphisms. In fact we use the complex version of this result, which, as pointed out by J. Rosenberg [36], is also relevant for the computation of the stable homotopy groups of commutative C^*-algebras. Building on this circle of ideas we develop in Section 3 some algebraic topology techniques in order to compute the homotopy classes of $*$-homomorphisms $C_0(X) \to C_0(Y) \otimes \mathscr{K}$. It turns out that

$$\mathrm{kk}(Y, X) := [C_0(X), C_0(Y) \otimes \mathscr{K}] = \lim_k [C_0(X), C_0(Y) \otimes M_k]$$

has a natural structure of abelian group with addition induced by the orthogonal sum of the homomorphisms. Moreover $\mathrm{kk}(Y, X)$ has good excision properties in both variables and this allows us to define the groups $\mathrm{kk}_n(Y, X)$ yielding a generalized homology-cohomology theory which can be regarded as the connective theory associated with the Kasparov KK-theory when the later is restricted to spaces. There is an obvious product structure on kk_n induced by the composition of homomorphisms which enables one to make many explicit computations. For instance if X and Y are torsion free spaces, then $\mathrm{kk}(Y, X)$ can be completely computed in terms of those group homomorphisms $\mathrm{H}^*(X, \mathbf{Z}) \to \mathrm{H}^*(Y, \mathbf{Z})$ which preserve some natural filtrations reminding of cyclic homology. The connection with K-theory is made with the aid of the natural map $\mathrm{kk}(Y, X) \to \mathrm{KK}(C_0(X), C_0(Y))$ which turns out to be an isomorphism provided that X and Y are $(n-2)$-connected finite CW-complexes of dimension $\leqslant n$.

The results of Section 3 become useful for our concrete purposes only after we know that there is a sequence $v(m)$ of integers which tends to infinity, such that the natural embedding

$$\mathrm{Hom}(C_0(X), M_m) \to \mathrm{Hom}(C_0(X), M_{m+1})$$

is a $v(m)$-homotopy equivalence (in fact one can take $v(m) = 2[(m/3)]$). This will imply that

$$\mathrm{kk}(Y, X) = [C_0(X), C_0(Y) \otimes M_m] = [C(X), C(Y) \otimes M_m]_1$$

provided that the dimension of Y is less than $v(m)$.

It is worth noting that this connectivity result extends the stability properties of vector bundles to cocycles of connective K-theory, i.e. to $*$-homomorphisms. The proof is quite intricate and we have defered it to Section 6.

The next problem concerning homomorphisms is how to reduce the study of $[\bigoplus_i C(X_i) \otimes M_{n_i}, \bigoplus_j C(Y_j) \otimes M_{m_j}]$ to the study of the simpler homotopy classes of the from $[C(X_i), C(Y_j) \otimes M_{k_j}]$. This completely nontrivial problem is discussed in Section 4. Since we use techniques based on stability properties of vector bundles and homomorphisms, our classification results apply only to those $*$-homomorphisms which are large in the sense that they "amplify" many times – with respect to the dimension of each Y_j – each matrix subalgebra M_{n_i} of $\bigoplus_i C(X_i) \otimes M_{n_i}$. This is a natural restriction if we want to obtain purely algebraic but complete invariants for the homotopy classes of $*$-homomorphisms. The precise definition of large homomorphisms is given in Section 2. The main topic of Section 2 is to give an intrinsic characterization of those inductive limits $A = \lim A_i$ with $A_i \in \mathscr{C}(n)$, which can be written as limits of inductive systems with all the bonding homomorphisms large. This is accomplished using the notion of ordered group with large denominators introduced in [31].

Having a rather satisfactory homotopy classification of $*$-homomorphisms we pass to the question of how this local invariants can be patched together to yield an invariant for both the diagram $A_1 \to A_2 \to \ldots$ and the inductive limit $A = \lim A_i$. This is the shape problem to which we devote Section 5.

Let us state informally some special cases of our results concerning shape classifications. Let $\mathscr{C}'_3(2n)$ be the category of the C^*-algebras of the form $\bigoplus_k C(X_k) \otimes M_{m_k}$ where the X_k are $(2n-2)$-connected finite CW-complexes of dimension $\leqslant 2n$ with $\tilde{\mathrm{K}}^0(X_k)$ torsion groups. Let $\mathscr{C}'_2(2n)$ be the category of the C^*-algebras of the form $C(X) \otimes M_m$ where X ranges over the $(2n-2)$-connected finite CW-complexes of dimension $\leqslant 2n$.

Let $A = \lim A_i$, $B = \lim B_i$ be unital C^*-algebras where $A_i, B_i \in \mathscr{C}'_j(2n$ ($j = 2, 3$ is fixed, $n \geqslant 1$).

Assume that both A and B have no nonzero finite dimensional representations and also that the group $\mathrm{K}_0(A)$ has no proper order ideals. Then the following assertions are equivalent:

1) $\mathrm{K}_0(A) \simeq \mathrm{K}_0(B)$ as scaled ordered groups and $\mathrm{K}_1(A) \simeq \mathrm{K}_1(B)$.

2) There is a diagram of C^*-algebras and $*$-homomorphisms:

$$\begin{tikzcd}[column sep=small]
A_{i_1} \arrow[rr, "\cdots"] \arrow[dr] & & A_{i_2} \arrow[rr, "\cdots"] \arrow[dr] & & A_{i_3} \arrow[r] \arrow[dr] & \cdots & \\
& B_{j_1} \arrow[ur] \arrow[rr, "\cdots"] & & B_{j_2} \arrow[ur] \arrow[rr, "\cdots"] & & B_{j_3} \arrow[r] & \cdots
\end{tikzcd}$$

with each triangle homotopy-commutative.

3) A is shape equivalent to B in the category of separable C^*-algebras in the sense of Blackadar [2].

The (hard) implication 1) $\Rightarrow$ 2) is somewhat surprising since we know by the work of Loring [27] that C^*-algebras as $C(S^1 \times S^1)$ and $C(S^2)$ are not semiprojective in the sense of Effros-Kaminker [17]. Also the general shape theory of Blackadar [2] does not give 2) even we assume A to be isomorphic to B. In order to handle such delicate situations we introduce the notion of $KK_{+,\Sigma}$-semiprojectivity as a K-theoretical analogue of semiprojectivity. Let us briefly describe the idea of the proof. Having an isomorphism $K_*(A) \simeq K_*(B)$ one first constructs a diagram as at the point 2) but in the category $KK_{+,\Sigma}$ (5.1.1 b)). At this stage the formalism based on $KK_{+,\Sigma}$-semiprojectivity (of Section 5) and large denominators (of Section 2) are used together with some results of [37]. Then one has to replace $KK_{+,\Sigma}$-homomorphisms by actual homomorphisms. This is done by first replacing them by kk-homomorphisms using Theorems 3.4.5–3.4.6 and 3.5.2 (where the main restriction on X_k actually arise!) and then by applying the stabilization techniques of Sections 4 and 6 (Theorems 4.2.8, 4.2.1?, 4.3.1, 4.3.2, 6.4.2, 6.4.4).

Related ideas are used to give a short proof (5.2.4) of a theorem of Effros and Kaminker concerning shape classification for inductive limits of Cuntz-Krieger algebras.

One may conclude from our computations the important role played by the connective K-theory in problems concerning homotopy theory. This is mainly due to the fact that it detects phenomena which are not seen by ordinary K-theory. What is however missing is a suitable continuous extension of connective K-theory to the category of C^*-algebras, which, among other things, would give a rather satisfactory shape invariant. We hope to discuss this problem in a future paper.

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1.2. PRELIMINARIES AND NOTATION

In this section we shall fix some notation and make some conventions and definitions to be used in the sequel.

1.2.1. Let $\mathcal{S}$ denote the category of all separable C^*-algebras and $*$-homomorphisms. If $A, B \in \mathcal{S}$ then we shall denote by $\mathrm{Hom}(A, B)$ (resp. $\mathrm{Hom}_1(A, B)$) the space of all (resp. all unital) $*$-homomorphisms $A \to B$ with the topology of pointwise-norm convergence. Accordingly we define $[A, B]$ (resp. $[A, B]_1$) to be the set of homotopy classes of homomorphisms in $\mathrm{Hom}(A, B)$ (resp. $\mathrm{Hom}_1(A, B)$).

For unital $A \in \mathcal{S}$ let 1_A denote its unit.

1.2.2. Given $A \in \mathcal{S}$ nonunital, let

$$0 \to A \to A^1 \xrightarrow{\mu} \mathbf{C} \to 0$$

be the unital extension of A. If X is a locally compact space (resp. compact), let $C_0(X)$ (resp. $C(X)$) denote the continuous complex functions vanishing at infinity on X (resp. all continuous complex functions on X). We may identify $C_0(X)^1$ with $C(X^1)$ where X^1 denotes the one-point compactification of X. We shall use the notation $C_0(X)$ even for compact spaces X with base point $x_0 \in X$ to mean $C_0(X) := C_0(X \setminus \{x_0\})$.

1.2.3. Let M_k denote the C^*-algebra of all $k \times k$ complex matrices and $\mathcal{K}$ the C^*-algebra of compact operators on an infinite separable Hilbert space. There are natural embeddings $M_k \hookrightarrow M_{k+1} \hookrightarrow \mathcal{K}$ such that $\mathcal{K} = \lim M_k$.

1.2.4. If X is a compact space with base point $x_0 \in X$ then the restriction map $\mathrm{Hom}_1(C(X), M_k) \to \mathrm{Hom}(C_0(X), M_k)$ is a homeomorphism of topological spaces. This is easily seen if we recall that for any $\varphi \in \mathrm{Hom}_1(C(X), M_k)$ there are $x_1, \ldots, x_r \in X$ and mutually orthogonal projections $p_1, \ldots, p_r \in M_k$ with $\sum p_i = 1_k$ such that $\varphi(f) = \sum_{i=1}^{r} f(x_i)p_i$ for each $f \in C(X)$. We say that $x_1, \ldots, x_r$ are the proper values of φ and $p_1, \ldots, p_r$ the spectral projections of φ.

1.2.5. If A is a unital $*$-algebra then $U(A)$ stands for the unitary group of A. For nonunital A let $U(A)$ be the subgroup of all $u \in U(A^1)$ with $\mu(u) = 1_{\mathbf{C}}$.

1.2.6. If A is a C^*-algebra we denote by $V(A)$ the semigroup of equivalence classes of projections in $A \otimes \mathcal{K}$, with orthogonal addition, and $\mathrm{K}_*(A) = \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ the K-groups of A. There is a canonical homomorphism from $V(A)$ to $\mathrm{K}_0(A)$. If $A \otimes \mathcal{K}$ has an approximate identity consisting of projections then $\mathrm{K}_0(A)$ can be identified with the Grothendieck group of $V(A)$. The image of $V(A)$ in $\mathrm{K}_0(A)$ is denoted by $\mathrm{K}_0(A)_+$. We denote by $\Sigma(A)$ the subset of $\mathrm{K}_0(A)$ corresponding to the projections of A. The triple $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, \Sigma(A))$ is a preordered scaled group ([3]).

1.2.7. There is a split extension

$$0 \to SA \to C(\mathrm{S}^1) \otimes A \to A \to 0$$

where $SA = C_0(\mathrm{S}^1) \otimes A$, which gives a natural isomorphism

$$\mathrm{K}_0(C(\mathrm{S}^1) \otimes A) \simeq \mathrm{K}_0(A) \oplus \mathrm{K}_0(SA) \simeq \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) = \mathrm{K}_*(A)$$

which commutes with inductive limits.

We define $K_*(A)_+$ to be the image of $K_0(C(S^1) \otimes A)_+$ in $K_*(A)$ under this isomorphism. It is clear that $K_*(A)_+$ is a subset of $(K_0(A)_+ \setminus \{0\}) \oplus K_1(A) \cup \{(0, 0)\}$. In general it is a proper subset (e.g. for $A = C(S^1) \oplus C(S^1)$). In a similar manner we define $\Sigma_*(A) \subset K_*(A)_+$ as corresponding to $\Sigma(C(S^1) \otimes A)$ under the above isomorphism. It is important to note that

$$K_*(\lim A_i)_+ = \lim K_*(A_i)_+ \quad \text{and} \quad \Sigma_*(\lim A_i) = \lim \Sigma_*(A_i).$$

We also define $\mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}$ to be the set of all homomorphisms of $\mathbf{Z}_2$-graded groups $K_*(A) \to K_*(B)$ which take $K_*(A)_+$ into $K_*(B)_+$ and $\Sigma_*(A)$ into $\Sigma_*(B)$.

1.2.8. For $A, B \in \mathcal{S}$ we shall consider the Kasparov groups $KK_n(A, B)$ ([26]). As a special case of the Kasparov product we have the pairing

$$KK_*(\mathbf{C}, A) \otimes KK_*(A, B) \to KK_*(\mathbf{C}, B)$$

which gives a natural map

$$\gamma: KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \qquad \text{(see [37])}.$$

It is useful to make the following notation

$$KK(A, B)_{+,\Sigma} = \{x \in KK(A, B) : \gamma(x) \in \mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}\}.$$

Notice that if $x \in KK(A, B)_{+,\Sigma}$ and $y \in KK(B, C)_{+,\Sigma}$ then $xy \in KK(A, C)_{+,\Sigma}$. As it follows from [37, Proposition 7.3] the map

$$KK(A, B)_{+,\Sigma} \to \mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}$$

is surjective for a large class of C^*-algebras.

1.2.9. For a compact space X let $\mathrm{Vect}_k(X)$ denote the set of isomorphism classes of complex vector bundles of rank k on X. In $\mathrm{Vect}_k(X)$ we have a distinguished element $[k]$ – the class of the trivial bundle of rank k. Let $\mathrm{Vect}(X) = \bigcup_k \mathrm{Vect}_k(X)$. We shall freely identify $\mathrm{Vect}(X)$ with the monoid of equivalence classes of idempotents in $C(X) \otimes \mathcal{K}$, i.e. with $V(C(X))$, and $K^0(X)$ with $K_0(C(X))$.

1.2.10. Recall that a map $f: (X, x_0) \to (Y, y_0)$ is an m-homotopy equivalence $(0 \leqslant m \leqslant \infty)$ if $f_*: \pi_q(X, x_0) \to \pi_q(Y, y_0)$ is an isomorphism for $0 \leqslant q \leqslant m - 1$ and an epimorphism for $q = m$. If $m = \infty$ then f is called a weak homotopy equivalence.

1.2.11. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called exact if for any space W, the sequence $[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{g_*} [W, Z]$ is an exact sequence of pointed sets, i.e. $\mathrm{image}(f_*) =$

$= g_*^{-1}(0)$ where 0 is the trivial element in $[W, Z]$. If this happens only for CW-complexes W, then the above sequence is called weak exact.

1.2.12. In order to simplify the terminology we shall mean by "category" a mathematical entity which satisfies all the axioms of the usual categories, except possibly the axiom which postulates the existence of identities. The term "subcategory" will be used in accordance with the above convention. Now if $\mathscr{C}$ is a "subcategory" of $\mathscr{S}$ then $\mathscr{A}\mathscr{C}$ will denote the class of all C^*-algebras which can be represented as inductive limits of countable inductive systems *(with injective bonding morphisms)* of C^*-algebras in $\mathscr{C}$. For instance if $\mathscr{F}$ is the category of finite dimensional C^*-algebras then $\mathscr{A}\mathscr{F}$ consists of all AF-C^*-algebras [7].

For the sake of brevity we shall use the terms algebra for C^*-algebra and homomorphism, or even morphism for $*$-homomorphism.

2. ORDERED K-THEORY AND LARGE DENOMINATORS

Recall that $\mathscr{C}(n)$ denotes the category of C^*-algebras of the form

$$\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}, \quad q > 0,$$

where $X_1, \ldots, X_q$ are arbitrary finite connected CW-complex with $\dim(X_i) \leqslant n$. The morphisms of $\mathscr{C}(n)$ are arbitrary C^*-algebras homomorphisms. Let $\mathscr{A}\mathscr{C}(n)$ be the class of C^*-algebras defined as in 1.2.12.

The methods we shall give in the next sections for computing the homotopy classes of $*$-homomorphisms use in an essential way some stability properties of vector bundles and $*$-homomorphisms. For this reasons our calculations apply only to those morphisms in $\mathscr{C}(n)$ which are large enough in the sense of Definition 2.1.8. Therefore it is natural to ask which inductive limits of C^*-algebras from $\mathscr{C}(n)$ can be written as limits of inductive systems with arbitrary large bonding morphisms and how can they be characterized in an intrinsic manner. The first part of this section is devoted to these and to related questions. The answers we offer are given in terms of K-theory groups. They are based on the notion of ordered group with large denominators introduced by Nistor [31] in order to settle similar questions for AF-algebras

The second part of this section deals with the states of the ordered group $K_0(A)$ for $A \in \mathscr{A}\mathscr{C}(n)$.

2.1. THE DIMENSION MAP

The main technical tool of this subsection is the dimension map associated with each description of $A \in \mathscr{A}\mathscr{C}(n)$ as the limit of an inductive system of C^*-algebras from $\mathscr{C}(n)$. We first introduce the dimension map for C^*-algebras in $\mathscr{C}(n)$.

For $A \in \mathscr{C}(n)$ of the form $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, we define $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$. Let $x_i \in X_i$, $1 \leqslant i \leqslant q$. The evaluation map $A \to r(A)$ given by $(f_i) \mapsto (f_i(x_i))$, $f_i \in C(X_i) \otimes M_{n_i}$, induces a split extension of groups

$$0 \to K_0'(A) \xrightarrow{i_A} K_0(A) \underset{\longleftarrow}{\overset{r_A}{\longrightarrow}} K_0(r(A)) \to 0$$

where $K_0'(A) := \ker(r_A)$. Notice that r_A does not depend on the choice of x_i in X_i since each X_i is connected. If $r(A) = \mathbf{C}^q$, $n_i = 1$, $1 \leqslant i \leqslant q$, then the above extension reduces to

$$0 \to K'(X) \to K^0(X) \to H^0(X, \mathbf{Z}) \to 0$$

where $X = X_1 \sqcup \ldots \sqcup X_q$, see [25], and this justifies our notation for $\ker(r_A)$.

We need the following stability properties of complex vector bundles (see [23, Chapter 8, Theorems 1.2, 1.5 and 2.6]). For $x \in \mathbf{R}$, let $\langle x \rangle$ denote the smallest integer t with $x \leqslant t$.

2.1.1. THEOREM. *Let X be a CW-complex of dimension $\leqslant n$.*

a) *If $E \in \mathrm{Vect}_k(X)$, $k \geqslant \langle (n-1)/2 \rangle$, then there is $F \in \mathrm{Vect}_{\langle (n-1)/2 \rangle}(X)$ such that E is isomorphic to $F \oplus [k - \langle (n-1)/2 \rangle]$.*

b) *If $E_1, E_2 \in \mathrm{Vect}_k(X)$, $k \geqslant \langle n/2 \rangle$, and $E_1 \oplus G$ is isomorphic to $E_2 \oplus G$ for some $G \in \mathrm{Vect}(X)$ then E_1 is isomorphic to E_2.*

c) *If $k \geqslant 0$ then the Grassmannian $G_k(k + \langle n/2 \rangle)$ classifies all vector bundles of rank k over X,* i.e. $\mathrm{Vect}_k(X) \simeq [X, G_k(k + \langle n/2 \rangle)]$.

The following corollary is a direct consequence of Theorem 2.1.1. See also [3, Example 6.10.3].

2.1.2. COROLLARY. *Let $A \in \mathscr{C}(n)$ be as above.*

a) *Let $a = (a_1, \ldots, a_q) \in K_0(A)$ and $r_A(a) = (s_1, \ldots s_q) \in K_0(r(A)) \simeq \mathbf{Z}^q$. If for each i, $1 \leqslant i \leqslant q$, we have either $a_i = 0$ or $s_i \geqslant n$ then $a \in K_0(A)_+$.*

b) *If $a \in K_0(A)_+$ and for each i, $1 \leqslant i \leqslant q$, the i^{th} component of $r_A(a)$ is no greater than $n_i - \langle n/2 \rangle$, then $a \in \Sigma(A)$,* i.e. *there is some projection in A whose* K-*theory class equals a.*

c) *Let $x = (a, b) \in K_*(A)$ with $a = (a_1, \ldots, a_q), b = (b_1, \ldots, b_q)$ and $r_A(a) = (s_1, \ldots, s_q)$. If for each $1 \leqslant i \leqslant q$, one has either $a_i = b_i = 0$ or $s_i \geqslant n+1$, then $x \in K_*(A)_+$ (*see 1.2.7*).*

d) *Let $x = (a, b) \in K_*(A)_+$. If for each $1 \leqslant i \leqslant q$, the i^{th} component of $r_A(a)$, is no greater that $n_i - \langle (n+1)/2 \rangle$ then $x \in \Sigma_*(A)$ (*see 1.2.7*).*

We shall prove below that the assignment $A \mapsto r(A)$ is functorial. To this purpose, given $A, B \in \mathscr{C}(n)$ and $\sigma \in \mathrm{Hom}(K_0(A), K_0(B))_{+,\Sigma}$, it is useful to consider the matrix of σ with respect to the decomposition

$$\mathrm{K}_0(A) = \mathrm{K}_0'(A) \oplus \mathrm{K}_0(r(A)), \quad \mathrm{K}_0(B) = \mathrm{K}_0'(B) \oplus \mathrm{K}_0(r(B)).$$

2.1.3. PROPOSITION. *If $\sigma\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is an homomorphism of (scaled) ordered groups, then its matrix with respect to the above decompositions is triangular: $\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ and $\gamma\colon \mathrm{K}_0(r(A)) \to \mathrm{K}_0(r(B))$ is a homomorphism of (scaled) ordered groups.*

Proof. If $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$ then $\mathrm{K}_0(r(A)) \simeq \mathbf{Z}^q$. The description of $r_A : \mathrm{K}_0(A) \to \mathrm{K}_0(r(A))$ is as follows: given $E_i \in \mathrm{Vect}(X_i)$, $1 \leqslant i \leqslant q$, $r_A([E_1], \ldots, [E_q]) = (\mathrm{rank}(E_1), \ldots, \mathrm{rank}(E_q))$ (recall that each X_i is connected). Now let $\sigma = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ and $x \in \mathrm{K}_0(A)$. Corollary 2.1.2 a), $kx + n[1_A] \in \mathrm{K}_0(A)_+$ for all $k \in \mathbf{Z}$. Since σ is an order preserving homomorphism we infer that $k\delta(x) + \gamma(n[1_A]) \in \mathrm{K}_0(r(B))_+$ for all $k \in \mathbf{Z}$. This implies $\delta(x) = 0$.

2.1.4. If we further decompose

$$\alpha = (\alpha_{ji})_{\substack{1 \leqslant j \leqslant h \\ 1 \leqslant i \leqslant q}} \colon \bigoplus_{i=1}^{q} \mathrm{K}_0'(A_i) \to \bigoplus_{j=1}^{h} \mathrm{K}_0'(B_j)$$

$$\beta = (\beta_{ji})\colon \bigoplus_{i=1}^{q} \mathrm{K}_0(r(A_i)) \to \bigoplus_{j=1}^{h} \mathrm{K}_0'(B_j)$$

$$\gamma = (\gamma_{ji})\colon \bigoplus_{i=1}^{q} \mathrm{K}_0(r(A_i)) \to \bigoplus_{j=1}^{h} \mathrm{K}_0(r(B_j))$$

where $A = \bigoplus_{i=1}^{q} A_i$, $B = \bigoplus_{j=1}^{h} B_j$, $A_i = C(X_i) \otimes M_{n_i}$, $B_j = C(Y_j) \otimes M_{m_j}$ then $\begin{pmatrix} (\alpha_{ji}) & (\beta_{ji}) \\ 0 & (\gamma_{ji}) \end{pmatrix}$ will be called the *standard picture* of σ.

2.1.5. The homomorphism γ associated in 2.1.3 to σ will be denoted by $r(\sigma)$. Proposition 2.1.3 shows that σ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{K}_0'(A) & \to & \mathrm{K}_0(A) & \to & \mathrm{K}_0(r(A)) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle r(\sigma)} & & \\ 0 & \to & \mathrm{K}_0'(B) & \to & \mathrm{K}_0(B) & \to & \mathrm{K}_0(r(B)) & \to & 0. \end{array}$$

Moreover it follows that the correspondence $\sigma \mapsto r(\sigma)$ (and even $\sigma \mapsto \alpha$) is functorial. For $A \in \mathscr{C}(n)$ the homomorphism $r_A : \mathrm{K}_0(A) \to \mathrm{K}_0(r(A))$ will be called the *dimension map* associated to A.

If $A \in \mathscr{A}\mathscr{C}(n)$ is the limit of the inductive system (A_i, φ_{ji}), $A_i \in \mathscr{C}(n)$, we shall denote by $r(A)$ the unique (up to isomorphism) AF-algebra determined by the scaled dimension group $\lim(\mathrm{K}_0(r(A_i)), r(\varphi_{ji})_*)$. There is also a surjective homomorphism $r_A : \mathrm{K}_0(A) \to \mathrm{K}_0(r(A))$ induced by the homomorphisms $r_{A_i} : \mathrm{K}_0(A_i) \to \mathrm{K}_0(r(A_i))$. Our notation is misleading since it is not clear whether the above AF-algebra depends only on A (and not on the approximating system (A_i, φ_{ji})). This is certainly true in certain cases pointed out in Section 6. In the general case we make the convention that $r(A)$ denotes a fixed AF-algebra arising as described above from a fixed approximating system of A. Note that $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+)$ is an ordered group since any $A \in \mathscr{A}\mathscr{C}(n)$ is stably unital and stably finite. Also, the epimorphism $r_A : \mathrm{K}_0(A) \to \mathrm{K}_0(r(A))$ is order preserving and does not vanish on the nonzero elements of $\mathrm{K}_0(A)_+$.

2.1.6. Definition ([31]). Let (G, G_+) be an ordered group. We say that G has *large denominators* if for any $a \geqslant 0$ and $k \in \mathbf{N}$ there are $b \in G_+$ and $m \in \mathbf{N}$ such that $kb \leqslant a \leqslant mb$.

2.1.7. Proposition ([31]). *If A is a simple* AF*-algebra, non stably isomorphic to $\mathscr{K}$, then* $\mathrm{K}_0(A)$ *has large denominators.*

2.1.8. Here we define the notion of m-large and m-full homomorphism.

Let $\mathbf{Z}^q$, $\mathbf{Z}^h$ have the standard orderings $\mathbf{Z}^q_+ = \mathbf{N}^q$, $\mathbf{Z}^h_+ = \mathbf{N}^h$ and the order units $u = (n_1, \ldots, n_q)$ and $v = (m_1, \ldots, m_h)$. Let $\gamma = [k_{ji}] : \mathbf{Z}^q \to \mathbf{Z}^h$ be a morphism of scaled ordered groups. This is equivalent to say that $k_{ij} \geqslant 0$ for all i, j and

$$t_j := m_j - \sum_{i=1}^{q} k_{ji} n_i \geqslant 0 \quad \text{for all } j.$$

The morphism γ is called *m-large* ($m \geqslant 0$) if satisfies the following two conditions:

a) if some $k_{ji} > 0$ then $k_{ji} \geqslant m$

b) if some $t_j > 0$ then $t_j \geqslant m$.

If, in addition, all $k_{ji} > 0$ then γ is called *m-full*. We extend the above definitions to the morphisms of (scaled) ordered groups $\sigma : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ with $A, B \in \mathscr{C}(n)$, by saying that σ is *m-large* (*m-full*) iff the morhism $r(\sigma)$ defined in 2.1.5 is m-large (m-full). By extension we say that a morphism $\varphi : A \to B$, $A, B \in \mathscr{C}(n)$ is *m-large* (*m-full*) iff $\mathrm{K}_0(\varphi)$ is m-large (m-full). Note that since $\mathscr{F} \subset \mathscr{C}(n)$ the above definitions also apply to morphisms of finite dimensional C^*-algebras. The product of an m-large morphism by a p-large morphism is mp-large. Also the product of an m-full morphism by a p-full morphism is an mp-full morphism.

2.1.9. REMARK. a) Let A be an AF-algebra and assume that $\mathrm{K}_0(A)$ has large denominators. Then any faithful inductive system of finite dimensional C^*-algebras $A_1 \to A_2 \to \ldots$ with $\lim A_i = A$ can be refined to an inductive system $A_{i_1} \to A_{i_2} \to \ldots$ with arbitrary large embeddings. The converse it is also true, (see [3]).

b) If A is as in Proposition 2.1.7, then the inductive system (A_i) can be refined to an inductive system with m-full embeddings for arbitrary large $m \in \mathbf{N}$ (see [15] and [31]).

The C^*-algebras in $\mathscr{A}\mathscr{C}(n)$ have similar properties (2.1.14).

2.1.10. LEMMA. *Let* $A \in \mathscr{A}\mathscr{C}(n)$ *and* $a,\ b \in \mathrm{K}_0(A)_+$ *such that* $r_A(a) \leqslant r_A(b)$ *and* $r_A(a)$ *belongs to the order ideal generated by* $r_A(b - a)$ *in* $\mathrm{K}_0(r(A))$. *If* $\mathrm{K}_0(r(A))$ *has large denominators then* $a \leqslant b$.

Proof. Let $A = \lim A_i$ with $A_i \in \mathscr{C}(n)$. Then $\mathrm{K}_0(A) = \lim \mathrm{K}_0(A_i)$ and $\mathrm{K}_0(r(A)) = \lim \mathrm{K}_0(r(A_i))$ in the category of ordered groups. The idea of the proof is to show that $b - a \in \mathrm{K}_0(A)$ comes from some element of some group $\mathrm{K}_0(A_j)$ satisfying the hypotheses of Corollary 2.1.2 a).

Let $x = r_A(a)$, $y = r_A(b)$, $z = r_A(b - a) \in \mathrm{K}_0(r(A))_+$. We have $0 \leqslant x \leqslant y$, $y = x + z$ and x belongs to the order ideal generated by z. If follows that y and z generate the same order ideal. On the other hand since $\mathrm{K}_0(r(A))$ has large denominators there are $w \in \mathrm{K}_0(r(A))_+$ and $m \in \mathbf{N}$ such that $nw \leqslant z \leqslant mw$. Combining the above data we obtain that y and w generate the same order ideal in $\mathrm{K}_0(r(A))$. Therefore we can find some $j \in \mathbf{N}$ such that $y,\ z \in \mathrm{K}_0(r(A_j))_+$ and they have the same support, i.e. their coordinates in $\mathbf{Z}^q \simeq \mathrm{K}_0(r(A_j))$ vanish simultaneously. Also we can assume j large enough such that $a,\ b \in \mathrm{K}_0(A_j)_+$ and $y - x - nw \in \mathrm{K}_0(r(A_j))_+$. Let $r_{A_j}(a) = (x_1 \ldots, x_q)$, $r_{A_j}(b) = (y_1, \ldots, y_q)$, $w = (w_1, \ldots, w_q) \in \mathbf{Z}^q$. We must have $y_i - x_i \geqslant nw_i$, $1 \leqslant i \leqslant q$, and moreover if some $w_i = 0$ then $y_i = x_i = 0$ since y and w have the same support. Therefore we may apply Corollary 2.1.2 a) to get $b - a \in \mathrm{K}_0(A_j)_+$.

2.1.11. PROPOSITION. *Let* $A \in \mathscr{A}\mathscr{C}(n)$. *Then* $\mathrm{K}_0(A)$ *has large denominators if and only if* $\mathrm{K}_0(r(A))$ *has large denominators.*

Proof. One implication is trivial since $r_A\colon \mathrm{K}_0(A) \to \mathrm{K}_0(r(A))$ is a surjective morphism of ordered groups. To prove the other assume that $\mathrm{K}_0(r(A))$ has larged denominators and fix $a \in \mathrm{K}_0(A)_+$ and $k \in \mathbf{N}$. By assumption there are $x \in \mathrm{K}_0(r(A))_+$ and $m \in \mathbf{N}$ such that $2kx \leqslant r_A(a) \leqslant mx$. Let $b \in \mathrm{K}_0(A)_+$ be such that $r_A(b) = x$. We wish to apply Lemma 2.1.10 in order to prove that $kb \leqslant a$ and $a \leqslant 2mb$. For the first inequality it is clear that $r_A(kb) \leqslant r_A(a)$ and we have to check only that $r_A(kb)$ belongs to the order ideal generated by $r_A(a - kb)$. But this is again obvious since $r_A(kb) = kx \leqslant r_A(a) - kx = r_A(a - kb)$. In the same way $r_A(a) \leqslant 2mr_A(b)$ and $r_A(a) \leqslant r_A(2mb - a)$ imply $a \leqslant 2mb$.

2.1.12. COROLLARY. *Let $A \in \mathcal{AC}(n)$ be such that $r(A)$ is not stably isomorphic to $\mathcal{K}$. Then* $K_0(A)$ *is simple if and only if* $K_0(r(A))$ *is simple.*

Proof. One implication is again trivial. To prove the other assume that $K_0(r(A))$ is simple. Let J be a nonzero ideal in $K_0(A)$. We shall prove that J contains any given positive element $a \in K_0(A)$. Indeed if $b \in J$, $b > 0$ then $2r_A(a) \leqslant mr_A(b)$ for some $m \in \mathbb{N}$, since $K_0(A)$ is simple. Therefore $r_A(a) \leqslant r_A(mb)$ and $r_A(a)$ belongs to the order ideal generated by $r_A(mb - a)$ since $r_A(a) \leqslant r_A(mb - a)$. By Proposition 2.1.7, $K_0(r(A))$ has large denominators and so we can apply Lemma 2.1.10 to get $a \leqslant mb$. Since $b \in J$ we must have $a \in J$.

We also have the following generalization of Proposition 2.1.7.

2.1.13. COROLLARY. *Let $A \in \mathcal{AC}(n)$ be a simple C^*-algebra such that $r(A)$ is not stably isomorphic to $\mathcal{K}$ (e.g. A is simple, unital and has no nonzero finite dimensional representation). Then* $K_0(A)$ *has large denominators.*

Proof. If A is simple then $K_0(A)$ is simple. By 2.1.12, $K_0(r(A))$ is simple so that $K_0(r(A))$ has large denominators by Proposition 2.1.7. Finally we apply Proposition 2.1.11 to obtain the desired result.

2.1.14. REMARK. Proposition 2.1.11–12 and Corollary 2.1.13 exhibit classes of C^*-algebras in $\mathcal{AC}(n)$ which display properties which are analogous with those in 2.1.9. For instance if $A \in \mathcal{AC}(n)$ and $K_0(A)$ has large denominators then any faithful inductive system of algebras in $\mathcal{C}(n)$, $A_1 \to A_2 \to \ldots$, such that $A = \lim A_i$, can be refined to an inductive system $A_{i_1} \to A_{i_2} \to \ldots$ with arbitrary large embeddings. The converse is also true. If A is as in Corollary 2.1.12 and $K_0(A)$ is simple, then these embeddings can be chosen m-full, for arbitrary large $m \in \mathbb{N}$.

2.2. SOME PROPERTIES RELATED TO LARGE DENOMINATORS

The results of this subsection are not used later in the paper but we find them enough interesting to be included here.

2.2.1. PROPOSITION. *Let $A \in \mathcal{AC}(n)$ and suppose that* $K_0(A)$ *has large denominators. Then*

a) *A has cancellation;*

b) $\pi_i(U(A)) \simeq K_{i+1}(A)$, $i \geqslant 0$;

c) *$\Sigma(A)$ is a generating, hereditary and directed subset of* $K_0(A)_+$ *(see* [15] *for definitions).*

Proof. Write $A = \lim A_j$ with $A_j \in \mathcal{C}(n)$ and n-large embeddings $A_j \to A_{j+1}$. Having in mind Theorem 2.1.1 and the fact that $U(k) \to U(k+1)$ is a $2k$-equivalence the proof goes along standard arguments. See [4] and [35] for related situations.

2.2.2. PROPOSITION. *If $A \in \mathcal{AC}(n)$, then for every state f on* $(K_0(A), K_0(A)_+)$ *there is a unique state f' on* $(K_0(r(A)), K_0(r(A))_+)$ *such that $f = f' \circ r_A$.*

Proof. It is enough to consider the case $A \in \mathscr{C}(n)$, when the proof is similar to the proof of Proposition 2.1.3. Indeed, as in 2.1.3, if f is regarded as a map from $K_0'(A) \oplus K_0(r(A))$ to $\mathbf{R}$ then the positivity of f implies that f vanishes on $K_0'(A)$. Hence f factors through r_A. The uniqueness holds since r_A is onto.

2.2.3. PROPOSITION. *Assume that $A \in \mathscr{A}\mathscr{C}(n)$ is unital. Then every state on $(K_0(A), K_0(A)_+, [1_A])$ is induced by some bounded trace state on A.*

Proof. Let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \ldots$, $A_i \in \mathscr{C}(n)$ be a faithful inductive system with unital embeddings such that $A = \lim A_i$. Define for each $i \geqslant 1$ a $*$-homomorphism $\gamma_i : r(A_i) \to r(A_{i+1})$ such that $K_0(\gamma_i) = r(K_0(\varphi_i))$. The AF-algebra $r(A)$ associated in 2.1.5 with the inductive system (A_i, φ_i) can be realized as $\lim(r(A_i), \gamma_i)$. Now let f be a state on $K_0(A)$ and let $f = f' \circ r_A$ be the factorization provided by Proposition 2.2.1. Since the statement of Proposition 2.2.3 holds for AF-algebras [3], there is some trace state σ on $r(A)$ such that f' is induced by σ. For each $i \geqslant 1$ we define the trace $\tau_i : A_i \to \mathbf{C}$ by $\tau_i = \sigma_i \circ ev_i$ where $ev_i : A_i \to r(A_i)$ is an evaluation map (2.1.2) and σ_i is the restriction of σ to $r(A_i)$. We shall prove that f is induced by any weak limit of the sequence (τ_i). More precisely let ω be a free ultrafilter on $\mathbf{N}$ and define a trace τ on the algebraic inductive limit of A_i by $\tau(a) = \lim_{\omega} \tau_i(a)$. Note that $\|\tau(a)\| \leqslant \|a\|$ since $\|\tau_i\| = \tau_i(1_{A_i}) = 1$. Therefore we can extend τ on A by continuity. Of course in general it is not true that $\tau|A_i = \tau_i$. However if e is any projection in A_i then $\tau(e) = \tau_i(e)$. To prove this equality it suffices to know that $\tau_{i+1}(e) = \tau_i(e)$ for any projection $e \in A_i$. But this follows from the following commutative diagram

$$\begin{tikzcd}
K_0(A_i) \arrow[r, "K_0(\varphi_i)"] \arrow[d] & K_0(A_{i+1}) \arrow[d] \\
K_0(r(A_i)) \arrow[r, "K_0(\gamma_i)"] \arrow[dr, "f_i'"'] & K_0(r(A_{i+1})) \arrow[d, "f_{i+1}'"] \\
& \mathbf{R}
\end{tikzcd}$$

where f_i' is induced by σ_i. Indeed it is easily checked that $f_i \circ (ev_i)_*([e]) = \tau_i(e)$ as a conseqeuence of the definition of τ_i. Therefore we obtain

$$f([e]) = f_i' \circ r_{A_i}([e]) = \tau_i(e) = \lim_{i \to \infty} \tau_i(e) = \tau(e).$$

It is also clear that the above equalities hold for all projections e in $A_i \otimes \mathscr{K}$. This completes the proof.

The following corollary proves, in a special case, a conjecture in [4].

2.2.4. COROLLARY. *Let $A \in \mathcal{AC}(n)$ be a unital simple C^*-algebra and let p, q be projections in A. If $\tau(p) < \tau(q)$ for every trace state τ of A, then $upu^* < q$ for some unitary $u \in U(A)$.*

Proof. It is useful to consider the following two cases.

a) $r(A)$ is stably isomorphic to $\mathcal{K}$.

b) $r(A)$ is not stably isomorphic to $\mathcal{K}$.

Since A is a unital simple C^*-algebra in $\mathcal{AC}(n)$ it can be proved that the first case can occur only if A is isomophic to some matrix algebra M_k. (The proof is easy and we omit it.) As our corollary is trivial for $A = M_k$ we turn our attention to the second case. Assuming b) we can apply Corollary 2.1.13 to find that $K_0(A)$ has large denominators.

On the other hand if $\tau(p) < \tau(q)$ for every trace state τ of A, then if follows (by Proposition 2.2.2) that $f' \circ r_A([p]) < f' \circ r_A([q])$ for every state f' of $K_0(r(A))$. Now $K_0(r(A))$ is a simple group which is also unperforated. Therefore it follows by [3, Theorem 6.8.5] that $K_0(r(A))$ has the strict ordering from its states (see also [15]). This implies $r_A([p]) < r_A([q])$ in $K_0(r(A))$. From Lemma 2.1.10 we derive $[p] < [q]$ in $K_0(A)$. (Note that $K_0(A)$ has no proper order ideals since A is simple.) After conjugating with suitable unitaries we may asume that p, q belong to some A_j and moreover $[p] < [q]$ in $K_0(A_j)$. Since $K_0(A)$ has large denominators by Theorem 2.1.1 it follows that p is homotopic in A_{j+k} (for k large enough) to some projection $p_1 < q$. But homotopic projections are unitarily equivalent.

3. CONNECTIVE KK-THEORY (FOR SPACES)

In this section we develop methods for computing the homotopy classes of $*$-homomorphisms $C_0(X) \to C_0(Y) \otimes \mathcal{K}$. We will introduce a bivariant functor $\mathrm{kk}(Y, X)$ which coresponds to such homotopy classes and which, as explained in the Subsection 3.3, defines the natural connective theory associated with Kasparov KK-theory. As noticed in the Introduction our results heavily depend on [39].

3.1. THE GROUPS kk_n

3.1.1. For A, B C^*-algebras, $\mathrm{Hom}(A, B)$ will denote the space of all $*$-homomorphisms $A \to B$ with the topology of pointwise-norm convergence. This is a pointed space — the base-point is the null homomorphism. We define $[A, B]$ to be the set of homotopy classes of homomorphism in $\mathrm{Hom}(A, B)$. If X, Y are pointed topological spaces then we define $\mathrm{Map}(X, Y)$ to be the space of all continuous base-point preserving maps of X into Y endowed with the compact-open topology. The space $\mathrm{Map}(S^1, X)$ is denoted by ΩX and is called the *space of loops in X* For $f \in \mathrm{Map}(X, Y)$, $\Omega f \in \mathrm{Map}(\Omega X, \Omega Y)$ is defined in the obvious way. If X, Y are compact then, via the Gelfand duality, $\mathrm{Map}(X, Y)$ can be identified with

$\mathrm{Hom}(C_0(Y), C_0(X))$. Also $[C_0(Y), C_0(X)]$ coincides with $[X, Y]$, the homotopy classes of maps in $\mathrm{Map}(X, Y)$. We shall use many times the obvious identification

$$\mathrm{Hom}(A, B \otimes C_0(X)) \simeq \mathrm{Map}(X, \mathrm{Hom}(A, B)).$$

This gives an isomorphism

$$[A, B \otimes C_0(X)] \simeq [X, \mathrm{Hom}(A, B)].$$

Assume that B is stable, $B \simeq B \otimes \mathscr{K}$. Then it is proved in [36] that $\mathrm{Hom}(A, B)$ is a commutative H-space with respect to the operation defined by letting $\varphi_1 + \varphi_2$ be the composite

$$A \xrightarrow{(\varphi_1, \varphi_2)} B \oplus B \hookrightarrow B \otimes M_2 \simeq B.$$

The homotopy unit is given by the null morphism. In this way $[A, B]$ is a commutative monoid.

3.1.2. Let W_0 be the category with base-vertex finite CW-complexes as objects and base-point preserving maps as morphisms. Let W_0^c be the full subcategory of W_0 consisting of connected spaces.

Following Segal [39], for each $X \in W_0$, we define $F(X) = \mathrm{Hom}(C_0(X), \mathscr{K})$ and $F_0^k(X) = \mathrm{Hom}(C_0(X), M_k)$. The natural embeddings $M_k \hookrightarrow M_{k+1}$, $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, induce embeddings $F_0^k(X) \hookrightarrow F_0^{k+1}(X)$. It is proved in [39] that the canonical map $\varinjlim_k F_0^k(X) \hookrightarrow F(X)$ is a homotopy equivalence, where the former space has the inductive limit topology. For $X = S^1$ this shows that the infinite unitary group $U(\infty)$ is homotopy equivalent to $U(\mathscr{K})$.

It is clear that each map $f: X \to Y$ induces a map $F(f): F(X) \to F(Y)$. If f_1 is homotopic to f_2, then $F(f_1)$ is homotopic to $F(f_2)$. If X is connected, then $F(X)$ is connected.

3.1.3. PROPOSITION. *If $X \in W_0^c$, $Y \in W_0$, then $[C_0(X), C_0(Y) \otimes \mathscr{K}]$ is an abelian group with respect to the direct sum of homomorphisms.*

Proof. If $X \in W_0^c$, then $F(X)$ is a path connected H-space and therefore we can apply [43, Chapter 10, Theorem 2.4] to get that $[Y, F(X)] \simeq [C_0(X), C_0(Y) \otimes \mathscr{K}]$ is a group. ▨

Another proof of this proposition will be available after we shall see that $F(X)$ is an infinite loop space. More precisely

$$F(X) \sim \Omega F(SX) \sim \dots \sim \Omega^k F(S^k X) \sim \dots \qquad \text{(see Corollary 3.1.7)}.$$

3.1.4. DEFINITION. If $X \in W_0^c$ and $Y \in W_0$ (see 3.1.2) then we define $kk(Y, X)$ to be the group $[C_0(X), C_0(Y) \otimes \mathcal{K}]$. More generally, for $n \in \mathbf{Z}$ we set

$$kk_n(Y, X) = \begin{cases} kk(S^n Y, X) & \text{if } n \geqslant 0 \\ kk(Y, S^{-n} X) & \text{if } n < 0. \end{cases}$$

This definition is extended for $X, Y \in W_0$ in 3.1.9 a).

Our next purpose is to find exact sequences for these groups. This requires the notion of quasifibration introduced by Dold and Thom [14].

Recall that a continuous map $p : E \to B$ between topological Hausdorff spaces is called *quasifibration* if for all points $b \in B$ and $e \in p^{-1}(b)$, the induced maps

$$p_* : \pi_q(E, p^{-1}(b), e) \to \pi_q(B, b)$$

are isomorphisms for all $q \geqslant 0$.

3.1.5. THEOREM (G. Segal [39]). *Let X be a compact connected space. If A is a path connected closed subspace of X and A is a neighbourhood deformation retract in X, then $F(X) \to F(X/A)$ is a quasifibration with fiber $F(A)$.*

For the purposes of homotopy theory the quasifibrations are as good as the fibrations. For instance, given a quasifibration $p : E \to B$ one can replace in the homotopy sequence of the pair $(E, F = p^{-1}(b))$, the groups $\pi_n(E, F)$ with $\pi_n(B)$, in order to obtain the homotopy exact sequence of the quasifibration:

$$\pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F).$$

As for fibrations even more is true.

3.1.4. PROPOSITION. *If $p : E \to B$ is a quasifibration, with fiber $F = p^{-1}(b_0)$, $b_0 \in B$, then for any CW-complex Y there is an exact sequence of groups $(k \geqslant 1)$ and sets $(k = 0)$:*

$$[Y, \Omega^{k+1}B] \to [Y, \Omega^k F] \to [Y, \Omega^k E] \to [Y, \Omega^k B] \to [Y, \Omega^{k-1}F] \to \dots$$

Proof. Let B^I be the space of all (free) paths in B and let $\mathrm{Cocyl}(p) = \{(u, e) \in B^I \times E : u(1) = p(e)\}$ be the mapping cocylinder of p. The map $p' : \mathrm{Cocyl}(p) \to B$ given by $p'(u, e) = u(0)$ is a fibration with fiber $W(p) = \{(u, e) \in B^I \times E : u(0) = b_0, u(1) = p(e)\}$. Moreover there is a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \| \\ W(p) & \longrightarrow & \mathrm{Cocyl}(p) & \xrightarrow{p'} & B \end{array}$$

where $i(e) :=$ (the trivial loop at $p(e)$, e) is a homotopy equivalence and j is induced by i. This is the standard construction used to prove that any continuous map is homotopy equivalent to a fibration (see [43, Chapter I, § 7]). Now the idea of our proof is to show that j is a weak equivalence. Assume that this was proved. Then it follows by Whitehead's theorem [43, Chapter IV, Theorem 7.17] that j induces an isomorphism $[Y, F] \simeq [Y, W(p)]$ for each CW-complex Y. Let us consider the following diagram with commutative squares

$$\begin{tikzcd}
{[Y, \Omega^{k+1}B]} \arrow[r, dashed, "\partial"] \arrow[d, equal] & {[Y, \Omega^k F]} \arrow[r] \arrow[d, "j_*"] & {[Y, \Omega^k E]} \arrow[r] \arrow[d, "i_*"] & {[Y, \Omega^k B]} \arrow[r, dashed] \arrow[d, equal] & {[Y, \Omega^{k-1}F]} \arrow[d] \\
{[Y, \Omega^{k+1}B]} \arrow[r, "\partial'"] & {[Y, \Omega^k W(p)]} \arrow[r] & {[Y, \Omega^k \mathrm{Cocyl}(p)]} \arrow[r] & {[Y, \Omega^k B]} \arrow[r] & {[Y, \Omega^{k-1} W(p)]}.
\end{tikzcd}$$

The bottom sequence is exact since p' is a fibration [43, Chapter I, Theorem 6.11*]. Therefore if the boundary maps ∂ are defined such that $\partial = j_*^{-1} \circ \partial'$ then the upper sequence is also exact. Finally let us prove that $j: F \to W(p)$ is a weak homotopy equivalence. Since the homotopy exact sequence of a quasifibration is natural in the sense of category theory we can use the first diagram in the proof to obtain the following commutative diagram

$$\begin{tikzcd}
\pi_{k+1}(B) \arrow[r] \arrow[d, equal] & \pi_k(F) \arrow[r] \arrow[d, "j_*"] & \pi_k(E) \arrow[r] \arrow[d, "i_*"] & \pi_k(B) \arrow[r] \arrow[d, equal] & \pi_{k-1}(F) \arrow[d] \\
\pi_{k+1}(B) \arrow[r] & \pi_k(W(p)) \arrow[r] & \pi_k(\mathrm{Cocyl}(p)) \arrow[r] & \pi_k(B) \arrow[r] & \pi_{k-1}(W(p)).
\end{tikzcd}$$

Using the five lemma it follows that $j_*: \pi_k(F) \to \pi_k(W(p))$ is an isomorphism for each $k \geqslant 0$. ▨

3.1.7. Suppose that $p: E \to B$ is a fibration with fiber F. The boundary map ∂ in the sequence below

$$\begin{tikzcd}
{[Y, \Omega B]} \arrow[r, "\partial"] \arrow[d, equal] & {[Y, F]} \arrow[r] & {[Y, E]} \arrow[r] & {} \\
{[SY, B]} \arrow[ur] & & &
\end{tikzcd}$$

is defined using the homotopy covering property of p. More precisely given a map $\varphi: SY \to B$ we solve the homotopy lifting problem with initial data

$$\begin{tikzcd}
Y \vee I \arrow[rr, "f"] \arrow[d, hook] & & E \arrow[d, "p"] \\
Y \times I \arrow[r] \arrow[urr, "\psi"] & SY \arrow[r, "\varphi"] & B
\end{tikzcd}$$

($Y \vee I = \{y_0\} \times I \cup Y \times \{0\}$, $f =$ the map onto the based point of E, $q =$ the natural quotient map) obtaining a map $\psi: Y \times I \to E$. Since $\varphi \circ q$ maps $Y \times \{1\}$ to the base point of B, ψ maps $Y \times \{1\}$ into F. This map, denoted by $\psi_1: Y \to F$, is well defined up to homotopy. By definition one takes $\partial[\varphi] = [\psi_1]$.

When p is only a quasifibration, the definition of ∂ is more involved as we have seen in the proof of Proposition 3.1.6. However for the maps $\varphi \in \mathrm{Map}(SY, B)$ which can be covered as in the above diagram the formula $\partial[\varphi] = [\psi_1]$ still holds. Indeed with the notation of 3.1.5 and ψ as above we have the following commutative diagram

$$\begin{tikzcd}
Y \vee F \arrow[rrr] \arrow[d, hook] & & & \mathrm{Cocyl}(p) \arrow[dd, dashed, "p'"] \\
 & & E \arrow[ur, "i"] \arrow[dr] & \\
Y \times F \arrow[urr, "\psi"] \arrow[rr] & & SY \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

which implies that $\partial'[\varphi] = j_*[\psi_1]$. By the very definition of ∂ we have $\partial'[\varphi] = j_*\partial[\varphi]$. Therefore $\partial[\varphi] = [\psi_1]$ since j_* is injective. $\square$

Recall that for a C^*-algebra A the suspension of A is defined by $SA = C_0(S^1) \otimes \otimes A$. If $\varphi \in \mathrm{Hom}(A, B)$ is a $*$-homomorphism, then its suspension $S\varphi \in \mathrm{Hom}(SA, SB)$ is defined by $S\varphi = \mathrm{id}(C_0(S^1)) \otimes \varphi$. More generally, given $\varphi_i \in \mathrm{Hom}(A_i, B_i)$, $i = 1, 2$, one can consider $\varphi_1 \otimes \varphi_2 \in \mathrm{Hom}(A_1 \otimes A_2, B_1 \otimes B_2)$.

3.1.8. COROLLARY. *The suspension map $\varphi \to S\varphi$ induces an weak equivalence $F(X) \to \Omega F(SX)$ for every $X \in \mathrm{W}_0^c$ (see 3.1.2).*

Proof. By definition $F(X) = \mathrm{Hom}(C_0(X), \mathscr{K})$. The map $F(X) \to \Omega F(SX)$ corresponds to the suspension map

$$\mathrm{Hom}(C_0(X), \mathscr{K}) \to \mathrm{Hom}(C_0(SX), C_0(S^1) \otimes \mathscr{K})$$

via the following identifications

$$\Omega F(SX) = \mathrm{Map}(S^1, \mathrm{Hom}(C_0(SX), \mathscr{K})) = \mathrm{Hom}(C_0(SX), C_0(S^1) \otimes \mathscr{K}).$$

Let $Y \in \mathrm{W}_0$. We shall prove that the induced map

$$S_*: [Y, F(X)] \to [Y, \Omega F(SX)]$$

is a bijection.

To this end we consider the quasifibration $p: F(CX) \to F(SX)$ with fiber $F(X)$ arising from the pair $X \subset CX =$ the cone over X, by Theorem 3.1.5. By Proposition 3.1.6 there is an exact sequence (of groups and pointed sets)

$$[Y,\ \Omega F(CX)] \to [Y,\ \Omega F(SX)] \xrightarrow{\partial} [Y,\ F(X)] \to [X,\ F(CX)].$$

Since $F(CX)$ is contractible (see the last part of 3.1.2) we find that the boundary map ∂ is a bijection. In what follows we check that S_* is a right inverse for ∂ and this will complete the proof of our corollary. Let $\varphi \in \mathrm{Map}(Y,\ F(X))$ and let us think $S\varphi$ as a map $SY \to F(SX)$. In the same way

$$\psi = \mathrm{id}(C_0(I)) \otimes \varphi \in \mathrm{Hom}(C_0(X \wedge I),\ C_0(Y \wedge I) \otimes \mathscr{K})$$

is regarded as a map $CY \to F(CX)$. Here $I = [0, 1]$ is pointed by 0 and the smash products $X \wedge I$, $Y \wedge I$ are identified with CX and CY respectively. Let $X \times I \to CI$ and $X \times I \to SX$ be the natural quotient maps.

Let $Y \vee I \to F(CX)$ be the map onto the null morphism. It easily checked that the following diagram is commutative

$$\begin{tikzcd}
Y \vee I \arrow[rr] \arrow[dd, hook] & & F(CX) \arrow[dd, "p"] \\
& CY \arrow[ur, "\psi"] & \\
Y \times I \arrow[ur] \arrow[r] & SY \arrow[r, "S\varphi"] & F(SX).
\end{tikzcd}$$

Using 3.1.7 it follows that $\partial[S\varphi] = [\psi_1]$ where ψ_1 is the restriction of ψ to $Y \times \{1\} \subset CY$ and $\mathrm{image}(\psi_1) \subset F(X) = p^{-1}(0)$. Finally, note that $\psi_1 : Y \to F(X)$ can be identified with φ. This proves that $\partial[S\varphi] = [\varphi]$ and hence S_* is a right inverse for ∂. ▨

3.1.9. Remarks. a) If $X \in W_0^c$, $Y \in W_0$, then the suspension map induces an isomorphism of groups

$$\mathrm{kk}(Y, X) \xrightarrow{\sim} \mathrm{kk}(SY, SX).$$

Consequently, the groups $\mathrm{kk}_n(Y, X)$ can be defined as $\mathrm{kk}_n(Y, X) = \lim\limits_{r\to\infty} \mathrm{kk}(S^{r+n}Y, S^rX)$ and this definition allows us to work even with non connected spaces.

b) The group operation on $\mathrm{kk}(Y, X) = [Y, F(X)]$ given by the (infinite) loop structure of $F(X)$ coincides with that given by the orthogonal sum of homomorphisms.

c) The groups kk_n are contravariant in the first variable and covariant in the second variable. Indeed each $g \in \mathrm{Map}(Y_2,\ Y_1)$ induces a map $\mathrm{Map}(Y_1,\ F(X)) \to \mathrm{Map}(Y_2, F(X))$ and then, for each $n \in \mathbf{Z}$, a homomorphism of groups $g^* : \mathrm{kk}_n(Y_1, X) \to \mathrm{kk}_n(Y_2,\ X)$. Concerning the second variable, each $f \in \mathrm{Map}(X_1,\ X_2)$ induces a map $F(X_1) \to F(X_2)$ and then for each $n \in \mathbf{Z}$, a homomorphism of groups $f_* : \mathrm{kk}_n(Y,\ X_1) \to \mathrm{kk}_n(Y,\ X_2)$.

The next result provides us with long exact sequences for the kk-groups.

3.1.10. PROPOSITION. *Let $i: A \hookrightarrow X$ be a pair in W_0^c and let $j: B \hookrightarrow Y$ be a pair in W_0. Let $p: X \to X/A$ and $q: Y \to Y/B$ denote the canonical identification maps. There are long exact sequences*

a) $\mathrm{kk}_{n+1}(Y, X) \xrightarrow{p_*} \mathrm{kk}_{n+1}(Y, X/A) \to \mathrm{kk}_n(Y, A) \xrightarrow{i_*} \mathrm{kk}_n(Y, X) \xrightarrow{p_*} \mathrm{kk}_n(Y, X/A)$;

b) $\mathrm{kk}_n(B, X) \xrightarrow{j^*} \mathrm{kk}_n(Y, X) \xrightarrow{q^*} \mathrm{k}_n(Y/B, X) \to \mathrm{kk}_{n+1}(B, X) \xrightarrow{j^*} \mathrm{kk}_{n+1}(Y, X), n \in \mathbf{Z}$.

Proof. a) Let $n \geqslant 0$. The sequence

$$[Y, \Omega^{n+1}F(X)] \to [Y, \Omega^{n+1}F(X/A)] \to [Y, \Omega^n F(A)] \to [Y, \Omega^n F(X)] \to [Y, \Omega^n F(X/A)]$$

is exact by Theorem 3.1.5 and Proposition 3.1.6. Then by definition

$$[Y, \Omega^n F(X)] = [S^n Y, F(X)] = \mathrm{kk}_n(Y, X).$$

For $n \leqslant 0$ the same argument works using the isomorphism

$$\mathrm{kk}(Y, X) \simeq \mathrm{kk}(SY, SX).$$

b) Let $n \geqslant 0$. We consider the coexact Puppe sequence associated with the pair $j: B \hookrightarrow Y$ (see [43, Chapter I, Theorem 6.22]):

$$S^n B \to S^n Y \to S^n(Y/B) \to S^{n+1}B \to S^{n+1}Y.$$

Therefore there is an exact sequence

$$[S^n B, F(X)] \leftarrow [S^n Y, F(X)] \leftarrow [S^n(Y/B), F(X)] \leftarrow [S^{n+1}B, F(X)] \leftarrow [S^{n+1}Y, F(X)].$$

It follows from the definition of the kk-groups that this is just the required exact sequence. For $n \leqslant 0$ one uses again the suspension isomorphism. ▨

3.2. SOME GENERALIZED HOMOLOGY (COHOMOLOGY) THEORIES AND THEIR SPECTRA

In this subsection we look at $\mathrm{kk}_n(Y, X)$ in the spirit of [42].

3.2.1. PROPOSITION. a) *For any fixed space $Y \in W_0$, the correspondence*

$$X \to \mathrm{k}_n^Y(X) := \mathrm{kk}_n(Y, X), \quad n \in \mathbf{Z},$$

defines a generalized (reduced) homology theory on the category W_0^c (see 3.1.2).

b) *For any fixed space $X \in W_0^c$, the correspondence*

$$Y = k_X^n(Y) := kk_{-n}(Y, X), \quad n \in \mathbf{Z}$$

defines a generalized (reduced) cohomology theory on the category W_0 *(see 3.1.2).*

Proof. a) Recall that a generalized reduced homology theory on W_0 is a sequence of covariant functors

$$h_n : W_0 \to \{\text{abelian groups}\}$$

together with a sequence of natural transformations

$$\sigma_n : h_n \to h_{n+1} \circ S, \quad S = \text{suspension}$$

verifying the following conditions:

1) If $f_0, f_1 \in W_0$ are homotopic maps, then $h_n(f_0) = h_n(f_1)$;

2) If $X \in W_0$ then $\sigma_n(X) : h_n(X) \cong h_{n+1}(SX)$;

3) If $i : A \hookrightarrow X$ is a pair in W_0, and if $p : X \to X/A$ is the identification map, then the sequence

$$h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(p)} h_n(X/A)$$

is exact.

We have to verify the above conditions for the functors k_n^Y. Now 1) it is easily checked since any homotopy f_t between f_0 and f_1 induces a homotopy $F(f_t)$ between $F(f_0)$ and $F(f_1)$ (see 3.1.2). The natural transformations σ_n are induced by the natural weak equivalence $\Omega F(SX) \sim F(X)$ described in Corollary 3.1.8. The third condition follows from the first exact sequence in Proposition 3.1.10.

b) The proof is similar but one needs the second exact sequence in Proposition 3.1.10. ▨

For future purposes it is useful to find the spectra of these theories, which exist by the Brown-Adams representability theorem. By definition, a spectrum E is a sequence of spaces E_n with base point, provided with structure maps, either

$$g_n : SE_n \to E_{n+1}$$

or their adjoints

$$g_n' : E_n \to \Omega E_{n+1}.$$

A spectrum E is an Ω-spectrum if g_n' is a weak equivalence for each n. Each spectrum E defines a homology theory

$$h_n(X, E) = \varinjlim_r [S^{n+r}, E_r \wedge X]$$

and a cohomology theory

$$h^n(X, E) = \varinjlim_r [S^r X, E_{n+r}]$$

(see [42]). If E is an Ω-spectrum then $h^n(X, E) = [X, E_n]$ because of the equivalence g_n'.

We are interested in the following two spectra:

1) $E_n(Y) = \mathrm{Map}(Y, F(S^n)) = \mathrm{Hom}(C_0(S^n), C_0(Y) \otimes \mathscr{K}), \quad Y \in W_0$

2) $F_n(X) = F(S^n X) = \mathrm{Hom}(C_0(S^n X), \mathscr{K}), \quad X \in W_0^c$.

The structure maps for both these spectra are defined by taking suspensions of homomorphisms. It follows by Remark 3.1.8 b) that both $E_n(Y)$ and $F_n(X)$ are Ω-spectra.

3.2.2. PROPOSITION. a) *On the category* W_0^c *the generalized homology theory* $k_n^Y(\cdot)$ *is given by the* Ω*-spectrum* $E_n(Y)$.

b) *On the category* W_0 *the generalized cohomology theory* $k_X^n(\cdot)$ *is given by the* Ω*-spectrum* $F_n(X)$.

Proof. b) The assertion is immediate since by definition.

$$k_X^n(Y) = \mathrm{kk}_{-n}(Y, X) = [Y, F(S^n X)].$$

a) First we extend the homology theory $k_n^Y(\cdot)$ on the category W_0 by setting $k_n'^Y(X) = \mathrm{kk}(S^{n+1}Y, SX) = k_{n+1}^Y(SX)$. By Remark 3.1.8 we have $k_n'^Y(X) \simeq k_n^Y(X)$ whenever X is connected. Let $h_n(\cdot, E(Y))$ be the generalized homology defined on W_0 by the spectrum $(E_n(Y))$. We want to prove that $h_n(\cdot, E(Y))$ is isomorphic to $k_n'^Y(\cdot)$. This will follow once we define a natural transformation of homology theories

$$T: h_*(\cdot, E(Y)) \to k_*'^Y(\cdot)$$

which induces an isomorphism on coefficients, i.e.

$$T: h_*(S^0, E(Y)) \simeq k_*'^Y(S_0).$$

For $X \in W_0$, let $T(X)$ be induced by the maps

$$\begin{array}{ccc} t_r: \mathrm{Hom}(C_0(S^r), C_0(Y) \otimes \mathscr{K}) \wedge X & \to & \mathrm{Hom}(C_0(S^r X), C_0(Y) \otimes \mathscr{K}) \\ \wr & & \wr \\ E_r(Y) \wedge X & & \mathrm{Map}(Y, F(S^r X)) \end{array}$$

where $t_r(\varphi \wedge x) = \varphi \otimes \varphi_x$ and $\varphi_x \in \mathrm{Hom}(C_0(X), \mathrm{C})$ is the homomorphism of evaluation at $x \in X$. More precisely, we use the following commutative diagram

$$\begin{array}{ccc}
[\mathrm{S}^{n+r}, E_r(Y) \wedge X] & \xrightarrow{\mathrm{S}'} & [\mathrm{S}^{n+r+1}, E_{r+1}(Y) \wedge X] \\
\downarrow{\scriptstyle (t_r)_*} & & \downarrow{\scriptstyle (t_{r+1})_*} \\
[\mathrm{S}^{n+r}, \mathrm{Map}(Y, F(\mathrm{S}^r X))] & & [\mathrm{S}^{n+r+1}, \mathrm{Map}(Y, F(\mathrm{S}^{r+1} X))] \\
\downarrow{\wr} & & \downarrow{\wr} \\
\mathrm{kk}(\mathrm{S}^{n+r} Y, \mathrm{S}^r X) & \xrightarrow{\simeq} & \mathrm{kk}(\mathrm{S}^{n+r+1} Y, \mathrm{S}^{r+1} X)
\end{array}$$

to define $T(X) = \varinjlim (t_r)_* : h_*(X, E(Y)) \to \mathrm{k}'^{Y}_*(X)$. Here S′ is the composite of the suspension map and the structure map $\mathrm{S}^1 \wedge E_r(Y) \to E_{r+1}(Y)$. One can check the naturality of T and, moreover, it is clear that $T(\mathrm{S}^0)$ is an isomorphism since $E_r(Y) \wedge \wedge \mathrm{S}^0 = \mathrm{Map}(Y, F(\mathrm{S}^r))$. ▨

3.3. WHY kk CAN BE REGARDED AS THE CONNECTIVE KK-THEORY

Recall that given a spectrum $E = (E_n)$, the homotopy groups of E are defined by $\pi_r(E) = \varinjlim_k [\mathrm{S}^{r+k}, E_k]$ where the arrows $[\mathrm{S}^{r+k}, E_k] \to [\mathrm{S}^{r+k+1}, E_{k+1}]$ are defined via suspensions and the structure maps $g_k : \mathrm{S}E_k \to E_{k+1}$.

3.3.1. DEFINITION ([1]). If E is a spectrum then the associated *connective spectrum* is a spectrum E^c together with a map of spectra $E^c \to E$ such that $\pi_r(E^c) \to \pi_r(E)$ is an isomorphism for $r \geqslant 0$ and $\pi_r(E^c) = 0$ for $r < 0$. E^c is uniquely determined by E up to a weak equivalence. If $h_*(\cdot\,, E)$ is the homology theory defined by the spectrum E, then the homology theory $h_*(\cdot\,, E^c)$ defined by the connective spectrum E^c is called the connective $h_*(\cdot\,, E)$-theory. One has a similar definition for the corresponding cohomology theories. In particular these definitions work for the topoogical K-theory. The Ω-spectrum $\mathrm{B}U$ of the complex K-theory is given by he sequence

$$\Omega U,\ U,\ \Omega U,\ U, \ldots$$

where U is the infinite unitary group $U = U(\infty) = \varinjlim_n U(n)$.

Let $F = (F_n)$ be the Ω-spectrum $F_n := F(\mathrm{S}^n) = \mathrm{Hom}(C_0(\mathrm{S}^n), \mathscr{K})$ if $n \geqslant 1$ and $F_0 = \Omega F_1$, with the structure maps

$$F_n \to \Omega F_{n+1}$$

given by suspending homomorphisms (see Corollary 3.1.7). Note that F is a ring spectrum with multiplication $\mu : F_n \wedge F_m \to F_{n+m}$, $\mu(\varphi, \psi) = \varphi \otimes \psi$. Therefore

$\pi_*(F)$ has a ring structure. It is a result of G. Segal [39] that F is the connective spectrum BU^c. We include a proof of this and the computation of the ring $\pi_*(F)$.

3.3.2. PROPOSITION. *The ring $\pi_*(F)$ is isomorphic to the polynomial ring* $\mathbf{Z}[t]$ *with* $\deg(t) = 2$; *t corresponds to the generator of $\pi_3(U) \simeq \mathbf{Z}$ regarded as a homomorphism $S \in \mathrm{Hom}(C_0(S^1), C_0(S^3) \otimes \mathscr{K})$.*

Proof. Note that each $\varphi \in \mathrm{Hom}(C_0(S^1), \mathscr{K})$ is given by some unitary $u \in U(\mathscr{K})$ and $U(\mathscr{K}) \sim U$. Therefore $F_1 \simeq U$. Let $r \geqslant 0$. By definition $\pi_r(F) = \lim\limits_{n\to\infty}[S^{r+n}, F_n] = \lim\limits_{n\to\infty}[S^{r+1}, \Omega^{n-1}F_n]$. Since F is an Ω-spectrum we find

$$\pi_r(F) = [S^{r+1}, U] = \begin{cases} \mathbf{Z} & \text{for } r \text{ even} \\ 0 & \text{for } r \text{ odd.} \end{cases}$$

For $r < 0$, $\pi_r(F) = \lim\limits_{n\to\infty}[S^{r+n}, F_n] = [S^0, F_r] = 0$ since F_r is connected. Thus $\pi_*(F) \simeq \mathbf{Z}[t]$ as groups. In order to determine the ring structure on $\pi_*(F)$ we observe that for

$$[\varphi] \in \pi_p(F) = [(C_0(S^1), \; C_0(S^{p+1}) \otimes \mathscr{K}]$$

and

$$[\psi] \in \pi_q(F) = [C_0(S^1), \; C_0(S^{q+1}) \otimes \mathscr{K}],$$

the element $\mu_*([\varphi], [\psi]) \in \pi_{p+q}(F) = [C_0(S^2), C_0(S^{p+q+2}) \otimes \mathscr{K}]$ is represented by

$$C_0(S^1) \otimes C_0(S^1) \xrightarrow{\varphi\otimes 1} C_0(S^{p+1}) \otimes \mathscr{K} \otimes C_0(S^1) \xrightarrow{1\otimes\psi} C_0(S^{p+1}) \otimes \mathscr{K} \otimes C_0(S^{q+1}) \otimes \mathscr{K}.$$

Therefore using the fact that the K_*-functor induces a bijection

$$[C_0(S^m), \; C_0(S^n) \otimes \mathscr{K}] \to \mathrm{Hom}(\mathrm{K}_*(C_0(S^m)), \; \mathrm{K}_*(C_0(S^n)))$$

for $n \geqslant m \geqslant 0$, the product

$$\mu_{2p,2q} : \pi_{2p}(F) \times \pi_{2q}(F) \to \pi_{2(p+q)}(F)$$

can be identified with the composition of group homomorphisms

$$\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \times \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \to \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}).$$

3.3.3. PROPOSITION (G. Segal). *$F = (F_n)$ is the connective spectrum associated with the spectrum of complex* K*-theory.*

Proof. Use the morphism S from Proposition 3.3.2 to define the maps $F_{n+2} \to \to \Omega^3 F_{n+3} = F_n$ by the rule $\varphi \mapsto \varphi \otimes S$. The compositions:

$$F_{2n+2} \to F_{2n} \to \ldots \to F_0 = \Omega U$$

$$F_{2n+1} \to F_{2n-1} \to \ldots \to F_1 = U$$

give a map of spectra $F \to \mathrm{B}U$ *which by* 3.3.2 *induces isomorphisms* $\pi_r(F) \to \pi_r(\mathrm{B}U)$ *for all* $r \geqslant 0$.

3.3.4. Remark. The generalized homology theory $k_n^{S^0}(\cdot) = \mathrm{kk}_n(S^0, \cdot)$ defined in Proposition 3.2.1 has the spectrum $F = (F_n)$ as was proved in Proposition 3.3.2. Therefore $\mathrm{k}_n^{S^0}(\cdot)$ is isomorphic to the reduced connective K-homology on W_0^c which is usually denoted by $\mathrm{k}_n(\cdot)$. In the same way the cohomology theory $\mathrm{kk}_{-n+1}(\cdot, S^1)$ is isomorphic to the reduced connective K-theory $\mathrm{k}^n(\cdot)$ on W_0. The formulae

$$\mathrm{kk}_{n-1}(S^1, X) = k_n(X), \quad \mathrm{kk}_{-n+1}(Y, S^1) \to \mathrm{k}^n(Y)$$

are important since they show that both the connective K-homology and cohomology can be realized via the homotopy of $*$-homomorphisms.

There are well-known similar formulae which relate the Kasparov KK-functor to the usual K-theory. We restrict our attention to the commutative case. Therefore starting with $\mathrm{KK}(C_0(X), C_0(Y) \otimes \mathscr{K}) = \mathrm{KK}(C_0(X), C_0(Y))$ one has

$$\mathrm{KK}_{n-1}(C_0(X), C_0(S^1)) \simeq \widetilde{\mathrm{K}}_n(X) = \text{the reduced K-homology}$$

$$\mathrm{KK}_{-n+1}(C_0(S^1), C_0(Y)) \simeq \widetilde{\mathrm{K}}^n(Y) = \text{the reduced K-theory}$$

for arbitrary $X, Y \in \mathrm{W}_0$.

Now it becomes apparent that kk is in a certain sense the connective KK-theory. In order to give a more precise statement we need some preparations.

First of all it is clear that there is an obvious natural transformation $\chi\colon \mathrm{kk}_* \to \to \mathrm{KK}_*$ since even for (arbitrary separable) C^*-algebras A, B the canonical map $\mathrm{Hom}(A, B) \to \mathrm{KK}(A, B)$ factors through $[A, B]$. Therefore χ is defined by the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(C_0(X), C_0(Y) \otimes \mathscr{K}) \arrow[dl] \arrow[dr] & \\
\mathrm{kk}(Y, X) = [C_0(X), C_0(Y) \otimes \mathscr{K}] \arrow[rr, "\chi"] & & \mathrm{KK}(C_0(X), C_0(Y)).
\end{tikzcd}$$

We shall see (3.3.6) that χ is induced by some map at the level of spectra.

3.3.5. PROPOSITION. *On the category* W_0 *the generalized homology theory* $X \to KK_n(C_0(X), C(Y))$ *is isomorphic to the generalized homology theory* $X \to h_n(X, G(Y))$ *defined by the spectrum* $G_n(Y) = \mathrm{Map}(Y, \Omega^{n+1}U)$.

Proof. This result is contained in [26, § 6, Theorem 4] since $G_n(Y)$ is weak homotopy equivalent to the space $\mathrm{Map}(Y, \widetilde{\mathscr{F}}_n)$ defined there, but for our purposes we prefer the description with $G_n(Y)$.

Thus we define $\tilde{T}(X): h_n(X, G(Y)) \to KK_n(C_0(X), C_0(Y))$, as being induced by the maps:

$$[S^{n+r}, X \wedge G_n(Y)] \xrightarrow{(t_r)_*} [C_0(S^1X), C_0(S^{2n+r+1}Y) \otimes \mathscr{K}] \to KK(C_0(X), C_0(S^rY))$$

via the identifications $U \sim U(\mathscr{K})$ and

$$G_n(Y) = \mathrm{Map}(Y, \Omega^{n+1}U) \simeq \mathrm{Hom}(C_0(S^1), C_0(S^{n+1}Y) \otimes \mathscr{K}).$$

The map t_r above is the same as the map t_r defined in the proof of Proposition 3.2.2. Moreover the proof is carried out in the same manner. ☐

3.3.6. REMARK. a) There is a commutative diagram

$$\begin{array}{ccc} kk_n(Y, X) & \xleftarrow[\sim]{T} & h_n(X, E(Y)) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \xi_*} \\ KK(C_0(X), C_0(Y)) & \xleftarrow[\sim]{\tilde{T}} & h_n(X, G(Y)) \end{array}$$

where $h_n(X, E(Y))$ and T were defined in the proof of Proposition 3.2.2 and ξ is the map of spectra

$$\xi: \mathrm{Map}(Y, F_n) \to \mathrm{Map}(Y, \Omega^{n+1}U)$$

induced by the map $F_n \to BU_n \sim \Omega^{n+1}U$ given in the proof of Proposition 3.3.3.

b) Let X' be an n-dual of X, for example a compact deformation retract of the complement of X embedded in S^{n+1}. By the naturality of the duality result stated in [42, Corollary 7.10] we have a commutative diagram

$$\begin{array}{ccccc} h_0(X, E(Y)) & \simeq & h^n(X', E(Y)) & = & k^n(X' \wedge Y) \\ \downarrow & & & & \downarrow \\ h_0(X, G(Y)) & \simeq & h^n(X', G(Y)) & = & \tilde{K}^n(X' \wedge Y). \end{array}$$

3.3.7. COROLLARY. *Let $X \in W_0^c$, $Y \in W_0$ and X' an n-dual of X. There is a commutative diagram*

$$\begin{array}{ccc} kk(Y, X) & \xrightarrow{\sim} & k^n(X' \wedge Y) \\ \downarrow{\scriptstyle \chi} & & \downarrow \\ kk(C_0(X), C_0(Y)) & \xrightarrow{\sim} & \tilde{K}^n(X' \wedge Y) \end{array}$$

where the right vertical arrow is induced by the map of spectra $F \to BU$ and the orizontal arrows are isomorphisms induced by natural transformations.

Proof. The assertion follows from the above remarks. ▨

Since any two duals are of the same stable homotopy type the above diagram does not depend on the choice of X'. The commutativity of this diagram gives a precise meaning to the assertion that kk is the connective KK (when restricted to spaces).

For a comparison result between kk and KK we refer to Theorem 3.4.5.

3.4. THE RELATION OF CONNECTIVE K-THEORY TO HOMOLOGY

The map $F_{n+2} \to F_n$ given by tensoring with S induces operations $k_n(X) \xrightarrow{S_*} k_{n+2}(X)$ and $k^{n+2}(X) \xrightarrow{S^*} k^n(X)$. If $K(\mathbf{Z}) = (K(\mathbf{Z}, n))$ is the Eilenberg-MacLane spectrum, there is a natural transformation of spectra $\eta\colon F \to K(\mathbf{Z})$ defined by maps $\eta_n : F_n \to K(\mathbf{Z}, n)$. To describe η_n one will use the realization of $K(\mathbf{Z}, n)$ as the infinite symmetric of S^n, $K(\mathbf{Z}, n) \sim P^\infty(S^n)$, due to Dold and Thom [14]. The corresponding map $F_n \to P^\infty(S^n)$ is given in 6.1.9.

The following result is due to L. Smith [40].

3.4.1. PROPOSITION. *For any finite* CW-*complex X, there is a natural exact triangle*

$$\begin{array}{ccc} k_*(X) & \xrightarrow{S_*} & k_*(X) \\ {\scriptstyle \partial_*}\nwarrow & & \swarrow{\scriptstyle \eta_*} \\ & \tilde{H}_*(X, \mathbf{Z}) & \end{array}$$

with $\deg S_* = 2$, $\deg \eta_* = 0$ *and* $\deg \partial_* = -3$.

We need also the dual triangle.

3.4.2. PROPOSITION. *For any finite* CW-*complex X, there is a natural exact triangle*

$$\begin{array}{ccc} k^*(X) & \xrightarrow{S^*} & k^*(X) \\ {\scriptstyle \delta_*}\nwarrow & & \swarrow \\ & \tilde{H}^*(X, \mathbf{Z}) & \end{array}$$

with $\deg S^* = -2$, $\deg \eta^* = 0$ *and* $\deg \delta_* = 3$.

Proof. Let W_n be the mapping fibre of $F_n \to F_{n-2}$, i.e. $W_n = \{(u, e) \in F_{n-2}^I \times F_n : u(0) = 0,\ u(1) = \text{the image of } e \text{ under the map } F_n \to F_{n-2}\}$. Then the Puppe exact sequence ([43, Chapter I, Theorem 6.11*])

$$\Omega^{q+1}F_{n-2} \to \Omega^q W_n \to \Omega^q F_n \to \Omega^q F_{n-2}$$

together with Proposition 3.3.2 imply

$$\pi_q(W_n) = \begin{cases} \mathbf{Z} & \text{if } q = n - 3 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently W_n is a $K(\mathbf{Z}, n-3)$-space and we get an exact sequence

$$F_{n-3} \to K(\mathbf{Z}, n-3) \to F_n \to F_{n-2}.$$

Passing to homotopy classes $[X, \cdot]$ we get the exact sequence

$$\to k^{n-3}(X) \xrightarrow{\eta^*} \tilde{H}^{n-3}(X, \mathbf{Z}) \to k^n(X) \xrightarrow{S^*} k^{n-2}(X)$$

which proves the statement.

3.4.3. Corollary. *Assume that X is a finite connected CW-complex of dimension n. Then*

$$k_q(X) = \begin{cases} 0 & \text{if } q \leqslant 0 \\ \tilde{H}_q(X, \mathbf{Z}) & \text{if } q = 1, 2 \\ \tilde{K}_q(X) & \text{if } q \geqslant n - 1. \end{cases}$$

Complementary informations are contained in the exact sequence

$$0 \to k_{n-2}(X) \xrightarrow{S_*} k_n(X) \xrightarrow{\eta_*} \tilde{H}_n(X, \mathbf{Z}) \to k_{n-3}(X) \to \ldots \to k_1(X) \xrightarrow{S_*} k_3(X) \xrightarrow{\eta_*}$$

$$\xrightarrow{\eta_*} \tilde{H}_3(X, \mathbf{Z}) \to 0.$$

With the identifications $k_q(X) \simeq K_q(X)$ *for* $q \geqslant n - 1$ *the isomorphism* $k_q(X) \overset{S_*}{\simeq} k_{q+2}(X)$ *is the Bott periodicity.*

Proof. If $q \leqslant 0$ then $k_q(X) = \varinjlim_r [S^{q+r}, X \wedge F_r] = 0$ since F_r is $(r-1)$-connected and X is 0-connected and hence $X \wedge F_r$ is r-connected by [42, 3.16]. It follows from Proposition 3.4.1 that $0 = k_{-1}(X) \to k_1(X) \to \tilde{H}_1(X, \mathbf{Z}) \to k_{-2}(X) = 0$

and $0 = \mathrm{k}_0(X) \to \mathrm{k}_2(X) \to \widetilde{\mathrm{H}}_2(X, \mathbf{Z}) \to \mathrm{k}_{-1}(X) = 0$ are exact sequences so that $\mathrm{k}_q(X) \overset{\eta_*}{\rightleftarrows} \widetilde{\mathrm{H}}_q(X, \mathbf{Z})$ for $q = 1, 2$. Since $\widetilde{\mathrm{H}}_q(X, \mathbf{Z}) = 0$ for $q \geqslant n + 1$ it follows that $\mathrm{k}_q(X) \overset{S_*}{\rightleftarrows} \mathrm{k}_{q+2}(X)$ for $q \geqslant n - 1$. Now let X' be a N-dual of X and choose $2m > N - q$. Using [42, Corollary 7.10] and Proposition 3.4.2 we may write $\mathrm{k}_{q+2m}(X) = \mathrm{k}^{N-q-2m}(X') = \widetilde{\mathrm{K}}^{N-q-2m}(X') = \widetilde{\mathrm{K}}_{q+2m}(X) = \widetilde{\mathrm{K}}_q(X)$. Therefore $\mathrm{k}_q(X) \simeq \mathrm{K}_q(X)$ for $q \geqslant n - 1$. A more natural proof of this isomorphism will follow from Theorem 3.4.5.

3.4.4. Corollary. *Assume that X is a finite connected* CW-*complex of dimension n. Then*

$$\mathrm{k}^q(X) = \begin{cases} \widetilde{\mathrm{K}}^q(X) & \text{if } q \leqslant 2 \\ \widetilde{\mathrm{H}}^q(X) & \text{if } q = n - 1,\ n \\ 0 & \text{if } q \geqslant n + 1. \end{cases}$$

Complementary informations are stored in the exact sequence

$$0 \to \mathrm{k}^3(X) \xrightarrow{S^*} \mathrm{k}^1(X) \overset{\eta^*}{\rightleftarrows} \widetilde{\mathrm{H}}^1(X, \mathbf{Z}) \to \mathrm{k}^4(X) \to \dots \to \mathrm{k}^n(X) \xrightarrow{S^*} \mathrm{k}^{n-2}(X) \xrightarrow{\eta^*}$$
$$\xrightarrow{\eta^*} \widetilde{\mathrm{H}}^{n-2}(X, \mathbf{Z}) \to 0.$$

With the identifications $\mathrm{k}^q(X) \simeq \widetilde{\mathrm{K}}^q(X)$ for $q \leqslant 2$ the isomorphism $\mathrm{k}^q(X) \overset{S^*}{\rightleftarrows} \mathrm{k}^{q-2}(X)$ is the Bott periodicity.

Proof. If $q \geqslant n + 1$, then $\mathrm{k}^q(X) = [X, F_q] = 0$, since F_q is $(q - 1)$-connected. If $0 \leqslant q \leqslant 1$, then $\mathrm{k}^q(X) = [X, F_q] = \widetilde{\mathrm{K}}^q(X)$, since $F_0 = \Omega U$, $F_1 = U$. As $\widetilde{\mathrm{H}}^q(X, \mathbf{Z}) = 0$ for $q \leqslant 0$, it follows from Proposition 3.4.2 that $\mathrm{k}^2(X) \rightleftarrows \mathrm{k}^0(X) = \widetilde{\mathrm{K}}^0(X)$. If $q < 0$, then

$$\mathrm{k}^q(X) = \varinjlim_r [\mathrm{S}^r X, F_{q+r}] = [\mathrm{S}^{-q} X, F_0] = \widetilde{\mathrm{K}}^0(\mathrm{S}^{-q} X) \simeq \widetilde{\mathrm{K}}^q(X).$$

Finally let us explain why we have a split extension

$$0 \to \mathrm{k}^3(X) \xrightarrow{S^*} \mathrm{k}^1(X) \overset{\eta^*}{\rightleftarrows} \widetilde{\mathrm{H}}^1(X, \mathbf{Z}) \to 0.$$

If we take $\mathrm{K}(\mathbf{Z}, 1) \simeq \mathrm{S}^1$ and $F_1 \simeq U$ then the map $\eta\colon U \to \mathrm{S}^1$ is the determinant map and the inclusion $\mathrm{S}^1 = U(1) \hookrightarrow U$ gives a multiplicative left inverse for η. ▨

3.4.5. THEOREM. *Let X, Y be finite CW-complexes and suppose that X is 0-connected and Y is m-connected ($m \geqslant -1$; if Y is not connected then $m = -1$). Then the cannonical map*

$$\chi\colon \mathrm{kk}_n(Y, X) \to \mathrm{KK}_n(C_0(X), C_0(Y))$$

is an isomorphism for $n \geqslant \dim(X) - m - 2$.

Proof. We shall compare the two homology theories

$$\mathrm{k}_n^Y(X) = \mathrm{kk}_n(Y, X) \quad \text{and} \quad \mathrm{K}_n^Y(X) = \mathrm{KK}_n(C_0(X), C_0(Y)).$$

If $X = \mathrm{S}^q$ with $1 \leqslant q \leqslant n + m + 2$ then $\mathrm{k}_n^Y(\mathrm{S}^q) = \mathrm{kk}_n(Y, \mathrm{S}^q) = \mathrm{k}^{q-n}(Y)$ and $\mathrm{K}_n^Y(\mathrm{S}^q) = \widetilde{\mathrm{K}}^{q-n}(Y)$. Using $\widetilde{\mathrm{H}}^j(Y, \mathbf{Z}) = 0$ for $j \leqslant m$ we infer from the exact triangle in Proposition 3.4.2 that $\mathrm{k}^{j+2}(Y) \overset{S*}{\rightleftarrows} \mathrm{k}^j(Y)$ for $j \leqslant m$. Since, for negative j, $\mathrm{k}^j(Y) \simeq \simeq \widetilde{\mathrm{K}}^j(Y)$, it follows that $\mathrm{k}^{q-n}(Y) \rightleftarrows \widetilde{\mathrm{K}}^{q-n}(Y)$ for all $q \leqslant n + m + 2$ and moreover this isomorphism can be identified with χ. It is clear that the same assertion is true if we substitute S^q with a finite wedge of q-spheres. Having this, the proof is carried out by induction on the dimension of X. If $\dim(X) = 1$, then X is homotopic to a wedge of circles, hence the assertion is true. Therefore assume that χ is an isomorphism for each connected X of dimension $\leqslant p$ and $n \geqslant \dim(X) - m - 2$. Now let X be a finite connected CW-complex of dimension $p + 1$. If X_p is the p-dimensional skeleton of X, then $X/X_p =: \mathrm{S}_{p+1}$ is a finite wedge of $(p+1)$-spheres. The pair $X_p \hookrightarrow X$ induces the following commutative diagram

$$\begin{array}{ccccccccccc}
\to \mathrm{k}_{n+1}^Y(\mathrm{S}_{p+1}) & \to & \mathrm{k}_n^Y(X_p) & \to & \mathrm{k}_n^Y(X) & \to & \mathrm{k}_n^Y(\mathrm{S}_{p+1}) & \to & \mathrm{k}_{n-1}^Y(X_p) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\to \mathrm{K}_{n+1}^Y(\mathrm{S}_{p+1}) & \to & \mathrm{K}_n^Y(X_p) & \to & \mathrm{K}_n^Y(X) & \to & \mathrm{K}_n^Y(\mathrm{S}_{p+1}) & \to & \mathrm{K}_{n-1}^Y(X_p) & \to & \dots
\end{array}$$

The inductive assumptions together with the five lemma imply that $\mathrm{k}_n^Y(X) \to \to \mathrm{K}_n^Y(X)$ is an isomorphism for $n \geqslant p + 1 - m - 2$. ∎

3.4.6. COROLLARY. *Let X, $Y \in \mathrm{W}_0^c$ have dimension n and assume that both X and Y are $(n-2)$-connected. Then*

$$[C_0(X), C_0(Y) \otimes \mathscr{K}] \rightleftarrows \mathrm{KK}(C_0(X), C_0(Y)).$$

Concerning the next result the referent pointed out that it can be easily deduced using the Atiyah-Hirzebruch spectral sequence or directly from the Dold's theorem which asserts that after tensoring with $\mathbf{Q}$ any generalized cohomology theory h^* becomes an ordinary cohomology theory with coefficients $\mathrm{h}^*(\text{point}) \otimes \mathbf{Q}$. However, we prefer to give a proof which is less elegant but specifies better the involved maps and requires no other identifications.

3.4.7. PROPOSITION. *If X is connected, then after tensoring with* $\mathbf{Q}$, *the exact triangle* 3.4.2 *splits into short exact sequences*

$$0 \to \mathrm{k}^{q+2}(X) \otimes \mathbf{Q} \to \mathrm{k}^q(X) \otimes \mathbf{Q} \to \widetilde{\mathrm{H}}^q(X, \mathbf{Q}) \to 0.$$

Proof. Since $\mathbf{Q}$ is flat as a $\mathbf{Z}$-module the triangle 3.4.2 remains exact after tensoring with $\mathbf{Q}$. Therefore we have the exact sequences

$$0 \to \ker(S^q \otimes 1) \to \mathrm{k}^q(X) \otimes \mathbf{Q} \xrightarrow{S^q \otimes 1} \mathrm{k}^{q-2}(X) \otimes \mathbf{Q} \to \operatorname{coker}(S^q \otimes 1) \to 0$$

$$0 \to \operatorname{coker}(S^{q-1} \otimes 1) \to \widetilde{\mathrm{H}}^{q-3}(X, \mathbf{Q}) \to \ker(S^q \otimes 1) \to 0.$$

This shows that if $a_q = \dim_{\mathbf{Q}} \mathrm{k}^q(X) \otimes \mathbf{Q}$, $b_q = \dim_{\mathbf{Q}} \widetilde{\mathrm{H}}^q(X, \mathbf{Q})$, $d_q = \dim_{\mathbf{Q}} \ker(S^q \otimes 1)$, $c_q = \dim_{\mathbf{Q}} \operatorname{coker}(S^q \otimes 1)$, then

$$d_q - a_q + a_{q-2} - c_q = 0 \quad \text{and} \quad c_{q-1} - b_{q-3} + d_q = 0.$$

According to 3.4.4, $a_q = b_q = 0$ for $q \geqslant n + 1$, $n = \dim(X)$, and $d_q = 0$ for $q \leqslant 3$. Also recall that $\mathrm{k}^2(X) \simeq \widetilde{\mathrm{K}}^2(X)$.

Using the isomorphism $\widetilde{\mathrm{K}}^0(X) \otimes \mathbf{Q} \simeq \widetilde{\mathrm{H}}^{\mathrm{even}}(X, \mathbf{Q})$ given by the Chern character we get

$$a_2 = \sum_{q \geqslant 1} b_{2q} = \sum_{q \geqslant 1} (c_{2q+2} + d_{2q+3}).$$

On the other hand from $a_q - a_{q-2} = d_q - c_q$ we infer

$$-a_2 = \sum_{q \geqslant 2} (a_{2q} - a_{2q-2}) = \sum_{q \geqslant 2} (d_{2q} - c_{2q}).$$

Therefore we must have

$$\sum_{q \geqslant 1} (c_{2q+2} + d_{2q+3}) = \sum_{q \geqslant 2} (c_{2q} - d_{2q}).$$

This is possible only if $d_q = 0$ for all $q \geqslant 4$. Since we have already seen that $d_q = 0$ for $q \leqslant 3$, it results that $\operatorname{Ker}(S^q \otimes 1) = 0$ for all $q \in \mathbf{Z}$. ▨

The above proposition shows that the image of the map $\widetilde{\mathrm{H}}^*(X, \mathbf{Z}) \to \mathrm{k}^{*+3}(X)$ is always a torsion group. In the case when $\mathrm{H}^*(X, \mathbf{Z})$ is torsion free the $\mathrm{k}^*(\mathrm{S}^0) = \mathbf{Z}[t]$-module structure of $\mathrm{k}^*(X)$ is completely described by the following

3.4.8. COROLLARY. *Let X be a finite connected* CW-*complex such that* $\widetilde{\mathrm{H}}^*(X, \mathbf{Z})$ *is torsion free. Then for each* $q \in \mathbf{Z}$ *there is an isomorphism* $\mathrm{k}^q(X) \to \bigoplus_{j \geqslant 0} \widetilde{\mathrm{H}}^{q+2j}(X, \mathbf{Z})$

such that the following diagram commutes

$$
\begin{array}{ccc}
k^{q+2}(X) & \xrightarrow{\quad S \quad} & k^{q}(X) \\
\downarrow & & \downarrow \\
\bigoplus\limits_{j\geq 1} \tilde{H}^{q+2j}(X, \mathbf{Z}) & \hookrightarrow & \bigoplus\limits_{j\geq 0} \tilde{H}^{q+2j}(X, \mathbf{Z}).
\end{array}
$$

Proof. For a finitely generated group G let $\mathrm{T}G$ be its torsion part and $\mathrm{L}G$ its free part. In the exact sequence

$$0 \to \ker S^q \to \mathrm{Tk}^q(X) \oplus \mathrm{Lk}^q(X) \to \mathrm{Tk}^{q-2}(X) \oplus \mathrm{Lk}^{q-2}(X) \to \operatorname{coker}(S^q) \to 0$$

$\ker S^q$ is a torsion group and $\operatorname{coker} S^q$ is a free group since it is a subgroup in $\tilde{H}^{q-2}(X, \mathbf{Z})$. If we assume that $\mathrm{Tk}^q(X) = 0$ then, as one can easily check, $\ker S^q = 0$ and $\mathrm{Tk}^{q-2}(X) = 0$. Since $k^n(X) = \tilde{H}^n(X, \mathbf{Z})$ and $k^{n-1}(X) = \tilde{H}^{n-1}(X, \mathbf{Z})$, $n = \dim X$, we can use an inductive argument to prove that $\ker(S^q) = 0$ for all q. Therefore we get split extension (non natural splittings)

$$0 \to k^q(X) \xrightarrow{S} k^{q-2}(X) \rightleftarrows \tilde{H}^{q-2}(X, \mathbf{Z}) \to 0,$$

hence the statement of the corollary. ▨

3.5. PRODUCTS

There are two fundamental operations with homomorphisms: composition and tensorization. One can use them to define various multiplicative structures on kk_*. It turns out that on this way one can reobtain *all* the products and pairings which can be introduced using the ring-spectrum structure of (F_n). We shall not develop this subject here, thus we limit our discussion to what is required in order to obtain a special Universal Coefficient Theorem for kk_*.

3.5.1. The composition on the homomorphisms induces a product

$$\mathrm{kk}(Y, X) \times \mathrm{kk}(Z, Y) \to \mathrm{kk}(Z, X).$$

More precisely for $\varphi \in \mathrm{Hom}(C_0(X), C_0(Y) \otimes \mathscr{K})$ and $\psi \in \mathrm{Hom}(C_0(Y), C_0(Z) \otimes \mathscr{K})$ we define $[\varphi] \cdot [\psi] = [\psi \otimes \mathrm{id}(\mathscr{K}) \circ \varphi]$. This product is *bilinear* and *associative*. For the bilinearity we refer to Theorem 3.1 d) in [36]. Also the associativity is a general fact which essentially follows from the associativity of the composition. The next computations are included just in order to make things clear.

Let A_i, $1 \leq i \leq 3$ be C^*-algebras and $\varphi_i \in \mathrm{Hom}(A_i, A_{i+1} \otimes \mathscr{K})$, $1 \leq i \leq 3$. Let 1 denote the identity morphism of $\mathscr{K}$. One has to check that

$$\varphi_3 \otimes 1 \otimes 1 \circ (\varphi_2 \otimes 1 \circ \varphi_1) = (\varphi_3 \otimes 1 \circ \varphi_2) \otimes 1 \circ \varphi_1,$$

but this follows from the equality

$$(\varphi_3 \otimes 1 \otimes 1) \circ (\varphi_2 \otimes 1) = (\varphi_3 \otimes 1 \circ \varphi_2) \otimes 1$$

which is straightforward.

3.5.2. Using the natural isomorphism $\mathrm{kk}(SY, SX) \simeq \mathrm{kk}(Y, X)$ we can extend the above product to a well defined product

$$\mathrm{kk}_n(Y, X) \times \mathrm{kk}_m(Z, Y) \to \mathrm{kk}_{n+m}(Z, X).$$

In particular we get the following facts:

a) $\mathrm{k}^*(X) = \bigoplus_{q \in \mathbf{Z}} \mathrm{k}^q(X)$ is endowed with a structure of left $\mathrm{k}^*(\mathrm{S}^0)$-module which coincides with that given by the operation $S^* : \mathrm{k}^{q+2}(X) \to \mathrm{k}^q(X)$.

b) The product $\mathrm{k}^*(X) \times \mathrm{kk}_*(Y, X) \to \mathrm{k}^*(Y)$ induces morphisms $\alpha_r : \mathrm{kk}_{-r}(Y, X) \to \to \mathrm{Hom}^r(\mathrm{k}^*(X), \mathrm{k}^*(Y))$ which actually map into $\mathrm{Hom}^r_{\mathrm{k}^*(\mathrm{S}^0)}(\mathrm{k}^*(X), \mathrm{k}^*(Y))$ since the product is associative. Here we use the notation $\mathrm{Hom}^r_{\mathrm{k}^*(\mathrm{S}^0)}(\mathrm{k}^*(X), \mathrm{k}^*(Y))$ for the group $\mathrm{k}^*(\mathrm{S}^0)$-morphisms $\varphi : \mathrm{k}^*(X) \to \mathrm{k}^*(Y)$ of degree r (i.e. $\varphi(\mathrm{k}^q(X)) \subset \mathrm{k}^{q+r}(Y)$ for all $q \in \mathbf{Z}$).

We want to prove that α_r is an isomorphism provided that $\mathrm{k}^*(X)$ is a free $\mathrm{k}^*(\mathrm{S}^0)$-module. The following discussion on products will enable us to use Adam's universal coefficient theorem [1].

3.5.3. Recall that (F_n) is a ring-spectrum with multiplication $\mu : F_n \wedge F_m \to F_{n+m}$ given by $\mu(\varphi, \psi) = \varphi \otimes \psi$. This allows one to construct four basic external products: an external product in k_*, an external product in k^*, and two slant products. Among them we are especially interested in the slant product

$$/ : \mathrm{k}^p(Y \wedge X) \times \mathrm{k}_q(Y) \to \mathrm{k}^{p-q}(X).$$

Let us recall the classical algebraic-topology definition of the slant product. For $a \in \mathrm{k}^p(Y \wedge X)$, $b \in \mathrm{k}_q(Y)$ represented by $f : Y \wedge X \to F_p$, $g : \mathrm{S}^{q+r} \to F_r \wedge Y$, respectively, a/b is the element represented by

$$\mathrm{S}^{q+r} \wedge X \xrightarrow{g \wedge 1_X} F_r \wedge Y \wedge X \xrightarrow{1_{F_r} \wedge f} F_r \wedge F_p \xrightarrow{\mu} F_{n+p}.$$

Our aim is to realize this slant product in terms of tensor products and composition of $*$-homomorphisms. Namely, tensoring to the right with $\mathrm{id}(C_0(X))$ gives a map

$$i_X : \mathrm{k}_q(Y) \to \mathrm{kk}_q(Y, Y \wedge X)$$

and using the product

$$k^p(Y \wedge X) \times kk_q(X, Y \wedge X) \to k^{p-q}(X)$$

(which is a special case of 3.5.2) we define

$$// : k^p(Y \wedge X) \times k_q(Y) \to k^{p-q}(X)$$

by the rule $a//b = a \cdot i_X(b)$.

The equivalence of the two products / and // is explained below.

We have two realizations of k_*:

i) $k_q(Y) = \varinjlim [S^{q+r}, F(S^r) \wedge Y]$ (via the spectrum $F_n = F(S^n)$)

and

ii) $k_q(Y) = \varinjlim [S^{q+r}, F(S^r \wedge Y)]$ (via the kk-groups).

We have seen (3.2.2 a)) that there is a natural isomorphism T between the two theories which is induced by the maps

$$t_r : F(S^r) \wedge Y \to F(S^r \wedge Y)$$

given by

$$t_r(\varphi \wedge y) = \varphi \otimes \varphi_y$$

where $\varphi_y \in \mathrm{Hom}(C_0(Y), \mathbf{C})$ is the evaluation map at y.

There is a similar situation concerning k^* but this time the isomorphism T is induced by the identification

$$\mathrm{Map}(Y, \mathrm{Hom}(C_0(X), \mathscr{K})) = \mathrm{Hom}(C_0(X), C_0(Y) \otimes \mathscr{K}) \qquad \text{(see 3.2.2 b))}$$

Now the proper statement about slant product is

$$T(a/b) = T(a)//T(b).$$

This equality follows from the identity

$$((t_r \circ g) \otimes \mathrm{id}(C_0(X)) \otimes \mathrm{id}(\mathscr{K})) \circ (\mathrm{id}(C_0(S^r)) \otimes f)(s \wedge x) = \varphi \otimes f(y \wedge x)$$

where $s \in S^{q+r}$, $x \in X$, $g(s) = \varphi \wedge y$.

It is worth noting that all the other products defined in terms of spectra admit similar realizations using the product 3.5.2.

With this preparation we are able to prove the following special Universal Coefficient Theorem.

3.5.4. THEOREM. *Let* $X \in \mathrm{W}_0^c$, $Y \in \mathrm{W}_0$ *and assume that* $\mathrm{H}^*(X, \mathbf{Z})$ *is torsion free. Then the map*

$$\alpha_r : \mathrm{kk}_{-r}(Y, X) \to \mathrm{Hom}^r_{\mathrm{k}^*(\mathrm{S}^0)}(\mathrm{k}^*(X), \mathrm{k}^*(Y)), \quad r \in \mathbf{Z}$$

is an isomorphism of groups.

Proof. We have seen that $\mathrm{k}^*_X(\cdot) = \mathrm{kk}_{-*}(\cdot, X)$ is a cohomology theory (see 3.2.1). The same is true for

$$\mathrm{h}^*_X(\cdot) := \mathrm{Hom}^*_{\mathrm{k}^*(\mathrm{S}^0)}(\mathrm{k}^*(X), \mathrm{k}^*(\cdot))$$

since by Corollary 3.4.8, $\mathrm{k}^*(X)$ is a free $\mathrm{k}^*(\mathrm{S}^0)$-module.

Since $\alpha_* : \mathrm{k}^*_X(\cdot) \to \mathrm{h}^*_X(\cdot)$ is a natural transformation of cohomology theories all we need is to prove that it induces an isomorphism on coefficients, i.e.

$$\alpha^0_* : \mathrm{k}_{-*}(X) \cong \mathrm{Hom}^*_{\mathrm{k}^*(\mathrm{S}^0)}(\mathrm{k}^*(X), \mathrm{k}^*(\mathrm{S}^0)).$$

By the very definition of α^0_* we have that

$$\alpha^0_*(b)(a) = a//b.$$

Here $//$ is a special case of $//$ in 3.5.3:

$$// : \mathrm{k}^p(X) \times \mathrm{k}_q(X) \to \mathrm{k}^{p-q}(\mathrm{S}^0).$$

According to the discussion in 3.5.3 this pairing can be identified with / which, in this special case, is just the Kronecker pairing for connective K-theory. With this identification α^0_* is an isomorphism by the Adam's universal coefficient Theorem ([1]).

3.5.5. THEOREM. *Let* X, Y *be finite connected* CW-*complexes without torsion in cohomology. There is an isomorphism* α *of* $\mathrm{kk}(Y, X) = [C_0(X), C_0(Y) \otimes \mathscr{K}]$ *into* $\mathrm{Hom}_{\mathbf{Z}}(\tilde{\mathrm{H}}^*(X, \mathbf{Z}), \tilde{\mathrm{H}}^*(Y, \mathbf{Z}))$. *The image of* α *consists of all group homomorphisms which preserve both the graduation even-odd of cohomology and the filtration* $F_m\tilde{\mathrm{H}}^* = \bigoplus_{q \geq m} \tilde{\mathrm{H}}^q$.

Proof. The theorem follows from Theorem 3.5.4 and Corollary 3.4.8.

4. HOMOTOPY COMPUTATIONS FOR LARGE HOMOMORPHISMS

In Section 3 we gave some methods for computing

$$[C_0(X), C_0(Y) \otimes \mathscr{K}] = \mathrm{kk}(Y, X).$$

The stability results of Section 6 will imply that the natural map

$$[C_0(X),\ C_0(Y) \otimes M_m] \to [C_0(X),\ C_0(Y) \otimes \mathcal{K}]$$

is a bijection, provided that m is large enough. Consequently in such cases we have

$$[C(X),\ C(Y) \otimes M_m]_1 = [C_0(X),\ C_0(Y) \otimes M_m] = \mathrm{kk}(Y, X)$$

so that one can make complete computations in many concrete situations.

This section is devoted to the more general problem of classifying up to homotopy the morphisms $A \to D$, where $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ and $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$ are C^*-algebras belonging to the category $\mathscr{C}(n)$ defined in Section 2. As explained below it actually suffices to compute

$$[A,\ C(Y) \otimes M_m] = [Y,\ \mathrm{Hom}(A,\ M_m)].$$

Briefly, our plan is as follows. First, we decompose $\mathrm{Hom}(A, M_m)$ into its connected components $B_m(\underline{k})$ parametrized by certain $(q + 1)$-uples integers $\underline{k} = (k_0, \ldots, k_q)$ Each component is the base space of a certain principal fiber bundle

$$\prod_{i=0}^{q} U(k_i) \to \prod_{i=1}^{q} \mathrm{Hom}_1(C(X_i), M_{k_i}) \times U(m) \to B_m(\underline{k})$$

so that the associated Puppe sequence will give some information on $[Y, B_m(\underline{k})]$. In order to reach a group structure on $[Y, B_m(\underline{k})]$ we embed the above bundle into a bundle of H-spaces. Using certain stability results for vector bundles and homomorphisms it is shown that this procedure does not affect the homotopy in small dimensions. This technique enables us to obtain complete algebraic invariants (ranging in kk and K-groups) for the homotopy classes of those homomorphisms from A to D which are $3(n + 3)/2$-large in the sense of 2.1.8 (see 4.2.8 and 4.2.1).

4.1. SOME FIBERINGS

4.1.1. Let $D_j = C(Y_j) \otimes M_{m_j}$. Since $\mathrm{Hom}(A, D)$ is the disjoint union of the $\mathrm{Hom}(A, D_j)$'s for $1 \leqslant j \leqslant h$ it suffices to consider the case when $D = C(Y) \otimes M_m$.

Let $B_m = \mathrm{Hom}(A, M_m) = \mathrm{Hom}\left(\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}, M_m\right)$ with X_i and n_i fixed throughout this section. Using the notation of Section 2, each $\varphi \in B_m$ induces a homomorphism of scaled ordered groups

$$r(\varphi) \in \mathrm{Hom}\left(\mathrm{K}_0\left(\bigoplus_{i=1}^{q} M_{n_i}\right), \ \mathrm{K}_0(M_m)\right)_{+,\Sigma} \subset \mathrm{Hom}(\mathrm{Z}^q, \mathrm{Z})$$

given by some integer vector $(k_1, \ldots, k_q)$. In fact k_i is the multiplicity of the embedding $(\varphi|M_{n_i})\colon M_{n_i} \to M_m$ hence $\sum_{i=1}^{q} k_i n_i \leqslant m$. If $k_0 = m - \sum_{i=1}^{q} k_i n_i$, then φ is unital if and only if $k_0 = 0$. Let $\underline{k} = (k_0\,; k_1, \ldots, k_q)$ and let $B_m(\underline{k})$ be the set of those φ in B_m with $r(\varphi) = (k_1, \ldots, k_q)$. $B_m(\underline{k})$ is a closed and open subspace of B_m and, as we shall see in 4.1.2, the $B_m(\underline{k})$ are exactly the connected components of B_m. Assume that $k_j = 0$ for some $j \geqslant 1$. Then $B_m(k) \subset \mathrm{Hom}(\bigoplus_{i \neq j} C(X_i) \otimes M_{n_i}, M_m)$. This fact allows us to make all the computations under the assumption that $k_j > 0$ for $1 \leqslant j \leqslant q$. We need some more notation and definitions. Let

$$U(\underline{k}) = U(k_0) \times U(k_1) \times \ldots \times U(k_q); \quad \text{for } k_0 = 0,\ U(k_0) = \text{a point.}$$

$$E(\underline{k}) = \mathrm{Hom}_1(C(X_1), M_{k_1}) \times \ldots \times \mathrm{Hom}_1(C(X_q), M_{k_q})$$

$$E_m(\underline{k}) = E(\underline{k}) \times U(m)$$

$$j^0 : U(\underline{k}) \to U(m), \quad \underline{w} = (w_0, w_1, \ldots, w_q) \in U(\underline{k})$$

$$j^0(\underline{w}) = w_0 \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})$$

where the above description of j^0 is given according to the unital embedding

$$M_{k_0} \oplus (M_{k_1} \otimes M_{n_1}) \oplus \ldots \oplus (M_{k_q} \otimes M_{n_q}) \hookrightarrow M_m, \quad m = k_0 + \sum_{i=1}^{q} k_i n_i .$$

Let $(\underline{\varphi}, u)$ denote a generic element of $E_m(\underline{k})$, that is $\underline{\varphi} = (\varphi_1, \ldots, \varphi_q)$ with $\varphi_i \in \mathrm{Hom}_1(C(X_i), M_{k_i})$ and $u \in U(m)$. We have a right continuous action of $U(\underline{k})$ on $E_m(\underline{k})$ given by:

$$(\underline{\varphi}, u)\underline{w} = (w_1^* \varphi_1 w_1, \ldots, w_q^* \varphi_q w_q\,;\, u j^0(\underline{w})).$$

We also define $p : E_m(\underline{k}) \to B_m(\underline{k})$ by

$$p(\underline{\varphi}, u) = u(0_{k_0} \oplus \varphi_1 \otimes \mathrm{id}(M_{n_1}) \oplus \ldots \oplus \varphi_q \otimes \mathrm{id}(M_{n_q}))u^* .$$

The next lemma describes the homomorphisms belonging to $B_m(\underline{k})$ in simpler terms.

4.1.2. LEMMA. a) *p is onto, hence $B_m(\underline{k})$ is connected;*

b) *$p(\underline{\varphi}, u) = p(\underline{\psi}, v)$ if and only if $(\underline{\varphi}, u)\underline{w} = (\underline{\psi}, v)$ for some $\underline{w} \in U(\underline{k})$.*

Proof. a) If $\alpha \in B_m(\underline{k})$ then there is some $u \in U(m)$ such that $\alpha\left(\bigoplus_{i=1}^{q} 1_{C(X_i)} \otimes a_i\right) = u(0_{k_0} \oplus (1_{k_1} \otimes a_1) \oplus \ldots \oplus (1_{k_r} \otimes a_r))u^*$ for all $a_i \in M_{n_i}$. Consequently if $\alpha' = u^*\alpha u$ then the algebra $\alpha'\left(\bigoplus_{i=1}^{q} C(X_i) \otimes 1_{n_i}\right)$ lies in the commutant of $0_{k_0} \oplus \left(\bigoplus_{i=1}^{q} 1_{k_i} \otimes M_{n_i}\right)$ in M_m. Since this commutant is equal to $C = M_{k_0} \oplus \left(\bigoplus_{i=1}^{q} M_{k_i} \otimes 1_{n_i}\right)$ it follows that there are $\varphi_i \in \mathrm{Hom}_1(C(X_i), M_{k_i})$, $1 \leqslant i \leqslant q$, such that $\alpha' = p(\underline{\varphi}, 1)$, hence $\alpha = p(\underline{\varphi}, u)$.

b) Let $\underline{a} = \bigoplus_{i=1}^{q} 1_{C(X_i)} \otimes a_i \in A$, $a_i \in M_{n_i}$. From $p(\underline{\varphi}, u)(\underline{a}) = p(\underline{\psi}, v)(\underline{a})$ we infer as above that u^*v is a unitary element in C so that $u^*v = j^0(\underline{w})$ for some $\underline{w} \in U(\underline{k})$. Therefore we have $p(\underline{\psi}, v) = p(\underline{\psi}, uj^0(\underline{w})) = p(\underline{\varphi}, u)$. The last equality implies $p(\underline{\psi}, j^0(\underline{w})) = p(\underline{\varphi}, 1)$ which means $w_i\psi_i w_i^* = \varphi_i$ for $1 \leqslant i \leqslant q$.

4.1.3. PROPOSITION. *The map $p: E_m(\underline{k}) \to B_m(\underline{k})$ is a principal fiber bundle with fibre $U(\underline{k})$.*

Proof. Since $U(\underline{k})$ is a Lie group freely acting on the compact space $E_m(\underline{k})$ it follows by a theorem of Gleason [22] that the quotient map onto the orbit space

$$E_m(\underline{k}) \to E_m(\underline{k})/U(\underline{k})$$

is a principal fibre bundle with fibre $U(\underline{k})$. By Lemma 4.1.2, p can be identified with ithis map. ∎

4.1.4. In order to discuss the unitary equivalence relation for homomorphisms t is useful to consider the continuous left action of $U(m)$ on $E_m(\underline{k})$ given by

$$v \cdot (\underline{\varphi}, u) = (\underline{\varphi}, vu), \quad \underline{\varphi} \in E(\underline{k}),\ u, v \in U(m).$$

Since this action commutes with the right action of $U(\underline{k})$ on $E_m(\underline{k})$, we get an action of $U(m)$ on $B_m(\underline{k})$ which is easily identified to

$$U(m) \times B_m(\underline{k}) \to B_m(\underline{k}), \quad (v, \psi) \mapsto v\psi v^*$$

since $p(v(\underline{\varphi}, u)) = vp(\underline{\varphi}, u)v^*$ for all $\underline{\varphi} \in E(\underline{k})$, $u, v \in U(m)$. In the case when each X_i is a point, the action of $U(m)$ on $B_m(\underline{k})$ corresponds to the action of $U(m)$ on the homogeneous space $U(m)/U(\underline{k})$.

4.2. EXACT SEQUENCES

4.2.1. Given a fibration $F \to E \to B$ the Puppe sequence

$$\to [SY, B] \to [Y, F] \to [Y, E] \to [Y, B]$$

is not always sufficient for the computation of $[Y, B]$ (without some additional structure). Letting aside the algebraic structure this is essentially due to the fact we do not know how to extend the above sequence to the right. However this can be done under certain favorable circumstances. For instance if $K \subset G$ are Lie groups, then the fibration $K \to G \to G/K$ can be extended to a homotopy exact sequence

$$K \xhookrightarrow{j} G \xrightarrow{\pi} G/K \xrightarrow{\varepsilon} \mathrm{B}K \xrightarrow{j'} \mathrm{B}G$$

where $\mathrm{B}K$, $\mathrm{B}G$ are the classifying spaces for the principal K, G-bundles, j' is the map naturally induced by the inclusion $j\colon K \hookrightarrow G$ and ε is a classifying map for the K-bundle $G \to G/K$. In the more general case when K acts freely on a compact space E we have to confine ourselves to a shorter exact sequence

$$K \xrightarrow{j} E \xrightarrow{\pi} E/K \xrightarrow{\varepsilon} \mathrm{B}K$$

where K is embedded in E as an orbit and ε is again a classifying map for the principal K-bundle $E \to E/K$. If these spaces happen to be H-groups, then passing to homotopy classes we get exact sequences of groups rather than pointed sets.

4.2.2. The case we deal with bears some resemblance with the above situation but is more involved since we need to embed our fibration into a fibration of H--spaces. First we complete the fibration given in Proposition 4.1.3 to an exact sequence

$$U(\underline{k}) \xrightarrow{j} E_m(\underline{k}) \xrightarrow{p} B_m(\underline{k}) \xrightarrow{\varepsilon} \mathrm{B}U(k).$$

All these spaces are pointed and the maps preserve the base points. If x_i^0 is the base point of X_i, then the homomorphism $\varphi_i^0(f) = f(x_i^0)\cdot 1_{k_i}$, $f \in C(X_i)$, is the base point of $\mathrm{Hom}_1(C(X_i), M_{k_i})$. Accordingly we distinguish $\varphi^0 = (\varphi_1^0, \dots, \varphi_q^0)$ in $E(\underline{k})$, $e^0 = (\varphi^0, 1)$ in $E_m(\underline{k})$ and $b^0 = p(e^0)$ in $B_m(\underline{k})$. The group $U(\underline{k})$ is pointed by its unit 1 so that we have a corresponding base point in $\mathrm{B}U(\underline{k})$.

Let $U(\underline{k}) \xrightarrow{j^0} U(m) \xrightarrow{p^0} B_m^0(\underline{k})$ be the fibration 4.1.3 in the special case when each space X_i reduces to a point. We define

$$i : U(m) \to E_m(\underline{k}) = E(\underline{k}) \times U(m), \quad i(u) = (\underline{\varphi}^0, u)$$

and

$$\rho\colon E_m(\underline{k}) \to U(m), \quad \rho(\underline{\varphi}, u) = u.$$

Note that both i and ρ are $U(\underline{k})$-equivariant since $\underline{w}^*\varphi^0\underline{w} = \varphi^0$ for any $\underline{w} \in U(\underline{k})$. Therefore there are natural maps i' and ρ' induced by i and ρ such that the following diagram is commutative:

$$\begin{tikzcd}
U(\underline{k}) \arrow[r, equal] \arrow[d, "j^0"'] & U(\underline{k}) \arrow[r, equal] \arrow[d, "j"'] & U(\underline{k}) \arrow[d, "j^0"'] \\
U(m) \arrow[r, "i"] \arrow[d, "p^0"'] & E_m(\underline{k}) \arrow[r, "\rho"] \arrow[d, "p"'] & U(m) \arrow[d, "p^0"'] \\
B_m^0(\underline{k}) \arrow[r, "i'"] & B_m(\underline{k}) \arrow[r, "\rho'"] & B_m^0(\underline{k}).
\end{tikzcd}$$

Using the naturality of the augumented Puppe sequence we derive the following diagram

$$\begin{tikzcd}
B_m^0(\underline{k}) \arrow[r, "i'"] \arrow[d, "\varepsilon^0"'] & B_m(\underline{k}) \arrow[r, "\rho'"] \arrow[d, "\varepsilon"] & B_m^0(\underline{k}) \arrow[d, "\varepsilon^0"] \\
\mathrm{B}U(\underline{k}) \arrow[r, equal] & \mathrm{B}U(\underline{k}) \arrow[r, equal] & \mathrm{B}U(\underline{k})
\end{tikzcd}$$

which commutes within homotopy. Therefore

$$\varepsilon \circ i' \sim \varepsilon^0 \quad \text{and} \quad \varepsilon^0 \circ \rho' \sim \varepsilon.$$

These factorizations allows us to prove the following

4.2.3. PROPOSITION. *There is an exact sequence*

$$U(\underline{k}) \xrightarrow{j} E_m(\underline{k}) \xrightarrow{p} B_m(\underline{k}) \xrightarrow{\varepsilon} \mathrm{B}U(\underline{k}) \xrightarrow{j'} \mathrm{B}U(m)$$

where j' is naturally induced by $j^0 : U(\underline{k}) \to U(m)$.

Proof. Given a space Y we have to check the exactness of the following sequence $[Y, B_m(\underline{k})] \xrightarrow{\varepsilon_*} [Y, \mathrm{B}U(\underline{k})] \xrightarrow{j'_*} [Y, \mathrm{B}U(m)]$ of pointed sets. Of course we shall use the exact sequence

$$[Y, B_m^0(\underline{k})] \xrightarrow{\varepsilon^0_*} [Y, \mathrm{B}U(\underline{k})] \xrightarrow{j'_*} [Y, \mathrm{B}U(m)].$$

First observe that $j'_* \circ \varepsilon_* = j'_* \circ (\varepsilon^0_* \circ \rho'_*) = 0$ since $j'_* \circ \varepsilon^0_* = 0$.

Now if $g \in \mathrm{Map}(Y, BU(\underline{k}))$ is such that $j' \circ g$ is null homotopic, then there exists $f \in \mathrm{Map}(Y, B_m^0(\underline{k}))$ such that $\varepsilon^0 \circ f$ is homotopic to g. Therefore $h = i' \circ f \in \mathrm{Map}(Y, B_m(\underline{k}))$ is such that $\varepsilon \circ h = (\varepsilon \circ i') \circ f \sim \varepsilon \circ f \sim g$.

4.2.4. We have reached the sequence 4.2.3 but it is not entirely satisfactory since it gives us only exact sequences of pointed sets. However after we pass to inductive limits in 4.2.3, natural group structures will be available. In what follows we shall describe this construction.

For any positive integer t let $t\underline{k} = (tk_0; tk_1, \ldots, tk_q)$ and let $j_t^0 : U(t\underline{k}) \to U(tm)$ be the corresponding embedding defined in 4.1.1, i.e. $j_t^0(w_0, w_1, \ldots, w_q) = w_0 \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})$.

For future purposes we need to describe j_t^0 using systems of matrix units. Thus if $(e^i_{x,y})$ and $(e_{a,b})$ are the usual systems of matrix units of M_{tk_i} and M_{tm} then

$$j_t^0(\underline{w}) = \sum_{i=0}^{q} \sum_{x,y=1}^{k_i} \sum_{r=1}^{n_i} w_i^{x,y} e_{h(i-1)+rt_ik_i+x,\, h(i-1)+rt_ik_i+y}$$

where: $w_i^{x,y}$ are the components of w_i, i.e. $w_i = \sum_{x,y} w_i^{x,y} e^i_{x,y}$, $\underline{w} = (w_0, w_1, \ldots, w_q)$,

$h(i) = \sum_{j=0}^{i} n_j k_j$ if $i \geqslant 0$ ($n_0 := 1$ if $k_0 \neq 0$, $n_0 := 0$ if $k_0 = 0$), $h(-1) = 0$.

Now we are going to define the bonding maps needed for inductive limits. The first is the canonical embedding $\alpha_t : U(t\underline{k}) \to U(s\underline{k})$,

$$\alpha_t(w_0, w_1, \ldots, w_q) = (w_0 \oplus 1_{k_0}, w_1, \oplus 1_{k_1}, \ldots, w_q \oplus 1_{k_q}).$$

In order to simplify the notation we set $s = t + 1$.

As it is easily seen there is a permutation matrix $v_t \in U(sm)$ such that if we define $\beta_t^0 : U(tm) \to U(sm)$ by $\beta_t^0(u) = v_t^*(u \oplus 1)v_t$ then the following diagram is commutative:

$$\begin{array}{ccc} U(t\underline{k}) & \xrightarrow{j_t^0} & U(tm) \\ {\scriptstyle \alpha_t}\downarrow & & \downarrow{\scriptstyle \beta_t^0} \\ U(s\underline{k}) & \xrightarrow[j_s^0]{} & U(sm). \end{array}$$

To be more precise one can take v_t to be the unitary which permutes the canonical basis of $\mathbf{C}^{sm}$ according to the permutation ξ of $\{1, 2, \ldots, sm\}$ which is defined below. For each $0 \leqslant i \leqslant q$ let

$$W_i = \{x = (t + 1)h(i - 1) + (t + 1)(r - 1)k_i + a : 1 \leqslant r \leqslant n_i,\ 1 \leqslant a \leqslant tk_i\}$$

and

$$V_i = \{y = (t + 1)h(i - 1) + (t + 1)(r - 1)k_i + tk_i + b : 1 \leqslant r \leqslant n_i,\ 1 \leqslant b \leqslant k_i\}.$$

These sets form a partition of $\{1, 2, \ldots, sm\}$. If x and y are generic elements of W_i and V_i as above we put

$$\xi(x) = th(i-1) + t(r-1)k_i + a$$

$$\xi(y) = th(q) + h(i-1) + (r-1)k_i + b.$$

Now define the second bonding map

$$\beta_t : E_{tm}(t\underline{k}) \to E_{sm}(s\underline{k})$$

by the rule

$$\beta_t(\underline{\varphi}, u) = (\underline{\varphi} \oplus \underline{\varphi}^0, \beta_t^0(u))$$

where $\underline{\varphi} = (\varphi_1, \ldots, \varphi_q) \in E(t\underline{k})$, $\underline{\varphi}^0 = (\varphi_1^0, \ldots \varphi_q^0) \in E(\underline{k})$

$$\underline{\varphi} \oplus \underline{\varphi}^0 = (\varphi_1 \oplus \varphi_1^0, \ldots, \varphi_q \oplus \varphi_q^0) \in E((t+1)\underline{k})$$

(see also the notation introduced in 4.1.1 and 4.2.2). Since β_t makes commutative the above diagram it follows that β_t is $U(t\underline{k})$-equivariant in the sense that

$$\beta_t((\underline{\varphi}, u), w) = \beta_t(\underline{\varphi}, u)\alpha_t(w).$$

Therefore β_t naturally induces a map $\gamma_t : B_{tm}(t\underline{k}) \to B_{sm}(s\underline{k})$. Also we consider the maps $\alpha_t' : BU(t\underline{k}) \to BU(s\underline{k})$ and $\beta_t' : BU(tm) \to BU(sm)$, naturally induced by the group homomorphisms α_t and β_t, so that the following diagram commutes within homotopy

$$\begin{tikzcd}
U(t\underline{k}) \arrow[r] \arrow[d, "\alpha_t"] & E_{tm}(t\underline{k}) \arrow[r] \arrow[d, "\beta_t"] & B_{tm}(t\underline{k}) \arrow[r] \arrow[d, "\gamma_t"] & BU(t\underline{k}) \arrow[r] \arrow[d, "\alpha_t'"] & BU(tm) \arrow[d, "\beta_t'"] \\
U(s\underline{k}) \arrow[r] & E_{sm}(s\underline{k}) \arrow[r] & B_{sm}(s\underline{k}) \arrow[r] & BU(s\underline{k}) \arrow[r] & BU(sm).
\end{tikzcd}$$

Given a space Y we pass to homotopy classes $[Y, \cdot]$ in the above diagram and then we take inductive limits. Since taking inductive limits preserves the exactness, we arrive at the following commutative diagram with exact rows:

4.2.5.

$$\begin{tikzcd}
{[Y, U(\underline{k})]} \arrow[r] \arrow[d, "\alpha_\infty"] & {[Y, E_m(\underline{k})]} \arrow[r] \arrow[d, "\beta_\infty"] & {[Y, B_m(\underline{k})]} \arrow[r] \arrow[d, "\gamma_\infty"] & {[Y, BU(\underline{k})]} \arrow[r] \arrow[d, "\alpha_\infty'"] & {[Y, BU(m)]} \arrow[d, "\beta_\infty'"] \\
\lim [Y, U(t\underline{k})] \arrow[r, "j_\infty"] & \lim [Y, E_{tm}(t\underline{k})] \arrow[r, "p_\infty"] & \lim [Y, B_{tm}(t\underline{k})] \arrow[r, "c_\infty"] & \lim [Y, BU(t\underline{k})] \arrow[r, "j'"] & \lim [Y, BU(tm)].
\end{tikzcd}$$

Using direct sums of unitaries, homomorphisms and respectively fiber-bundles one can introduce obvious abelian semigroup structures on each set

$$\varinjlim [Y, F_t] \quad \text{where } F_t = U(t\underline{k}), E_{tm}(t\underline{k}), B_{tm}(t\underline{k}), \mathrm{B}U(t\underline{k}), \mathrm{B}U(tm).$$

In order to check that the addition operations given by direct sums are well defined one has to apply several times the $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$-trick in order to join by a continuous path in $\mathrm{Map}(Y, F_t)$ two objects a and a' of the form $a = b \oplus b_0 \oplus c \oplus c_0$, $a' = b \oplus c \oplus b_0 \oplus c_0$. Also one will notice that each map $F_t \to F_{t+1}$ is homotopic to the map $a \to a \oplus a^0$ where $a \in F_t$ and a^0 is the base point of F_1.

It is also clear that the maps j_*, p_*, ε_* and j'_* preserve the direct sums so that the bottom row in the above diagram is (at least) an exact sequence of pointed semigroups.

Having in mind the definition of K-groups and kk-groups we can make the following identifications:

$$\lim [Y, U(t\underline{k})] = \coprod_{i=0}^{q} \lim [Y, U(tk_i)] = \begin{cases} \mathrm{K}^1(Y)^q & \text{if } k_0 = 0 \\ \mathrm{K}^1(Y)^{q+1} & \text{if } k_0 > 0 \end{cases}$$

$$\lim [Y, E_{tm}(t\underline{k})] = \coprod_{i=1}^{q} \lim [Y, F^{tk_i}(X_i)] \times \lim [Y, U(tm)] = \coprod_{i=1}^{q} \mathrm{kk}(Y, X_i) \times \mathrm{K}^1(Y)$$

$$\lim [Y, \mathrm{B}U(t\underline{k})] = \coprod_{i=0}^{q} \lim [Y, \mathrm{B}U(tk_i)] = \begin{cases} \tilde{\mathrm{K}}^0(Y)^q & \text{if } k_0 = 0 \\ \tilde{\mathrm{K}}^0(Y)^{q+1} & \text{if } k_0 > 0 \end{cases}$$

$$\lim [Y, \mathrm{B}U(tm)] = \tilde{\mathrm{K}}^0(Y).$$

Since all these are abelian groups, using 4.2.5 it follows that $\varinjlim [Y, B_{tm}(t\underline{k})]$ is also an (abelian) group. In this way we get the following exact sequence of groups:

4.2.6.

$$\mathrm{K}^1(Y)^{\tilde{q}} \xrightarrow{j_*} \coprod_{i=1}^{q} \mathrm{kk}(Y, X_i) \times \mathrm{K}^1(Y) \xrightarrow{p_*} \lim [Y, B_{tm}(t\underline{k})] \xrightarrow{\varepsilon_*} \tilde{\mathrm{K}}^0(Y)^{\tilde{q}} \xrightarrow{j'_*} \tilde{\mathrm{K}}^0(Y)$$

where $\tilde{q} = q$ if $k_0 = 0$ and $\tilde{q} = q + 1$ if $k_0 > 0$.

The homomorphisms j_* and j'_* can be easily described if we recall the definition of $j : U(\underline{k}) \to E(\underline{k}) \times U(m)$, namely

$$j(\underline{w}) = (\underline{\varphi}^0,\ w_0 \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})).$$

Therefore if $k_0 = 0$ then

$$j_*(y_1, \ldots, y_q) = (0, n_1 y_1 + \ldots + n_q y_q),$$

$\left(\text{the null component corresponds to } \prod_{i=0}^{q} \mathrm{kk}(Y, X_i)\right)$

$$j'_*(x_1, \ldots, x_q) = n_1 x_1 + \ldots + n_q x_q$$

$$\operatorname{coker} j_* = \prod_{i=1}^{q} \mathrm{kk}(Y, X_i) \times \left(\mathrm{K}^1(Y) \Big/ \sum_{i=1}^{q} n_i \mathrm{K}^1(Y)\right)$$

$$\ker j'_* = \left\{(x_1, \ldots, x_q) \in \tilde{\mathrm{K}}^0(Y)^q : \sum_{i=1}^{q} n_i x_i = 0\right\}.$$

For $k_0 > 0$

$$j_*(y_0, y_1, \ldots, y_q) = (0, y_0 + n_1 y_1 + \ldots + n_q y_q)$$

$$j'_*(x_0, x_1, \ldots, x_q) = x_0 + n_1 x_1 + \ldots + n_q x_q$$

$$\operatorname{coker} j_* = \prod_{i=1}^{q} \mathrm{kk}(Y, X_i)$$

$$\ker j'_* = \left\{(x_0, x_1, \ldots, x_q) \in \tilde{\mathrm{K}}^0(Y)^{q+1} : x_0 + \sum_{i=1}^{q} n_i x_i = 0\right\}.$$

For spaces X_i reducing to a point, the sequence 4.2.6 becomes

$$\mathrm{K}^1(Y)^{\tilde{q}} \xrightarrow{j_*^0} \mathrm{K}^1(Y) \xrightarrow{p_*^0} \lim\,[Y, B^0_{tm}(t\underline{k})] \xrightarrow{\varepsilon_*^0} \tilde{\mathrm{K}}^0(Y)^{\tilde{q}} \xrightarrow{j'_*} \tilde{\mathrm{K}}^0(Y)$$

which gives the middle term up to an extension:

$$0 \to \operatorname{coker} j_*^0 \to \lim\,[Y, B^0_{tm}(t\underline{k})] \to \ker j'_* \to 0.$$

4.2.7. Proposition. *There is a natural isomorphism*

$$\theta : \lim\,[Y, B_{tm}(t\underline{k})] \to \lim\,[Y, B^0_{tm}(t\underline{k})] \times \prod_{i=1}^{q} \mathrm{kk}(Y, X_i).$$

Proof. The construction given in 4.2.4—6 equally applies to the following diagram which commutes within homotopy:

$$\begin{tikzcd}
U(\underline{k}) \arrow[r, "j^0"] \arrow[d, equal] & U(m) \arrow[r, "p^0"] \arrow[d, "i"] & B^0_m(\underline{k}) \arrow[r, "\varepsilon^0"] \arrow[d, "i'"] & \mathrm{B}U(\underline{k}) \arrow[r, "j'"] \arrow[d, equal] & \mathrm{B}U(m) \arrow[d, equal] \\
U(\underline{k}) \arrow[r, "j"] \arrow[d, equal] & E_m(\underline{k}) \arrow[r, "p"] \arrow[d, "\rho"] & B_m(\underline{k}) \arrow[r, "\varepsilon"] \arrow[d, "\rho'"] & \mathrm{B}U(\underline{k}) \arrow[r, "j'"] \arrow[d, equal] & \mathrm{B}U(m) \arrow[d, equal] \\
U(\underline{k}) \arrow[r, "j^0"] & U(m) \arrow[r, "p^0"] & B^0_m(\underline{k}) \arrow[r, "\varepsilon^0"] & \mathrm{B}U(\underline{k}) \arrow[r, "j'"] & \mathrm{B}U(m).
\end{tikzcd}$$

In this way we arrive at the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} j^0_* \arrow[r, "p^0_*"] \arrow[d, "i_*"] & \lim[Y, B^0_{tm}(t\underline{k})] \arrow[r, "\varepsilon^0"] \arrow[d, "i'_*"] & \ker j'_* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \prod \mathrm{kk}(Y, X_i) \times \operatorname{coker} j^0_* \arrow[r, "p_*"] \arrow[d, "\rho_*"] & \lim[Y, B_{tm}(t\underline{k})] \arrow[r, "\varepsilon_*"] \arrow[d, "\rho'_*"] & \ker j'_* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{coker} j^0_* \arrow[r, "p^0_*"] & \lim[Y, B^0_{tm}(t\underline{k})] \arrow[r, "\varepsilon^0_*"] & \ker j'_* \arrow[r] & 0.
\end{tikzcd}$$

Next we define $\nu : \lim[Y, B_{tm}(t\underline{k})] \to \prod_{i=1}^{q} \mathrm{kk}(Y, X_i)$, by

$$\nu(x) = x - i'_*\rho'_*(x) \in p_*\left(\prod_{i=1}^{q} \mathrm{kk}(Y, X_i) \times \{0\}\right).$$

(Note that $\varepsilon_*\nu(x) = 0$ and $\rho'_*\nu(x) = 0$.)

Now we can define

$$\theta : \lim[Y, B_{tm}(t\underline{k})] \to \lim[Y, B^0_{tm}(t\underline{k})] \times \prod_{i=1}^{q} \mathrm{kk}(Y, X_i)$$

by $\theta(x) = (\rho'_*(x), \nu(x))$. Using the commutativity of the above diagram it is easily seen that $\theta'(y, z) = i'_*(y) + p_*(z)$ is an inverse for θ hence θ is an isomorphism.

4.2.8. THEOREM. *Let Y, X_i, $1 \leqslant i \leqslant q$, be finite connected* CW*-complexes and let $n \geqslant \dim(Y)$. Assume that each nonzero component of $\underline{k} = (k_0; k_1, \ldots, k_q)$ is greater or equal than $3(n+3)/2$. Then*

a) *There is an isomorphism*

$$\theta = (\rho'_*, \nu): [Y, B_m(\underline{k})] \to [Y, B^0_m(\underline{k})] \times \prod_{i=1}^{q} \mathrm{kk}(Y_i, X).$$

b) *There is an exact sequence of abelian groups*

$$0 \to \operatorname{coker}(j_*) \to [Y, B^0_m(\underline{k})] \to \ker(j'_*) \to 0.$$

c) *If* $\varphi, \psi \in \mathrm{Map}(Y, B_m(\underline{k}))$ *then* $\nu[\varphi] = \nu[\psi]$ *and* $\varepsilon_*[\varphi] = \varepsilon_*[\psi]$ *if and only if there is some* $u \in \mathrm{Map}(Y, U(m))$ *such that* $[\varphi] = [u\psi u^*]$.

Proof. a) The natural embeddings $U(s) \hookrightarrow U(s+1)$ and $BU(s) \hookrightarrow BU(s+1)$ are $2s$-equivalences [23]. Moreover, it follows by Theorem 6.4.2 that the embedding $E(\underline{k}) \hookrightarrow E(2\underline{k})$ is a $(n+1)$-equivalence. Using these facts it follows from the last diagram in 4.2.4 (via a Five Lemma argument) that the map

$$\gamma_t : B_{tm}(t\underline{k}) \to B_{(t+1)m}((t+1)\underline{k})$$

is a $(n+1)$-equivalence for any $t \geqslant 1$.

Therefore by Whitehead Theorem, the map

$$\gamma_* : [Y, B_m(\underline{k})] \to \lim [Y, B_{tm}(t\underline{k})]$$

is a bijection since $\dim(Y) \leqslant n$. This map is used to transfer the group structure on $[Y, B_m(\underline{k})]$ and we shall identify x with $\gamma_*(x)$ for every x in $[Y, B_m(\underline{k})]$. Accordingly, the maps ρ'_*, ν and ε_* may be seen as maps from $[Y, B_m(\underline{k})]$. Finally, after these identifications the assertion follows from 4.2.7.

b) Similar to a).

c) The proof is divided into two parts.

In the first part we prove the statement assuming that the following assertion is true:

ASSERTION. *If* $\varphi, \psi \in \mathrm{Map}(Y, B^0_m(\underline{k}))$ *then* $\varepsilon^0_*(\varphi) = \varepsilon^0_*(\psi)$ *if and only if* $\psi = u^*\varphi u$ *for some* $u \in \mathrm{Map}(Y, U(m))$.

In the second part we prove the above assertion.

The results of a) and b) will be used several times.

Let $\varphi, \psi \in \mathrm{Map}(Y, B_m(\underline{k}))$ such that $\nu[\varphi] = \nu[\psi]$ and $\varepsilon_*[\varphi] = \varepsilon_*[\psi]$. The condition $\nu[\varphi] = \nu[\psi]$ shows that φ and ψ have the same kk-component. Let θ' be the inverse of θ as in the proof of 4.2.7. Then there are $y_1, y_2 \in [Y, B_m(\underline{k})]$ and $z \in \prod_{i=1}^{q} \mathrm{kk}(Y_i, X)$ such that $[\varphi] = \theta'(y_1, z) = i_*(y_1) + p_*(z)$ and $[\psi] = \theta'(y_2, z) = i_*(y_2) + p_*(z)$. Choose $\alpha_1, \alpha_2 \in \mathrm{Map}(Y, B^0_m(\underline{k}))$ and $\beta \in \mathrm{Map}(Y, B_m(\underline{k}))$ such that $[\alpha_i] = y_i$ and

$[\beta] = p_*(z)$ and let φ^0 be the map $Y \to B^0_m(\underline{k})$ which takes Y to the base point of $B^0_m(\underline{k})$. The equations $[\varphi] = \theta'(y_1, z)$ and $[\psi] = \theta'(y_2, z)$ imply that $\varphi \oplus \varphi^0$ is homotopic to $\alpha_1 \oplus \beta$ and $\psi \oplus \varphi^0$ is homotopic to $\alpha_2 \oplus \beta$ as maps from Y to $B_m(2\underline{k})$. On the other hand since $\varepsilon^*[\varphi] = \varepsilon^*[\psi]$ we must have $\varepsilon_0^*[\alpha_1] = \varepsilon_0^*[\alpha_2]$. According to the Assertion $\alpha_2 = u\alpha_1 u^*$ for some $u \in \mathrm{Map}[Y, U(m)]$. Putting the above facts together we have the following sequence of homotopies:

$$\psi \oplus \varphi^0 \sim \alpha_2 \oplus \beta = (u \oplus 1_m)\alpha_1 \oplus \beta(u^* \oplus 1_m) \sim u \oplus 1(\varphi \oplus \varphi^0)u^* \oplus 1 =$$

$$= u\varphi u^* \oplus \varphi^0.$$

By the main stability result 6.4.2 we must have $\psi \sim u\varphi u^*$. Conversely, assume $\psi \sim u\varphi u^*$. As above let $\varphi \oplus \varphi^0 \sim \alpha_1 \oplus \beta_1$, $\psi \oplus \varphi^0 \sim \alpha_2 \oplus \beta_2$ where α_i corresponds to the kk-components and $\beta_i \in [Y, B_m(\underline{k})]$. We have

$$\alpha_2 \oplus \beta_2 \sim \psi \oplus \varphi^0 \sim u \oplus 1(\varphi \oplus \varphi^0)u^* \oplus 1 \sim 1 \oplus u(\alpha_1 \oplus \beta_1)1 \oplus u^* \sim$$

$$\sim \alpha_1 \oplus u\beta_1 u^*.$$

The direct sum decomposition provided by 4.2.7 and 4.2.8 a) shows that $\alpha_1 \sim \alpha_2$ and $u\beta_1 u^* \sim \beta_2$. Consequently $\nu[\varphi] = [\alpha_1] = [\alpha_2] = \nu[\psi]$ and $\varepsilon^0_*[\varphi] = \varepsilon^0_*(u\beta_1 u^*) = = \varepsilon^0_*[\beta_2] = \varepsilon^0_*[\psi]$.

The proof of the Assertion relies on some general facts. Recall that if $K \hookrightarrow G$ are Lie groups then we have an exact sequence:

$$K \to G \xrightarrow{p} G/K \xrightarrow{\varepsilon^0} BK \to BG.$$

The left action of G on G/K induces a left action of $\mathrm{Map}(Y, G)$ on $\mathrm{Map}(Y, G/K)$. Now the fact is that if $f^1, f^2 \in \mathrm{Map}(Y, G/K)$ then $\varepsilon^0 \circ f^1$ is homotopic to $\varepsilon^0 \circ f^2$ iff $f^2 = gf^1$ for some $g \in \mathrm{Map}(Y, G)$. A proof of this folklore type result is included below. The Assertion corresponds to the case $K = U(k)$, $G = U(m)$.

Let ξ^j, $j = 1, 2$ be the induced bundle over f^j of the bundle $p: G \to G/K$. Since $\varepsilon^0 \circ f^j$ is a classifying map for ξ^j, ξ^1 is isomorphic to ξ^2 iff $\varepsilon^0 \circ f^1 \sim \varepsilon^0 \circ f^2$. Therefore we have to prove that ξ^1 is isomorphic to ξ^2 iff $f^2 = g \cdot f^1$ for some $g \in \mathrm{Map}(Y, G/K)$. To prove this equivalence we shall work with G-cocycles (= systems of transition functions for principal G-bundles) (see [23]).

First assume $\xi^1 \simeq \xi^2$. Then there is an open cover $(U_\alpha)_\alpha$ of Y, $g^j_\alpha \in \mathrm{Map}(U_\alpha, G)$, $j = 1, 2$, such that g^j_α lifts $f^j|U_\alpha$, i.e. $p(g^j_\alpha(x)) = f^j(x)$ for all $x \in U_\alpha$. For any α, β with $U_\alpha \cap U_\beta \neq \emptyset$ define $h^j_{\alpha\beta} \in \mathrm{Map}(U_\alpha \cap U_\beta, K)$ by $h^j_{\alpha\beta}(x) = = g^j_\alpha(x)^{-1} g^j_\beta(x)$, $x \in U_\alpha \cap U_\beta$.

The system $(U_\alpha, h^j_{\alpha\beta})$ is called a G-cocycle associated with the G-bundle ξ^j. There is an equivalence relation in the set of G-cocycles such that two bundles are isomorphic if and only if any two G-cocycles associated with them are equivalent ([23]). The equivalence relation for G-cocycles takes a simpler form when they correspond to the same covering of the base space. Thus in our case $\xi^1 \sim \xi^2$ iff there exist $h_\alpha \in \mathrm{Map}(U_\alpha, G)$ such that $h^2_{\alpha\beta}(x) = h_\alpha^{-1}(x)h^1_{\alpha\beta}(x)h_\beta(x)$ for $x \in U_\alpha \cap U_\beta$. This implies

$$g^2_\alpha(x)h_\alpha(x)g^1_\alpha(x)^{-1} = g^2_\beta(x)h_\beta(x)g^1_\beta(x)^{-1} \quad \text{for } x \in U_\alpha \cap U_\beta.$$

Therefore we can define $g \in \mathrm{Map}(Y, G)$ by

$$(g|U_\alpha)(x) := g^2_\alpha(x)h_\alpha(x)g^1_\alpha(x)^{-1}$$

and is easily seen that $f^2 = gf^1$.

The converse is almost contained in the above arguments.

If $f^2 = gf^1$ and $g^1_\alpha \in \mathrm{Map}(U_\alpha, G)$ is chosen as above then we may take $g^2_\alpha :=$ $:= (g|U_\alpha)g^1_\alpha$. Therefore $h^2_{\alpha\beta}(x) = g^1_\alpha(x)^{-1}g(x)g(x)^{-1}g^1_\beta(x) = h^1_{\alpha\beta}(x)$ for $x \in U_\alpha \cap U_\beta$ and so ξ^1 is isomorphic to ξ^2.

4.2.9. Remark. Define $B_\infty = \lim B_{tm}(t\underline{k})$. Since B_∞ is a weak H space it follows that the action of $\pi_1(B_\infty)$ on $[Y, B_\infty]$ is trivial. As the embedding $B_m(\underline{k}) \to B_\infty$ is a $(n + 1)$-equivalence it is easily seen that $\pi_1(B_m(\underline{k}))$ acts trivially on $[Y, B_m(\underline{k})]$. Consequently $[Y, B_m(\underline{k})]$ coincides with the free homotopy classes $[Y, B_m(\underline{k})]_{\mathrm{free}} =$ $= [A, C(Y)\otimes M_m]_{\underline{k}}$.

4.2.10. Let $A = \oplus\, C(X_i) \otimes M_{n_i}$, $D = C(Y) \otimes M_m$ as above and let $\gamma \in \mathrm{Hom}(\mathrm{K}_0(r(A)), \mathrm{K}_0(r(D)))_{+,\Sigma}$. Define $[A, D]_\gamma = \{\varphi \in [A, D] : r(\varphi) = \gamma\}$. It is clear thatd $[A, D]$ is the disjoint union of the $[A, D]_\gamma$. Theorem 4.2.8 computes $[A, D]_\gamma$ provided that γ is $3(n + 3)/2$-large. To make the result more clear we give below more concrete formulae for the maps ν, ρ'_*, ε_*. The reader would have in mind the isomorphism $[Y, B^0_m(\underline{k})] \simeq [r(A), C(Y) \otimes M_m]_\gamma$, $\gamma = (k_1, \ldots, k_q)$. Thus ρ'_* is just the map $[A, D]_\gamma \to [r(A), D]_\gamma$ given by $[\varphi] \mapsto [\varphi|r(A)]$, where $\varphi|r(A)$ denotes the restriction of φ to $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$ regarded as a subalgebra of A. Let e^i be a minimal projection in M_{n_i} and for $\varphi \in \mathrm{Hom}(A, D)$ let $\varphi^i \in \mathrm{Hom}(C_0(X_i), C(Y) \otimes M_m)$ be given be the composition

$$C_0(X_i) \otimes e^i \hookrightarrow A \xrightarrow{\varphi} D.$$

With this notation ν can be identified with the map

$$[A, D]_\gamma \to \prod_{i=1}^{q} \mathrm{kk}(Y, X_i), \quad [\varphi] \to ([\varphi^1], \ldots, [\varphi^q]).$$

Finally $\varepsilon_*: [A, D] \to K^0(Y)^{\tilde{q}}$

$$\tilde{q} = \begin{cases} q & \text{if } k_0 = 0 \\ q + 1 & \text{if } k_0 > 0 \end{cases}$$

is essentially the map $K_0(\varphi|r(A)) \in \mathrm{Hom}(K_0(r(A)), K_0(D))$. More precisely, let $[\varphi] \in [A, D]_\gamma$, $\gamma = (k_1, \ldots, k_q)$ and $x_i \oplus [k_i] \in \tilde{K}^0(Y) \oplus K^0(pt)$ be the K-theory class of the vector bundle $\varphi(e^i)$. Then we have

$$\varepsilon_*[\varphi] = (x_1, \ldots, x_q) \quad \text{if } k_0 = 0$$

$$\varepsilon_*[\varphi] = \left(m - k_0 - \sum_{i=1}^{q} n_i x_i, x_1, \ldots, x_q\right) \quad \text{if } k_0 > 0.$$

Consequently, if $\varphi, \psi \in \mathrm{Hom}(A, D)$ then $K_0(\varphi|r(A)) = K_0(\psi|r(A))$ if and only if $r(\varphi) = r(\psi)$ and $\varepsilon_*(\varphi) = \varepsilon_*(\psi)$.

The following theorem is essentially a reformulation of 4.2.8.

4.2.11. THEOREM. *Let* $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$ *where* X_i, Y_j *are finite connected* CW*-complexes and* $\dim(Y_j) \leqslant n$ *for all* $1 \leqslant j \leqslant h$. *Let* φ, $\psi \in \mathrm{Hom}(A, D)$ *be* $3(n + 3)/2$*-large. Then*

a) $[\varphi] = [\psi]$ *if and only if* $[\varphi^{j,i}] = [\psi^{j,i}]$ *in* $\mathrm{kk}(Y_j, X_i)$ *for all* i, j *and* $[\varphi|r(A)] = [\psi|r(A)]$ *in* $[r(A), D]$;

b) $[\varphi] = [u\psi u^*]$ *for some unitary* $u \in U(D)$ *if and only if* $[\varphi^{j,i}] = [\psi^{j,i}]$ *in* $\mathrm{kk}(Y_j, X_i)$ *for all* i, j *and* $K_0(\varphi|r(A)) = K_0(\psi|r(A))$.

(For $\varphi \in \mathrm{Hom}(A, D)$, $\varphi^{j,i}$ stands for the composition

$$C_0(X_i) \otimes e^i \hookrightarrow A \xrightarrow{\varphi} D \to C(Y) \otimes M_{m_j}.)$$

4.3. HOMOTOPY AND K-THEORY

Throughout this section we let A, D stand for two fixed C^*-algebras in $\mathscr{C}(n)$

$$A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}, \quad D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}.$$

We gave in 4.2 complete invariants based on kk and K-theory for the large morphisms belonging to $[A, D]$. In order to use our computations for shape classifications it is useful to point out some cases when K-theory suffices for computing $[A, D]$.

This is done by comparing kk with KK. A related problem is to describe the image of $[A, D]$ in $KK[A, D]$ using order concepts.

Let $\mathscr{X}(n)$ be the class of finite connected CW-complexes of dimension $\leqslant n$ whose total cohomology is torsion free and supported in two dimensions having distinct parity. Thus for given $X \in \mathscr{X}(n)$ there are $p, q \in \mathbf{N}$ depending on X such that $\mathrm{H}^{\mathrm{even}}(X, \mathbf{Z}) = \mathrm{H}^p(X, \mathbf{Z})$, $\mathrm{H}^{\mathrm{odd}}(X, \mathbf{Z}) = \mathrm{H}^q(X, \mathbf{Z})$ and both these groups are torsion free.

4.3.1. THEOREM. *Let A, D as above and assume that $X_i, Y_j \in \mathscr{X}(n)$ for all i, j. Then*

a) *If $\sigma \in \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(D))_{+,\Sigma}$ is $3(n+3)/2$-large then there is some $\varphi \in \mathrm{Hom}(A, D)$ such that $\mathrm{K}_*(\varphi) = \sigma$.*

b) *If $\varphi_1, \varphi_2 \in \mathrm{Hom}(A, D)$ are $3(n+3)/2$-large and $\mathrm{K}_*(\varphi_1) = \mathrm{K}_*(\varphi_2)$ then there is $u \in U(D)$ such that*

$$\varphi_1 \text{ is homotopic to } u\varphi_2 u^*.$$

Proof. a) Let $A_0 = \bigoplus_i C_0(X_i) \otimes M_{n_i}$, $A_1 = \bigoplus_i M_{n_i}$. We can suppose that $D = C(Y) \otimes M_m$ so that we take $D_0 = C_0(Y) \otimes M_m$ and $D_1 = M_m$. Let $\sigma = (\sigma^0, \sigma^1)$ where $\sigma^0 \colon \mathrm{K}_0(A) \to \mathrm{K}_0(D)$ and $\sigma^1 \colon \mathrm{K}_1(A) \to \mathrm{K}_1(D)$.

Let $\sigma^0 = \begin{pmatrix} \alpha^0 & \beta \\ 0 & \underline{k} \end{pmatrix}$ and $\sigma^1 = \begin{pmatrix} \alpha^1 & 0 \\ 0 & 0 \end{pmatrix}$ be the matrix description of σ^0 and σ^1 corresponding to the decompositions $\mathrm{K}_*(A) = \mathrm{K}_*(A_0) \oplus \mathrm{K}_*(A_1)$, $\mathrm{K}_*(D) = \mathrm{K}_*(D_0) \oplus \mathrm{K}_*(D_1)$ (see 2.1.3). We have:

$$\alpha^0 = (\alpha_1^0, \ldots, \alpha_q^0) \colon \mathrm{K}_0(A_0) \simeq \bigoplus_i \tilde{\mathrm{K}}^0(X_i) \to \tilde{\mathrm{K}}^0(Y) \simeq \mathrm{K}_0(D_0)$$

$$\beta = (\beta_1, \ldots, \beta_q) \colon \mathrm{K}_0(A_1) = \mathbf{Z}^q \to \tilde{\mathrm{K}}^0(Y), \quad \beta_i \in \tilde{\mathrm{K}}^0(Y)$$

$$\underline{k} = (k_1, \ldots, k_q) \colon \mathbf{Z}^q \to \mathbf{Z}.$$

Since $\sigma \geqslant 0$ we have that if some $k_i = 0$ then $\alpha_i^0 = 0$, $\alpha_i^1 = 0$ and $\beta_i = 0$ for the same index i. The condition $\sigma(\Sigma(A)) \subset \Sigma(D)$ is equivalent to $\sum_i k_i n_i \leqslant m$. This is easily seen using Corollary 2.1.2 b) and the fact that σ is large enough. Since σ is $3(n+3)/2$-large if $k_i \neq 0$ then $k_i \geqslant 3(n+3)/2$ and so $[C_0(X_i), C_0(Y) \otimes M_{k_i}] \simeq \mathrm{kk}(Y, X_i)$ by Corollary 6.4.4. On the other hand the cohomological conditions on X_i, Y imply that $\mathrm{kk}(Y, X_i) \simeq \mathrm{Hom}(\tilde{\mathrm{K}}^*(X_i), \tilde{\mathrm{K}}^*(Y))$ by the results of Section 3.

Therefore we can find $\varphi_i \in \mathrm{Hom}_1(C(X_i), C(Y) \otimes M_{k_i})$ such that $\mathrm{K}'_*(\varphi_i) = (\alpha_i^0, \alpha_i^1)$. Define $\varphi' \in \mathrm{Map}(Y, \mathrm{Hom}(A, M_m)) = \mathrm{Hom}(A, D)$ by

$$\varphi'(y) = p(\varphi_1(y), \ldots, \varphi_q(y);\ 1) \qquad \text{(see 4.1.1).}$$

Then

$$\mathrm{K}_*(\varphi') = \left(\begin{pmatrix} \alpha^0 & 0 \\ 0 & \underline{k} \end{pmatrix}, \begin{pmatrix} \alpha^1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Aside φ', we need another morphism φ'' constructed as follows.

Let $\beta_0 = -\sum_{i=1}^{q} \beta_i$. Then $(\beta_0, \beta_1, \ldots, \beta_q) \in \mathrm{Ker} j_* = \text{image } \varepsilon_*^0 \subset \tilde{\mathrm{K}}^0(Y)^{\tilde{q}}$. Therefore there is $\psi \in \mathrm{Map}(Y, B_m^0(\underline{k})) \simeq \mathrm{Hom}(\bigoplus_{i=1}^{q} M_{n_i}, C(Y) \otimes M_m)$ such that $\varepsilon_*(\psi) = (\beta_0, \beta_1, \ldots, \beta_q)$ (see 4.2.8 b)). If $\mathrm{ev}: A \to A_1$ is any evaluation morphism, then using 4.2.10 it can be verified that $\varphi'' := \psi \circ \mathrm{ev}: A \to B$ is such that $\mathrm{K}_*(\varphi'') = \left(\begin{pmatrix} 0 & \beta \\ 0 & \underline{k} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$ (see [13] for a similar computation).

Finally, we choose $[\varphi] \in [Y, B_m(\underline{k})]$ such that its image in $[Y, B_{2m}(2\underline{k})]$ is equal to $[\varphi' \oplus \varphi'']$. The above computations show that $\mathrm{K}_*(\varphi) = \sigma$.

b) Let

$$\mathrm{K}_*(\varphi_i) = \begin{pmatrix} \alpha(\varphi_i) & \beta(\varphi_i) \\ 0 & r(\varphi_i) \end{pmatrix} : \mathrm{K}_*(A_0) \oplus \mathrm{K}_*(A_1) \to \mathrm{K}_*(D_0) \oplus \mathrm{K}_*(D_1).$$

We have $\beta(\varphi_1) = \beta(\varphi_2)$ and $r(\varphi_1) = r(\varphi_2)$ since $\mathrm{K}_*(\varphi_1) = \mathrm{K}_*(\varphi_2)$.

The following commutative diagram

$$\begin{array}{ccc} [A, D] & \longrightarrow & \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(D)) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \alpha} \\ \oplus \mathrm{kk}(Y, X_i) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{K}_*(A_0), \mathrm{K}_*(D_0)) \end{array}$$

shows that $\alpha(\varphi_1) = \alpha(\varphi_2)$ implies $\nu(\varphi_1) = \nu(\varphi_2)$. Now we can apply Theorem 4.2.11 b) in order to derive the desired conclusion.

The following result concerns spaces which may have torsion in K_0 but we have to make some restrictions.

4.3.2. THEOREM. *Let* $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$ *where* X_i, Y_j *are* $(n-2)$*-connected finite* CW*-complexes of dimension* $\leqslant n$ *and* n *is even.*

a) *If* $\sigma \in \mathrm{KK}(A, D)_{+,\Sigma}$ *is* $3(n+3)/2$*-full* (2.1.8) *then there is some* $\varphi \in \mathrm{Hom}(A, D)$ *such that* $[\varphi]_{\mathrm{KK}} = \sigma$.

b) *If* $\varphi_1, \varphi_2 \in \mathrm{Hom}(A, D)$ *are* $3(n+3)/2$*-full and* $[\varphi_1]_{\mathrm{KK}} = [\varphi_2]_{\mathrm{KK}}$ *then* φ_1 *is homotopic to* $u\varphi_2 u^*$ *for some* $u \in U(D)$.

($\sigma \in \mathrm{KK}(A, D)_{+,\Sigma}$ is called m-full if its image in $\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(D))_{+,\Sigma}$ is m-full.)

Proof. a) Let A_0, A_1, D_0, D_1 be as in the proof of 4.3.1. We shall analyse the components of σ corresponding to the decomposition

$$\mathrm{KK}(A, D) = \begin{pmatrix} \mathrm{KK}(A_0, D_0) & \mathrm{KK}(A_1, D_0) \\ \mathrm{KK}(A_0, D_1) & \mathrm{KK}(A_1, D_1) \end{pmatrix}.$$

Let $n = 2s$. By hypothesis each X_i is $(2s-2)$-connected. It follows that the $(2s-1)$-dimensional skeleton of X_i is homotopic to a wedge of $(2s-1)$-spheres. Since $\dim X_i \leqslant 2s$ this easily implies that $K^1(X_i)$ is free. Using the Universal Coefficient Theorem for KK [37], we get

$$\mathrm{KK}(A_0, D_1) = \mathrm{Hom}(\mathrm{K}_0(A_0), \mathrm{K}_0(D_1)) \text{ since } \mathrm{K}_1(A_0) \text{ is free and } \mathrm{K}_1(D_1) = 0;$$

$$\mathrm{KK}(A_1, D_0) = \mathrm{Hom}(\mathrm{K}_0(A_1), \mathrm{K}_0(D_0)) \text{ since } \mathrm{K}_1(A_1) = 0 \text{ and } \mathrm{K}_0(A_1) \text{ are free;}$$

$$\mathrm{KK}(A_1, D_1) = \mathrm{Hom}(\mathrm{K}_0(A_1), \mathrm{K}_0(D_1)).$$

Since $\sigma \in \mathrm{KK}(A, D)_+$ it follows by Proposition 2.1.3 that its image in $\mathrm{KK}(A_0, D_1) = \mathrm{Hom}(\mathrm{K}_0(A_0), \mathrm{K}_0(D_1))$ is 0. Thus we get

$$\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & \underline{k} \end{pmatrix}$$

where $\alpha = (\alpha_1, \ldots, \alpha_q)$, $\alpha_i \in \mathrm{KK}(C_0(X_i), C_0(Y)) \simeq \mathrm{kk}(Y, X_i)$ (see 3.4.6). From this point the proof is accomplished by analogy with the proof of Theorem 4.3.1 since β and $\underline{k}$ have the same meaning as there. However, one may wonder why we have asked σ to be full. This is because in general the presence of torsion in K_0 may prevent the implication $k_i = 0 \Rightarrow \alpha_i = 0$ to be true.

b) Similar to b) in 4.1.8 but use the following commutative diagram:

$$\begin{array}{ccc} [A, D] & \longrightarrow & \mathrm{KK}(A, D) \\ {\scriptstyle \nu}\downarrow & & \downarrow \\ \oplus\, \mathrm{kk}(Y, X_i) & \xrightarrow[\chi]{\simeq} & \mathrm{KK}(A_0, D_0). \end{array}$$

5. SHAPE THEORY

In this section we use the homotopy computations from the previous section by giving shape classification results for certain inductive limits of C^*-algebras.

5.1. SEMIPROJECTIVITY

In this subsection we extend the notion of semiprojectivity for C^*-algebras introduced by Effros and Kaminker [17] to a more general setting which allow us to develop a satisfactory shape theory for a larger class of C^*-algebras.

Let $\mathscr{S}$ denote the category of separable C^*-algebras. We start with a covariant functor $T: \mathscr{S} \to \mathscr{D}$ with values in a category $\mathscr{D}$ having the same objects as $\mathscr{S}$, such that $T(A) = A$ for each C^*-algebra A in $\mathscr{S}$. We have in mind two basic examples.

5.1.1. EXAMPLES. a) Let $\mathscr{H}$ be the category of separable C^*-algebras and homotopy classes of homomorphisms, and let $H: \mathscr{S} \to \mathscr{H}$ be the homotopy functor, i.e. H preserves the objects and takes the homomorphisms into their classes of homotopy equivalence: $\varphi \mapsto H(\varphi) = [\varphi]$.

b) Let $KK_{+,\Sigma}$ be the category whose objects are separable C^*-algebras and for which the morphisms from A to B are elements of $KK(A, B)_{+,\Sigma}$ (1.2.8). The law of composition is the Kasparov product (cf. [21]). There is a canonical functor $KK_{+,\Sigma}: \mathscr{S} \to KK_{+,\Sigma}$ since each $\varphi \in \mathrm{Hom}(A, B)$ defines in a natural way an element $[\varphi]_{KK} \in KK(A, B)_{+,\Sigma}$ and this assignment preserves the products ([26]).

Let $\mathscr{C}$ denote a fixed subcategory of $\mathscr{S}$. By a $\mathscr{C}$-inductive system (A_i, α_{ji}) we shall mean a diagram of objects and morphisms in $\mathscr{C}$

$$A_1 \xrightarrow{\alpha_{21}} A_2 \xrightarrow{\alpha_{32}} \cdots$$

By definition, for $i < j$, $\alpha_{ji} := \alpha_{j,j-1} \circ \ldots \circ \alpha_{i+1,i}$.

Of course the above definition makes sense in any category.

If (A_i, α_{ji}) denote a $\mathscr{C}$-inductive system and $A = \varinjlim A_i \in \mathscr{S}$ is the associated inductive limit C^*-algebra then we have canonical maps in $\mathscr{S}$, $\alpha_i: A_i \to A$ such that $\alpha_j \circ \alpha_{ji} = \alpha_i$. For each $S \in \mathscr{S}$ we have an inductive system of sets

$$(\mathrm{Hom}_{\mathscr{D}}(S, A_i), T(\alpha_{ji})_*)$$

where $T(\alpha_{ji})_*(\beta) = T(\alpha_{ji}) \circ \beta$. The maps

$$T(\alpha_i)_*: \mathrm{Hom}_{\mathscr{D}}(S, A_i) \to \mathrm{Hom}_{\mathscr{D}}(S, A)$$

given by $T(\alpha_i)_*(\beta) = T(\alpha_i) \circ \beta$ induce a natural map T_* from the set theoretic inductive limit $\lim(\mathrm{Hom}_{\mathscr{D}}(S, A_i), T(\alpha_{ji})_*)$ to $\mathrm{Hom}_{\mathscr{D}}(S, A)$.

5.1.2. DEFINITION. A C^*-algebra $S \in \mathscr{S}$ is called *T-semiprojective relative to* $\mathscr{C}$ if the natural map $T_*: \lim \mathrm{Hom}_{\mathscr{D}}(S, A_i) \to \mathrm{Hom}_{\mathscr{D}}(S, \lim A_i)$ is bijective for any $\mathscr{C}$-inductive system (A_i, α_{ji}).

Equivalently S is T-semiprojective relative to $\mathscr{C}$ if and only if the following conditions are fulfiled:

a) For every $\gamma \in \mathrm{Hom}_{\mathscr{D}}(S, A)$ there are j and $\alpha \in \mathrm{Hom}(S, A_j)$ such that $T(\alpha_j) \circ \alpha = \gamma$.

b) If $\alpha, \beta \in \mathrm{Hom}_{\mathscr{D}}(S, A_i)$ and $T(\alpha_i) \circ \alpha = T(\alpha_i) \circ \beta$ then there is $j > i$ such that $T(\alpha_{ji}) \circ \alpha = T(\alpha_{ji}) \circ \beta$.

5.1.3. EXAMPLES. a) $S \in \mathscr{S}$ is H-semiprojective relative to $\mathscr{S}$ if and only if it is semiprojective in the sense of Effros and Kaminker (see the examples in [17], [2], [28]).

b) Let $\mathscr{C}$ be the subcategory of $\mathscr{S}$ consisting of commutative C^*-algebras. If X is an ANR-space then $C(X)$ is H-semiprojective relative to $\mathscr{C}$ (see [29]).

c) $C(S^2)$ and $C(S^1 \times S^1)$ are not H-semiprojective relative to $\mathscr{S}$ (see [27]).

d) Propositions 5.1.4 and 5.1.6 give some criteria for $KK_{+,\Sigma}$-semiprojectivity.

Recall from the introduction that $\mathscr{C}(n)$ denotes the category of the C^*-algebras of the form $\underset{k}{\oplus} C(X_k) \otimes M_{n_k}$ (finite sums) where X_i are finite connected CW-complexes of dimension $\leqslant n$. We let $\mathscr{C}'(n)$ denote the subcategory of $\mathscr{C}(n)$ having the same objects as $\mathscr{C}(n)$ but only 2-large homomorphisms.

5.1.4. PROPOSITION. *Let* $A = \overset{q}{\underset{k=1}{\oplus}} C(X_k) \otimes M_{n_k} \in \mathscr{C}(n)$ *and assume that the semigroup* $K_0(A)_+$ *is finitely generated or equivalently that each* $\tilde{K}^0(X_k)$ *is a finite group. Then* A *is* $KK_{+,\Sigma}$*-semiprojective relative to* $\mathscr{C}'(n)$.

Proof. Let $B = \varinjlim(B_i, \beta_{ji})$ with $B_i, \beta_{ji} \in \mathscr{C}'(n)$. We have to prove that

$$KK(A, B)_{+,\Sigma} = \varinjlim KK(A, B_i)_{+,\Sigma}.$$

As we shall see below all the difficulties come from the ordering. Indeed, since $K_0(A)$ is finitely generated and $K_*(B) = \lim K_*(B_i)$ we have

$$\mathrm{Hom}(K_*(A), K_*(B)) = \lim \mathrm{Hom}(K_*(A), K_*(B_i)).$$

Moreover, since A is nuclear it follows by [37, Theorem 1.4 and Proposition 7.13] that

$$KK(A, B) = \lim KK(A, B_i).$$

Now recall from 1.2.8 that $KK(A, B)_{+,\Sigma} = \{x \in KK(A, B) : \gamma(x) \in \mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}\}$.

Therefore the canonical map $\lim KK(A, B_i)_{+,\Sigma} \to KK(A, B)_{+,\Sigma}$ is injective. Also it is easy to show (using set-theoretic arguments related to inductive limits and the naturality of γ) that the surjectivity of the above map reduces to the surjectivity of the canonical map

$$\lim \mathrm{Hom}(K_0(A), K_0(B_i))_{+,\Sigma} \to \mathrm{Hom}(K_0(A), K_0(B))_{+,\Sigma}.$$

Therefore what we need is to prove that this last map is onto. To begin with, we need some notation.

Let A_k denote the subalgebra $C(X_k) \otimes M_{n_k}$ let e_k be a minimal projection in M_{n_k} and let J_k be a finite set of generators for $K_1(A_k)$. Given $\sigma = (\sigma^0, \sigma^1) \in \mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}$, $\sigma \neq 0$, we can find $i \geqslant 0$ and a morphism of $\mathbf{Z}_2$-graded groups $\rho = (\rho^0, \rho^1) : K_*(A) \to K_*(B_i)$ such that

a) $K_*(\beta_i) \circ \rho = \sigma$ where β_i is the embedding $B_i \to B$.

Note that if $x \in K_*(A)_+$ is given we can replace ρ by $\beta_{ji} \circ \rho$, for large enough j, in order to get $\rho(x) \in K_*(B_j)_+$. This is possible since $K_*(B)_+ = \lim K_*(B_i)_+$. Consequently by increasing i we may assume that ρ also satisfies

b) $\rho^0(K_0(A)_+) \subset K_0(B_i)_+$ (recall that $K_0(A)_+$ is finitely generated)

and

c) $(2n[e_k], x_k) \in K_*(B_i)_+$ for any $1 \leqslant k \leqslant q$ and $x_k \in J_k$.

(Note that $(2n[e_k], x_k) \in K_*(A)_+$ by Corollary 2.1.2c).)

A related argument shows that we can assume

d) $\rho^0(\Sigma(A)) \subset \Sigma(B_i)$

since $\Sigma(A)$ is a finite subset of $K_0(A)_+$ and $\Sigma(B) = \lim \Sigma(B_i)$. Finally, we may also assume that

e) ρ^0 is $2n$-large

since otherwise we replace ρ by $\beta_{i+n,i} \circ \rho$ which is $2n$-large because $\beta_{i+n,i}$ is so and ρ^0 is order preserving (see 2.1.8).

With these choice we shall prove that $\rho \in \mathrm{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$ and this will complete the proof. As a first step we show that ρ is order preserving. As B_i belongs to $\mathscr{C}(n)$, it has the form $B_i = \bigoplus_{j=1}^{h} D_j$ where $D_j = C(Y_j) \otimes M_{m_j}$. Writing $A = \bigoplus_{k=1}^{q} A_k$ we may describe ρ as a matrix of morphisms $\rho = ((\rho_{jk}^0), (\rho_{jk}^1))$ where $\rho_{jk}: K_*(A_k) \to K_*(D_j)$. The condition b) implies that ρ_{jk}^0 is order preserving and so we can consider its standard picture defined in 2.1.3–2.1.4, $\rho_{jk}^0 = \begin{pmatrix} \alpha_{jk} & \beta_{jk} \\ 0 & \gamma_{jk} \end{pmatrix}$. Of course $\gamma_{jk} \geqslant 0$ and a simple calculation shows that if some $\gamma_{jk} = 0$ then $\alpha_{jk} = 0$, $\beta_{jk} = 0$ and $\rho_{jk}^1 = 0$ for the same indices j and k. This remark is essential in what follows.

Now let $a \in K_*(A)_+$. We wish to apply Corollary 2.1.2c) in order to prove that $\rho(a) = K_0(B_i)_+$. To this purpose we need the following coordinate description of a and $\rho(a)$:

$$a = (a_1, \ldots, a_q), \quad a_k = ((a_k', t_k), a_k^1) \in K_0(A_k) \oplus K_0(r(A_k)) \oplus K_1(A_k), \qquad 1 \leqslant k \leqslant q$$

$$\rho(a) = (b_1, \ldots, b_h), \quad b_j = ((b_j', s_j), b_j^1) \in K_0(D_j) \oplus K_0(r(D_j)) \oplus K_1(D_j), \qquad 1 \leqslant j \leqslant h.$$

Since ρ is order preserving we must have $s_j = \sum \gamma_{jk} t_k \geqslant 0$.

If some $s_j = 0$, then $\gamma_{jk} t_k = 0$ for each k and we will prove that $b_j = 0$.

Let $I = \{k: \gamma_{jk} = 0\}$ and $J = \{k : t_k = 0\}$. We have already seen that $\alpha_{jk} = 0$, $\beta_{jk} = 0$ and $\rho_{jk}^1 = 0$ for each $k \in I$. Also it is clear that $a_k' = 0$ and $a_k^1 = 0$ for each

$k \in J$ since $a \geqslant 0$. Therefore

$$b_j = \sum_k \left(\begin{pmatrix} \alpha_{jk} & \beta_{jk} \\ 0 & \gamma_{jk} \end{pmatrix} \begin{pmatrix} a_k' \\ t_k \end{pmatrix}, \rho_{jk}^1(a_k^1) \right) = 0.$$

If $s_j > 0$, then $\gamma_{jk} t_k \neq 0$ for some k and so $s_j \geqslant 2n$ since $\gamma_{jk} \geqslant 2n$ because ρ^0 is $2n$-large. Thus we may apply Corollary 2.1.2 c) to get $\rho(a) \geqslant 0$. It remains to show that $\rho(\Sigma_*(A)) \subset \Sigma_*(B)$. We shall use the point d) of the same corollary. The condition d) satisfied by ρ^0 implies that $0 \leqslant s_j \leqslant m_j$. There are three cases to be considered for each index j:

(i) if $s_j = 0$ then $b_j = 0$ since $b_j \geqslant 0$;

(ii) if $0 < s_j < m_j$ then $s_j \leqslant m_j - 2n \leqslant m_j - \langle (n+1)/2 \rangle$ since ρ^0 is $2n$-large;

(iii) if $s_j = m_j$, or equivalently $\sum_{k=1}^{q} \gamma_{jk} t_k = m_j$, then $t_k = n_k$ for each $k \in J =$ $= \{p : \gamma_{jp} \neq 0\}$, since otherwise $\sum_{k=1}^{q} \gamma_{jk} n_k > m_j$ which contradicts $\rho^0[1_A] \in \Sigma(B_i)$.

As $a_k \in \Sigma_*(A_k)$ we must have $a_k = [1_{A_k}]$ for each $k \in J$. Therefore b_j coincides with the j^{th} component of $\rho^0[1_A]$ and $b_j \in \Sigma_*(D_j)$ by condition d).

5.1.5. One cannot drop the assumption on $K_0(A)_+$ made in 5.1.4. For instance it follows by [27] that $C(S^3)$ and $C(S^1 \times S^1)$ are not $KK_{+,\Sigma}$-semiprojective in $\mathscr{C}'(2)$.

Let $\mathscr{C}''(n)$ be the subcategory of $\mathscr{C}(n)$ with objects of the form $C(X) \otimes M_m$ and 2-large (possibly nonunital) $*$-homomorphisms.

5.1.6. Proposition. *Any C^*-algebra $A \in \mathscr{C}''(n)$ is $KK_{+,\Sigma}$-semiprojective in $\mathscr{C}''(n)$.*

Proof. Let $A = C(X) \otimes M_m$, $B_i = C(Y_i) \otimes M_{m_i} \in \mathscr{C}''(n)$ and let $B = \varinjlim(B_i, \beta_{ji})$ each β_{ji} being 2-large. Like in the proof of 5.1.4 for a given $\sigma \in \mathrm{Hom}(K_*(A), K_*(B))_{+,\Sigma}$, $\sigma \neq 0$, we have to find $i \geqslant 0$ and $\rho \in \mathrm{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$ such that $\sigma = K_*(\beta_i) \circ \rho$. (Here β_i is the embedding $B_i \to B$.)

Since $K_*(A)$ is finitely generated there is no problem to find $\rho = (\rho^0, \rho^1)$: $: K_*(A) \to K_*(B_i)$ satisfying

a) $K_*(\beta_i) \circ \rho = \sigma$.

As $\Sigma(r(A))$ is a finite set, $\Sigma(B) = \lim \Sigma(B_i)$ and $\sigma(\Sigma_*(A)) \subset \Sigma_*(B)$, by increasing i (if necessary) we may assume that

b) $\rho^0(\Sigma(r(A))) \subset \Sigma(B_i)$.

We will prove that if ρ satisfies the conditions a) and b) and is $2n$-large, then $\rho \in \mathrm{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$. In proving this is convenient to use the formalism of Section 2. Let $\rho^0 = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ be the matrix description of ρ^0 with respect to the decompositions

$$K_0(A) = K_0'(A) \oplus K_0(r(A)), \quad K_0(B_i) = K_0'(B_i) \oplus K_0(r(B_i)).$$

We claim that $\delta = 0$. This is equivalent to say that there is a certain map γ such that the following diagram is commutative:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho^0} & K_0(B_i) \\ \downarrow{\scriptstyle r_A} & & \downarrow{\scriptstyle r_{B_i}} \\ K_0(r(A)) & \xrightarrow{\gamma} & K_0(r(B_i)). \end{array}$$

Since it is clear that γ exists iff $r_{B_i}\rho^0(a) = 0$ for all $a \in K_0(A)$ such that $r_A(a) = 0$, we have to deal with this last implication. Consider the comutative diagram

$$\begin{array}{ccc} K_0(B_i) & \xrightarrow{K_0(\beta_i)} & K_0(B) \\ \downarrow{\scriptstyle r_{B_i}} & & \downarrow{\scriptstyle r_B} \\ K_0(r(B_i)) & \xrightarrow{r(K_0(\beta_i))} & K_0(r(B)) \end{array}$$

where $r(B)$ is the matroid C^*-algebra arising from the inductive system (B_i, β_{ji}) as described in 2.1.5. Since $r(K_0(\beta_i))$ is injective, the equalities

$$r(K_0(\beta_i))r_{B_i}\rho^0 = r_B K_0(\beta_i)\rho^0 = r_B\sigma^0$$

shows that our claim reduces to the implication

$$r_A(a) = 0 \Rightarrow f(a) = 0, \text{ where by definition } f = r_B\sigma^0.$$

Now if $r_A(a) = 0$ then $a \in K_0'(A)$ and so $ka + (n+1)[1_A] \in K_0(A)_+$ for each $k \in \mathbf{Z}$. Since the morphism f is order preserving we have $kf(a) + (n+1)f([1]_A) \in$ $\in K_0(r(B))_+$ for any $k \in \mathbf{Z}$ and this is possible only if $f(a) = 0$. To derive the last implication one can use an embedding of $(K_0(r(B)), K_0(r(B))_+)$ in $(\mathbf{R}, \mathbf{R}_+)$. Thus we get $\rho^0 = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$. Let us observe that $\gamma > 0$. Indeed, if γ would be less or equal to zero then b) would imply $\gamma = 0$, $\beta = 0$ and so $\rho^0[1_A] = 0$. But this gives a contradiction since $\sigma^0[1_A] = 0$ would imply $\sigma^0 = 0$ because σ^0 is order preserving.

Once we know that $\gamma > 0$ we replace ρ by $\beta_{i+n,i} \circ \rho$ to reach

c) ρ is $2n$-large.

From now on the proof is similar to the last part of the proof of 5.1.4. Like there, we need coordinate descriptions of a and $\rho(a)$:

$$a = ((a',\ t),\ a^1) \in \mathrm{K}_0(A) \oplus \mathrm{K}_0(r(A)) \oplus \mathrm{K}_1(A)$$

$$\rho(a) = ((\alpha(a') + \beta(t),\ \gamma t),\ \rho^1(a^1)) \in \mathrm{K}_0(B_i) \oplus \mathrm{K}_0(r(B_i)) \oplus \mathrm{K}_1(B_i).$$

If $a \in \mathrm{K}_*(A)_+$ and $a \neq 0$ then $\gamma t \geqslant 2n \geqslant n+1$ and so $\rho(a) \in \mathrm{K}_*(B_i)_+$ by Corollary 2.1.1 c). Therefore we checked that ρ is order preserving. To complete the proof we must show that $\rho(\Sigma_*(A)) \subset \Sigma_*(B_i)$. If $a \in \Sigma_*(A)$, $a \neq 0$, then $t \leqslant m$ and $\gamma t \leqslant m_i$ since $\gamma t = r_{B_i}\rho^0(a', t) \in \Sigma(r(B_i))$ by condition b).

Next we shall consider two cases:

i) if $\gamma t = m_i$ then $t = m$ since $t \leqslant m$ and $\gamma m \leqslant m_i$.
As $a \in \Sigma_*(A)$, $A = C(X) \otimes M_m$ it follows that $a = [1_A]$ and so $\rho(a) \in \Sigma(B)$ by b).

ii) if $\gamma t < m_i$ then $\gamma t \leqslant m_i - 2n \leqslant m_i - \langle(n+1)/2\rangle$ since γ is $2n$-large. Therefore $\rho(a) \in \Sigma_*(B_i)$ by Corollary 2.1.2 d).

5.1.7. Having the notion of semiprojectivity defined in 5.1.2 we can construct a "formal" shape theory following the pattern of the (shape) approaches in [29] and [17]. First we need the category inj-$\mathscr{D}$ associated with $\mathscr{D}$. The objects of inj-$\mathscr{D}$ are all the inductive systems $(A_i,\ \alpha_{ji})$ in $\mathscr{D}$. A map of systems $\varphi\colon (A_i,\ \alpha_{ji}) \to (B_i,\ \beta_{ji})$ consists of a sequence of integers $\varphi(1) < \varphi(2) < \ldots$ and a collection of morphisms $\varphi_i\colon A_i \to B_{\varphi(i)}$ in $\mathscr{D}$ such that each square of the diagram

$$\begin{array}{ccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \ldots \\
\Big\downarrow{\scriptstyle\varphi_1} & & \Big\downarrow{\scriptstyle\varphi_2} & & \\
B_{\varphi(1)} & \longrightarrow & B_{\varphi(2)} & \longrightarrow & \ldots
\end{array}$$

commutes. Two maps of systems $\underline{\varphi}, \underline{\psi}\colon (A_i, \alpha_{ji}) \to (B_i, \beta_{ji})$ are said to be equivalent provided for each $i > 0$ there is an $j \geqslant \varphi(i), \psi(i)$ such that $\beta_{j\varphi(i)} \circ \varphi_i = \beta_{j\psi(i)} \circ \psi_i$. Morphisms $\underline{\underline{\varphi}}\colon (A_i,\ \alpha_{ji}) \to (B_i, \beta_{ji})$ in inj-$\mathscr{D}$ are equivalence classes of maps of systems. Let (A_i, α_{ji}), (B_i, β_{ji}) be faithful inductive systems in $\mathscr{C}$ (i.e. we assume that all α_{ji} and β_{ji} are injective) and let $A = \lim A_i$, $B = \lim B_i$, $A, B \in \mathscr{S}$. Given an homomorphism of C^*-algebras $\varphi\colon A \to B$ we say that a map of inductive systems in $\mathscr{D}$

$$\underline{\varphi}\colon (A_i,\ T(\alpha_{ji})) \to (B_i,\ T(\beta_{ji}))$$

is associated with φ if for all $i > 0$ diagram

$$\begin{array}{ccc}
A_i & \xrightarrow{T(\alpha_i)} & A \\
\Big\downarrow{\scriptstyle\varphi_i} & & \Big\downarrow{\scriptstyle T(\varphi)} \\
B_{\varphi(i)} & \xrightarrow{T(\beta_{\varphi(i)})} & B
\end{array}$$

is commutative in $\mathscr{D}$.

The following three propositions are crucial for any shape theory based on semiprojectives. They were proven in [17] for the special case of the homotopy functor H. The proofs from there are set theoretical (in their essence) and can be easily modified to work in our more general setting.

5.1.8. PROPOSITION. *Let $A = \lim A_i$, $B = \lim B_i$ as above and assume that the A_i are T-semiprojective relative to $\mathscr{C}$. Then any $*$-homomorphism $\varphi: A \to B$ has an associated system map $\underline{\varphi}$.*

5.1.9. PROPOSITION. *Suppose that A, B are as above and that $\varphi, \psi: A \to B$ have associated inductive systems maps $\underline{\varphi}, \underline{\psi}: (A_i, T(\alpha_{ji})) \to (B_i, T(\beta_{ji}))$. If $T(\underline{\varphi}) = T(\underline{\psi})$ then $\underline{\varphi}$ and $\underline{\psi}$ are equivalent and therefore they define the same morphism in* inj-$\mathscr{D}$.

5.1.10. PROPOSITION. *Suppose that $A = \lim A_i$ and $B = \lim B_i$ where A_i and B_i are T-semiprojective relative to $\mathscr{C}$. If A is isomorphic to B in $\mathscr{D}$ then $(A_i, T(\alpha_{ji}))$ is isomorphic to $(B_i, T(\beta_{ji}))$ in* inj-$\mathscr{D}$.

5.2. SHAPE THEORY

If $\mathscr{C}$ is a subcategory of $\mathscr{S}$ recall that $\mathscr{A}\mathscr{C}$ denotes the class of all C^*-algebras in $\mathscr{S}$ which can be written as inductive limits of faithful inductive systems from $\mathscr{C}$.

The next definition points out the crucial property which must enjoy a category $\mathscr{C}$ of C^*-algebras in order to give sense for shape invariants.

5.2.1. DEFINITION. a) $\mathscr{C}$ is called a *shape category* if and only if the following implication holds.

If (A_i, α_{ji}) and (B_i, β_{ji}) are faithful inductive systems in $\mathscr{C}$, which have homotopic inductive limits: $\lim A_i \sim \lim B_i$, then $(A_i, [\alpha_{ji}])$ is isomorphic to $(B_i, [\beta_{ji}])$ in the category inj-$\mathscr{H}$ (see 5.1.1 a) and 5.1.7).

b) Let $\mathscr{C}$ be a shape category and $A = \lim(A_i, \alpha_{ji}) \in \mathscr{A}\mathscr{C}$. *The shape invariant of A*, denoted by $\mathrm{Sh}_{\mathscr{C}}(A)$, is the class of isomorphism of $(A_i, [\alpha_{ji}])$ in inj-$\mathscr{H}$.

5.2.2. REMARKS. a) If the objects of $\mathscr{C}$ are H-semiprojective in $\mathscr{C}$ then it follows by Proposition 5.1.10 that $\mathscr{C}$ is a shape category.

b) If $\mathscr{C}$ is a shape category and $A, B \in \mathscr{A}\mathscr{C}$ are such $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$ then it follows by [2, Theorem 4.8] that A and B have the same shape in the sense of the general theory in [2].

It is interesting that one can exhibit shape categories $\mathscr{C}$ of C^*-algebras without proving that its objects are H-semiprojective in $\mathscr{C}$. As we shall see in certain cases, it is enough to look for $\mathrm{KK}_{+,\Sigma}$-semiprojectivity. This was in fact one of the starting points of our paper. Let $\mathscr{N}$ denote the category introduced in [37] as being the smallest full subcategory of the separable nuclear C^*-algebras which contain the separable type I C^*-algebras and is closed under stable isomorphism, inductive limits, extensions and crossed products by $\mathbf{R}$ and $\mathbf{Z}$.

5.2.3. THEOREM. *Let $\mathscr{C}$ be a "subcategory" of $\mathscr{N}$ satisfying the following conditions:*

a) *each $A \in \mathscr{C}$ is $\mathrm{KK}_{+,\Sigma}$-semiprojective relative to $\mathscr{C}$;*

b) *if A, $B \in \mathscr{C}$ and $\sigma \in \mathrm{KK}(A, B)_{+,\Sigma}$, then there is $\varphi \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ such that $[\varphi]_{\mathrm{KK}} = \sigma$;*

c) *if φ, $\psi \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and $[\varphi]_{\mathrm{KK}} = [\psi]_{\mathrm{KK}}$ then there is an inner automorphism η of B such that $\eta \circ \psi$ belongs to $\mathscr{C}$ and φ is homotopic to $\eta \circ \psi$.*

Then $\mathscr{C}$ is a shape category. Moreover if A, $B \in \mathscr{A}\mathscr{C}$ then $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$ iff $\mathrm{K}_*(A) \simeq \mathrm{K}_*(B)$ as scaled ordered groups.

Proof. Let (A_i, α_{ji}), (B_i, β_{ji}) be inductive systems in $\mathscr{C}$ and let $A = \lim A_i$, $B = \lim B_i$. In order to prove the theorem it suffices to show that the following are equivalent:

1) There is an isomorphism in $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))_{+,\Sigma}$;

2) A is isomorphic to B in the category $\mathrm{KK}_{+,\Sigma}$;

3) $(A_i, [\alpha_{ji}])$ is isomorphic to $(B_i, [\beta_{ji}])$ in the category inj-$\mathscr{H}$.

We have

1) $\Leftrightarrow$ 2) by Proposition 7.3 in [37] which applies since A, $B \in \mathscr{N}$.

3) $\Rightarrow$ 1) is immediate as noticed in [2].

It remains to prove 2) $\Rightarrow$ 3). By hypothesis A_i and B_i are $\mathrm{KK}_{+,\Sigma}$-semiprojective. Using Proposition 5.1.10 it follows from 2) that $(A_i, [\alpha_{ji}]_{\mathrm{KK}})$ is isomorphic to $(B_i, [\beta_{ji}]_{\mathrm{KK}})$ in inj-$\mathrm{KK}_{+,\Sigma}$. Such an isomorphism gives a diagram in $\mathrm{KK}_{+,\Sigma}$

$$\begin{tikzcd}
A_{i} \arrow[r] \arrow[dr, "\sigma_1"'] & \cdots \arrow[r] & A_{i_2} \arrow[r] \arrow[dr, "\sigma_2"'] & \cdots \arrow[r] & A_{i_3} \arrow[r] & \cdots \\
& B_{j_1} \arrow[r] \arrow[ur, "\tau_1"] & \cdots \arrow[r] & B_{j_2} \arrow[r] \arrow[ur, "\tau_2"] & \cdots \arrow[r] & {}
\end{tikzcd}$$

where the triangles commutes. Using condition b) we may assume that there are suitable homomorphisms φ_k, ψ_k, $k \geq 1$, in $\mathscr{C}$, such that

$$\sigma_k = [\varphi_k]_{\mathrm{KK}} \quad \text{and} \quad \tau_k = [\psi_k]_{\mathrm{KK}}.$$

Therefore $[\psi_k \circ \varphi_k]_{\mathrm{KK}} = [\alpha_{i_{k+1}i_k}]_{\mathrm{KK}}$ and $[\varphi_{k+1} \circ \psi_k]_{\mathrm{KK}} = [\beta_{j_{k+1}j_k}]_{\mathrm{KK}}$.

Based on condition c) we can find inductively two sequences of inner automorphisms $\gamma_k \in \mathrm{Aut}(B_{j_k})$ and $\delta_k \in \mathrm{Aut}(A_{i_{k+1}})$ such that if $\varphi_k' = \gamma_k \circ \varphi_k$ and $\psi_k' = \delta_k \circ \psi_k$ then $\psi_k' \circ \varphi_k'$ is homotopic to $\alpha_{i_{k+1}i_k}$ and $\varphi_{k+1}' \circ \psi_k$ is homotopic to $\beta_{j_{k+1}j_k}$. This means exactly that $(A_i, [\alpha_{ji}])$ and $(B_i, [\beta_{ji}])$ are isomorphic in inj-$\mathscr{H}$.

5.2.4. So far we have been very formalistic. That is why some comments on our technique for approaching shape problems are perhaps in order. Now the fundamental problem is to relate in inj-$\mathscr{H}$ two inductive systems whose limits are homotopic. More generally what can be said if the limits have the same K-theory groups (including order and scale if necessary)? The main difficulty in solving this problem within the homotopy category $\mathscr{H}$ comes from the possible absence of H-semiprojectivity. The point is that for certain categories $\mathscr{C} \subset \mathscr{S}$ we can overcome this difficulty by taking a devious route provided by the $\mathrm{KK}_{+,\Sigma}$-functor. Diagrammaticaly this is as follows:

$$\begin{array}{ccc} & A \simeq B \text{ in } \mathscr{H} \;\dashrightarrow\; (A_i) \simeq (B_i) \text{ in inj-}\mathscr{H} & \\ \mathrm{K}_*(A) \simeq \mathrm{K}_*(B) & \Big\Downarrow{\scriptstyle (1)} \qquad\qquad\qquad\qquad \Big\Uparrow{\scriptstyle (3)} & \\ \Big\| {\scriptstyle (1')} & & \\ & A \simeq B \text{ in } \mathrm{KK}_{+,\Sigma} \xRightarrow{(2)} (A_i) \simeq (B_i) \text{ in inj-}\mathrm{KK}_{+,\Sigma}, & \end{array}$$

where $A = \lim A_i$, $B = \lim B_i$.

Very informally the implications indicated by arrows hold since:

(1) KK is homotopic functor;

(1′) This implication is a result in [37];

(2) A_i, B_j are semiprojective in $\mathrm{KK}_{+,\Sigma}$;

(3) $\mathrm{KK}(A_i, B_j)_{+,\Sigma} = [A_i, B_j]$ or assume the weaker condition of 5.2.3c).

Let $\mathcal{O}$ be the category of Cuntz-Krieger algebras and proper (i.e. nonunital) $*$-homomorphisms. We shall illustrate the power of our formalism by giving a short proof for a theorem of Effros and Kaminker [18], stating that two algebras in $\mathscr{A}\mathcal{O}$ are shape equivalent iff they have isomorphic K-groups. Therefore, we shall discuss each arrow in the above diagram. Let A, $B \in \mathscr{A}\mathcal{O}$ such that $\mathrm{K}_*(A) \simeq \mathrm{K}_*(B)$.

(1′) Each $O_a \in \mathcal{O}$ is nuclear. Therefore $\mathscr{A}\mathcal{O} \subset \mathscr{N}$ and we can use [37, Proposition 7.3] to get that A is KK-equivalent to B.

(2) Since $\mathrm{K}_*(O_a)$ is finitely generated, it follows by [37, Theorem 1.4. and Proposition 7.13] that the functor $\mathrm{KK}(O_a, \cdot)$ is continuous. In our terms this means exactly that each $O_a \in \mathcal{O}$ is KK-semiprojective.

(3) As a variant of the computation in [9] it is proved in [18] that $[O_a, O_b] \simeq \simeq \mathrm{KK}(O_a, O_b)$ although the result is stated in a slightly different form.

Note that the above proof based on KK-semiprojectivity avoids the splitting principle for progroups proved in [18].

5.3. SEVERAL SHAPE CATEGORIES

Our main result concerning shape calculations is the following list of shape categories which verify the conditions of Theorem 5.2.3.

5.3.1. Recall that W_0^c denotes the category of finite connected CW-complexes. For each $n \geqslant 1$ we define the following four classes of spaces included in W_0^c.

$\mathcal{X}_1(n)$ consists of all spaces $X \in W_0^c$ of dimension $\leqslant n$ for which there are $p, q \in \mathbf{N}$, depending on X, such that $H^{\mathrm{even}}(X, \mathbf{Z}) = H^p(X, \mathbf{Z})$, $H^{\mathrm{odd}}(X, \mathbf{Z}) = H^q(X, \mathbf{Z})$ and these groups are torsion free.

Note that $S^p \vee S^q \in \mathcal{X}_1(n)$ provided that $p, q \leqslant n$ and $p - q \equiv 1 \pmod 2$.

$\mathcal{X}_2(2n)$ consists of all spaces $X \in W_0^c$ of dimension $\leqslant 2n$ which are $(2n-2)$-connected. Note that any finite connected CW-complex of dimension $\leqslant 2$ belongs to $\mathcal{X}_2(2)$.

$\mathcal{X}_3(2n)$ consists of those spaces $X \in \mathcal{X}_2(2n)$ with the property that $H^{2n}(X, \mathbf{Z})$ is a torsion group. Note that $\mathcal{X}_3(2)$ contains the non-orientable manifolds of dimension 2.

$\mathcal{X}_4(2n-1)$ consists of finite wedges of $(2n-1)$-spheres. Starting with the above classes of spaces we define the "categories" of C^*-algebras $\mathcal{C}_1(n)$, $\mathcal{C}_2(2n)$, $\mathcal{C}_3(2n)$, $\mathcal{C}_4(2n-1)$ as follows.

The objects of $\mathcal{C}_1(n)$ have the generic form $C(X) \otimes M_m$ where $X \in \mathcal{X}_1(n)$ and $m \in \mathbf{N}$. The set $\mathrm{Hom}_{\mathcal{C}_1(n)}(A, B)$ of morphisms from A to B consists of all $3(n+3)/2$-large $*$-homomorphisms (2.1.8).

The objects of $\mathcal{C}_2(2n)$ have the generic form $C(X) \otimes M_m$ where $X \in \mathcal{X}_2(2n)$ and $m \in \mathbf{N}$. The set $\mathrm{Hom}_{\mathcal{C}_2(2n)}(A, B)$ consists of all $3(n+3)/2$-large $*$-homomorphisms from A to B.

The objects of $\mathcal{C}_3(2n)$ have the generic form $\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ where $X_i \in \mathcal{C}_3(2n)$ and $n_i, q \in \mathbf{N}$. The set $\mathrm{Hom}_{\mathcal{C}_3(2n)}(A, B)$ of morphisms from A to B consists of all $3(n+3)/2$-full $*$-homomorphisms (2.1.8).

The objects of $\mathcal{C}_4(2n-1)$ have the generic form $\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ where $X_i \in \mathcal{X}_4(2n-1)$ and $n_i, q \in \mathbf{N}$. The set $\mathrm{Hom}_{\mathcal{C}_4(2n-1)}(A, B)$ consists of all $3(n+2)/2$-large $*$-homomorphisms.

5.3.2. Remarks. a) Let $\mathcal{C}$ be one of the categories $\mathcal{C}_1(n)$, $\mathcal{C}_2(2n)$, $\mathcal{C}_4(2n-1)$. Let (A_i, φ_{ji}) be an inductive sistem of C^*-algebras such that $A_i \in \mathcal{C}$ but φ_{ji} are not assumed to be large in any sense. Let $A = \lim(A_i, \varphi_{ji})$. The associated AF-algebra $r(A)$ defined in Section 2 depends only on A. This can be proved using Propositions 5.1.4, 5.1.6 in conjunction with Proposition 2.1.3. Moreover, by the results of Section 2 it follows that A can be represented as the limit of an inductive system with $3(n+3)/2$-large bonding maps if and only if $K_0(A)$ has large denominators, or equivalently, if and only if $K_0(r(A))$ has large denominators.

b) Let $A = \lim(A_i, \varphi_{ji})$ where $A_i \in \mathcal{C}_3(2n)$ but the embeddings φ_{ji} are not assumed to be large or full. Like above it can be shown that $r(A)$ depends only on A. If $r(A)$ is not stably isomorphic to $\mathcal{K}$ then A can be represented as the limit of an inductive system with $3(n+3)/2$-full morphisms if and only if $K_0(r(A))$ is simple.

5.3.3. THEOREM. *Let $\mathscr{C}$ be one of the categories $\mathscr{C}_1(n)$, $\mathscr{C}_2(2n)$, $\mathscr{C}_3(2n)$, $\mathscr{C}_4(2n-1)$. Then $\mathscr{C}$ is a shape category. Moreover for A, $B \in \mathscr{A}\mathscr{C}$ (1.2.12) the following assertions are equivalent:*

a) $\mathrm{K}_*(A) \simeq \mathrm{K}_*(B)$ *as* $\mathbf{Z}_2$*-graded scaled ordered groups;*

b) $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$*;*

c) *A and B have the same shape invariant in the sense of Blackadar* [2].

Proof. If follows by Propositions 5.1.4, 5.1.6 and Theorems 4.3.1, 4.3.2, that the category $\mathscr{C}$ satisfies the conditions of Theorem 5.2.3. Therefore a) $\Leftrightarrow$ b). The second remark of 5.2.3 gives b) $\Rightarrow$ c), while c) $\Rightarrow$ a) is a general fact based on the continuity of K-theory.

5.3.4. COROLLARY. *Let $\mathscr{C}$ as avove and $A, B \in \mathscr{A}\mathscr{C}$. The following assertions are equivalent:*

a) $\mathrm{K}_*(A) \simeq \mathrm{K}_*(B)$ *as* $\mathbf{Z}_2$*-graded groups;*

b) $\mathrm{Sh}_{\mathscr{C}}(A \otimes \mathscr{K}) = \mathrm{Sh}_{\mathscr{C}}(B \otimes \mathscr{K})$*;*

c) *$A \otimes \mathscr{K}$ and $B \otimes \mathscr{K}$ have the same shape invariant in the sense of Blackadar.*

Proof. If $A \in \mathscr{A}\mathscr{C}$ then $A \otimes \mathscr{K} \in \mathscr{A}\mathscr{C}$ and $\Sigma_*(A \otimes \mathscr{K}) = \mathrm{K}_*(A)_+$.

5.3.5. REMARKS. a) Let $\mathscr{C}$ be a shape category. Let $A = \lim(A_i, \alpha_{ji})$, $B = \lim(B_i, \beta_{ji})$ with $A_i, B_i, \alpha_{ji}, \beta_{ji} \in \mathscr{C}$. Then $\mathrm{Sh}_{\mathscr{C}}(A) = \mathrm{Sh}_{\mathscr{C}}(B)$ means exactly that there is a diagram

$$\begin{tikzcd}
A_{i_1} \arrow[rr] \arrow[drr] & & \dots \arrow[rr] & & A_{i_2} \arrow[rr] \arrow[drr] & & \dots \arrow[rr] & & A_{i_3} \arrow[rr] \arrow[drr] & & \dots \\
& & B_{j_1} \arrow[rr] \arrow[urr] & & \dots \arrow[rr] & & B_{j_2} \arrow[rr] \arrow[urr] & & \dots \arrow[rr] & & B_{j_3}
\end{tikzcd}$$

such that each triangle commutes within homotopy.

The existence of such a diagram is not automatically assured even if $A = B$ unless $\mathscr{C}$ is a shape category or has other related properties, since it is not evident that inductive systems having the same limit are related in inj-$\mathscr{H}$.

b) If $A \in \mathscr{A}\mathscr{C}$ for $\mathscr{C} = \mathscr{C}_1(n)$, $\mathscr{C}_2(2n)$, $\mathscr{C}_3(2n)$ then $\mathrm{K}_0(A)$ is simple (as an ordered group) and it can be proved that

$$\mathrm{K}_*(A)_+ = \{(0,0)\} \cup (\mathrm{K}_0(A)_+ \setminus \{0\}) \oplus \mathrm{K}_1(A).$$

Consequently, the assertion a) of Theorem 5.3.2 is equivalent to

a′) $\mathrm{K}_0(A) \simeq \mathrm{K}_0(B)$ as scaled ordered groups and $\mathrm{K}_1(A) \simeq \mathrm{K}_1(B)$.

However, for $\mathscr{C} = \mathscr{C}_4(2n-1)$, a) cannot be replaced by a′) as it is shown below.

c) For $p, q \geqslant 2$, let $A(p, q)$ be the C^*-algebra arising as the inductive limit of

$$\to C(\mathrm{S}^1) \otimes M_{p^n} \xrightarrow{\varphi_n} C(\mathrm{S}^1) \otimes M_{p^{n+1}} \longrightarrow \cdots$$

where the embedding φ_n is given by the rule

$$\varphi_n(f)(z) = \begin{pmatrix} f(z^q) & & & \\ & f(z_0) & & \\ & & \ddots & \\ & & & f(z_0) \end{pmatrix}$$

$f \in C(S^1) \otimes M_{p^n}$, $z \in S^1$ and z_0 is the base point of S^1.

It is easily seen that $K_0(A(p, q)) \simeq Z[1/p]$ with the order induced by the embedding $Z[1/p] \subset Q \subset R$ and $K_1(A(p, q)) = Z[1/q]$. Let

$$A = A(2, 3) \oplus A(3, 2) \quad \text{and} \quad B = A(2, 2) \oplus A(3, 3).$$

It is clear that $K_0(A) \simeq K_0(B)$ as ordered scaled groups and $K_1(A) \simeq K_1(B)$. However *A is not shape equivalent to B* since arithmetical reasons prevent $K_*(A)$ to be isomorphic to $K_*(B)$ as Z_2-graded groups. To prove this, observe that any isomorphisms

$$\varphi : K_0(A) = Z[1/2] \oplus Z[1/3] \to Z[1/2] \oplus Z[1/3] = K_0(B)$$

$$\psi : K_1(A) = Z[1/2] \oplus Z[1/3] \to Z[1/3] \oplus Z[1/2] = K_1(B)$$

are of the from

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & \psi_{12} \\ \psi_{21} & 0 \end{pmatrix}.$$

This follows since the only morphism $Z[1/p] \to Z[1/q]$ is the trivial one (provided that p, q are distinct primes). Also it is easily seen that $(a, b; x, y) \in Z[1/2] \oplus Z[1/3] \oplus \oplus Z[1/2] \oplus Z[1/3] = K_*(A)$ belongs to $K_*(A)_+$ iff $a > 0$, $b > 0$ or $a > 0$, $b = 0$, $y = 0$ or $a = 0$, $b > 0$, $x = 0$. A similar description holds for $K_*(B)$. Therefore $(\varphi, \psi)(1, 0; 1, 0) = (\varphi_{11}(1), 0, 0, \psi_{21}(1))$ is not positive since $\psi_{21}(1) \neq 0$ (recall that ψ_{21} is an isomorphism).

The categories for which we succeded in shape computations are rather limited. There are two essential difficulties to be overcome in order to extend the above results to larger categories. The first one is the absence of $(KK_{+,\Sigma})$-semiprojectivity in $\mathscr{C}(n)$ even for nice algebras like $C(S^2)$ or $C(S^1 \times S^1)$. The second one is the limited power of K-theory in homotopy computations (see Section 4). For instance for $A_i = C(S^1 \times S^3) \otimes M_{n_i}$ $(i = 1, 2)$ the canonical map $[A_1, A_2] \to \mathrm{Hom}(K_*(A_1), K_*(A_2))_{+,\Sigma}$ is not surjective. Having this it is easy to construct inductive limit C^*-algebras having the same (scaled, ordered) K-theory but for which do not exist diagrams as in 5.3.5 a) (see 5.3.6 below).

If the connective K-theory would extend to a continuous theory on a larger category of C^*-algebras one would have a powerful tool in shape problems.

5.3.6. Let $A_i = C(S^1 \vee S^3) \times M_{9^i}$. Using Theorems 3.5.5 and 4.2.11 it is easily seen that the image of the map

$$[A_i, A_{i+1}] \to \mathrm{Hom}(K_1(A_i), K_1(A_{i+1})) \simeq M_2(\mathbf{Z})$$

is exactly

$$\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,\, b,\, c \in \mathbf{Z}\right\}.$$

Let $u = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, $x = vu = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$, $y = uv = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.
Choose $\varphi_i, \psi_i \in \mathrm{Hom}(A_i, A_{i+1})$ such that $K_1(\varphi_i) = x$ and $K_1(\psi_i) = y$ and define $A = \lim(A_i, \varphi_i)$, $B = \lim(A_i, \psi_i)$. It is clear that $\underline{K_0(A) \simeq K_0(B)}$ as ordered scaled group. Moreover the commutative diagram

$$\begin{tikzcd}
K_1(A_1) \arrow[rr, "x"] \arrow[dr, "u"'] & & K_1(A_2) \arrow[rr, "x"] \arrow[dr, "u"'] & & K_1(A_3) \arrow[rr, "x"] \arrow[dr, "u"'] & & {} \\
& K_1(A_2) \arrow[ur, "v"] \arrow[rr, "y"'] & & K_1(A_3) \arrow[ur, "v"] \arrow[rr, "y"'] & & K_1(A_4) &
\end{tikzcd}$$

shows that $\underline{K_1(A) \simeq K_1(B)}$.

However, the inductive systems (A_i, φ_i) and (A_i, ψ_i) *are not isomorphic in inj-$\mathscr{H}$*. Indeed, if there would exist a homotopy commutative diagram as in 5.3.5 a), then passing to K_1-groups we should find a commutative diagram of the form

$$\begin{tikzcd}
K_1(A_{i_1}) \arrow[rr, "x^{n_1}"] \arrow[dr, "z_1"'] & & K_1(A_{i_2}) \arrow[rr, "x^{n_2}"] \arrow[dr, "z_3"'] & & K_1(A_{i_3}) \arrow[dr] & \\
& K_1(A_{j_1}) \arrow[ur, "z_2"] \arrow[rr, "y^{m_1}"'] & & K_1(A_{j_2}) \arrow[ur, "z_4"] \arrow[rr, "y^{m_2}"'] & & {}
\end{tikzcd}$$

with $z_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \in M_2(\mathbf{Z})$ and $n_i, m_i \geqslant 2$. But the equalities $z_2 z_1 = x^{n_1}$, $z_3 z_2 = y^{m_1}$, $z_4 z_3 = x^{n_2}$ can not hold simultaneously since there do not exist $c_1, c_2, c_3 \in \mathbf{Z}$ such that $c_2 c_1 = 2^{n_1}$, $c_3 c_2 = 3^{m_1}$, $c_4 c_3 = 2^{n_2}$.

6. STABILITY PROPERTIES OF HOMOMORPHISMS

Let X be a finite connected CW-complex with base point $x_0 \in X$ and let $F^k(X) = \mathrm{Hom}_1(C(X), M_k)$. There is a natural embedding $\alpha_k \colon F^k(X) \to F^{k+1}(X)$ given by the orthogonal sum with the morphism $f \mapsto f(x_0)$. The main result of this section asserts that this embedding is a $2[k/3]$-homotopy equivalence for any X as above and $k \geqslant 3$ (see Theorem 6.4.2).

Basically, the idea of the proof is the following one. As a first step it is proved that $\pi_1(F^k(X)) = \pi_1(F^{k+1}(X))$ and this is done via $\pi_1(P^k(X))$. A key fact here is the comparison theorem between $F^k(X)$ and the symmetric product $P^k(X)$. Next, the main result is proved for $X = \bigvee S^1$ (and this is the most difficult part). Finally, the induction over the numers of cells of dimension $\geqslant 2$ is carried out.

6.1. A COMPARISON THEOREM BETWEEN $F^k(X)$ AND THE SYMMETRIC PRODUCT $P^k(X)$

For $k \geqslant 1$, the k-fold symmetric product of X, denoted by $P^k(X)$ is defined by $P^k(X) = X^k/\mathfrak{S}_k$ where X^k denotes the k-fold cartesian product of X with itself and $\mathfrak{S}_k$ denotes the symmetric group on k objects regarded as acting on X^k by permuting the coordinates:

$$\sigma(x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \quad \sigma \in \mathfrak{S}_k.$$

If $\underline{x} = (x_1, \ldots, x_k)$, then we shall use the notation $[\underline{x}] = [x_1, \ldots, x_k]$ for a generic element of $P^k(X)$.

There is a natural embedding $\beta_k \colon P^k(X) \to P^{k+1}(X)$ given by

$$\beta_k[x_1, \ldots, x_k] = [x_0, x_1, \ldots, x_k]$$

which is used to define $P^\infty(X) = \lim P^k(X)$.

Dold and Thom [14] have introduced the notion of quasifibration which enabled them to prove that $\pi_n(P^\infty(X)) = \tilde{\mathrm{H}}_n(X, \mathbf{Z})$.

We shall study maps which are not quasifibrations but which have similar properties up to some dimension. Therefore it is natural to make the following

6.1.1. DEFINITION. A continuous map $p \colon E \to B$ between topological Hausdorff spaces is called *m-quasifibration*, $(0 \leqslant m \leqslant \infty)$, if for all points $b \in B$ and $e \in p^{-1}(b)$ the induced maps $p_* \colon \pi_q(E, p^{-1}(b), e) \to \pi_q(B, b)$ are isomorphisms for $0 \leqslant q \leqslant m - 1$ and epimorphisms for $q = m$. (For $m = \infty$ one obtains the definition of the quasifibration.)

A careful inspection of the proof of Satz 2.2 in [14] shows that one has the following

6.1.2. THEOREM. *Let $0 \leqslant m \leqslant \infty$, $p: E \to B$ continuous and $\mathcal{U} = (U_i)_{i \in L}$ an open covering of B such that*

a) *For each $i \in L$, $p: p^{-1}(U_i) \to U_i$ is a m-quasifibration,*

b) *Each nonvoid intersection $U_i \cap U_j$ can be written as a union of elements in $\mathcal{U}$.*
Then p is m-quasifibration.

6.1.3. REMARK. If $p: E \to B$ is a m-quasifibration then it follows from the homotopy sequence of the pair $p^{-1}(b) \subset E$ that there is an exact sequence:

$$\pi_m(p^{-1}(b)) \to \pi_m(E) \to \pi_m(E, p^{-1}(b)) \to \pi_{m-1}(p^{-1}(b)) \to \pi_{m-1}(E) \to \pi_{m-1}(B) \to$$

$$\to \pi_{m-2}(p^{-1}(b)) \to \dots .$$

Therefore if $p: E \to B$ is an m-quasifibration with connected E and $p^{-1}(b)$ is m-connected for some $b \in B$, then p is an m-equivalence.

In this section we shall meet several times continuous maps $p: E \to B$ which are surjective and satisfy the following conditions:

6.1.4. a) B is locally contractible, i.e. each $x \in B$ has a fundamental system of open neighbourhoods $(U_j)_{j \geqslant 1}$ together with continuous homotopies $h_j: U_j \times I \to U_j$ such that $h_{j,0} = \mathrm{id}(U_j)$, the image of $h_{j,1} = \{x\}$ and $h_{j,t}(x) = x$ for all $t \in I$, $j \geqslant 1$ (by definition $h_{j,t}(y) = h_j(y, t)$).

b) Each homotopy h_j lifts to a homotopy

$$H_j: p^{-1}(U_j) \times I \to p^{-1}(U_j), \quad p \circ H_{j,t} = h_{j,t} \circ p \quad \text{for all } t \in I,$$

such that $H_{j,0} = \mathrm{id}(p^{-1}(U_j))$, the image of $H_{j,1} \subset p^{-1}(x)$ and $H_{j,t}(y) = y$ for all $y \in p^{-1}(x)$, $t \in I$, $j \geqslant 1$. ($H_{j,t}(y) = H_j(y, t)$.)

c) If $x' \in U_j$ and $H'_{j,1}$ denotes the restriction of $H_{j,1}$ at $p^{-1}(x')$, $H'_{j,1}: p^{-1}(x') \to$ $\to p^{-1}(x)$, then for any $y' \in p^{-1}(x')$, $y = H_{j,1}(y')$,

$$\pi_q(H'_{j,1}): \pi_q(p^{-1}(x'), y') \to \pi_q(p^{-1}(x), y)$$

is an isomorphism for $0 \leqslant q \leqslant m - 1$ and an epimorphism for $q = m$.

d) For all $x \in B$, $p^{-1}(x)$ is 0-connected.

Note that if each fibre $p^{-1}(x)$ is m-connected then c) is automatically satisfied. Moreover, if there are triangulations $|K| \simeq E$ and $|L| \simeq B$ such that modulo these identifications $p: E \to B$ is induced by a simplicial proper map $K \to L$, then by standard techniques with barycentric coordinates it is easily seen that a) and b) are satisfied.

6.1.5. PROPOSITION. *If $p: E \to B$ is surjective and satisfies the conditions a), b) c), d) from above then p is an $(m + 1)$-quasifibration (*cf. *Hilfssatz* 2.10 *in* [14]*).*

Proof. For each $x \in B$ let $(U_j^x)_{j\geq 1}$, $(h_j^x)_{j\geq 1}$, $(H_j^x)_{j\geq 1}$ having the properties a) – d). It is clear that the family $U = (U_j^x)_{x\in X, j\geq 1}$ satisfies the condition b) of Theorem 6.1.2. Moreover we shall prove that $p: p^{-1}(U_j^x) \to U_j^x$ is a $(m+1)$-quasifibration for each $x \in X$ and $j \geq 1$. Since U_j^x is contractible it is enough to check that

$$\pi_q(p^{-1}(U_j^x), p^{-1}(x'), y') = 0$$

whenever $0 \leq q \leq m$, $x' \in U_j^x$ and $y' \in p^{-1}(x')$. Now there is a commutative diagram

$$\begin{array}{ccccccc}
\pi_q(p^{-1}(x'), y') & \to & \pi_q(p^{-1}(U_j^x), y') & \to & \pi_q(p^{-1}(U_j^x), p^{-1}(x'), y') & \to & \pi_{q-1}(p^{-1}(x'), y') \\
\downarrow{\scriptstyle \pi_q(H'^x_{j,1})} & & \downarrow{\scriptstyle \pi_q(H^x_{j,1})} & & & & \\
\pi_q(p^{-1}(x), y) & \to & \pi_q(p^{-1}(U_j^x), y). & & & &
\end{array}$$

The first left vertical arrow is an isomorphism for $q \leq m-1$ and an isomorphism for $q = m$ by 6.1.4c). The second vertical arrow is an isomorphism for all q since $p^{-1}(U_j^x)$ is 0-connected. The bottom orizontal arrow is an isomorphism by 6.1.4b). Since the upper sequence is exact it follows that $\pi_q(p^{-1}(U_j^x), p^{-1}(x'), y') = 0$ for all $0 \leq q \leq m$.

6.1.6. We need also the following formalism which enables us to describe the the stratification of $\mathrm{Hom}_1(C(X), M_k)$ given by the multiplicities of the proper values of the homomorphisms.

Let $\mathscr{L}$ be the set of all disjoint partitions of the set $A = \{1, 2, \ldots, k\}$. A generic element $I \in \mathscr{L}$ is described by $I = (I_1, \ldots, I_m)$ so that A is the union of I_j and $I_i \cap I_j = \emptyset$ whenever $i \neq j$. $\mathscr{L}$ becomes a lattice with the order: $J \leq I$ iff the partition I is finer than J.

The symmetric groups $\mathfrak{S}_k$ act by order preserving automorphisms on $(\mathscr{L}, \leq)$ by the rule:

$$\sigma(I) = (\sigma(I_1), \ldots, \sigma(I_m)), \quad \sigma \in \mathfrak{S}_k.$$

If $x = (x_1, \ldots, x_k) \in X^k$ then $I(x)$ denotes the partition of A which corresponds to the following equivalence relation: if $i, j \in A$ then $i \sim j$ iff $x_i = x_j$. That is i and j are contained in the same I_r iff $x_i = x_j$.

Let $e_1, \ldots, e_k$ be the canonical minimal projections in M_k; $e_i e_j = \delta_{ij} e_i$ and $e_1 + \ldots + e_k = 1_k$. For $I = (I_1, \ldots, I_m) \in \mathscr{L}$ we define

$$e(I_r) = \sum_{i \in I_r} e_i.$$

Each $\sigma \in \mathfrak{S}_k$ gives an isometric endomorphism of $\mathbf{C}^k$

$$\sigma(\lambda_1, \ldots, \lambda_k) = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}).$$

If σ is regarded as an element of $U(k)$ we have $\sigma^* e_i \sigma = e_{\sigma(i)}$, hence $\sigma^* e(I_r)\sigma = e(\sigma(I_r))$. Let $A(I)$ be the C^*-algebra of those elements in M_k which commute with all the projections $e(I_1), \ldots, e(I_n)$. We let $U(I)$ denote the unitary group of of $A(I)$. If $J \leqslant I$ then $U(I) \subset U(J)$. Therefore there is a natural map

$$U(k)/U(I) \to U(k)/U(J)$$

which is easily seen to be a 2-equivalence (consider the homotopy sequences of the two fibrations).

6.1.7. We define $\psi: X^k \times U(k)] \to \mathrm{Hom}_1(C(X), M_k)$ as follows: if $u \in U(k)$ and $\underline{x} = (x_1, \ldots, x_k) \in X^k$, $I(\underline{x}) = (I_1(\underline{x}), \ldots, I_m(\underline{x})) \in \mathscr{L}$, $i_r \in I_r(\underline{x})$, $1 \leqslant r \leqslant m$, then

$$\psi(\underline{x}, u)(f) \to u\left(\sum_{r=1}^{m} f(x_{i_r})e(I_r(\underline{x}))\right)u^*, \quad \text{for all } f \in C(X).$$

For the sake of brevity $\psi(\underline{x}, u)$ will be denoted with $[\underline{x}, u]$ and sometimes with $[\underline{x}, e]$ where $e = (ue(I_1)u^*, \ldots, ue(I_m)u^*)$ is the list of the spectral projections of the homomorphism $\psi(\underline{x}, u)$. Also is clear that $I(\underline{x})$ gives the multiplicities of the proper values of $\psi(\underline{x}, u)$.

6.1.8. Proposition. *The map $\psi: X^k \times U(k) \to \mathrm{Hom}_1(C(X), M_k)$ is continuous and surjective. Moreover $\psi(\underline{x}, u) = \psi(\underline{y}, v)$ iff there is $\sigma \in \mathfrak{S}_k$ such that $\sigma(\underline{x}) = \underline{y}$ and $u^* v\sigma \in U(I(\underline{x}))$.*

Proof. The continuity of ψ is obvious and ψ is surjective by the spectral theorem. Finally, equal homomorphisms must have the same proper values counted with multiplicities and the same spectral projections corresponding to equal proper values, whence the second part of the statement.

6.1.9. Let $\mathrm{pr}: X^k \times U(k) \to X^k$ be the projection onto the first factor and let $\psi_0: X^k \to P^k(X)$ be the canonical map $\psi_0(\underline{x}) = [\underline{x}]$.

We define the map $p: F^k(X) \to P^k(X)$ by asking the following diagram to be commutative

$$\begin{array}{ccc} X^k \times U(k) & \xrightarrow{\mathrm{pr}} & X^k \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi_0} \\ F^k(X) & \xrightarrow{p} & P^k(X). \end{array}$$

That is $p[\underline{x}, u] = [\underline{x}]$ and this formula is correct by Proposition 6.1.8. Note that $p^{-1}[\underline{x}] \simeq U(k)/U(I(\underline{x}))$.

Since $F^k(X)$ is homeomorphic to $F_0^k(X) = \mathrm{Hom}(C_0(X), M_k)$, p will induce a map $\eta: \mathrm{Hom}(C_0(X), M_k) \to P^k(X)$ which naturally extends to $\eta: \varinjlim \mathrm{Hom}(C_0(X), M_k) \to \varinjlim P^k(X) = P^\infty(X)$.

Now the natural embedding $\varinjlim F_0^k(X) \to F(X) = \mathrm{Hom}(C_0(X), \mathscr{K})$ is a homotopy equivalence so that up to homotopy $\eta: \mathrm{Hom}(C_0(X), \mathscr{K}) \to P^\infty(X)$.

6.1.10. THEOREM. *The map $p: F^k(X) \to P^k(X)$ is a 3-quasifibration for any $1 \leqslant k \leqslant \infty$.*

Proof. We shall prove that p verifies the hypotheses of Proposition 6.1.5 for $m = 2$. Let $x = (x_1, \ldots, x_k) \in X^k$ and choose $V_0 = V_1 \times \ldots \times V_k$, (depending on x), such that V_i are open neighbourhoods of x_i small enough to assure that $x_i \neq x_j$ iff $V_i \cap V_j = \emptyset$, and $x_i = x_j$ iff $V_i = V_j$. These choices imply that for any $\sigma_1, \sigma_2 \in \mathfrak{S}_k$, $\sigma_1(V_0) \cap \sigma_2(V_0) \neq \emptyset$ iff $\sigma_1(I_r(x)) = \sigma_2(I_r(x))$. Moreover if $\underline{y} \in \sigma(V_0)$ for some $\sigma \in \mathfrak{S}_k$, then $I(\sigma(\underline{x})) \leqslant I(y)$. Since X is locally contractible, shrinking the V_i if necessary, we can find $h^\circ: V_0 \times I \to I$, $h^\circ = (h_1, \ldots, h_k)$, where $h_i: V_i \times I \to V_i$ with $h_i = h_j$ iff $V_i = V_j$ are such that $h_0^\circ = \mathrm{id}(V_0)$, the image of $h_1^\circ = \{x\}$ and $h_t^\circ(x) = x$ for all $t \in I$.

Starting with h° we define for each $\sigma \in \mathfrak{S}_k$, $h_\sigma^\circ: \sigma(V_0) \times I \to \sigma(V_0)$ by $h_\sigma^\circ = (h_{\sigma(1)}, \ldots, h_{\sigma(k)})$. Finally we let V be the union of all $\sigma(V_0)$ over $\sigma \in \mathfrak{S}_k$ and define the homotopy $h: V \times I \to V$ such $h\,\sigma(V_0) = h_\sigma^\circ$. By construction $h_t(\sigma(\underline{y})) = \sigma(h_t(\underline{y}))$ and $I(h_t(\underline{y})) \leqslant I(\underline{y})$ for $\underline{y} \in V$ and $t \in I$.

After these preparations we can reach the condition 6.1.4 a), b), c), d).

a) Let V as above and let $U = \psi_0(V)$ be the corresponding open neighbourhood of $[x]$ in $P^k(X)$. We define $\tilde{h}: U \times I \to U$ by $\tilde{h}_t([\underline{y}]) = [h_t(\underline{y})]$ and the couple $(U, \tilde{h})$ satisfies 6.1.4 a).

b) Since V is $\mathfrak{S}_k$-equivariant, $p^{-1}(U) = \psi(V \times U(k))$. We define $\tilde{H}: p^{-1}(U) \times I \to p^{-1}(U)$ by $\tilde{H}_t[\underline{y}, u] = [h_t(\underline{y}), u]$. Let us check that $\tilde{H}$ is well defined. If $[\underline{y}, u] = [\underline{z}, v]$ then there is some $\sigma \in \mathfrak{S}_k$ such that $\sigma(\underline{y}) = \underline{z}$ and $u^* v\sigma \in U(I(\underline{y}))$ (6.1.8). It follows that $\sigma(h_t(\underline{y})) = h_t(\underline{z})$ since h_t is $\mathfrak{S}_k$-equivariant and $u^* v\sigma \in U(I(\underline{y})) \subset U(I(h_t(\underline{y})))$ since $I(h_t(\underline{y})) \leqslant I(\underline{y})$. Hence $[h_t(\underline{y}), u] = [h_t(\underline{z}), v]$. It is clear that $p \circ \tilde{H}_t = \tilde{h}_t \circ p$ and $p^{-1}(U)$, $\tilde{H}$ satisfy 6.1.4 b).

c) There is a commutative diagram

$$\begin{array}{ccc} p^{-1}[\underline{x}'] & \longrightarrow & p^{-1}[\underline{x}] \\ \downarrow \wr & & \downarrow \wr \\ U(k)/U(I(\underline{x}')) & \longrightarrow & U(k)/U(I(\underline{x})) \end{array}$$

where $\underline{x}' \in V_0$ and $U(I(\underline{x}')) \subset U(I(\underline{x}))$ since $I(\underline{x}) \leqslant I(\underline{x}')$.

If $I, J \in \mathscr{L}$ and $J \leqslant I$ then $U(k)/U(I) \to U(k)/U(J)$ is a 2-equivalence as noticed earlier.

d) $p^{-1}[\underline{x}] \simeq U(k)/U(I(\underline{x}))$ is connected.

6.1.11. COROLLARY. *p: $F^k(X) \to P^k(X)$ is a 3-equivalence for any $1 \leqslant k \leqslant \infty$.*

Proof. If $\underline{x} = (x_1, \ldots, x_k)$ and $x_1 = x_2 = \ldots = x_k$ then $U(I(\underline{x})) = U(k)$ so that $p^{-1}[\underline{x}]$ reduces to a point. Since, by Theorem 6.1.10, p is a 3-quasifibration it follows by Remark 6.1.3 that p is a 3-equivalence.

6.1.12. Note that the following diagram is commutative

$$\begin{array}{ccc} F^k(X) & \longrightarrow & F^{k+1}(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ P^k(X) & \longrightarrow & P^{k+1}(X). \end{array}$$

6.2. THE EMBEDDINGS $P^k(X) \to P^{k+1}(X)$

If X is a finite connected CW-complex then it can be proved that the embedding β_k: $P^k(X) \to P^{k+1}(X)$, $\beta_k[\underline{x}] = [x_0\underline{x}]$, is a k-equivalence ($k \geqslant 2$). Since we do not need this result in full generality we shall prove in this paragraph only the following weaker result.

6.2.1. PROPOSITION. *If $k \geqslant 2$ then $\pi_1(P^k(X)) \to \pi_1(P^{k+1}(X))$ is an isomorphism.*

This proposition can be regarded as a first step to the main connectivity theorem since by Corollary 6.1.11 $\pi_1(P^k(X)) \simeq \pi_1(F^k(X))$. The proof of 6.2.1 requires certain preliminaries.

6.2.2. For $l, s \geqslant 1$ let $T_{l,s}$ denote the set of all $(a_1, \ldots, a_l) \in ([-1, 1]^s)^l$ such that some a_i has at least one coordinate equal to ± 1. We define $D_{l,s}$ to be the image of $T_{l,s}$ in $P^l([-1, 1]^s)$, i.e. $D_{l,s} = T_{l,s}/\mathfrak{S}_l$. Note that $D_{1,1} \simeq S^0$ and $D_{l,s}$ is connected if $(l, s) \neq (1,1)$.

6.2.3. LEMMA. *If $(l, s) \neq (1, 1), (1, 2)$, then $D_{l,s}$ is simply connected.*

Proof. $T_{l,s}$ can be identified to S^{ls-1} and the quotient map φ: $S^{ls-1} \to D_{l,s}$ is a 1-quasifibration. Since there are points $x \in D_{l,s}$ such that $\varphi^{-1}(x)$ is a singleton it follows that $\varphi_*: \pi_1(S^{ls-1}) \to \pi_1(D_{l,s})$ is onto.

6.2.4. Now we are going to describe $P^{k+1}(X) \setminus P^k(X)$ using the cell structure of X. Let $X = e_1 \cup e_2 \cup \ldots \cup e_N$ be a cell decomposition of X with $\dim e_i \leqslant \dim e_{i+1}$. Since we are interested into homotopy questions, we may assume that X has a single vertex e_1 and so $\dim e_2 \geqslant 1$. Recall that if $(x_1, \ldots, x_k) \in X^k$ then its class in $P^k(X)$ is denoted by $[x_1, \ldots, x_k]$.

Now $P^{+1}(X)$ has a decomposition

$$P^{k+1}(X) = P^k(X) \cup \bigcup_{\substack{j_1,\dots,j_{k+1}\\ 1\leqslant j_i\leqslant N}} [e_{j_1} \times e_{j_2} \times \dots \times e_{j_{k+1}}].$$

Since $[e_{j_1} \times \dots \times e_{j_{k+1}}] = [e_{j_{\sigma(1)}} \times \dots \times e_{j_{\sigma(k+1)}}]$ for all $\sigma \in \mathfrak{S}_{k+1}$ we have

$$\bigcup_{\substack{j_1,\dots,j_{k+1}\\ j\leqslant j_i\leqslant N}} [e_{j_1} \times \dots e_{j_{k+1}}] = \bigcup_{1\leqslant j_1\leqslant \dots \leqslant j_{k+1}\leqslant N} [e_{j_1} \times \dots \times e_{j_{1+k}}].$$

For given $1 \leqslant j_1 \leqslant \dots \leqslant j_{k+1} \leqslant N$, let $t(1), t(2), \dots, t(r)$ be determined by the conditions

$$j_1 = \dots = j_{t(1)} < j_{t(1)+1} = \dots = j_{t(1)+t(2)} < \dots = j_{t(1)+\dots+t(r)}$$

and put $J = \{j_1, \dots, j_{k+1}\}$

$$J_i = \{j_s : t(1) + \dots + t(i-1) + 1 \leqslant s \leqslant t(1) + \dots + t(i)\} \quad 1 \leqslant i \leqslant r.$$

Define

$$E(J) = [e_{j_1} \times \dots \times e_{j_{k+1}}] = \bigtimes_{q\in J} e_q/\mathfrak{S}_{k+1} \subset P^{k+1}(X),$$

$$E(J)_i = \bigtimes_{q\in J_i} e_q/\mathfrak{S}_{t(i)}.$$

Then

$$E(J) = E(J_1) \times \dots \times E(J_r).$$

Note that $e_{j_1} \times \dots \times e_{j_{k+1}}$ (resp. $\bigtimes_{q\in J_i} e_q$) has a natural cone structure (as a product of balls) over the bord $\partial(e_{j_1} \times \dots \times e_{j_{k+1}})$ (resp. over $\partial(\bigtimes_{q\in J_i} e_q)$). Since the action of $\mathfrak{S}_{k+1}$ (resp. of $\mathfrak{S}_{t(i)}$) is compatible with this cone structure, the quotient space $E(J)$ (resp. $E(J)_i$) has a natural cone structure over $\partial E(J)$ (resp. over $\partial E(J)_i$). But obviously $\partial E(J)_i$ can be identified with $D_{t(i),d(i)}$, where $d(i) = \dim e_{j'}$ and $j' = j_{t(1)} + \dots + j_{t(i)}$. By the general formula $\bigtimes_{i=1}^{r} \operatorname{Cone} X_i = \operatorname{Cone}\left(\mathop{*}_{i=1}^{r} X_i\right)$ where $X_1 * X_2$ denotes the join operation between spaces ([23]), we obtain that

$$E(J) = \bigtimes_{i=1}^{r} \operatorname{Cone} D_{t(i),d(i)} = \operatorname{Cone}\left(\mathop{*}_{i=1}^{r} D_{t(i),d(i)}\right).$$

In particular the bord $\partial E(J)$ can be seen by this identification as $\mathop{*}_{i=1}^{r} D_{t(i),d(i)}$.

Consequently, $P^{k+1}(X)$ is obtained from $P^k(X)$ by glueing cones of the type $\mathrm{Cone}\left(\mathop{*}\limits_{i=1}^{r} D_{t(i),d(i)}\right)$ along their bords $\mathop{*}\limits_{i=1}^{r} D_{t(i),d(i)}$.

6.2.5. We are now in position to prove Proposition 6.2.1. First recall [30] that if X_i are m_i-connected then $X_1 * X_2$ is $(m_1 + m_2 + 2)$-connected. Having this property if follows by 6.2.2. and 6.2.3 that $\partial E(J) = \mathop{*}\limits_{i=1}^{r} D_{l(i),d(i)}$ is simply connected provided that $k \geqslant 2$. Therefore $\partial E(J)$ is homotopy equivalent to a CW--complex with a single vertex and no cells of dimension one [30]. This shows that up to homotopy $P^{k+1}(X)$ is obtained by attaching to $P^k(X)$ cells of dimension $\geqslant 3$ and this does not change the fundamental group.

6.3. THE EMBEDDING $F^k(S^1 \vee \ldots \vee S^1) \to F^{k+1}(S^1 \vee \ldots \vee S^1)$

Throughout this paragraph X will denote a finite cluster of standard circles. The main result here is that the embedding

$$F^k(X) \to F^{k+1}(X)$$

is a $2[k/3]$-equivalence (not depending on the number of the circles entering in X).

6.3.1. Let $X = S_1 \vee \ldots \vee S_m$ where each $S_i \simeq S^1$ and let $\eta_i\colon S^1 \to X$ denote the inclusion onto the i^{th} factor. In order to analyse $F^k(X)$ it is useful to consider the following filtration

$$F^k(X)_0 \subset \ldots \subset F^k(X)_l \subset F^k(X)_{l+1} \subset \ldots \subset F^k(X) \qquad 0 \leqslant l \leqslant k$$

where $F^k(X)_l$ consists of those $[\underline{x}, e] \in F^k(X)$ for which $\underline{x} = (x_1, \ldots, x_k)$ has at most l coordinates x_i which are not equal to x_0. Here the base point x_0 is chosen to be the common point of the circles S_i in X.

For $0 \leqslant l \leqslant k$ let $A(l)$ denote the set of all ordered multi-indices $\underline{a} = (a_1, \ldots \ldots, a_m)$ such that $a_i \in \mathbf{N}\setminus\{0\}$ and $\sum a_i = l$. If $\underline{a} \in A(l)$ then we shall denote by $F(\underline{a}, l)$ the set of those homomorphisms in $F_k(X)$ which for any $1 \leqslant i \leqslant m$ have exactly a_i proper values (counted with multiplicities) belonging to $S_i \setminus \{x_0\}$.

It is easily seen that

$$F^k(X)_l \setminus F^k(X)_{l-1} = \bigcup_{\underline{a}\in A(l)} F(\underline{a}, l)$$

gives the decomposition of $F^k(X)_l\setminus F^k(X)_{l-1}$ into its connected components.

6.3.2. Let $k \geqslant 1$. It is useful to define $F^k(Y)$ even for noncompact spaces Y. Having in mind Proposition 6.1.8 we shall define $F^k(Y)$ to be the space $Y^k \times U(k)$ factorized to the equivalence relation: $(\underline{y}, u) \sim (\underline{z}, v)$ iff $\sigma(\underline{y}) = \underline{z}$ and $u^* v\sigma \in$

$\in U(I(\underline{y}))$ for some $\sigma \in \mathfrak{S}_k$. Of course if Y happens to be compact then $F^k(Y) \simeq \simeq \mathrm{Hom}_1(C(Y), M_k)$. We need later to know that $F^k((-1,1))$ is homeomorphic to $\mathbf{R}^{k^2}$. Using the notation of 6.1.6, 6.1.7 and arguments similar to those in 6.1.8, one can check that

$$[\underline{x}, u] \to u\left(\sum_{r=1}^{m} \tan\left(\frac{\pi}{2} x_{i_r}\right) e(I_r(\underline{x}))\right) u^*$$

defines an homeomorphism $F^k((-1, 1))$ onto the subspace of $M_k \simeq \mathbf{C}^{k^2}$ consisting of all self-adjoint matrices, which in its turn is homeomorphic to $\mathbf{R}^{k^2}$.

6.3.3. For any $\underline{a} \in A(l)$ let $B(\underline{a}, l)$ denote the homogeneous spaces

$$U(k)/U(a_1) \times \ldots \times U(a_m) \times U(k-l).$$

There is a well defined map $p_{\underline{a}}$: $F(\underline{a}, l) \to B(\underline{a}, l)$ which we are going to describe below. The space $B(\underline{a}, l)$ can be identified with the space of all ordered $(m+1)$-uples $(p_1, \ldots, p_m, p_0)$ of mutually orthogonal self-adjoint projections acting on $\mathbf{C}^k$ such that $\dim p_j = a_j$, $1 \leqslant j \leqslant m$ and $\dim p_0 = k - l$. Given an homomorphism $\varphi \in F(\underline{a}, l)$ we define $p_{\underline{a}}(\varphi) = (p_1, \ldots, p_m, p_0)$ where for $1 \leqslant j \leqslant m$, p_j is equal to the sum of all spectral projections corresponding to the proper values of φ which lies in $\mathrm{S}_j \setminus \{x_0\}$ and $p_0 = 1_k - (p_1 + \ldots + p_m)$ is the projection corresponding to x_0.

It is not hard to see that $p_{\underline{a}}$: $F(\underline{a}, l) \to B(\underline{a}, l)$ is a fiber bundle with fiber isomorphic to

$$F^{a_1}((-1, 1)) \times \ldots \times F^{a_m}((-1, 1)).$$

Moreover $F(\underline{a}, l)$ admits a canonical structure of C^∞-manifold relative to which $p_{\underline{a}}$: $F(\underline{a}, l) \to B(\underline{a}, l)$ becomes a C^∞-differentiable fiber bundle. This is due to the fact that $B(\underline{a}, l)$ and $\bigtimes_{i=1}^{m} F^{a_j}((-1, 1)) \simeq \mathbf{R}^{\Sigma a_j^2}$ have natural C^∞-structure relative to which the clutching maps arising from the local trivialization of the above fiber bundle are smooth.

There is a canonical section for $p_{\underline{a}}$, $s_{\underline{a}}$: $B(\underline{a}, l) \to F(\underline{a}, l)$ defined as follows. Let μ be the natural map μ: $([-1, 1], \{-1, 1\}) \to (\mathrm{S}^1, x_0)$, $\mu(t) = \exp(2\pi i t)$ and let φ_j: $[-1, 1] \to X$ be given by $\varphi_j = \eta_j \circ \mu$ (6.3.1). For any $(p_1, \ldots, p_m, p_0) \in B(\underline{a}, l)$ we define $s_{\underline{a}}(p_1, \ldots, p_m, p_0)$ to be the homomorphism $\varphi \in F^k(X) = \mathrm{Hom}_1(C(X), M_k)$ given by

$$\varphi(f) = f(x_0)p_0 + \sum_{j=0}^{m} f(\varphi_j(0))p_i \quad \text{for any } f \in C(X).$$

The image of $s_{\underline{a}}$ can be identified with a smooth submanifold of codimension $\sum_{i=1}^{m} a_i^2$ in $F(\underline{a}, l)$ denoted by $\operatorname{im} s_{\underline{a}}$.

The previous facts enable us to prove the following

6.3.4. PROPOSITION. *The embedding* $F^k(X)_{l-1} \to F^k(X)_l$ *is a* $(l-1)$*-equivalence,* $k \geqslant 2$, $l \geqslant 1$.

Proof. The embedding

$$F^k(X)_{l-1} = F^k(X)_l \setminus \bigcup_{\underline{a} \in A(l)} F(\underline{a}, l) \hookrightarrow F^k(X)_l \setminus \bigcup_{\underline{a} \in A(l)} \operatorname{im} s_{\underline{a}}$$

admits a deformation retract so it is a homotopy equivalence. To have a good image for this deformation retract one can look at the following topologically equivalent situation. Consider a hermitian vector bundle E together with a continuous map $: SE = \{v \in E : \|v\| = 1\} \to Y$ and the inclusion map $SE \hookrightarrow BE = \{v \in E : \|v\| \leqslant 1\}$. Then there is a deformation retract for the embedding of Y in $Y \cup_f BE \setminus \{\text{zero section}\}$ which acts on $BE \setminus \{\text{the zero section}\}$ by pushing out the vectors $(t, v) \mapsto \mapsto v/(t\|v\| + 1 - t)$.

Consequently it is enough to look at the map

$$F^k(X)_l \setminus \bigcup_{\underline{a} \in A(l)} \operatorname{im} s_{\underline{a}} \hookrightarrow F^k(X)_l.$$

Now each $\operatorname{im} s_{\underline{a}}$ has an open neighbourhood $V_{\underline{a}}$ with smooth differential structure such that $\operatorname{im} s_{\underline{a}}$ has codimension $\sum a_j^2 \geqslant \sum a_j = l$ in $V_{\underline{a}}$. Therefore we may use the aproximation theorem of continuous maps by differentiable maps and the Transversality Theorem [20] to see that the above inclusion is an $(l-1)$-equivalence.

6.3.5. Let $l(k) = [2k/3]$. By Proposition 6.3.4 each of the following inclusions

$$F^k(X)_{l(k)} \subset F^k(X)_{l(k)+1} \subset \ldots \subset F^k(X)_{k-1} \subset F^k(X)$$

is an $l(k)$-equivalence so that $F^k(X)_{l(k)} \hookrightarrow F^k(X)$ is an $l(k)$-equivalence. We shall consider the following commutative diagram

$$\begin{tikzcd} F^k(X)_{l(k+1)} \arrow[rr, hook, "\alpha"] \arrow[dr, "p_0"'] & & F^{k+1}(X)_{l(k+1)} \arrow[dl, "p_1"] \\ & P^{l(k+1)}(X) & \end{tikzcd}$$

which is induced by the diagram 6.1.12 after we identify $P^{l(k+1)}(X)$ with its image in $P^{k+1}(X)$. Therefore α is the restriction of α_k, p_0 is the restriction of $\beta_k \circ p_k$ and p_1

is the restriction of p_{k+1}. Each $\underline{x} \in X^{l(k+1)}$ defines a partition $I(\underline{x}) = (I_1, \dots I_t^*)$ of $\{1, 2, \dots, l(k+1)\}$ as in 6.1.6. The marked subset I_t^* corresponds to those indices i for which $x_i = x_0$. If $\underline{x}' = (x, x_0 \dots x_0) \in X^k$ then the partition of $\{1, \dots, k\}$ associated to $\underline{x}'$ is $I(\underline{x}') = (I_1, \dots, I_{t-1}, I_t)$ where $I_t = I_t^* \cup \{l(k+1)+1, \dots, k\}$. Similarly if $\underline{x}'' = (\underline{x}', x_0) \in X^{k+1}$ then $I(\underline{x}'') = (I_1, \dots, I_{t-1}, I_t \cup \{k+1\})$. Therefore by 6.1.9 $p_0^{-1}[\underline{x}] = U(k)/U(I(\underline{x}'))$ and $p_1^{-1}[\underline{x}] = U(k+1)/U(I(\underline{x}''))$. Let m_i, $1 \leqslant i \leqslant t$ be the cardinal of each I_i. We can identitify $U(I(\underline{x}'))$ with $U(m_1) \times \dots \times U(m_t)$ and $U(I(\underline{x}''))$ with $U(m_1) \times \dots \times U(m_t + 1)$ in such a way that the inclusion of $U(I(\underline{x}))$ in $U(I(\underline{x}''))$ induced by $U(k) \hookrightarrow U(k+1)$ corresponds to the embedding

$$(u_1, \dots, u_{t-1}, u_t) \to \left(u_1, \dots, u_{t-1}, \begin{pmatrix} u_t & 0 \\ 0 & 1 \end{pmatrix}\right), \quad \text{where } u_i \in U(m_i).$$

6.3.6. Lemma. *The map* $\alpha : p_0^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ *is a* $2[k/3]$*-equivalence.*

Proof. The homotopy exact sequences associated to the fibrations $U(k) \to \to p_0^{-1}[\underline{x}]$ and $U(k+1) \to p_1^{-1}[\underline{x}]$ together with the map $p_0^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ (which can be identified with α), induced by the embedding $U(I(x')) \hookrightarrow U(I(x''))$, give the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\pi_q(U(m_1) \times \dots \times U(m_t)) & \to & \pi_q(U(k)) & \to & \pi_q(p_0^{-1}[\underline{x}]) & \to & \pi_{q-1}(U(m_1) \times \dots \times U(m_t)) & \to & \pi_{q-1}(U(k)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_q(U(m_1) \times \dots \times U(m_t+1)) & \to & \pi_q(U(k+1)) & \to & \pi_q(p_1^{-1}[\underline{x}]) & \to & \pi_{q-1}(U(m_1) \times \dots \times U(m_t+1)) & \to & \pi_{q-1}(U(k+1))
\end{array}$$

Since the embedding $U(k) \to U(k+1)$ is a $2k$-equivalence and $k \geqslant m_t \geqslant k - l(k+1)$ we can apply the five lemma to obtain that $p_0^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ is a $2(k - l(k+1))$-equivalence. The proof is complete since $k - l(k+1) = k - [2(k+1)/3] = [k/3]$.

6.3.7. Lemma. *If* $k \geqslant 2$ *then each map in the following diagram*

$$\begin{array}{ccc}
\pi_1(F^k(X)_{l(k+1)}) & \xrightarrow{\alpha_*} & \pi_1(F^{k+1}(X)_{l(k+1)}) \\
& {\scriptstyle p_{0*}} \searrow \quad \swarrow {\scriptstyle p_{1*}} & \\
& \pi_1(P^{l(k+1)}(X)) &
\end{array}$$

is an isomorphism.

Proof. The result follows from Corollary 6.1.11 and Propositions 6.2.1, 6.3.5, 6.3.8. Let $f: (A, a) \to (B, b)$ be a continuous map between topological spaces such that $f_*: \pi_1(A, a) \to \pi_1(B, b)$ is an isomorphism. Assume that A, B are semi-locally simply connected, connected and locally path-connected and let $\omega_A: (\tilde{A}, \tilde{a}) \to (A, a)$,

$\omega_B\colon (\tilde{B}, \tilde{b}) \to (B, b)$ be realizations of their universal covering spaces. If $\tilde{f}\colon (\tilde{A}, \tilde{a}) \to (\tilde{B}, \tilde{b})$ is the unique continuous lifting of f, $\omega_B \circ \tilde{f} = f \circ \omega_A$, then it is easy to check that for every $\tilde{x} \in \tilde{B}$ the restriction of ω_A to $\tilde{f}^{-1}(\tilde{x})$ gives an homeomorphism between $\tilde{f}^{-1}(\tilde{x})$ and $f^{-1}(\omega_B(\tilde{x}))$.

6.3.9. In order to simplify notation, let $A = F^k(X)_{l(k+1)}$, $B = F^{k+1}(X)_{l(k+1)}$, $C = P^{l(k+1)}(X)$. Consequently the diagram from 6.3.5 becomes

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "p_0"'] & & B \arrow[dl, "p_1"] \\ & C & \end{tikzcd} .$$

If we pass at the universal covering spaces as in 6.3.8 we get the following commutative diagram

$$\begin{tikzcd} \tilde{A} \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\tilde{p}_0"'] & & \tilde{B} \arrow[dl, "\tilde{p}_1"] \\ & C & \end{tikzcd} .$$

By Lemma 6.3.7, $\pi_1(\alpha)$, $\pi_1(p_0)$ and $\pi_1(p_1)$ are isomorphisms. Therefore given $\tilde{x} \in \tilde{C}$, $x \in C$ such that $\omega_C(\tilde{x}) = x$, we can use 6.3.8 to identify $p_0^{-1}(x) \to p_1^{-1}(x)$ with $\tilde{p}_0^{-1}(\tilde{x}) \to \tilde{p}_1^{-1}(\tilde{x})$ hence the later map is also a $2[k/3]$-equivalence.

Let $\tilde{D}$ be the space obtained by collapsing to (distinct) points all the subsets of the form $\tilde{p}_1^{-1}(\tilde{x}) \cap \tilde{\alpha}(\tilde{A})$ with $\tilde{x} \in \tilde{C}$, i.e. $\tilde{D} = \tilde{B}/\sim$ where by definition $\tilde{b}_1 \sim \tilde{b}_2$ iff $\tilde{b}_1, \tilde{b}_2 \in \tilde{\alpha}(\tilde{A})$ and $\tilde{p}_1(b_1) = \tilde{p}_1(\tilde{b}_2)$. If $\tilde{\beta}\colon \tilde{B} \to \tilde{D}$ is the induced quotient map then $\tilde{p}_1$ factors through $\tilde{\beta}$, i.e. $\tilde{p}_1 = \tilde{p}_2 \circ \tilde{\beta}$, as in the following diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "\tilde{\alpha}"] \arrow[dr, "\tilde{p}_0"'] & \tilde{B} \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{p}_1"] & \tilde{D} \arrow[dl, "\tilde{p}_2"] \\ & \tilde{C} & \end{tikzcd} .$$

It is easy to see that $\tilde{\beta}^{-1}\tilde{\beta}(\tilde{b})$ is equal to $\tilde{b}$ if $\tilde{b} \notin \tilde{\alpha}(\tilde{A})$ and to $\tilde{\alpha}(\tilde{A}) \cap \tilde{p}_1^{-1}(\tilde{p}_1(\tilde{b})) = \tilde{\alpha}(\tilde{p}_0^{-1}(\tilde{p}_1(\tilde{b})))$ if $\tilde{b} \in \tilde{\alpha}(\tilde{A})$.

6.3.10. Proposition. *The unique map* $\tilde{p}_2\colon \tilde{D} \to \tilde{C}$ *satisfying* $\tilde{p}_1 = \tilde{p}_2 \circ \tilde{\beta}$ *is a* $(2[k/3] + 1)$*-quasifibration.*

Proof. We want to apply Proposition 6.1.5 for $m = 2[k/3]$ so that we have to check that $\tilde{p}_2$ satisfies to 6.1.4 a)–d). To reach c) and d) will suffice to prove that each fiber $\tilde{p}_2^{-1}(\tilde{x})$ is $2[k/3]$-connected. Now it follows from the definition of $\tilde{p}_2$ that $\tilde{p}_2^{-1}(\tilde{x}) = \tilde{p}_1^{-1}(\tilde{x})/\tilde{\alpha}(\tilde{p}_0^{-1}(\tilde{x}))$. Since $\tilde{\alpha}: \tilde{p}_0^{-1}(\tilde{x}) \to \tilde{p}_1^{-1}(\tilde{x})$ is a $2[k/3]$-equivalence we can apply the first part of the next lemma in order to get that $\tilde{p}_2^{-1}(x)$ is $2[k/3]$-connected.

In virtue of the discution from the end of 6.1.4 in order to achieve the condition a) and b) it is enough to show that the map $\tilde{p}_2: \tilde{D} \to \tilde{C}$ can be identified with some simplicial map between simplicial complexes. In order to perform this identification one will use the following general facts about simplicial complexes.

1. Let X be a finite simplicial complex. There is a triangulation $\varphi_1: |K_1| \to X^k$ such that for each $I \in \mathscr{L}$ (see 6.1.6) $\varphi_1^{-1}(\{\underline{x} \in X^k : I(\underline{x}) = I\})$ is the space of some subcomplex of K_1 and the action of $\mathfrak{S}_k$ on X^k is induced by some action by simplicial maps of $\mathfrak{S}_k$ on K_1.

2. There is a triangulation $\varphi_2: |K_2| \to U(k)$ such that the subspaces $U(I)$ (see 6.1.6) correspond to some simplicial subcomplexes and the action of $\mathfrak{S}_k$ on $U(k)$, $(\sigma, u) \mapsto \sigma u \sigma^*$, is induced by some action by simplicial maps of $\mathfrak{S}_k$ on K_2 (see A. Verona, Stratified mappings-structure and triangulability, *Lecture Notes in Math.*, No. 1102).

3. Let $p: K \to K_0$ be a simplicial map between finite simplicial complexes and et $L \subset K$ be a simplicial subcomplex such that $p(L) =$ some vertex of K_0. Then here are subdivisions K' of K and K'_0 of K_0 and a simplicial complex K_L such that here exists a commutative diagram

$$\begin{tikzcd} K' \arrow[rr, "p'"] \arrow[dr] & & K'_0 \\ & K_L \arrow[ur, "p_L"'] & \end{tikzcd}$$

(p' induced by p) with the property that the map $|p_L|: |K_L| \to |K'_0|$ can be identified with $|p|^{\wedge}: |K|/|L| \to |K_0|$ ($|p|^{\wedge}$ is induced by $|p|$).

4. Let $p: K \to K_0$ be a simplicial map between finite simplicial complexes and let G be a finite group which acts on K by simplicial maps such that $p(g \cdot v) = p(v)$ for each vertex $v \in K$ and $g \in G$. Then there are suitable divisions K' of K and K'_0 of K and a simplicial complex K_G such that there exists a commutative diagram of simplicial complexes:

$$\begin{tikzcd} K' \arrow[rr, "p'"] \arrow[dr] & & K'_0 \\ & K_G \arrow[ur, "p_G"'] & \end{tikzcd}$$

with the property that the map $|p_G|: |K_G| \to |K_0'|$ can be identified with $|p|^\wedge : |K|/G \to |K_0|$ ($|p|^\wedge$ is induced by $|p|$).

5. Let $p: K \to K'$ be a simplicial map between finite simplicial complexes. There is a commutative diagram of simplicial complexes

$$\begin{tikzcd} \tilde{K} \arrow[r, "\tilde{p}"] \arrow[d] & \tilde{K}' \arrow[d] \\ K \arrow[r, "p"] & K \end{tikzcd}$$

such that the map $|\tilde{K}| \xrightarrow{|\tilde{p}|} |\tilde{K}'|$ can be identified with $|K|^\sim \xrightarrow{|p|^\sim} |K'|^\sim$ (see 6.3.8).

6.3.11. LEMMA. *Let i: $Y \hookrightarrow Z$ be a* NDR-*pair of path connected spaces and let* $m \geqslant 0$.

a) *If i is an m-equivalence then Z/Y is m-connected.*

b) *Assume that both Y and Z are simply connected. If Z/Y is m-connected then i is an m-equivalence.*

Proof. The lemma is a direct consequence of the theorems of Van-Kampen, Hurewicz and Whitehead [43].

6.3.12. PROPOSITION. $\alpha: A \to B$ *is a* $2[k/3]$-*equivalence.*

Proof. If $\tilde{x} \in \tilde{C}$ is such that $\omega_C(\tilde{x})$ is the base point of $C = P^{l(k+1)}(X)$ then $\tilde{p}_2^{-1}(\tilde{x})$ reduces to a point. If follows by Proposition 6.3.10 and Remark 6.1.3 that $\tilde{p}_2$ is a $(2[k/3] + 1)$-equivalence.

There is a continuous map $\tilde{s}: \tilde{C} \to \tilde{D}$ such that $\tilde{p}_2 \circ \tilde{s} = \mathrm{id}(\tilde{C})$. (For each $\tilde{x} \in \tilde{C}$ let $y \in \tilde{p}_0^{-1}(\tilde{x})$ and define $\tilde{s}(\tilde{x}) = \tilde{\beta}\tilde{\alpha}(y)$.) Since $\pi_q(\tilde{p}_2) \circ \pi_q(\tilde{s}) = \mathrm{id}$ we find that $\tilde{s}$ is a $2[k/3]$-equivalence.

Now $\tilde{D}$ and $\tilde{C}$ was chosen such that

$$\tilde{\beta}: (\tilde{B}, \tilde{\alpha}(\tilde{A})) \to (\tilde{D}, s(\tilde{C}))$$

is a relative homeomorphism of NDR-pairs [43]. Using Lemma 6.3.11 it results that $\tilde{\alpha}$ is a $2[k/3]$-equivalence since $\tilde{s}$ it is so.

Finally, since $\pi_1(\alpha)$ is an isomorphism (by 6.3.7), we get that α is a $2[k/3]$-equivalence.

6.3.13. THEOREM. *The embedding $F^k(X) \to F^{k+1}(X)$ is a* $2[k/3]$-*equivalence.*

(Recall $X = S^1 \vee \ldots \vee S^1$.)

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} F^{k+1}(X)_{l(k+1)} & \xrightarrow{\left[\frac{2(k+1)}{3}\right]} & F^{k+1}(X) \\ {\scriptstyle 2\left[\frac{k}{3}\right]}\uparrow & & \uparrow \\ F^{k}(X)_{l(k+1)} & \xrightarrow{\left[\frac{2(k+1)}{3}\right]} & F^{k}(X) \end{array}$$

in which the orizontal arrows are $[(2(k+1)/3)]$-equivalences as it was noticed in 6.3.5. The left arrow is a $2[k/3]$-equivalence by Proposition 6.3.12. Now the result follows since $[(2(k+1)/3)] \geq 2[k/3]$.

6.4. THE MAIN CONNECTIVITY THEOREM

The last step towards the central result of this section is the following Mayer--Vietoris type result.

6.4.1. THEOREM. *Let A, B be connected, locally connected and semilocally simply connected spaces and $f: A \to B$ a continuous map. Let $(U_i)_i, (V_i)_i$, $1 \leq i \leq r$ be open coverings of A and respective B and define*

$$U_I = \bigcap_{i\in I} U_i \quad \text{and} \quad V_I = \bigcap_{i\in I} V_i \quad \textit{for } I \subset \{1, 2, \ldots, r\}.$$

Suppose that there is $m \geq 1$ such that for each nonvoid $I \subset \{1, 2, \ldots, r\}$ the following conditions are fulfiled:

a) *U_I and V_I are nonvoid and connected;*

b) *The embeddings $U_I \hookrightarrow A$ and $V_I \hookrightarrow B$ are 1-equivalences;*

c) *$f(U_I) \subset V_I$;*

d) *$f: U_I \to V_I$ is an m-equivalence.*

Then $f: A \to B$ is an m-equivalence.

Proof. If $m = 1$ then the proof is accomplished by applying several times Van Kampem theorem.

If $m \geq 2$ using the same theorem we get that $\pi_1(f)$ is an isomorphism. Let $\omega_A : \tilde{A} \to A$, and $\omega_B : \tilde{B} \to B$ be the universal covering spaces for A and B, and fix $\tilde{f}: \tilde{A} \to \tilde{B}$ a lifting for f, $\omega_B \circ \tilde{f} = f \circ \omega_A$. Let $\tilde{U}_I = \omega_A^{-1}(U_I)$, $\tilde{V}_I = \omega_B^{-1}(V_I)$ and note that $\tilde{f}(\tilde{U}_I) \subset \tilde{V}_I$. We want to prove that each $\tilde{f}: \tilde{U}_I \to \tilde{V}_I$ is an m-equivalence. Since $\pi_q(\tilde{f}|\tilde{U}_1)$ identifies with $\pi_q(f|U_I)$ for $q \geq 2$ we have only to show that $\pi_1(\tilde{f}|\tilde{U}_I)$ is an isomorphism. Using b) and the functoriality of the homotopy, exact sequences for fibrations we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(\tilde{U}_I) \arrow[r] \arrow[d] & \pi_1(U_I) \arrow[r] \arrow[d] & \pi_0(F) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi_1(\tilde{A}) \arrow[r] & \pi_1(A) \arrow[r] & \pi_0(F) \arrow[r] & 0 \end{tikzcd}$$

where F is the fibre over the base point. Note that we have $\pi_0(\tilde{U}_I) = 0$ since $\pi_1(U_I) \to \pi_1(A)$ is surjective and of course $\pi_1(\tilde{A}) = 0$. Therefore we obtain the following exact sequence

$$0 \to \pi_1(\tilde{U}_I) \to \pi_1(U_I) \to \pi_1(A) \to 0$$

and a similar sequence for V_I. We can compare this sequences by the following commutative diagram induced by f

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(\tilde{U}_I) \arrow[r] \arrow[d, "\pi_1(\tilde{f})"] & \pi_1(U_I) \arrow[r] \arrow[d, "\pi(f)"] & \pi_1(A) \arrow[r] \arrow[d, "\pi_1(f)"] & 0 \\ 0 \arrow[r] & \pi_1(\tilde{V}_I) \arrow[r] & \pi_1(V_I) \arrow[r] & \pi_1(B) \arrow[r] & 0 \end{tikzcd}$$

in order to obtain that $\pi_1(\tilde{f}|\tilde{U}_I)$ is an isomorphism. Once we know that $\tilde{f}|\tilde{U}_I$ is an m-equivalence we can apply the Whitehead theorem to find that $\tilde{f}_* : \mathrm{H}_q(\tilde{U}_I) \to \mathrm{H}_q(\tilde{V}_I)$ is an isomorphism for $q < m$ and an epimorphism for $q = m$. Now the usual Mayer-Vietoris argument gives the same conclusions for $\tilde{f}_*: \mathrm{H}_q(\tilde{A}) \to \mathrm{H}_q(\tilde{B})$. Since $\tilde{A}$, $\tilde{B}$ are simply connected we can apply the converse of Whitehead theorem to get that $\tilde{f}: \tilde{A} \to \tilde{B}$ is an m-equivalence. Since we know that $\pi_1(f)$ is an isomorphism this shows that $f: A \to B$ is an m-equivalence. ∎

6.4.2. THEOREM. *The natural embedding*

$$\alpha_k: \mathrm{Hom}_1(C(X), M_k) \to \mathrm{Hom}_1(C(X), M_{k+1}), \quad k \geqslant 3,$$

is a $2[k/3]$*-homotopy equivalence for any finite connected* CW*-complex* X.

Proof. For $n \geqslant 1$ and $r \geqslant 0$ define $W(n, r)$ as follows:

$W(n, 0)$ is the class of all finite connected CW-complexes of dimension $n - 1$;

$W(n, r)$ is the class of all finite connected CW-complexes of dimension n having exactly $r \geqslant 1$ cells of dimension n.

If $X \in W(2,0)$ then $\dim(X) = 1$ hence X is homotopic to a finite cluster of circles. Therefore α_k: $F^k(X) \to F^{k+1}(X)$ is a $2[k/3]$-equivalence by Theorem 6.3.13. The theorem will be proved if we show the following implication which allow an inductive argument.

If α_k: $F^k(X) \to F^{k+1}(X)$ is a $2[k/3]$-equivalence for any space in $W(n, r-1)$, $r \geqslant 1$, then the same is true for any space in $W(n, r)$. Therefore let us fix $n \geqslant 2$, $r \geqslant 1$ and $X \in W(n, r)$. Let e be a cell of dimension n in X and choose $(k+2)$-distinct points, in the interior of e, no one of them being equal to x_0. For any nonvoid $I \subset \{1, 2, \ldots \ldots, k+2\}$ let $\alpha_I = \{x_i : i \in I\}$ and define U_I (resp. V_I) to be the set of all homeomorphisms in $F^k(X)$ (resp. $F^{k+1}(X)$) which have no proper values in α_I or equivalently $U_I = F^k(X \setminus \alpha_I)$ and $V_I = F^{k+1}(X \setminus \alpha_I)$. Note that $U_I = \bigcap_{i \in I} U_i$ and $V_I = \bigcap_{i \in I} V_i$ and let $U_i = U_{\{i\}}$, $V_i = V_{\{i\}}$. We want to show that U_i, V_i and f satisfy the hypotheses of Theorem 6.4.3. First of all, it is clear that U_i, V_i are open and each (intersection) U_I, V_I is nonvoid. Since any homeomorphism in $F^{k+1}(X)$ has at most $k+1$ proper values, it follows that both the famillies $(U_i)_i$ and $(V_i)_i$ cover X. Now let us look closely at each condition asked by 6.4.1.

a) We have that U_I and V_I are nonvoid. They are also connected since $X \setminus \alpha_I$ is connected $(\dim e \geqslant 2)$.

b) It follows by Proposition 6.2.1 in conjunction with Corollary 6.1.11 that the vertical arrows is the following commutative diagram induce isomorphisms at π_1 :

$$\begin{array}{ccc} F^k(X \setminus \alpha_I) & \longrightarrow & F^k(X) \\ \downarrow & & \downarrow \\ P^k(X \setminus \alpha_I) & \longrightarrow & P^k(X) \\ \downarrow & & \downarrow \\ P^\infty(X \setminus \alpha_I) & \longrightarrow & P^\infty(X). \end{array}$$

Now according to [14] the map

$$\pi_1(P^\infty(X \setminus \alpha_I)) \to \pi_1(P^\infty(X))$$

can be identified with the map between the one dimensional homology groups

$$\mathrm{H}_1(X \setminus \alpha_I) \to \mathrm{H}_1(X)$$

(induced by $X \setminus \alpha_I \hookrightarrow X$) which clearly is surjective.

c) $f(U_I) \subset V_I$ since $x_0 \notin \alpha_I$.

d) $X \setminus \alpha_I$ is homotopic to a space in $W(n, r-1)$ so that f: $U_I = F^k(X \setminus \alpha_I) \to \to F^{k+1}(X \setminus \alpha_I) = V_I$ is a $2[k/3]$-equivalence by hypothesis.

Finally the above considerations show that we can apply Theorem 6.4.1 to get that $\alpha_k\colon F^k(X) \to F^{k+1}(X)$ is a $2[k/3]$-equivalence.

6.4.3. REMARK. An inspection of our proof shows that this result can be improved especially for lower values of k or by imposing certain restriction on X. In any case it seems to us be quite remarkable that there is a rather large inferior bound, which tends to infinity when k does, for the order of the connectivity of the pair $(F^{k+1}(X), F^k(X))$ and this does not depend on X but only on k.

6.4.4. COROLLARY. *If Y is a finite* CW-*complex of dimension less than* $2[k/3]$ *then the natural map*

$$[C_0(X), C_0(Y) \otimes M_k] \to \mathrm{kk}(Y, X)$$

is a bijection.

Proof. Since $F^k(X) = \mathrm{Hom}_1(C(X), M_k)$ is homeomorphic to $F_0^k(X) = \mathrm{Hom}(C_0(X), M_k)$, it follows by Theorem 6.4.2 that the inclusion $F_0^k(X) \hookrightarrow F_0^{k+1}(X)$ is a $2[k/3]$-equivalence. Moreover, we know that $\lim F_0^k(X)$ is homotopic to $F(X)$ (3.1.2) and so $F_0^k(X) \to F(X)$ is a $2[k/3]$-equivalence. Consequently, the map $[Y, F_0^k(X)] \to [Y, F(X)]$ is one-to-one whenever $\dim Y < 2[k/3]$.

6.4.5. REMARK. Let X, Y, k as above. Then $\pi_1(F_0^k(X))$ acts trivially on $[Y, F_0^k(X)]$. Therefore $[C_0(X), C_0(Y) \otimes M_k] \simeq [C(X), C(Y) \otimes M_k]_1 \simeq \mathrm{kk}(Y, X)$.

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