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ON INVARIANT OPERATOR RANGES OF ABELIAN STRICTLY CYCLIC ALGEBRAS

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1. INTRODUCTION

If $\mathscr{A}$ is a norm closed unital algebra of operators on a Hilbert space $\mathbf{H}$ and $T \in B(\mathbf{H})$ satisfies $AT\mathbf{H} \subseteq T\mathbf{H}$ for all $A \in \mathscr{A}$, then T induces a bounded homomorphism $\Phi_T : \mathscr{A} \to B(\mathbf{H})$ defined by $\Phi_T(A) = T_0^{-1}AT$, where T_0 is the restriction of T to $(\ker T)^{\perp}$ (see [3]). It is proved in [8] that if T and S have the same range hen Φ_T is completely bounded if and only if Φ_S is completely bounded. It is also proved that the set of invariant ranges that induce completely bounded homomorphisms is a sublattice of the lattice of all invariant ranges under the operations of linear span and intersection, and we denote this sublattice by $\mathrm{Lat}_{\mathrm{cb}}\mathscr{A}$.

In this paper we obtain a characterization of $\mathrm{Lat}_{\mathrm{cb}}\mathscr{A}$ when $\mathscr{A}$ is a particular type of abelian strictly cyclic algebra. In particular, if $\mathscr{A}$ is the commutant of a strictly cyclic weighted shift with a monotonically decreasing weight sequence, then $\mathrm{Lat}_{\mathrm{cb}}\mathscr{A}$ is characterized.

2. ABELIAN STRICTLY CYCLIC ALGEBRAS AND WEAK HILBERT-SCHMIDT MAPS

An *abelian strictly cyclic algebra* (ASCA) of operators on a Hilbert space $\mathbf{H}$ is an abelian subalgebra $\mathscr{A}$ of $B(\mathbf{H})$ such that there exists a *strictly cyclic vector* $x_0 \in \mathbf{H}$ for $\mathscr{A}$, that is, a vector such that

$$\mathscr{A}x_0 = \{Ax_0 \mid A \in \mathscr{A}\} = \mathbf{H}.$$

The study of such algebras was initiated in [6] by Lambert, and many of his results appear in Shield's survey article [12] (also see [7]). In particular, if $\mathscr{A}$ is an ASCA with strictly cyclic vector x_0, then $\mathscr{A}$ is maximal abelian and the map $\Psi : \mathscr{A} \to \mathbf{H}$ defined by $\Psi(A) = Ax_0$ is a Banach space isomorphism of $\mathscr{A}$ onto $\mathbf{H}$. We may use this bijection to carry the multiplication in $\mathscr{A}$ over to $\mathbf{H}$; given $x, y \in \mathbf{H}$, define $\varphi(x,y) \equiv$

$\equiv \Psi(\Psi^{-1}(x)\Psi^{-1}(y)) = \Psi^{-1}(x)\Psi^{-1}(y)x_0$. Thus φ is an abelian multiplication with identity on **H**. If M_x is defined by $M_x(y) \equiv \varphi(x, y)$, then we may write $\mathscr{A} = \{M_x | x \in \mathbf{H}\}$. Conversely, if we are given a φ that is an abelian continuous multiplication with identity on **H**, then $\{M_x | x \in \mathbf{H}\}$ is an ASCA. We will write $\mathscr{A} \sim (\mathbf{H}, \varphi)$ to indicate this relationship between **H**, φ, and $\mathscr{A}$. When viewed in this manner, the invertible elements of **H** are the strictly cyclic vectors of the algebra $\mathscr{A}$, and the ideals in **H** are the invariant linear manifolds. The ranges of operators in $\mathscr{A}$ $(=\mathscr{A}')$ are the principle ideals, and the invariant operator ranges for $\mathscr{A}$ are the "para-closed" ideals (using the terminology of [3]).

If **H**, **K**, and **M** are Hilbert spaces, and $\varphi: \mathbf{H} \times \mathbf{K} \to \mathbf{M}$ is a bounded bilinear form, then φ is called a *weak Hilbert-Schmidt map* if there exist an operator $T \in B(\mathbf{H} \otimes \mathbf{K}, \mathbf{M})$ such that $\varphi(x, y) = T(x \otimes y)$ for all $x \in \mathbf{H}$ and $y \in \mathbf{K}$ (see [4]). If $\{e_i | i = 0,1, \dots\}$ is an orthonormal basis of **H** and $\{f_i | i = 0,1, \dots\}$ is an orthonormal basis of **K**, then a bounded bilinear form φ is a weak Hilbert-Schmidt map if and only if

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} |\langle \varphi(e_i, f_k), u\rangle|^2 < \infty,$$

for all $u \in \mathbf{M}$. When this is the case, define

$$\|\varphi\|_{\mathrm{HS}} = \sup_{\|u\| \leqslant 1} \sqrt{\sum_{ij} |\langle \varphi(e_i, f_j), u\rangle|^2},$$

thus with T as above, $\|\varphi\|_{\mathrm{HS}} = \|T\|$. This definition is independent of the choice of basis.

1. LEMMA. *Suppose* (α_{ij}) *is a lower triangular matrix of complex numbers such that*

$$M \equiv \sup_{i \geqslant 0} \sum_{j=0}^{i} |\alpha_{ij}|^2 < \infty .$$

If **H** *is a separable Hilbert space with orthonormal basis* $\{e_0, e_1, e_2, e_3, e_4, \dots\}$, *then*

$$\varphi(a, b) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \langle a, e_j\rangle\langle b, e_{i-j}\rangle \alpha_{ij} \right) e_i$$

defines a weak Hilbert-Schmidt mapping of $\mathbf{H} \times \mathbf{H}$ *into* **H** *with* $\|\varphi\|_{\mathrm{HS}} = M^{1/2}$.

Proof. Note that $\varphi(e_r, e_s) = \alpha_{r+s\,r} e_{r+s}$; thus

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |\langle \varphi(e_r, e_s), u\rangle|^2 = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |\langle e_{r+s}, u\rangle|^2 |\alpha_{r+s\,r}|^2 =$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} |\langle e_i, u\rangle|^2 |\alpha_{ij}|^2 \leqslant M \sum_{i=0}^{\infty} |\langle e_i, u\rangle|^2 .$$

It follows that $\|\varphi\|_{\mathrm{HS}} \leqslant M^{1/2}$.

On the other hand

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\langle\varphi(e_r, e_s), e_k\rangle|^2 = \sum_{j=0}^{k}|\alpha_{kj}|^2 \quad \text{so } \|\varphi\|_{\mathrm{HS}} \geqslant M^{1/2}.$$ ▨

Let $\{w_i\}$ be a monotonically decreasing sequence in $(0, \infty)$, define $\beta_n = 1$ when $n = 0$, and $\beta_n = w_0 w_1 \dots w_{n-1}$ otherwise, and for $i \leqslant k$, define $\beta(k, i) = \beta_k/\beta_{k-i}\beta_i$. For convenience (and with no loss of generality) we assume $w_0 = 1$. A weighted unilateral shift A with weight sequence $\{w_i\}$ has a strictly cyclic commutant if and only if

$$\sup_k \sum_{i=0}^{k} \beta(k, i)^2 < \infty \qquad \text{(see [5] or [12]).}$$

When this is the case, a bounded bilinear form such that $\{A\}' \sim (\mathbf{H}, \varphi)$ may be written as

$$\varphi(x, y) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\langle x, e_j\rangle\langle y, e_{i-j}\rangle\beta(i, j)\right)e_i,$$

where $\{e_i \mid i = 0, 1, \dots\}$ is the orthonormal basis relative to which the weighted shift A is defined (thus $Ae_i = w_i e_{i+1}$). It follows easily that M_{e_0} is the identity operator (e_0 is the multiplicative identity) and $A = M_{e_1}$. It follows from Lemma 1 that φ is a weak Hilbert-Schmidt map.

We now return to the situation of an arbitrary abelian strictly cyclic algebra $\mathscr{A}$. Let $\Phi\colon \mathscr{A} \to B(\mathbf{H}^{(\infty)})$ be the infinite inflation map, $\Phi(A) = A^{(\infty)}$, and define

$$\mathfrak{M}_\Phi \equiv \{T \in B(\mathbf{H}^{(\infty)}, \mathbf{H}) \mid AT = T\Phi(A) \text{ for all } A \in \mathscr{A}\}.$$

Every operator $T \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$ may be viewed as a row operator matrix $T = (B_0, B_1, B_2, \dots)$ with $B_i \in B(\mathbf{H})$ $(i = 0, 1, \dots)$. Since $\mathscr{A}$ is maximal abelian, it is clear that $T \in \mathfrak{M}_\Phi$ if and only if $T \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$ and $B_i \in \mathscr{A}$ for all i. It follows that given $T \in \mathfrak{M}_\varphi$, there exists a sequence $\{x_i\}$ in $\mathbf{H}$ such that $T = (M_{x_0}, M_{x_1}, M_{x_2}, \dots)$. The following lemma identifies sequences $\{x_i\}$ in $\mathbf{H}$ that induce operators $(M_{x_0}, M_{x_1}, M_{x_2}, \dots)$ in $B(\mathbf{H}^{(\infty)}, \mathbf{H})$.

2. Lemma. *Assume* $\mathscr{A} \sim (\mathbf{H}, \varphi)$ *and* $\{e_i \mid i = 0, 1, \dots\}$ *is an orthonormal basis of* $\mathbf{H}$. *Then* φ *is a weak Hilbert-Schmidt map if and only if*

$$(M_{e_0}, M_{e_1}, M_{e_2}, \dots) \in B(\mathbf{H}^{(\infty)}, \mathbf{H}).$$

When this is the case we have

$$\mathfrak{M}_\Phi = \{(M_{De_0}, M_{De_1}, M_{De_2}, \dots) \mid D \in B(\mathbf{H})\}.$$

Proof. Let $U: \mathbf{H}^{(\infty)} \to \mathbf{H} \otimes \mathbf{H}$ be the operator defined by

$$U(\oplus x_i) = \sum x_i \otimes e_i,$$

thus U is unitary. Assume φ is a weak Hilbert-Schmidt map and suppose $Y \in B(\mathbf{H} \otimes \mathbf{H}, \mathbf{H})$ with $\varphi(x, y) = Y(x \otimes y)$ for every $x, y \in \mathbf{H}$. Then

$$YU = (M_{e_0}, M_{e_1}, M_{e_2}, \ldots).$$

and $(M_{e_0}, M_{e_1}, M_{e_2}, \ldots) \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$. The converse is a reversal of this argument, and we have established the first assertion of the lemma.

Assume that φ is a weak Hilbert-Schmidt map and $D \in B(\mathbf{H})$. Define $\psi: \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ by $\psi(x, y) = (x, Dy)$. Then ψ is a weak Hilbert-Schmidt map and we may choose $X \in B(\mathbf{H} \otimes \mathbf{H}, \mathbf{H})$ such that $X(x \otimes y) = \psi(x, y) = \varphi(x, Dy)$. Thus

$$XU = (M_{De_0}, M_{De_1}, M_{De_2}, \ldots) \in B(\mathbf{H}^{(\infty)}, \mathbf{H}).$$

It follows that $(M_{De_0}, M_{De_1}, M_{De_2}, \ldots) \in \mathfrak{M}_\Phi$ for all $D \in B(\mathbf{H})$.

Assume now that $(M_{x_0}, M_{x_1}, M_{x_2}, \ldots) \in \mathfrak{M}_\Phi$. There exists $D \in B(\mathbf{H})$ with $De_i = x_i$ if and only if for every $u \in \mathbf{H}$, $\sum |\langle u, x_i \rangle|^2 < \infty$. Suppose that $u \in \mathbf{H}$; to prove that $\sum |\langle u, x_i \rangle|^2 < \infty$, it suffices to show that for every complex sequence $\{\beta_i\}$ in ℓ^2, $\sum \langle x_i, u \rangle \beta_i$ converges. Since $\mathscr{A}$ contains the scalars, for each β_i there exists $y_i \in \mathbf{H}$ such that

$$M_z(y_i) = \varphi(y_i, z) = \varphi(z, y_i) = M_{y_i}(z) = \beta_i z \quad (\text{for every } z \in \mathbf{H}).$$

The operator norm of M_{y_i} is equivalent to the Hilbert space norm of y_i (see [12]), thus $\oplus y_i \in \mathbf{H}^{(\infty)}$, and it follows that $\sum x_i \beta_i = (M_{x_0}, M_{x_1}, M_{x_2}, \ldots)(\oplus y_i)$. Thus $\sum x_i \beta_i$ is norm convergent in $\mathbf{H}$. In particular, $\langle \sum x_i \beta_i, u \rangle = \sum \langle x_i, u \rangle \beta_i$ converges. ∎

3. THE MAIN THEOREM

Assume $T\mathbf{H}$ is an invariant range for the abelian strictly cyclic algebra $\mathscr{A}$, with $\mathscr{A} \sim (\mathbf{H}, \varphi)$. Then for all $x, y \in \mathbf{H}$, $\varphi(x, Ty) \in T\mathbf{H}$, and it makes sense to define $\varphi_T: \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ by $\varphi_T(x, y) \equiv T_0^{-1}(\varphi(x, Ty))$. It follows that

$$\varphi_T(x, y) = T_0^{-1} M_{Ty}(x) = T_0^{-1} M_x(Ty) = \Phi_T(M_x)(y).$$

Notice also that

$$\|T_0^{-1}(\varphi(x, Ty))\| = \|T_0^{-1}(M_x Ty)\| = \|\Phi_T(M_x)(y)\| \leq$$

$$\leq \|\Phi_T\|\|M_x\|\|y\| \leq K\|x\|\|y\|,$$

where K is a constant. It follows that φ_T is a bounded bilinear map.

3. THEOREM. *Assume that* $\varphi: \mathbf{H}\times\mathbf{H} \to \mathbf{H}$ *is a weak Hilbert-Schmidt map,* $\mathscr{A} \sim (\mathbf{H}, \varphi)$ *is an abelian strictly cyclic algebra, and* $T\mathbf{H}$ *is an invariant operator range for* $\mathscr{A}$. *Then the following are equivalent.*

(i) $T\mathbf{H} \in \mathrm{Lat}_{\mathrm{cb}}\mathscr{A}$;

(ii) φ_T *is a weak Hilbert-Schmidt map;*

(iii) $T\mathbf{H} = \mathrm{ran}(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)$;

(iv) *There exists* $\mathbf{M} \in \mathrm{Lat}\mathscr{A}^{(\infty)}$ *such that* $T\mathbf{H} = P\mathbf{M}$, *where* $P(\oplus x_i) = x_0$.
In particular, $\mathrm{Lat}_{\mathrm{cb}}\mathscr{A} = \{T\mathbf{H} \mid T \in B(\mathbf{H}^{(\infty)}, \mathbf{H});\ AT = TA^{(\infty)}$ *for all* $A \in \mathscr{A}\}$.

Proof. (i) implies (ii). By Lemma 3, $(M_{e_0}, M_{e_1}, M_{e_2}, \ldots) \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$. Thus,

$$\Omega = \begin{pmatrix} M_{e_0} & M_{e_1} & M_{e_2} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in B(\mathbf{H}^{(\infty)}).$$

Since we are assuming that Φ_T is completely bounded, we have that the range of $T^{(\infty)}$ must be invariant under Ω and $(T^{(\infty)})_0^{-1}\Omega T^{(\infty)} \in B(\mathbf{H}^{(\infty)})$ (see [8]). Define $U: \mathbf{H}\otimes\mathbf{H} \to \mathbf{H}^{(\infty)}$ on elementary tensors by $U(x \otimes y) = \oplus\langle x, e_i\rangle y$; thus U extends to a unitary operator. Let P be as in the statement of the theorem; $P(\oplus x_i) = x_0$. It follows that $P(T^{(\infty)})_0^{-1}\Omega T^{(\infty)}U \in B(\mathbf{H} \otimes \mathbf{H}, \mathbf{H})$, and

$$P(T^{(\infty)})_0^{-1}\Omega T^{(\infty)}U(x \otimes y) = T_0^{-1} \sum (\langle x, e_i\rangle M_{e_i}(Ty)) =$$

$$= T_0^{-1}\varphi(x, Ty) = \varphi_T(x, y),$$

so φ_T is a weak Hilbert-Schmidt map.

(ii) implies (iii). We assert that the inclusion

$$T\mathbf{H} \subset (M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)\mathbf{H}^{(\infty)}$$

always holds. As in the proof of Lemma 3, given any sequence $\{\beta_i\}$ in ℓ^2, there exists a sequence $\{y_i\}$ in $\mathbf{H}$ such that $\varphi(y_i, x) = \varphi(x, y_i) = \beta_i x$ and $\oplus y_i \in \mathbf{H}^{(\infty)}$. Thus,

$$(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)(\oplus y_i) = \sum M_{Te_i}(y_i) = \sum \varphi(Te_i, y_i) =$$

$$= \sum \beta_i Te_i = T(\sum \beta_i e_i),$$

and the assertion follows.

To prove the reverse inclusion, choose $X, Y \in B(\mathbf{H} \otimes \mathbf{H}, \mathbf{H})$ such that $Y(x \otimes y) = \varphi_T(x, y) = T_0^{-1}(\varphi(x, Ty))$ and $X(x \otimes y) = \varphi(x, Ty)$; it follows that $X = TY$. Suppose U is defined as in Lemma 3; $U(\oplus x_i) = \sum x_i \otimes e_i$. Then $XU = TYU$, and we assert that $XU = (M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)$. Assume $\oplus x_i \in \mathbf{H}^{(\infty)}$. Then

$$XU(\oplus x_i) = X(\sum x_i \otimes e_i) = \sum \varphi(x_i, Te_i) =$$

$$= \sum M_{Te_i}(x_i) = (M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)(\oplus x_i).$$

We have established that $(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots) = TYU$, hence

$$(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots)\mathbf{H}^{(\infty)} \subset T\mathbf{H}.$$

(iii) implies (iv). Since $(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots) \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$, we have that $\operatorname{graph}(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots) = \{(\oplus x_i, \sum M_{Te_i}(x_i)) : \oplus x_i \in \mathbf{H}^{(\infty)}\}$ is a closed subspace of $\mathbf{H}^{(\infty)} \oplus \mathbf{H}$. Let $U: \mathbf{H}^{(\infty)} \oplus \mathbf{H} \to \mathbf{H}^{(\infty)}$ be the unitary operator defined by $U((\oplus x_i, y)) = \oplus z_i$, where $z_0 = y$, and $z_n = x_{n-1}$ when $n \geqslant 1$. Define $M = U(\operatorname{graph}(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots))$, so $\mathbf{M}$ is a closed subspace of $\mathbf{H}^{(\infty)}$ and it is clear that $P\mathbf{M} = \operatorname{ran}(M_{Te_0}, M_{Te_1}, M_{Te_2}, \ldots) = \operatorname{ran} T$. It is easy to prove that $\mathbf{M} \in \operatorname{Lat} \mathscr{A}^{(\infty)}$.

(iv) implies (i). Assume that $\mathbf{M} \in \operatorname{Lat} \mathscr{A}^{(\infty)}$ and let $Q \in B(\mathbf{H}^{(\infty)})$ such that $Q\mathbf{H}^{(\infty)} = \mathbf{M}$. Then $APQ = PQ\Phi_Q(A^{(\infty)})$ for every $A \in \mathscr{A}$, and $PQ\mathbf{H}^{(\infty)} \in \operatorname{Lat}_{cb}\mathscr{A}$ since the map $A \to \Phi_Q(A^{(\infty)})$ is completely bounded.

The last assertion in the theorem follows from Lemma 3 together with the equivalence of (i) and (iii). ∎

4. INVARIANT RANGES OF A DONOGHUE ALGEBRA

If A is a weighted shift with a monotonically decreasing square summable weight sequence, then we call A a *Donoghue operator*, and we call $\{A\}'$ a *Donoghue algebra*. It is well known that if $\mathscr{A}$ is a Donoghue algebra, then $\mathscr{A}$ is strictly cyclic, and if φ is the bilinear form such that $\mathscr{A} \sim (\mathbf{H}, \varphi)$, then φ is a weak Hilbert-Schmidt map (see [6], [13] and Lemma 2). In [9] it was first proved that $\mathscr{A}$ is the norm closed algebra generated by the polynomials in A, thus every element of $\mathscr{A}$ is either an invertible operator or a compact operator (the compact ones are in fact Hilbert-Schmidt operators). In particular, the ranges of operators in $\mathscr{A}'$ $(=\mathscr{A})$ are all compact operator ranges, with the exception of $\mathbf{H}$. It follows that none of the non-trivial invariant subspaces are in the lattice generated by the set of ranges of operators from $\mathscr{A}$ (see [2] or [11] for a description of the invariant subspaces). We summarize these remarks in the following proposition.

4. PROPOSITION. *Assume* $\mathscr{A}$ *is a Donoghue algebra,* $\mathscr{A} \sim (\mathbf{H}, \varphi)$*, and let* $\mathscr{L}$ *denote the lattice generated by* $\{\operatorname{ran} M_z \mid z \in \mathbf{H}\}$*. Then* $\mathscr{L}$ *is a proper sublattice o* $\mathrm{Lat}_{\mathrm{cb}}\mathscr{A}$.

This proposition shows that the ranges of commuting operators may be a very small part of $\mathrm{Lat}_{\mathrm{ct}}\mathscr{A}$ when $\mathscr{A}$ is the commutant of a single operator. This is very different from the situation when $\mathscr{A}$ is the commutant of a normal operator; in this case the ranges of commuting operators are all of $\mathrm{Lat}_{1/2}\mathscr{A}$. Ong proved (see [10]) that if A is a normal operator, and if $\mathscr{A} = \{A\}'$, then every element of $\mathrm{Lat}_{1/2}\mathscr{A}$ can be written as $T\mathbf{H}$ for some $T \in \mathscr{A}$. Note that $\mathscr{A}'$ is the von Neumann algebra generated by A and $\mathscr{A}' \subset \mathscr{A}$. We assert that what we actually have in this situation is

$$\mathrm{Lat}_{1/2}\,\mathscr{A} = \{\operatorname{ran} T \mid T \in \mathscr{A}'\} = \mathrm{Lat}_{\mathrm{cb}}\mathscr{A}.$$

Since $\mathscr{A}$ is the commutant of an abelian von Neumann algebra, it must contain a MASA $\mathfrak{M}$ which satisfies the inclusion relation $\mathscr{A}' \subset \mathfrak{M} \subset \mathscr{A}$. Thus, by the theorem in [1], $T\mathbf{H}$ is the range of an operator in the weakly closed convex hull of the projections in $\mathscr{A}'$.

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