J. OPERATOR THEORY
24 (1990), 277–299

SCATTERING FOR DISSIPATIVE HYPERBOLIC SYSTEMS OF NONCONSTANT DEFICIT

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0. INTRODUCTION

In this paper we study the scattering theory for hyperbolic systems with dissipative boundary conditions. We consider the problem

$$(0.1)\qquad \begin{cases} \left(\partial_t - \sum_{j=1}^{n} A_j(x)\partial_{x_j} - B(x)\right)u(t, x) = 0 & \text{in } R_t^+ \times \Omega, \\ A(x)u(t, x) = 0 & \text{on } R_t^+ \times \partial\Omega, \\ u(0, x) = f(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$, is an open domain with bounded complement and smooth boundary $\partial\Omega$; $A_j(x) \in C^1(\overline{\Omega}; \operatorname{Hom}\mathbf{C}^d)$, $j = 1, \ldots, n$, are Hermitian $d \times d$ matrices for each $x \in \overline{\Omega}$, $B(x) \in C(\overline{\Omega}; \operatorname{Hom}\mathbf{C}^d)$ and $A(x) \in C^1(\partial\Omega; \operatorname{Hom}\mathbf{C}^d)$. The system (0.1) is a short-range perturbation of the system

$$(0.2)\qquad \begin{cases} \left(\partial_t - \sum_{j=1}^{n} A_j^0\partial_{x_j}\right)u(t, x) = 0 & \text{in } \mathbf{R}_t \times \mathbf{R}_x^n, \\ u(0, x) = f(x) & \text{in } \mathbf{R}^n, \end{cases}$$

where A_j^0, $j = 1, \ldots, n$, are constant Hermitian $d \times d$ matrices.

Let

$$A^0(\xi) = \sum_{j=1}^{n} A_j^0\xi_j, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \setminus 0.$$

It is well known that the eigenvalues $\lambda_j(\xi)$, $j = 1, \ldots, d$, of the matrix $A^0(\xi)$ are continuous real-valued functions positively homogeneous of degree one. Clearly, they can be divided into three groups:

(a) nonvanishing in $\mathbf{R}^n \setminus 0$;

(b) vanishing in $\mathbf{R}^n\setminus 0$ without being identicaly zero;

(c) identically zero eigenvalues.

The system (0.2) is said to be of constant deficit (or strongly propagative) if rank $A^0(\xi)$ is constant for all $\xi\in\mathbf{R}^n\setminus 0$, and to be of nonconstant deficit — otherwise. Clearly, (0.2) is of nonconstant deficit if and only if there are eigenvalues $\lambda_j(\xi)$ of type (b).

The case of strongly propagative systems is relatively well studied (see [5], [7], [11], [14], [17], [28]). Especially, we wish to mention [7] where the scattering for systems of the form (0.1) is studied under short-range perturbations and coercive estimates by using a suitable form of Enss' method.

On the other hand, there are only few works treating systems of nonconstant deficit and most of them are related to Cauchy problems in $\mathbf{R}_t\times R^n_x$. In [10] and [24] a system of the form

$$(0.3)\qquad \begin{cases}\left(\partial_t - E(x)^{-1}\sum_{j=1}^n A_j^0\partial_{x_j}\right)u(t,x)=0 & \text{in } \mathbf{R}_t\times\mathbf{R}^n_x,\\ u(0,x)=f(x) & \text{in } \mathbf{R}^n,\end{cases}$$

is considered under short-range perturbation of the matrix $E(x)$ and some conditions on the $\lambda_j(\xi)$ of type (b). In [24] Tamura proves the completeness of the wave operators, while in [10] another description of the ranges of the wave operators is given. In [18] Ralston studied system on $\mathbf{R}_t\times\mathbf{R}^n_x$ under compact perturbation of the coeficients and the hypothesis that there is only one eigenvalue of the principal symbol $\sum_{j=1}^n A_j(x)\xi_j$ which may vanish in $\mathbf{R}^n\times(\mathbf{R}^n\setminus 0)$. Then the completeness of the wave operators is established under the assumption that the projections on the x-space of the null bicharacteristics corresponding to this eigenvalue escape to infinity.

In this paper we consider the case when the system (0.2) is of nonconstant deficit. Moreover, we assume that the solutions to (0.1) are expressed by a contraction semigroup $V(t)=e^{tG}$ on the Hilbert space $H=L^2(\Omega;\mathbf{C}^d)$ with generator G precisely defined in the next section. Let $U_0(t)=e^{tG_0}$ be the unperturbed unitary group on the Hilbert space $H_0=L^2(\mathbf{R}^n;\mathbf{C}^d)$ with generator $G_0=\sum_{j=1}^n A_j^0\partial_{x_j}$, $D(G_0)=\{f\in H_0: G_0f\in H_0\}$, where the derivatives are taken in distribution sense. We are interested in the existence of the wave operators W_- and W defined by

$$(0.4)\qquad W_-f=\lim_{t\to+\infty}V(t)JU_0(-t)f \quad \text{for } f\in H_0'\stackrel{\text{def}}{=}(\operatorname{Ker}G_0)^\perp,$$

and

$$Wf = \lim_{t \to +\infty} U_0(-t) J^* V(t) f \quad \text{for } f \in H_b^\perp. \tag{0.5}$$

where $J\colon H_0 \to H$ is the operator $Jf = f \upharpoonright_\Omega$, J^* is the adjoint to J, and $H_b^\perp$ is the orthogonal complement in H of the space H_b spanned by the eigenfunctions of G with purely imaginary eigenvalues. The main problem is to prove the existence of W. One of the aims of this work is to prove this under the following hypothesis of local energy decay

$$\begin{cases} \text{For any bounded subdomain } \Omega' \text{ of } \Omega \text{ and any } f \in H_b^\perp \\ \text{we have } \lim_{t \to +\infty} \inf \|V(t)f\|_{L^2(\Omega'; \mathbf{C}^d)} = 0. \end{cases} \tag{0.6}$$

It is worth noticing that the implication "local energy decay $\Rightarrow$ existence of Wf for all $f \in H_b^\perp$" is quite a nontrivial statement even for the wave equation with dissipative boundary conditions as well as for some simple examples of first order systems (see [13], [17]). In our case the proof of such a statement becomes much more complicated in view of the very general assumptions under which we work. To prove the existence of W we require some natural restrictions on the eigenvalues $\lambda_j(\xi)$ of type (b), which are the same as those in [10] and [24].

For a complete investigation of the problem concerning the existence of W, however, it is important to know when the hypothesis (0.6) is fulfilled. For example, it is not hard to see that (0.6) holds under te following assumption

$$\begin{cases} \text{There exists a dense subset } \mathscr{E} \text{ of } H \text{ so that for any} \\ f \in \mathscr{E},\ \varphi \in C_0^\infty(\mathbf{R}^n),\ t \geqslant 0, \text{ we have } \|\varphi V(t) f\|_1 \leqslant C \\ \text{with } C \text{ independent of } t, \text{ where } \|\cdot\|_1 \text{ denotes the norm} \\ \text{in the Sobolev space } H^1(\Omega; \mathbf{C}^d). \end{cases} \tag{0.7}$$

Although an assumption like this is difficult to be verified, it is known to hold for semigroups with generators satisfying coercive estimates, as well as for unperturbed group $U_0(t)$. So, one may expect that, while the coercive estimates are typical for systems of constant deficit, (0.7) may hold for some class of systems of nonconstant deficit.

In [18] Ralston showed that for some systems of nonconstant deficit there is a very close link between the hypothesis (0.6) and the escape to infinity of the projections on the x-space of the null bicharacteristics of $\sum_{j=1}^{n} A_j(x)\partial_{x_j}$. Perhaps such a link exists for a more general class of systems of nonconstant deficit including exterior problems. We also wish to mention [19] where an uniform decay of local energy is obtained for some systems of nonconstant deficit under the assumption that the projections of all bicharacteristics escape to infinity.

Another purpose of this work is to give some characterization of the ranges of the wave operators $W_\pm$ in the case of conservative systems, i.e. when $V(t)$ is an unitary group, without assuming the hypothesis (0.6). Here W_+ is the operator defined by letting $t \to -\infty$ in the definition of W_-. We reduce this problem to the one of describing of those $f \in H$ for which Wf, defined by the limit at (0.5), exists.

Below we shall sketch the idea of our proof of the existence of the operator W. At first, let us fix our notations. Given two Hilbert spaces X and Y, $\mathscr{L}(X, Y)$ denotes the space of all linear bounded operators acting from X into Y. If $Y = X$ we shall write $\mathscr{L}(X)$ instead of $\mathscr{L}(X,X)$. Moreover, given a set $\mathscr{M} \subset \mathbf{R}^n$, $\chi(\mathscr{M})$ denotes the characteristic function of $\mathscr{M}$. We also set $F(z) = -iz(z+i)^{-2}$. Following [7] we would like to construct bounded on H_0 operators P_R^{out} and P_R^{in} depending on large parameters $R, M \gg 1$ and satisfying the following conditions:

$$(0.8)\qquad \begin{cases} \|P_R^{\text{out(in)}}\|_{\mathscr{L}(H_0)} \leqslant C_M \quad \text{for all } R, M \\ \text{with } C_M \text{ independent of } R. \end{cases}$$

$$(0.9)\qquad F(-iG_0) = P_R^{\text{out}} + P_R^{\text{in}} + L_{R,M} + L_M,$$

with operators $L_{R,M}$ and L_M, L_M independent of R, satisfying the following

$$(0.10)\qquad \lim_{M\to\infty} \|L_M\|_{\mathscr{L}(H_0)} = 0,$$

$$(0.11)\qquad \begin{cases} \|L_{R,M}f\|_{H_0} \leqslant C_{M,N}R^{-N}\|f\|_{H_0} + C'_M\|\chi(|x| \leqslant 4R)f\|_{H_0} \\ \text{for all } f \in H_0 \text{ and all integers } N \geqslant 1. \end{cases}$$

Moreover, with some $\delta > 0$ which depends on M but is independent of R and t, we have

$$(0.12)\qquad \|\chi(|x| \leqslant \delta(t+R))U_0(t)P_R^{\text{out}}\|_{\mathscr{L}(H_0)} \leqslant C'_{M,N}(t+R)^{-N},$$

and

$$(0.13)\qquad \|\chi(|x| \leqslant \delta(t+R))U_0(t)^*P_R^{\text{in}*}\|_{\mathscr{L}(H_0)} \leqslant C'_{M,N}(t+R)^{-N},$$

for all $t \geqslant 0$, $R \gg 1$ and all integers $N \geqslant 1$.

Now the existence of W can easily be derived from the existence of such operators P_R^{out} and P_R^{in} combined with (0.6) (see [7]). So, the main part of this paper is devoted to the construction of these operators as well as to the proof of (0.8)—(0.13). Note that the construction given in [7] can not be applied to the case of nonconstant deficit. One of the reasons is the following: if $\varphi \in C_0^\infty(\mathbf{R}\setminus 0)$, then in the case of constant deficit we have that the entries of the matrix-valued function $\varphi(A^0(\xi))$

belong to $C_{0}^{\infty}(\mathbf{R}^n)$, a fact essentially used in [7]; in the case of nonconstant deficit, however, these entries do not have compact supports. Therefore, we propose another construction based on an application of suitable pseudodifferential operators. To prove the above estimates we essentially use the fact that the pseudodifferential operators of class $\mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$ are bounded on $L^2(\mathbf{R}^n)$ (see [25]). Here some additional difficulties arise when one tries to arrange all the desired properties (0.8) – (0.13). For example, the operators P_R^{out} and P_R^{in} do not take symmetrical part at (0.12) and (0.13), a fact which effects to the construction of P_R^{out} and P_R^{in}.

The paper is organized as follows. In Section 1 we introduce our assumptions and formulate the main results. In Section 2 we discuss the existence of the operators W_- and W, provided that there exist operators P_R^{out} and P_R^{in} with the properties described above. In Section 3 we state without proof some standard facts from the theory of pseudodifferential operators. In Section 4 we construct operators P_R^{out} and P_R^{in} satisfying (0.8)—(0.13).

Acknowledgments. The author would like to thank Vesselin Petkov, Vladimir Georgiev and Plamen Stefanov for the helpful discussions during the preparation of this work.

1. ASSUMPTIONS AND RESULTS

Our first assumption is:

$$(1.1)\qquad \begin{cases}\text{There exist constants } \rho_0,\ \varepsilon_0 > 0 \text{ and } C \geqslant 0 \text{ so that}\\ A_j(x) = A_j^0,\ j = 1, \dots, n,\ |B(x)| \leqslant C|x|^{-1-\varepsilon_0} \text{ for } |x| \geqslant \rho_0.\end{cases}$$

The following assumption means that the solutions to (0.1) can be expressed by a contraction semigroup on an appropriate Hilbert space.

$$(1.2)\qquad \begin{cases}\text{The operator } \sum_{j=1}^{n} A_j(x)\partial_{x_j} + B(x) \text{ with domain}\\ \{f \in C^1_{(0)}(\bar{\Omega}; \mathbf{C}^d) \colon A(x)f(x) = 0 \text{ on } \partial\Omega\} \text{ has an}\\ \text{unique closed extension } G \text{ in the Hilbert space}\\ H = L^2(\Omega; \mathbf{C}^d), \text{ which is a generator of a con-}\\ \text{traction semigroup } V(t) = e^{tG},\ t \geqslant 0.\end{cases}$$

The work of Rauch [20] shows that this assumption is fulfilled for uniformly characteristic boundary $\partial\Omega$, i.e. when rank $\sum_{j=1}^{n} A_j(x)\nu_j(x)$ is constant on each connected component of $\partial\Omega$. Here $\nu(x)$ denotes the unit outward normal vector to $\partial\Omega$ at x. We refer the reader to [17] and [20] for more details.

Our first result is

THEOREM 1.1. *Assume* (1.1) *and* (1.2) *fulfilled. Then the operator* W_-, *defined by* (0.4) *on the Hilbert space* H_0', *exists and* $\operatorname{Ran} W_- \subset H_b^\perp$.

REMARK. Note that our proof can easily be extended in order to establish the existence of W_- under more general short-range perturbations of the matrices $A_j(x)$, $j = 1, \dots, n$.

To prove the existence of the operator W we need to impose some restrictions on the $\lambda_j(\xi)$ of type (b). It is well known that there exists a conic closed subset Σ of $\mathbf{R}^n \setminus 0$ with Lebesgue measure zero so that $\lambda_j(\xi)$ have constant multiplicities on $(\mathbf{R}^n \setminus 0) \setminus \Sigma$ and the $\lambda_j(\xi)$ of type (b) do not vanish in $(\mathbf{R}^n \setminus 0) \setminus \Sigma$ (see [1], Appendix A). Hence, without loss of generality we can suppose that $\lambda_1(\xi), \dots, \lambda_k(\xi), \lambda_{k+1}(\xi) \equiv 0$, $1 \leqslant k \leqslant d$, are all different eigenvalues of $A^0(\xi)$ on $(\mathbf{R}^n \setminus 0) \setminus \Sigma$. Clearly, the first k eigenvalues are either of type (a) or of type (b) and are nonvanishing in $(\mathbf{R}^n \setminus 0) \setminus \Sigma$. Now, we write down

$$A^0(\xi) = \sum_{j=1}^{k} \lambda_j(\xi) \Pi_j(\xi), \quad \xi \in \mathbf{R}^n \setminus 0, \tag{1.3}$$

with $\Pi_j(\xi)$ orthogonal projections. We set $\Pi_{k+1}(\xi) = \mathrm{Id} - \sum_{j=1}^{k} \Pi_j(\xi)$ where Id stands for the identity $d \times d$ matrix. It is well known that $\Pi_j(\xi)$ are continuous matrix-valued functions, homogeneous of degree zero. Our next assumption is:

$$\left\{\begin{array}{l} \text{For each } \lambda_j(\xi),\ 1 \leqslant j \leqslant k, \text{ of type (b) there exists an open} \\ \text{conic neighbourhood } \Sigma_j \text{ of the set } \{\xi : \lambda_j(\xi) = 0\} \\ \text{so that } \lambda_j(\xi),\ \Pi_j(\xi) \in C^\infty(\Sigma_j) \text{ and} \\ |\nabla_\xi \lambda_j(\xi)| \neq 0 \text{ for all } \xi \in \Sigma_j . \end{array}\right. \tag{1.4}$$

Now our key result is the following

THEOREM 1.2. *Assume* (1.1), (1.2), (1.4), (0.6) *fulfilled and let* $f \in H_b^\perp$. *Then for any* $\varepsilon > 0$ *there exists* $T_\varepsilon > 0$ *so that*

$$\sup_{t \geqslant 0} \| (J^* V(t) - U_0(t) J^*) V(T_\varepsilon) f \|_{H_0} \leqslant \varepsilon. \tag{1.5}$$

As an easy consequence of this theorem we obtain

THEOREM 1.3. *Under the same assumptions as in Theorem* 1.2, *the operator* W, *defined by* (0.5) *on the Hilbert space* $H_b^\perp$, *exists and* $\operatorname{Ran} W \subset H_0'$.

COROLLARY 1.4. *Under the same assumptions as in Theorem* 1.2, *the scattering operator* $S = WW_-$ *is a well defined bounded operator on the Hilbert space* H_0'.

Theorem 1.3 enables one to describe the range of the wave operator W_+ defined by

$$W_+ f = \lim_{t\to+\infty} V(t)^* J U_0(t) f, \quad f \in H_0'.$$

We refer the reader to [7] for more details.

In what follows in this section we shall assume that the operator G, defined at (1.2), is antiself-adjoint, i.e. $V(t)$ is an unitary group. Besides, we shall not assume the hypothesis (0.6) fulfilled. Note that in this case $H_b^\perp$ coincides with the continuous space of G. Our aim is to describe the range of $W_\pm$, the wave operators defined above. Introduce the sets

$$H_\pm \stackrel{\text{def}}{=} \{f \in H: \liminf_{t\to\pm\infty} \|V(t)f\|_{L^2(\Omega';\mathbf{C}^d)} = 0 \text{ for any bouned subdomain } \Omega' \text{ of } \Omega\}.$$

It is easy to see that $H_\pm$ are closed subsets of $H_b^\perp$ but it does not follow from the definition that they are linear spaces. In fact, it turns out that this is true. Our main result for conservative systems is the following

THEOREM 1.5. *Assume* (1.1), (1.4) *fulfilled. Assume also that the operator* G *defined at* (1.2) *is antiself-adjoint. Then* $\operatorname{Ran} W_\pm = H_\pm$.

REMARK. A little modification of our proof establishes Theorems 1.2 – 1.5 for systems with short-range perturbations of the matrices $A_j(x)$ under the extra assumption that there exists a dense subset $\mathscr{D}$ of H so that for some $\alpha \in C^\infty(\mathbf{R}^n)$ vanishing in a neighbourhood of $\mathbf{R}^n \setminus \Omega$ and equal to 1 outside another one, we have $\|\alpha V(t) f\|_1 \leqslant C$ for all $f \in \mathscr{D}$, $t \geqslant 0$, with C independent of t, where $\|\cdot\|_1$ denotes the norm in the Sobolev space $H^1(\mathbb{R}^n; \mathbf{C}^d)$.

2. EXISTENCE OF THE OPERATORS W_- AND W

In this section we shall prove our main theorems assuming that there exist operators P_R^{out} and P_R^{in} satisfying (0.8) – (0.13). In fact, for the analysis of W_- (or W_+) this is not necessary. Since our analysis of W_- and W closely follows that in [7] and [17], we shall only sketch the main points of our proof.

We begin by the proof of Theorem 1.1. The only difficulties which arise here are caused by the arbitrariness of the $\lambda_j(\xi)$ of type (b). The existence of W_- easily follows from the following

LEMMA 2.1. *There exists a family of bounded operators P_ε, $0 < \varepsilon \leqslant 1$, on H_0 so that* $\operatorname{s-lim}_{\varepsilon\to 0} P_\varepsilon = P_0$, *$P_0$ being the orthogonal projection onto H_0', and for some $\delta_\varepsilon > 0$, all $f \in C_0^\infty(\mathbf{R}^n; \mathbf{C}^d)$, $|t| \geqslant 1$ and all integers $N \geqslant 1$, we have*

$$\|\chi(|x| \leqslant \delta_\varepsilon|t|)U_0(t)P_\varepsilon f\|_{H_0} \leqslant C_{\varepsilon,f,N}|t|^{-N}.$$

Below we shall sketch the proof of Lemma 2.1. Let $\mathscr{F}$ denote the d-dimensional Fourier transform. Clearly, $-\mathrm{i}G_0 = \mathscr{F}^{-1}A^0(\xi)\mathscr{F}$ and therefore

$$\varphi(-\mathrm{i}G_0) = \mathscr{F}^{-1}\varphi(A^0(\xi))\mathscr{F} \quad \text{for all } \varphi \in C(\mathbf{R}). \tag{2.1}$$

Moreover, it follows from (1.3) that

$$\varphi(A^0(\xi)) = \sum_{j=1}^{k+1} \varphi(\lambda_j(\xi))\Pi_j(\xi), \tag{2.2}$$

with φ as above. It is easy also to see that

$$P_0 = \mathscr{F}^{-1}\sum_{j=1}^{k} \Pi_j(\xi)\mathscr{F}. \tag{2.3}$$

Now for $\varepsilon > 0$ choose a function $\chi_0^\varepsilon(\xi) \in C^\infty(\mathbf{R}^n)$, $0 \leqslant \chi_0^\varepsilon \leqslant 1$, such that $\chi_0^\varepsilon = 0$ for $|\xi| \leqslant \varepsilon$ and $\chi_0^\varepsilon = 1$ for $|\xi| \geqslant 2\varepsilon$. Let $\lambda_j(\xi)$ be of type (b). Then, since, as it was mentioned in the previous section, the $\lambda_j(\xi)$ vanishes on a set of Lebesque measure zero only and is a continuous function in ξ, there exists a constant $C_{\varepsilon,j} > 0$ so that

$$\operatorname{mes}\{\xi : |\xi| = 1,\ |\lambda_j(\xi)| \leqslant C_{\varepsilon,j}\} \leqslant \varepsilon^2. \tag{2.4}$$

Denoting by $\chi_j^\varepsilon(\xi)$ the characteristic function of the set $\{\xi : |\lambda_j(\xi)| \geqslant C_{\varepsilon,j}|\xi|\}$, we define P_ε by

$$P_\varepsilon = \mathscr{F}^{-1}\chi_0^\varepsilon(\xi)\Big(\sum_{(a)} \Pi_j(\xi) + \sum_{(b)} \chi_j^\varepsilon(\xi)\Pi_j(\xi)\Big)\mathscr{F}, \tag{2.5}$$

where the sign $\sum_{(a)}$ (resp. $\sum_{(b)}$) means a sum over all indices j for which $\lambda_j(\xi)$ are of type (a) (resp. (b)). Clearly, $\sum_{(a)} + \sum_{(b)} = \sum_{j=1}^{k}$. Now, Lemma 2.1 follows from (2.1) – (2.5) combined with an integration-by-parts argument.

Now we turn to the analysis of W. Choose a function $\theta(x) \in C^\infty(\mathbf{R}^n)$ such that $\theta = 0$ for $|x| \leqslant \rho_0 + 1$ and $\theta = 1$ for $|x| \geqslant \rho_0 + 2$. We shall compare the solutions to the perturbed problem with those to the unperturbed one by the following Duhamel type formula:

$$V(t)\theta f - \theta U_0(t)f = \int_0^t V(t-s)\left(\theta B(x) + \sum_{j=1}^n (\partial_{x_j}\theta)A_j^0\right)U_0(s)f\,ds \tag{2.6}$$

and its adjoint

$$V(t)^*\theta f - \theta U_0(t)^*f = -\int_0^t V(t-s)^*\left(\theta B(x)^* + \sum_{j=1}^n (\partial_{x_j}\theta)A_j^0\right)U_0(s)^*f\,ds \tag{2.6*}$$

which hold for all $f \in H_0$. Using (2.6) we shall prove the following

LEMMA 2.2. *For any integer $m \geqslant 1$ we have*

$$\lim_{R\to+\infty} \|((G-1)^{-m} - \theta(G_0-1)^{-m})\chi(|x| \geqslant R)\|_{\mathscr{L}(H_0,H)} = 0. \tag{2.7}$$

Proof. By the resolvent formula

$$(G-1)^{-m} = \frac{1}{m!}\int_0^\infty e^{-t}t^{m-1}V(t)\,dt,$$

for large R and all $f \in H_0$, we obtain

$$((G-1)^{-m} - \theta(G_0-1)^{-m})\chi(|x| \geqslant R)f =$$

$$= \frac{1}{m!}\int_0^\infty e^{-t}t^{m-1}(V(t)\theta - \theta U_0(t))\chi(|x| \geqslant R)f\,dt = \tag{2.8}$$

$$= \int_{\left(\frac{R}{2}-\rho_0-2\right)/v_m}^\infty \dots + \int_0^{\left(\frac{R}{2}-\rho_0-2\right)/v_m} \dots = I_1 + I_2,$$

where $v_m = \max_{1\leqslant j\leqslant k, |\xi|=1} |\lambda_j(\xi)|$ is the maximal speed of propagation of the solutions to the unperturbed problem. Obviously,

$$\|I_1\|_H \leqslant C_1\|f\|_{H_0} \int_{\left(\frac{R}{2}-\rho_0-2\right)/v_m}^\infty e^{-t}t^{m-1}\,dt = o(R)\|f\|_{H_0} \tag{2.9}$$

with $o(R) \to 0$ as $R \to +\infty$.

To estimate the norm of I_2 observe that by a finite-speed-of-propagation argument we have $U_0(s)\chi(|x| \geqslant R)f = 0$ for $|x| \leqslant \frac{R}{2} + \rho_0 + 2$ and $0 \leqslant s \leqslant \left(\frac{R}{2} - \rho_0 - 2\right)/v_m$. Hence, using (2.6) and taking into account the assumption (1.1), for $0 \leqslant t \leqslant \left(\frac{R}{2} - \rho_0 - 2\right)/v_m$, we obtain

$$\|(V(t)\theta - \theta U_0(t))\chi(|x| \geqslant R)f\|_H \leqslant \int_0^t \|B(x)U_0(s)\chi(|x| \geqslant R)f\|_{H_0} \mathrm{d}s \leqslant$$

$$\leqslant C_2\left(\frac{R}{2} + \rho_0 + 2\right)^{-1-\varepsilon_0} \int_0^t \|U_0(s)\chi(|x| \geqslant R)f\|_{H_0} \mathrm{d}s \leqslant$$

$$\leqslant C_3\left(\frac{R}{2} + \rho_0 + 2\right)^{-1-\varepsilon_0}\left(\frac{R}{2} - \rho_0 - 2\right)\|f\|_{H_0} = \mathrm{o}(R)\|f\|_{H_0}$$

with a new $\mathrm{o}(R) \to 0$ as $R \to +\infty$. Hence

$$\text{(2.10)} \qquad \|I_2\|_H \leqslant \mathrm{o}(R)\|f\|_{H_0} \frac{1}{m!} \int_0^{\left(\frac{R}{2} - \rho_0 - 2\right)/v_m} \mathrm{e}^{-t} t^{m-1} \mathrm{d}t \leqslant \mathrm{o}(R)\|f\|_{H_0}.$$

Now (2.7) follows from (2.8) – (2.10) and the proof is complete.

Below we shall sketch the proof of Theorem 1.2. Recall that $F(z) = -iz(z + i)^{-2} = -i(z + i)^{-1} - (z + i)^{-2}$. According to a result of Simon (see [23], Section 9, (Lemma 2), the set $\{F(-iG)g : g \in H_b^\perp\}$ is dense in $H_b^\perp$. Hence it suffices to prove (1.5) for $f = F(-iG)g$ with $g \in H_b^\perp$. Now, by Lemma 2.2 and (0.9) – (0.11) we obtain

$$\|(J^* V(t) - U_0(t)J^*)V(s)F(-iG)g\|_{H_0} \leqslant$$

$$\text{(2.11)} \qquad \leqslant \|(J^* V(t) - U_0(t)J^*)\theta P_R^{\mathrm{out}}\theta V(s)g\|_{H_0} + C_1\|P_R^{\mathrm{in}}\theta V(s)g\|_{H_0} +$$

$$+ \mathrm{o}(R)\|g\|_H + \mathrm{o}(M)\|g\|_H + C_1\|\chi(|x| \leqslant 4R)V(s)g\|_H,$$

with $\mathrm{o}(M) \to 0$ as $M \to +\infty$ independent of s, t, R, g, with $\mathrm{o}(R) \to 0$ as $R \to +\infty$ independent of s, t, g, and finally with C_1 independent of s, t, R and g. By an easy computation, using (2.6) together with (0.8), (0.12) and (1.1), one can estimate from above the first term at the right-hand side of (2.11) by

$$\begin{aligned} &\|(V(t)\theta - \theta U_0(t))P_R^{\text{out}}\theta V(s)g\|_H + C_2\|(1-\theta)P_R^{\text{out}}\theta V(s)g\|_H + \\ (2.12)\qquad & \\ &+ C_2\|(1-\theta)U_0(t)P_R^{\text{out}}\theta V(s)g\|_{H_0} \leqslant C_3(R^{-1}+R^{-\varepsilon_0})\|g\|_H \end{aligned}$$

with C_3 independent of s, t, R and g.

Similarly, using (2.6)* together with (0.8), (0,13) and (1.1), we obtain

$$\begin{aligned} \|P_R^{\text{in}}\theta V(s)g\|_{H_0} &\leqslant \|P_R^{\text{in}}(\theta V(s) - U_0(s)\theta)g\|_{H_0} + \|P_R^{\text{in}}U_0(s)\theta g\|_{H_0} \leqslant \\ &\leqslant \|(V(s)^*\theta - \theta U_0(s)^*)P_R^{\text{in}*}\|_{\mathscr{L}(H_0,H)}\|g\|_H + \\ (2.13)\qquad &+ \|\chi(|x| \leqslant \delta R)U_0(s)^*P_R^{\text{in}*}\|_{\mathscr{L}(H_0)}\|\theta g\|_{H_0} + \\ &+ \|P_R^{\text{in}}U_0(s)\chi(|x| \geqslant \delta R)\theta g\|_{H_0} \leqslant C_4(R^{-1} + R^{-\varepsilon_0})\|g\|_H + \\ &+ C_4\|\chi(|x| \geqslant \delta R)\theta g\|_{H_0}, \end{aligned}$$

with C_4 independent of s, R and g.

Now (1.5) follows immediately from (2.11) – (2.13) combined with the assumption (0.6). This completes the proof of Theorem 1.2. We refer the reader to [7] and [17] for the proof of the implication "Theorem 1.2 ⇒ Theorem 1.3".

In the rest of this section we shall deal with the proof of Theorem 1.5. We shall analyse the range of the operator W_+. The analysis for W_- is similar. Obviously, for any bounded subdomain Ω' of Ω, we have

$$(2.14)\qquad \lim_{t\to+\infty} \|V(t)f\|_{L^2(\Omega';\mathbb{C}^d)} = 0 \quad \text{for all } f \in \operatorname{Ran} W_+ .$$

Hence, $\operatorname{Ran} W_+ \subset H_+$ and since W_+ is an isometry we deduce that $\operatorname{Ran} W_+$ is a closed linear subset of H_+. Thus, assuming $\operatorname{Ran} W_+ \neq H_+$ leads to $f = f_1 + f_2$ for some $f \in H_+$ with $f_1 \in \operatorname{Ran} W_+$, $f_2 \perp \operatorname{Ran} W_+$ and $f_2 \neq 0$. Now, by (2.14) we deduce $f_2 \in H_+$ and our assumption yields

$$(2.15)\qquad (W_+g, f_2)_H = 0 \quad \text{for all } g \in H_0' \text{ and some } f_2 \in H_+,\ f_2 \neq 0.$$

Now, it remains to see that Wf, defined as a limit at (0.5), exists for all $f \in H_+$ and $\operatorname{Ran} W \subset H_0'$. Indeed, if we assume this fulfilled, (2.15) yields $(g, Wf_2)_{H_0} = 0$ for all $g \in H_0'$. Hence $Wf_2 = 0$, and since W is an isometry, we conclude $f_2 = 0$ which contradicts our assumption.

We shall finish our analysis showing that W exists as an operator acting from H_+ into H_0'. To this end, it suffices to prove that (1.5) holds for each $f \in H_+$. Let us fix $f \in H_+$. Since $H_+ \subset H_0^\perp \subset (\operatorname{Ker} G)^\perp$, we can approximate f in H by functions of the form $\varphi(-iG)f$ with $\varphi \in C_0^\infty(\mathbf{R}\setminus 0)$. Hence, it suffices to establish (1.5) for $\varphi(-iG)f$. Now, replacing the function F in the definition of the operator P_R^{out} and P_R^{in} by φ, we obtain new operators P_R^{out} and P_R^{in} which again satisfy (0.8)–(0.13) with φ instead of F. Then, taking into account that $f \in H^+$, one can prove (1.5) for $\varphi(-iG)f$ in a similar manner as above. This completes the proof of Theorem 1.5.

REMARK. Note that an analysis similar to that above can not be carried out in the case of a contraction semigroup since then, if $\varphi \in C_0^\infty(\mathbf{R}\setminus 0)$, the operator $\varphi(-iG)$ does not make sense as a bounded operator on H commuting with e^{tG}.

3. SOME PRELIMINARIES

In this section we shall state only some standard facts from the theory of pseudodifferential operators, which will be important for our analysis in the next section. At first, let us recall that a function $a(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ belongs to the space $S^0_{0,0}(\mathbf{R}^n \times \mathbf{R}^n)$ of symbols if and only if

$$\sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| = C_{\alpha,\beta} < +\infty \quad \text{for all } \alpha, \beta. \tag{3.1}$$

The collection of the pseudodifferential operators with such symbols will be denoted by $\mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$. According to Theorem 1.3 in Chapter 8 of [25], if $a(x, \xi)$ satisfies (3.1), the pseudodifferential operator $a(x, D)$ is bounded on $L^2(\mathbf{R}^n)$ and

$$\|a(x, D)\|_{\mathscr{L}(L^2(\mathbf{R}^n))} \leqslant C_n \max_{|\alpha|+|\beta|\leqslant s_n} \{C_{\alpha,\beta}\}. \tag{3.2}$$

Now, let us consider an operator $\mathscr{A}$ defined by the following oscillatory integral

$$(\mathscr{A}f)(x) = \iint e^{i\langle x-y,\xi\rangle} b(x, y, \xi) f(y) d\xi\, dy \tag{3.3}$$

with an amplitude $b(x, y, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$ satisfying the estimate

$$\sup_{x,y,\xi} (1 + |y|)^{n+1} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi)| = C_{\alpha,\beta,\gamma} < +\infty \tag{3.4}$$

for all multiindeces α, β, γ. It is easy to see that actually $\mathscr{A} \in \mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$ with symbol

$$a(x, \xi) = e^{-i\langle x,\xi\rangle} \mathscr{A}(e^{i\langle\cdot,\xi\rangle}),$$

and moreover

$$\sup_{x,\xi}|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant c_{n,\alpha,\beta} \max_{|\alpha'|+|\beta'|+|\gamma'|\leqslant s_{n,\alpha,\beta}} \{C_{\alpha',\beta',\gamma'}\} \tag{3.5}$$

for all multiindeces α, β with constants $c_{n,\alpha,\beta}$ and $s_{n,\alpha,\beta}$ independent of $C_{\alpha',\beta',\gamma'}$. Now it follows immediately from (3.2) that $\mathscr{A}$ is a bounded operator on $L^2(\mathbf{R}^n)$ and

$$\|\mathscr{A}\|_{\mathscr{L}(L^2(\mathbf{R}^n))} \leqslant C_n' \max_{|\alpha|+|\beta|+|\gamma|\leqslant s_n'} \{C_{\alpha,\beta,\gamma}\}. \tag{3.6}$$

Using (3.6) one can easily prove the following

LEMMA 3.1. *Let $a(x,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ satisfy* (3.1). *Then for $R \gg 1$ and any integer N we have*

$$\|\chi(|x| \leqslant R)a(x,D)^*\chi(|x| \geqslant 2R)\|_{\mathscr{L}(L^2(\mathbf{R}^n))} \leqslant C_N R^{-N} \max_{|\alpha|+|\beta|\leqslant s_N} \{C_{\alpha,\beta}\}$$

with constants C_N and s_N depending only on N and n.

At the end of this section note that if $\varphi \in C_0^\infty(\mathbf{R})$, then $\varphi(-iG_0) = \mathscr{F}^{-1}\varphi(A^0(\xi))\mathscr{F}$ is a matrix-valued operator with entries of the class $\mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$. Indeed, an easy application of the formula

$$\varphi(A^0(\xi)) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{itA^0(\xi)}\hat{\varphi}(t)\,dt$$

yields the following

LEMMA 3.2. *For any $\varphi \in C_0^\infty(\mathbf{R})$ the entries $\varphi_{ij}(\xi)$ of the matrix $\varphi(A^0(\xi))$ belong to $C^\infty(\mathbf{R}^n)$ and*

$$\sup_\xi |\partial_\xi^\alpha \varphi_{ij}(\xi)| = C_\alpha < +\infty, \quad i,j = 1,\dots,d,$$

for all multiindices α.

4. THE OPERATORS P_R^{out} AND P_R^{in}

This section is devoted to the construction of operators P_R^{out} and P_R^{in} satisfying (0.8)–(0.13). We shall carry out this construction in several steps.

Step 1. Let $\varepsilon > 0$ be a small parameter to be chosen later on. For each $\lambda_j(\xi)$, $1 \leqslant j \leqslant k$, of type (b) we choose homogeneous of degree zero functions $\eta_j^\sigma(\xi)$, $\tilde{\eta}_j^\sigma(\xi) \in C^\infty(\mathbf{R}^n \setminus 0)$, where $\sigma = \pm$ or 0, so that $\eta_j^\pm = 1$ on $\{\xi : \pm\lambda_j(\xi) \geqslant 2\varepsilon|\xi|\}$,

$\eta_j^{\pm} = 0$ on $\{\xi : \pm\lambda_j(\xi) \leqslant \varepsilon|\xi|\}$, $\eta_j^0 = 1 - \eta_j^+ - \eta_j^-$; $\tilde{\eta}_j^{\pm} = 1$ on $\left\{\xi : \pm\lambda_j(\xi) \geqslant \frac{\varepsilon}{2}|\xi|\right\}$, $\tilde{\eta}_j^{\pm} = 0$ on $\left\{\xi : \pm\lambda_j(\xi) \leqslant \frac{\varepsilon}{3}|\xi|\right\}$, $\tilde{\eta}_j^0 = 1$ on $\left\{\xi : |\lambda_j(\xi)| \leqslant \frac{5}{2}\varepsilon|\xi|\right\}$ and $\tilde{\eta}_j^0 = 0$ on $\{\xi : |\lambda_j(\xi)| \geqslant 3\varepsilon|\xi|\}$. In fact, the functions $\tilde{\eta}_j^\sigma$ are chosen so that $\eta_j^\sigma\tilde{\eta}_j^\sigma$ is identically equal to η_j^σ on $\mathbf{R}^n\setminus 0$ for all $\sigma = \pm, 0$. Now, since the function $\lambda_j(\xi)$ is continuous, we can find $\varepsilon = \varepsilon' > 0$ so that $\operatorname{supp}\eta_j^0 \subset \operatorname{supp}\tilde{\eta}_j^0 \subset \Sigma_j$ where Σ_j is the set introduced in the assumption (1.4).

For $\lambda_j(\xi)$ of type (a) the sign of $\lambda_j(\xi)$ does not depend on $\xi \in \mathbf{R}^n\setminus 0$, and then we set $\eta_j^0 = \eta_j^{-\sigma} = \tilde{\eta}_j^0 = \tilde{\eta}_j^{-\sigma} = 0$ and $\eta_j^\sigma = \tilde{\eta}_j^\sigma = 1$ where $\sigma = \operatorname{sign}\lambda_j(\xi)$.

Introduce the operator $\Pi_{\eta^\sigma}(D) = \mathscr{F}^{-1}\Pi_{\eta^\sigma}(\xi)\mathscr{F}$, $\sigma = \pm, 0$ where

$$\Pi_{\eta^\sigma}(\xi) = \sum_{j=1}^{k} \eta_j^\sigma(\xi)\Pi_j(\xi).$$

Similarly, we define the operator $\Pi_{\tilde{\eta}^\sigma}(D)$. Clearly,

$$\Pi_{\eta^\sigma}(D)\Pi_{\tilde{\eta}^\sigma}(D) = \Pi_{\eta^\sigma}(D). \tag{4.1}$$

Step 2. For a large parameter $M \gg 1$ choose functions $\psi_M^\pm, \tilde{\psi}_M^\pm \in C_0^\infty(\mathbf{R})$ such that $\psi_M^\pm = 0$ outside the set $\{z : 1/M \leqslant \pm z \leqslant M\}$, $\psi_M^\pm = 1$ on $\{z : 2/M \leqslant \pm z \leqslant M/2\}$, $\tilde{\psi}_M^\pm = 0$ outside $\{z : 1/(3M) \leqslant \pm z \leqslant 3M\}$ and $\tilde{\psi}_M^\pm = 1$ on $\{z : 1/(2M) \leqslant \pm z \leqslant 2M\}$. Then we set $\psi_M^0 = \psi_M^+ + \psi_M^-$ and $\tilde{\psi}_M^0 = \tilde{\psi}_M^+ + \tilde{\psi}_M^-$. Clearly,

$$\psi_M^\sigma\tilde{\psi}_M^\sigma = \psi_M^\sigma \text{ on } \mathbf{R}, \quad \sigma = \pm, 0. \tag{4.2}$$

Step 3. For a small parameter $\delta_1 > 0$ choose functions $q^\pm \in C^\infty(\mathbf{R})$, $0 \leqslant q^\pm \leqslant 1$, such that $q^+(z) = 1$ for $z \leqslant -\delta_1$, $q^+(z) = 0$ for $z \geqslant \delta_1$, and $q^- = 1 - q^+$.

Step 4. Let $\varphi(x) \in C^\infty(\mathbf{R}^n)$ be such that $\varphi(x) = 0$ for $|x| \leqslant 1$ and $\varphi(x) = 1$ for $|x| \geqslant 2$. For a large parameter $R \gg 1$ set $\varphi_R(x) = \varphi(x/R)$. Clearly, $\varphi_R(x) = 0$ for $|x| \leqslant R$, $\varphi_R(x) = 1$ for $|x| \geqslant 2R$, and

$$\sup_x |\partial_x^\alpha \varphi_R(x)| \leqslant C_\alpha R^{-|\alpha|} \quad \text{for all } \alpha \tag{4.3}$$

with C_α independent of R.

Step 5. Choose a function $\theta_1(\xi) \in C^\infty(\mathbf{R}^n)$ such that $\theta_1 = 0$ for $|\xi| \leqslant 1/(5Mr_m)$ and $\theta_1 = 1$ for $|\xi| \geqslant 1/(4Mr_m)$ where r_m is the maximal speed of propagation of the unperturbed system. It is easy to see that

$$\theta_1(\xi)\psi_M^\sigma(A^0(\xi)) = \psi_M^\sigma(A^0(\xi)) \text{ on } \mathbf{R}^n, \quad \sigma = \pm, 0. \tag{4.4}$$

Introduce the following matrix-valued functions:

$$Q_1^\sigma(x, \xi) = \varphi_R(x) q^\sigma\left(\left\langle \frac{x}{|x|}, \frac{\xi}{|\xi|} \right\rangle\right) \theta_1(\xi) \mathrm{Id},$$

$$Q_2^\sigma(x, \xi) = \sum_{(b)} \varphi_R(x) q^\sigma\left(\left\langle \frac{x}{|x|}, \frac{\nabla_\xi \lambda_j(\xi)}{|\nabla_\xi \lambda_j(\xi)|} \right\rangle\right) \theta_1(\xi) \eta_j^0(\xi) \Pi_j(\xi)$$

where $\sigma = \pm$. In view of the assumption (1.4) we have $Q_j^\sigma(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n; \mathrm{Hom}\,\mathbf{C}^d)$, $j = 1, 2$. Moreover, the functions $Q_j^\sigma(x, \xi)$ are homogeneous of degree zero in ξ for $|\xi| \geq 1/(4Mv_m)$.

Step 6. For a small parameter $\delta_2 > 0$ define the sets

$$K_{\delta_2}^{j,\sigma} = \{(x, \xi)\colon \text{there is } (x', \xi') \in \operatorname{supp} Q_j^\sigma \text{ so that } |x - x'| \leq \delta_2 R$$
$$\text{and } |\xi - \xi'| \leq \delta_2\}, \quad \sigma = \pm, \ j = 1, 2,$$

where R is the same as in Step 4. Then we choose functions $h_j^\sigma(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that $h_j^\sigma = 1$ on $K_{\delta_2}^{j,\sigma}$, $h_j^\sigma = 0$ outside $K_{2\delta_2}^{j,\sigma}$, and

$$\sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta h_j^\sigma(x, \xi)| = C_{\alpha,\beta} < +\infty \tag{4.5}$$

with $C_{\alpha,\beta}$ independent of R. Now we set $Q_j^{\sigma,\mathrm{in}} = h_j^\sigma Q_j^{\sigma *}$ where $Q_j^{\sigma *}(x, \xi)$ is the symbol of the formally adjoint operator to $Q_j^\sigma(x, D)$.

Now we are ready to define the operators P_R^{out} and P_R^{in}. We set

$$P_{1,R}^{\mathrm{out}} = \sum_{\sigma = \pm} F(-\mathrm{i}G_0) \psi_M^\sigma(-\mathrm{i}G_0) \Pi_{\eta\sigma}(D) Q_1^\sigma(x, D)^* \tilde{\psi}_M^\sigma(-\mathrm{i}G_0) \Pi_{\tilde{\eta}\sigma}(D),$$

$$P_{1,R}^{\mathrm{in}} = \sum_{\sigma = \pm} F(-\mathrm{i}G_0) \psi_M^\sigma(-\mathrm{i}G_0) \Pi_{\eta\sigma}(D) Q_1^{-\sigma,\mathrm{in}}(x, D) \tilde{\psi}_M^\sigma(-\mathrm{i}G_0) \Pi_{\tilde{\eta}\sigma}(D),$$

$$P_{2,R}^{\mathrm{out}} = F(-\mathrm{i}G_0) \psi_M^0(-\mathrm{i}G_0) Q_2^+(x, D)^* \tilde{\psi}_M^0(-\mathrm{i}G_0) \Pi_{\tilde{\eta}0}(D),$$

$$P_{2,R}^{\mathrm{in}} = F(-\mathrm{i}G_0) \psi_M^0(-\mathrm{i}G_0) Q_2^{-,\mathrm{in}}(x, D) \tilde{\psi}_M^0(-\mathrm{i}G_0) \Pi_{\tilde{\eta}0}(D),$$

where F is the function introduced in the introduction. Then we set $P_R^{\mathrm{out}} = P_{1,R}^{\mathrm{out}} + P_{2,R}^{\mathrm{out}}$, $P_R^{\mathrm{in}} = P_{1,R}^{\mathrm{in}} + P_{2,R}^{\mathrm{in}}$. Clearly, when there are no eigenvalues $\lambda_j(\xi)$ of type (b) we have $P_{2,R}^{\mathrm{out}} = P_{2,R}^{\mathrm{in}} = 0$.

Proof of (0.8). Clearly, $Q_j^\sigma(x, \xi) \in S^0(\mathbf{R}^n \times \mathbf{R}^n)$, the space of matrix-valued symbols of degree zero. Since the map $S^0 \ni a(x, \xi) \to a^*(x, \xi) \in S^0$ is continuous (see [9]), $a^*(x, \xi)$ being the symbol of the adjoint operator to $a(x, D)$, by (4.3) and (4.5) we conclude that $Q_j^{\sigma,\mathrm{in}}(x, \xi) \in S^0_{0,0}(\mathbf{R}^n \times \mathbf{R}^n)$ uniformly in $R \gg 1$. The same is true for $Q_j^\sigma(x, \xi)$. Now (0.8) follows directly from (3.2).

Taking into account (4.1), (4.2), (4.4) together with the identity

$$F(-\mathrm{i}G_0) \sum_{\sigma=\pm,0} \Pi_{\eta^\sigma}(D) = F(-\mathrm{i}G_0),$$

one can easily find that (0.9) holds with $L_M = (F(1-\psi_M^0))(-\mathrm{i}G_0)$ and $L_{R,M} = L'_{R,M} + L''_{R,M}$, where

$$L'_{R,M} = \sum_{\sigma=\pm,0} (F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta^\sigma}(D)(\varphi_R - 1)\tilde{\psi}_M^\sigma(-\mathrm{i}G_0)\Pi_{\tilde{\eta}^\sigma}(D),$$

$$L''_{R,M} = \sum_{\sigma=\pm} (F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta^\sigma}(D)Q_1^{\sigma,0}(x, D)\tilde{\psi}_M^\sigma(-\mathrm{i}G_0)\Pi_{\tilde{\eta}^\sigma}(D) +$$

$$+ (F\psi_M^0)(-\mathrm{i}G_0)Q_2^{-,0}(x, D)\tilde{\psi}_M^0(-\mathrm{i}G_0)\Pi_{\tilde{\eta}^0}(D).$$

Here $Q_j^{\sigma,0}(x, D)$, $j = 1, 2$, $\sigma = \pm$, denote the pseudodifferential operators with symbols $Q_j^{\sigma,0}(x, \xi) = (h_j^\sigma(x, \xi) - 1)Q_j^{\sigma*}(x, \xi)$.

Proof of (0.10). By the spectral calculus we get

$$\|L_M\|_{\mathscr{L}(H_0)} \leqslant \sup_{z\in\mathbf{R}}|(F(1 - \psi_M^0))(z)| \leqslant C/M$$

with a constant C independent of M.

Proof of (0.11). Given $f \in H_0$ we have

$$\|L'_{R,M}f\|_{H_0} \leqslant C_1\|f\|_{H_0} \sum_{\sigma=\pm,0} \|\chi(|x| \leqslant 2R)\tilde{\psi}_M^\sigma(-\mathrm{i}G_0)\Pi_{\tilde{\eta}^\sigma}(D)\cdot$$

(4.6)
$$\cdot\chi(|x| \geqslant 4R)\|_{\mathscr{L}(H_0)} + C_1\|\chi(|x| \leqslant 4R)f\|_{H_0}.$$

Now, setting $\tilde{\eta}'_j = 1 - \tilde{\eta}_j^+ - \tilde{\eta}_j^-$, $j = 1, \ldots, k$, we have

(4.7)
$$\tilde{\psi}_M^\sigma(-\mathrm{i}G_0)\Pi_{\tilde{\eta}^\sigma}(D) = \tilde{\psi}_M^\sigma(-\mathrm{i}G_0) - \tilde{\psi}_M^\sigma(-\mathrm{i}G_0)\Pi_{\tilde{\eta}'}(D),$$

Notice that if $\lambda_j(\xi)$ is of type (a), the corresponding $\tilde{\eta}'_j$ and $\tilde{\eta}^0{}_j$ are identically zero, while for $\lambda_j(\xi)$ of type (b) we have $\operatorname{supp}\tilde{\eta}'_j \subset \operatorname{supp}\tilde{\eta}_j^0 \subset \Sigma_j$, Σ_j being the set introduced in (1.4). Hence, by assumption (1.4), Lemma 3.2 and (4.7) we deduce $\tilde{\psi}^{\sigma}_{M}(-\mathrm{i}G_0)\Pi_{\tilde{\eta}\sigma}(D) \in \mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$. Now, by Lemma 3.1 we obtain

$$\|L'_{R,M}f\|_{H_0} \leqslant C_N R^{-N}\|f\|_{H_0} + C_1\|\chi(|x| \leqslant 4R)f\|_{H_0} \tag{4.8}$$

for any integer $N \geqslant 1$ with constants C_N and C_1 independent of R.

Next, we are going to estimate the norm of $L''_{R,M}$. Clearly,

$$\|L''_{R,M}\|_{\mathscr{L}(H_0)} \leqslant C_2 \sum_{\sigma=\pm,j=1,2} \|Q_j^{\sigma,0}(x, D)\|_{\mathscr{L}(H_0)} \tag{4.9}$$

with C_2 independent of R. To estimate the norms at the right-hand side of (4.9) we shall use that

$$Q_j^{\sigma*}(x, \xi) = \mathrm{e}^{\mathrm{i}\langle D_x, D_\xi\rangle}\overline{Q_j^{\sigma}(x, \xi)}, \quad j = 1, 2, \sigma = \pm, \tag{4.10}$$

where $D_x = -\mathrm{i}\partial_x$, $D_\xi = -\mathrm{i}\partial_\xi$ (see [9]). Passing to new coordinates $x' = R^{-\frac{1}{2}}x$, $\xi' = R^{\frac{1}{2}}\xi$, by (4.10) we obtain

$$\tilde{Q}_j^{\sigma*}(x', \xi') = \mathrm{e}^{\mathrm{i}\langle D_{x'}, D_{\xi'}\rangle}\overline{\tilde{Q}_j^{\sigma}(x', \xi')}, \quad j = 1, 2, \sigma = \pm, \tag{4.11}$$

where the functions $\tilde{Q}_j^{\sigma}$ (resp. $\tilde{Q}_j^{\sigma*}$) are obtained by replacing x and ξ by $R^{\frac{1}{2}}x'$ and $R^{-\frac{1}{2}}\xi'$, respectively, in Q_j^{σ} (resp. $Q_j^{\sigma*}$). Let l be the Euclidean distance between a fixed point $(x'_0, \xi'_0) \notin \operatorname{supp} \tilde{Q}_j^{\sigma}$ and $\operatorname{supp} \tilde{Q}_j^{\sigma}$. Then Theorem 7.6.5 in [8] implies

$$|\tilde{Q}_j^{\sigma*}(x'_0, \xi'_0)| \leqslant C_N l^{-N} \sum_{|\alpha|+|\beta| \leqslant s+N} \sup_{x',\xi'} |\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\tilde{Q}_j^{\sigma}(x', \xi')| \tag{4.12}$$

for a fixed $s > n + 1$ and any integer $N \geqslant 1$.

On the other hand, by (4.3) one can easily obtain

$$\begin{aligned} \sup_{x',\xi'}|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\tilde{Q}_j^{\sigma}(x', \xi')| &\leqslant R^{\frac{|\alpha|}{2}} \sup_{x,\xi}|\partial_x^{\alpha}\partial_\xi^{\beta}Q_j^{\sigma}(x, \xi)| \leqslant \\ &\leqslant C_{\alpha,\beta}R^{-\frac{|\alpha|}{2}} \leqslant C_{\alpha,\beta} \end{aligned} \tag{4.13}$$

for all multiindices α, β with $C_{\alpha,\beta}$ independent of R. By (4.12) and (4.13) we obtain

$$|\tilde{Q}_j^{\sigma*}(x'_0, \xi'_0)| \leqslant C_N l^{-N} \tag{4.14}$$

with a new C_N independent of R.

Now let $(x'_0, \xi'_0) \in \operatorname{supp}(1 - \tilde{h}^\sigma_j(x', \xi'))$, where the function $\tilde{h}^\sigma_j$ is defined in the same manner as $\tilde{Q}^\sigma_j$ or $\tilde{Q}^{\sigma*}_j$. It is easy to see by the definition of h^σ_j that then $l \geqslant \delta_2 R^{\frac{1}{2}}$ Hence, by (4.14),

(4.15) $$|\tilde{Q}^{\sigma*}_j(x'_0, \xi'_0)| \leqslant C_N R^{-\frac{N}{2}}.$$

Thus we deduce by (4.15) and (4.5),

$$\sup_{x,\xi} |(1 - h^\sigma_j(x, \xi))Q^{\sigma*}_j(x, \xi)| =$$

$$= \sup_{x',\xi'} |(1 - \tilde{h}^\sigma_j(x', \xi'))\tilde{Q}^{\sigma*}_j(x', \xi')| \leqslant C_N R^{-\frac{N}{2}}.$$

Similarly, using the fact that the operators $e^{i\langle D_x, D_\xi\rangle}$ and $\partial^\alpha_x \partial^\beta_\xi$ commute, we obtain

(4.16) $$\sup_{x,\xi} |\partial^\alpha_x \partial^\beta_\xi((1 - h^\sigma_j(x, \xi))Q^{\sigma*}_j(x, \xi))| \leqslant C_{N,\alpha,\beta} R^{-\frac{N}{2}}$$

for all multiindices α, β with constants $C_{N,\alpha,\beta}$ independent of R. Now, by (3.2) and (4.16) we deduce

(4.17) $$\|Q^{\sigma,0}_j(x, D)\|_{\mathscr{L}(H_0)} \leqslant C_N R^{-\frac{N}{2}}, \quad j = 1, 2, \ \sigma = \pm,$$

for any integer N with C_N independent of R.

Now (0.11) follows from (4.8), (4.9) and (4.17).

Proof of (0.12). At first, recall that the operators P^{out}_R and P^{in}_R depend on two large parameters R, $M \gg 1$ and two small ones δ_1, $\delta_2 > 0$. We shall show that δ_1 and δ_2 can be chosen depending on M only so that (0.12) holds with some $\delta > 0$ which may depend on M but is indepedent of R and t. Clearly, the estimate (0.12) is reduced to the following ones:

(4.18) $$\|\chi(|x| \leqslant \delta(t + R))U_0(t)(F\psi^\sigma_M)(-iG_0)\Pi_{2,\sigma}(D)Q^\sigma_1(x, D)^*\|_{\mathscr{L}(H_0)} \leqslant$$
$$\leqslant C_N(t + R)^{-N}, \quad \sigma = \pm,$$

(4.19) $$\|\chi(|x| \leqslant \delta(t + R))U_0(t)(F\psi^0_M)(-iG_0)Q_2(x, D)^*\|_{\mathscr{L}(H_0)} \leqslant$$
$$\leqslant C_N(t + R)^{-N}.$$

Suppose that $\delta < 1/2$. To prove the estimates above we shall consider two cases.

Case 1. $\delta(t + R) + v_m t \leqslant \frac{R}{2}$.

Clearly, in this case it sufficies to establish (4.18) and (4.19) with terms of the form $C_N R^{-N}$ at the right-hand sides. The finite speed of propagation yields

$$\chi(|x| \leqslant \delta(t + R))U_0(t) = \chi(|x| \leqslant \delta(t + R))U_0(t)\chi(|x| \leqslant \delta(t + R) + v_m t).$$

This together with the identity $Q_1^\sigma(x, D)^* = Q_1^\sigma(x, D)^*\chi(|x| \geqslant R)$ lead to the following estimate from above of the left-hand side of (4.18):

$$\left\| \chi\left(|x| \leqslant \frac{R}{2}\right)(F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta\sigma}(D)Q_1^\sigma(x, D)^*\chi(|x| \geqslant R) \right\|_{\mathscr{L}(H_0)}. \tag{4.20}$$

Now, since

$$\psi_M^\sigma(-\mathrm{i}G_0)\Pi_{\eta\sigma}(D) = \psi_M^\sigma(-\mathrm{i}G_0) - \psi_M^\sigma(-\mathrm{i}G_0)\Pi_{\eta 0}(D),$$

by the assumption (1.4) and Lemma 3.2 we conclude that

$$Q_1^\sigma(x, D)\Pi_{\eta\sigma}(D)(\bar{F}\psi_M^\sigma)(-\mathrm{i}G_0) \in \mathrm{OPS}^0_{0,0}(\mathbf{R}^n)$$

uniformly with respect to R. Hence, applying Lemma 3.1 leads to the desired estimate for (4.20), (4.19) is proved similarly.

Case 2. $\delta(t + R) + v_m t \geqslant \frac{R}{2}$.

In this case we need to prove (4.18) and (4.19) with terms $C_N t^{-N}$ at the right-hand sides. Setting $\delta' = \delta\left(1 + (\delta + v_m) \Big/ \left(\frac{1}{2} - \delta\right)\right)$ we estimate from above the left-hand side of (4.18) by

$$\begin{aligned} &\|\chi(|x| \leqslant (\delta' + v_m)t)(F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta\sigma}(D)Q_1^\sigma(x, D)^*\chi(|x| \geqslant 2(\delta' + v_m)t)\|_{\mathscr{L}(H)} + \\ &+ \|\chi(|x| \leqslant \delta' t)U_0(t)(F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta\sigma}(D)Q_1^\sigma(x, D)^* \cdot \\ &\cdot \chi(|x| \leqslant 2(\delta' + v_m)t)\|_{\mathscr{L}(H_0)}. \end{aligned} \tag{4.21}$$

We deal with the first term of (4.21) similarly to the previous case. To estimate the second term, given $f \in H_0$, we write the function

$$u(t, x) = U_0(t)(F\psi_M^\sigma)(-\mathrm{i}G_0)\Pi_{\eta\sigma}(D)Q_1^\sigma(x, D)^*\chi(|x| \leqslant 2(\delta' + v_m)t)f(x)$$

as an oscilatory integral:

$$u(t, x) = (2\pi)^{-n} \sum_{j=1}^{k} \int\int e^{i\langle x-y,\xi\rangle + it\lambda_j(\xi)} (F\psi_M^\sigma)(\lambda_j(\xi)) \cdot$$

$$\cdot \eta_j^\sigma(\xi) \Pi_j(\xi) Q_1^\sigma(y, \xi) \chi(|y| \leqslant 2(\delta' + v_m)t) f(y) d\xi dy.$$

We shall estimate the integral above by passing to the polar coordinates $\xi = \rho w$, $\rho \in \mathbf{R}$, $|w| = 1$. At first, notice that there is $v_0 > 0$, depending on ε', so that if $(\rho, w) \in \operatorname{supp} \psi_M^\sigma(\rho\lambda_j(w))\eta_j^\sigma(w)$, then $\sigma\lambda_j(w) \geqslant v_0$ and $1/(Mv_m) \leqslant \rho \leqslant M/v_0$. Moreover, for (w, y) in the support of the integrand we have $\sigma\langle y, w\rangle \leqslant \delta_1 |y|$ and $|y| \leqslant 2(\delta' + v_m)t$. Hence, taking $\delta' \leqslant \frac{v_0}{2}$ and $\delta_1 \leqslant v_0/(8\delta' + 8v_m)$, for these (w, y) and for $|x| \leqslant \delta' t$ we obtain

$$|\langle x - y, w\rangle + t\lambda_j(w)| \geqslant \frac{v_0}{4} t. \tag{4.22}$$

Now, integrating by parts N times with respect to ρ and using (4.22) we get

$$\sup_{|x| \leqslant \delta' t} |u(t, x)| \leqslant C_N' t^{-N} \sum_{j=1}^{k} \int_{|y| \leqslant 2(\delta' + v_m)t} \int_{|w|=1} \int_{1/(Mv_m)}^{M/v_0} \cdot$$

$$\cdot |\partial_\rho^N (\rho^{n-1} (F\psi_M^\sigma)(\rho\lambda_j(w)))| |f(y)| d\rho dw dy \leqslant$$

$$\leqslant C_N'' t^{-N} \int_{|y| \leqslant 2(\delta' + v_m)t} |f(y)| dy \leqslant C_N''' t^{-N + \frac{n}{2}} \|f\|_{H_0}.$$

Hence

$$\|\chi(|x| \leqslant \delta' t) u(t, x)\|_{H_0} \leqslant c_n t^{\frac{n}{2}} \sup_{|x| \leqslant \delta' t} |u(t, x)| \leqslant C_N t^{-N+n} \|f\|_{H_0}. \tag{4.23}$$

Now (4.18) follows from (4.23) and (4.21) at once.

Next we turn to the proof of (4.19). Choose a function $\chi(x) \in C^\infty(\mathbf{R}^n)$ such that $\chi = 1$ for $|x| \leqslant \delta' t$, $\chi = 0$ for $|x| \geqslant \delta' t + 1$, and $|\partial_x^\alpha \chi(x)| \leqslant C_\alpha$ with C_α independent of t. Then we write the operator $\mathscr{A} = \chi(x) U_0(t) (F\psi_M^0)(-iG_0) Q_2^+(x, D)^*$ in the form

$$(\mathscr{A} f)(x) = (2\pi)^{-n} \sum_{(b)} \int\int e^{i\varphi_j^+(x,y,\xi,t)} a_j^+(x, y, \xi) f(y) d\xi\, dy, \tag{4.24}$$

where $\varphi_j^+ = \langle x-y, \xi\rangle + t\lambda_j(\xi)$ and

$$a_j^+ = \chi(x)(F\psi_M^0)(\lambda_j(\xi))\eta_j^0(\xi)\Pi_j(\xi)q^+\left(\frac{y}{|y|}, \frac{\nabla_\xi\lambda_j(\xi)}{|\nabla_\xi\lambda_j(\xi)|}\right)\varphi_R(y).$$

The assumption (1.4) guarantees that $a_j^+ \in C^\infty(\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n)$ and with some $\mu > 0$,

$$|\nabla_\xi\lambda_j(\xi)| \geqslant \mu \quad \text{for all } \xi \in \operatorname{supp}\psi_M^0(\lambda_j(\xi))\eta_j^0(\xi).$$

Therefore, for $(x, y, \xi) \in \operatorname{supp} a_j^+$, $\delta_1 \leqslant 1/2$ and $\delta' \leqslant \mu/12$ we have

$$|\nabla_\xi\varphi_j^+| \geqslant C(|y| + t). \tag{4.25}$$

This estimate enables us to integrate by parts with respect to ξ in the integral at the right-hand side of (4.24). Indeed, setting $L_j = |\nabla_\xi\varphi_j^+|^{-2}\langle\nabla_\xi\varphi_j^+, \nabla_\xi\rangle$, we obtain

$$(\mathscr{A}f)(x) = (2\pi)^{-n}\sum_{(b)}\iint e^{i\langle x-y,\xi\rangle}(e^{it\lambda_j(\xi)}L_j^{*N}a_j^+)f(y)\,d\xi dy$$

for any integer N. Now, taking into account (4.25) one can easily deduce

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(e^{it\lambda_j(\xi)}L_j^{*N}a_j^+)| \leqslant C_{N,\alpha,\beta,\gamma}t^{-N+n+1+|\gamma|}(1+|y|)^{-n-1} \tag{4.26}$$

for all (x, y, ξ) and all multiindices α, β, γ. Hence, by (3.6) and (4.26) we conclude

$$\|\mathscr{A}\|_{\mathscr{L}(H_0)} \leqslant C_N t^{-N+s_n} \tag{4.27}$$

with an integer s_n depending on n only. Since N is arbitrary, (4.27) implies (4.19) in this case. The proof of (0.12) is complete.

Proof of (0.13). We would like to prove this estimate in a similar manner as (0.12). To this end, we need the following estimates:

(4.28) $|\langle x-y, w\rangle - t\lambda_j(w)| \geqslant Ct$, $\quad 1 \leqslant j \leqslant k$, and $C_1 \leqslant \rho \leqslant C_2$, $C_1 > 0$,

for $|x| \leqslant \delta' t$, $|y| \leqslant 2(\delta' + v_m)t$, $(\rho, y, w) \in \operatorname{supp}\tilde\psi_M^0(\rho\lambda_j(w))(\tilde\eta_j^0\Pi_j)(w)Q_1^{-\sigma,\mathrm{in}}(y, \rho w)$; and for each $\lambda_j(\xi)$ of type (b),

$$|\nabla_\xi(\langle x-y, \xi\rangle - t\lambda_j(\xi))| \geqslant C(|y| + t) \tag{4.29}$$

for $|x| \leqslant \delta' t$, $(y, \xi) \in \operatorname{supp}\tilde\psi_M^0(\lambda_j(\xi))(\tilde\eta_j^0\Pi_j)(\xi)Q_2^{-,\mathrm{in}}(y, \xi)$.

Taking δ, δ_1 and δ_2 sufficiently small, one can easily establish (4.28) and (4.29). Then the same technique as in the proof of (0.12) is applied. The proof of (0.13) is complete.

Partially supported by Bulgarian Ministry of Science and Education under Grant 52/1989.

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Received April 13, 1989.