J. OPERATOR THEORY
24(1990), 301–396

MINIMUM INDEX FOR SUBFACTORS AND ENTROPY

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INTRODUCTION

Jones [12] constructed the index theory for pairs of a type II_1 factor and a subfactor of it using the coupling constant and Umegaki's conditional expectation [27]. Pimsner and Popa [22] developed the entropy $H(M|N)$ of a finite von Neumann algebra M relative to a subalgebra N of it, which was defined by Connes and Størmer [7] for finite dimensional algebras. When M is a type II_1 factor and N is a subfactor of it, Pimsner and Popa exactly estimated $H(M|N)$ in terms of Jones' index. The general relation between $H(M|N)$ and Jones' index $[M: N]$ is given by $H(M|N) \leqslant \log[M:N]$, and several characterizations for the equality $H(M|N) = \log]M:N]$ were established in [22] as a consequence of the exact estimate of $H(M|N)$. The complete computation of the entropy in the finite dimensional case is also contained in [22].

On the other hand, Kosaki [16] extended Jones' index theory to that for conditional expectations between arbitrary factors based on Connes' spatial theory [5] and Haagerup's theory on operator valued weights [8]. For von Neumann algebras $M \supseteq N$, let $\mathscr{E}(M, N)$ denote the set of all faithful normal conditional expectations from M onto N. When $M \supseteq N$ are factors, Kosaki's index of $E \in \mathscr{E}(M, N)$ is defined by $\operatorname{Index} E = E^{-1}(1)$ where E^{-1} is the operator valued weight from N' to M' determined by the equation of spatial derivatives $\mathrm{d}(\varphi\circ E)/\mathrm{d}\psi = \mathrm{d}\varphi/\mathrm{d}(\psi\circ E^{-1})$ with faithful normal semifinite weights φ on N and ψ on M'. In [16], the analysis analogous to that in [12] was done and, among other things, the following restriction on index values was obtained: $\operatorname{Index} E \in \{4\cos^2(\pi/n): n \geqslant 3\} \cup [4, \infty]$.

In [9], given a pair of factors $M \supseteq N$, we uniquely characterized $E_0 \in \mathscr{E}(M, N)$ whose index is the minimum of $\{\operatorname{Index} E: E \in \mathscr{E}(M, N)\}$. In particular, when $M \supseteq N$ are type II_1 factors, it follows from this characterization that Umegaki's conditional expectation $M \to N$ has the minimum index if and only if the equality $H(M|N) = \log[M:N]$ holds.

The first aim of this paper is to discuss general properties of the minimum index for a pair of a factor and one of its subfactors. The second aim is to introduce the entropy of Pimsner and Popa's type for arbitrary von Neumann algebras and to establish the relation between the entropy and the minimum index.

In Section 1 of this paper, we collect several preliminary results concerning the correspondence $E \mapsto E^{-1}$ for $E \in \mathcal{E}(M, N)$. These may be of interest by themselves. In Section 2, we present several properties of the minimum index $[M:N]_0 := \min\{\operatorname{Index} E : E \in \mathcal{E}(M, N)\}$ $(= \infty$ if $\mathcal{E}(M, N) = \emptyset)$ for a pair of factors $M \supseteq N$. In particular, we obtain the formulas of the minimum indices for tensor products and for crossed products. Some similar results have been independently obtained by Longo [18] whose method is different from ours.

In Section 3, taking account of Pimsner and Popa's estimate of $H(M|N)$, we introduce the entropy $K_\varphi(M|N)$ of a von Neumann algebra M relative to a von Neumann subalgebra N of it and a faithful normal state φ on M, where $E \in \mathcal{E}(M, N)$ with respect to φ exists. More precisely, $K_\varphi(M|N)$ is defined as the minus sign of the relative entropy of $\varphi|N' \cap M$ and $\varphi \cdot (E^{-1}|N' \cap M)$. The relative entropy of normal positive functionals was first studied by Umegaki [28] on semifinite von Neumann algebras and was extended by Araki [1], [2] to the case on general von Neumann algebras. Section 3 contains some basic properties of the entropy $K_\varphi(M|N)$.

In Section 4, we establish decomposition theorems for the entropy $K_\varphi(M|N)$, which reduce the computation of $K_\varphi(M|N)$ to the case of factors. Similar reduction theorems for Pimsner and Popa's entropy were obtained by Kawakami and Yoshida [13], [14]. Furthermore we present the estimate of $K_\varphi(M|N)$ in terms of Kosaki's index, which is completely analogous to that of $H(M|N)$ for type II_1 factors [22]. It follows as a corollary that our entropy coincides with Pimsner and Popa's one when M is a type II_1 factor and φ is the trace. When $K_\varphi(M|N) < \infty$, the conditions of atomicness of $Z(M)$ $(= M' \cap M)$, $Z(N)$ and $N' \cap M$ are shown to be equivalent.

In Section 5, for von Neumann algebras $M \supseteq L \supseteq N$, we establish the inequality $K_\varphi(M|N) \leqslant K_\varphi(M|L) + K_\varphi(L|N)$ in full generality together with the examination of the equality. Also we show that $K_\varphi(M|N) \geqslant K_\varphi(M|L)$ holds under some assumptions. But another fundamental inequality $K_\varphi(M|N) \geqslant K_\varphi(L|N)$ remains open.

Finally, in Section 6, we establish the relationship between the minimum index and the entropy for a pair of factors $M \supseteq N$ with $[M:N]_0 < \infty$. For $E \in \mathcal{E}(M, N)$, we use the notation $K_E(M|N)$ instead of $K_\varphi(M|N)$ because of the independence of φ with $\varphi \circ E = \varphi$. Then we show that $K_E(M|N) \leqslant \log[M:N]_0$ for any $E \in \mathcal{E}(M, N)$, and moreover that E has the minimum index if and only if the equality $K_E(M|N) = \log[M:N]_0$ holds. The set of all values $K_E(M|N)$, $E \in \mathcal{E}(M, N)$, is determined. Also the chain rule $[M:N]_0 = [M:L]_0\,[L:N]_0$ for factors $M \supseteq L \supseteq N$ is characterized by means of the entropy.

1. AUXILIARY RESULTS

In this paper, von Neumann algebras are always assumed to be σ-finite. Let M be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and N be a von Neumann subalgebra of M. Let $P(M, N)$ denote the set of all faithful normal semifinite operator valued weights from M_+ to $\hat{N}_+$, where $\hat{N}_+$ is the extended positive part of N (see [8]). Let $P(M) = P(M, \mathbf{C})$, the set of all faithful normal semifinite weights on M. Also we denote by $\mathscr{E}(M, N)$ the set of all faithful normal conditional expectations from M onto N, and let $\mathscr{E}(M) = \mathscr{E}(M, \mathbf{C})$, the set of all faithful normal states on M. The bijective correspondence $T \in P(M, N) \mapsto T^{-1} \in P(N', M')$ is uniquely determined by the following equation of spatial derivatives:

$$\frac{\mathrm{d}\varphi\circ T}{\mathrm{d}\psi} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi\circ T^{-1}}$$

with any $\varphi \in P(N)$ and $\psi \in P(M')$ (see [8, Theorem 6.13] and [24, 12.11]).

If $E \in \mathscr{E}(M, N)$, then $E^{-1}(1) \in Z(M)^{\wedge}_{+}$ where $Z(M)$ is the center of M. When $M \supseteq N$ is a pair of a factor and one of its subfactors, Kosaki's index [16] of $E \in \mathscr{E}(M, N)$ is defined by $\operatorname{Index} E = E^{-1}(1)$. Note [16, Theorem 2.2] that $\operatorname{Index} E$ is determined independently of the choice of $\mathscr{H}$, and for any isomorphism α of M, $\operatorname{Index}(\alpha\circ E\circ\alpha^{-1})$ of $\alpha\circ E\circ\alpha^{-1} \in \mathscr{E}(\alpha(M), \alpha(N))$ is equal to $\operatorname{Index} E$.

In this section, we collect several auxiliary results concerning the correspondence $E \in \mathscr{E}(M, N) \mapsto E^{-1} \in P(N', M')$, which will be used in later sections.

Given $T \in P(M, N)$, let $(N' \cap M)_T$ denote the centralizer of T, i.e.

$$(N' \cap M)_T = \{x \in N' \cap M : \sigma_t^T(x) = x,\ t \in \mathbf{R}\},$$

where $\sigma_t^T = \sigma_t^{\varphi\circ T} \mid N' \cap M$ for any $\varphi \in P(N)$ (see [8]). In particular, the centralizer of $\varphi \in P(M)$ is denoted by M_φ. It is known [3, Corollaire 3.10] that if $E \in \mathscr{E}(M, N)$ and $x \in M$, then $x \in (N' \cap M)_E$ if and only if $E(xy) = E(yx)$ for all $y \in M$. Hence we easily see that $Z(M) \subseteq (N' \cap M)_E$ and $Z(N) \subseteq (N' \cap M)_E$.

For a projection e in M or M', let $z_M(e)$ denote the central support in $Z(M)$ of e.

LEMMA 1.1. *Let $\varphi \in P(M)$ and $\psi \in P(M')$. If a is a strictly positive self-adjoint operator affiliated with M_φ, then*

$$\frac{\mathrm{d}a^{1/2}\varphi a^{1/2}}{\mathrm{d}\psi} = a^{1/2}\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}a^{1/2} = \frac{\mathrm{d}\varphi}{\mathrm{d}a^{-1/2}\psi a^{-1/2}},$$

where $a^{1/2}\varphi a^{1/2} = \varphi(a^{1/2}\cdot a^{1/2})$.

Proof. For the case $a \geqslant 1$, taking a sequence $\{a_n\}$ of invertible elements in $(M_\varphi)_+$ with $a_n \uparrow a$, we have $a_n^{1/2}\varphi a_n^{1/2} \uparrow a^{1/2}\varphi a^{1/2}$ by [19, Proposition 4.2], so that

$$a_n^{1/2}\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}a_n^{1/2} = \frac{\mathrm{d}a_n^{1/2}\varphi a_n^{1/2}}{\mathrm{d}\psi} \uparrow \frac{\mathrm{d}a^{1/2}\varphi a^{1/2}}{\mathrm{d}\psi}$$

by [5, Proposition 8 and Corollary 15]. On the other hand, because [5, Theorem 9] implies

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right)^{it} x \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right)^{-it} = x, \quad x \in M_\varphi,$$

we have $a_n^{1/2}(\mathrm{d}\varphi/\mathrm{d}\psi)a_n^{1/2} \uparrow a^{1/2}(\mathrm{d}\varphi/\mathrm{d}\psi)a^{1/2}$. Hence $\mathrm{d}a^{1/2}\varphi a^{1/2}/\mathrm{d}\psi = a^{1/2}(\mathrm{d}\varphi/\mathrm{d}\psi)a^{1/2}$. For the general case, using the spectral decomposition of a, we can take commuting positive selfadjoint operators $b, c \geqslant 1$ affiliated with M_φ such that $a = bc^{-1}$. Since b and c are affiliated with the centralizer of $c^{-1/2}\varphi c^{-1/2}$ (see [19, Theorem 4.6]), the above case shows

$$\frac{\mathrm{d}a^{1/2}\varphi a^{1/2}}{\mathrm{d}\psi} = b^{1/2}\frac{\mathrm{d}c^{-1/2}\varphi c^{-1/2}}{\mathrm{d}\psi}b^{1/2}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} = c^{1/2}\frac{\mathrm{d}c^{-1/2}\varphi c^{-1/2}}{\mathrm{d}\psi}c^{1/2}.$$

Hence the first equality is obtained. Next we see by [5, Theorem 9] that a is affiliated with M_ψ as well. Therefore, the second equality follows from the first together with [5, Theorem 9] by interchanging φ and ψ. ▨

PROPOSITION 1.2. *Let $T \in P(M, N)$. If a is a strictly positive selfadjoint operator affiliated with $(N' \cap M)_T$, then $a^{1/2}Ta^{1/2} \in P(M, N)$ and $(a^{1/2}Ta^{1/2})^{-1} = a^{-1/2}T^{-1}a^{-1/2}$ where $a^{1/2}Ta^{1/2} = T(a^{1/2}\cdot a^{1/2})$.*

Proof. Since T has a unique normal extension to a map $\hat{M}_+ \to \hat{N}_+$, it is easy to check that $a^{1/2}Ta^{1/2} \in P(M, N)$. For $\varphi \in P(N)$ and $\psi \in P(M')$, because $(N' \cap M)_T \subseteq M_{\varphi\circ T}$ and $(N' \cap M)_T = (N' \cap M)_{T^{-1}} \subseteq (N')_{\psi\circ T^{-1}}$ by [8, Theorem 6.13], we have by Lemma 1.1

$$\frac{\mathrm{d}\varphi\circ a^{1/2}Ta^{1/2}}{\mathrm{d}\psi} = \frac{\mathrm{d}a^{1/2}(\varphi\circ T)a^{1/2}}{\mathrm{d}\psi} = a^{1/2}\frac{\mathrm{d}\varphi}{\mathrm{d}\psi\circ T^{-1}}a^{1/2} =$$

$$= \frac{\mathrm{d}\varphi}{\mathrm{d}a^{-1/2}(\psi\circ T^{-1})a^{-1/2}} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi\circ a^{-1/2}T^{-1}a^{-1/2}},$$

as desired. ▨

LEMMA 1.3. *Let* $\varphi \in P(M)$, $\psi \in P(M')$, *and* e *be a nonzero projection in* M_φ. *Define* $\varphi_e \in P(M_e)$ *by* $\varphi_e = \varphi \mid M_e$ and $\psi^e \in P(M'_e)$ *by* $\psi^e(xe) = \psi(xz_M(e))$, $x \in M'_+$. *Then*

$$\frac{d\varphi_e}{d\psi^e} = \frac{d\varphi}{d\psi}\Big|_{e\mathscr{H}}, \quad \frac{d\psi^e}{d\varphi_e} = \frac{d\psi}{d\varphi}\Big|_{e\mathscr{H}}.$$

Proof. Note that ψ^e is well defined because $x \in M'z_M(e) \mapsto xe \in M'e$ is an isomorphism. By definition of the spatial derivative, it is easy to see that $d\varphi_p/d\psi_p = (d\varphi/d\psi) \mid p\mathscr{H}$ for every central projection p in M. So we may assume $z_M(e) = 1$. Then the lemma can be proved in such a way as mentioned in [24, p. 99]. ▨

Let $E \in \mathscr{E}(M, N)$ and e be a nonzero projection in $(N' \cap M)_E$. Then $E(e)^{-1}e$ is well defined as a positive selfadjoint operator affiliated with $(N' \cap M)_E$, because $E(e) \in Z(N)$ and the support of $E(e)$ contains e. So define

$$E_e(x) = E(x)E(e)^{-1}e, \quad x \in M_e.$$

Since $E_e(e) = e$, we have $E_e \in \mathscr{E}(M_e, N_e)$.

The next proposition extending [16, Proposition 4.2] will be very useful.

PROPOSITION 1.4. *If* $E \in \mathscr{E}(M, N)$ *and* e *is a nonzero projection in* $(N' \cap M)_E$, *then* $E_e^{-1} \in P(N'_e, M'_e)$ *is given by*

$$E_e^{-1}(x) = E^{-1}(E(e)x)e, \quad x \in (N'_e)_+.$$

Moreover if E *is the conditional expectation with respect to* $\varphi \in \mathscr{E}(M)$, *then* E_e *is that with respect to* $\varphi \mid M_e$.

Proof. Let $\varphi \in P(N)$ and $\psi \in P(M')$. Define $\varphi^e \in P(N_e)$ by $\varphi^e(xe) = \varphi(xz_N(e))$, $x \in N_+$, and $\psi^e \in P(M'_e)$ by $\psi^e(xe) = \psi(xz_M(e))$, $x \in M'_+$. Also let $(\varphi\circ E)_e = \varphi\circ E \mid M_e$ and $(\psi\circ E^{-1})_e = \psi\circ E^{-1} \mid N'_e$. Then $(\varphi\circ E)_e \in P(M_e)$ and $(\psi\circ E^{-1})_e \in P(N'_e)$ because $e \in M_{\varphi\circ E}$ and $e \in M_{\psi\circ E^{-1}}$ by [8, Theorem 6.13]. Since

$$\varphi^e(E_e(x)) = \varphi(E(x)E(e)^{-1}z_N(e)) =$$

$$= \varphi(E(E(e)^{-1/2}xE(e)^{-1/2})), \quad x \in (M_e)_+,$$

we get

$$\varphi^e\circ E_e = (E(e)^{-1/2}e)(\varphi\circ E)_e(E(e)^{-1/2}e).$$

Now define $T(x) = E^{-1}(E(e)x)e$ for $x \in N'_e$. Then $T \in \mathscr{E}(N'_e, M'_e)$ is easily seen. Since

$$(\psi\circ E^{-1})_e(x) = \psi(E^{-1}(x)z_M(e)) =$$

$$= \psi^e(E^{-1}(x)e), \quad x \in (N'_e)_+,$$

we get

$$(E(e)^{1/2}e)(\psi\circ E^{-1})_e(E(e)^{1/2}e) =$$

$$= (E(e)^{1/2}e)(\psi^e\circ(E^{-1}(\cdot)e))(E(e)^{1/2}e) = \psi^e\circ T.$$

Furthermore, we have by Lemma 1.3

$$\frac{\mathrm{d}(\varphi\circ E)_e}{\mathrm{d}\psi^e} = \frac{\mathrm{d}\varphi\circ E}{\mathrm{d}\psi}\Big|_{e\mathscr{H}} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi\circ E^{-1}}\Big|_{e\mathscr{H}} = \frac{\mathrm{d}\varphi^e}{\mathrm{d}(\psi\circ E^{-1})}.$$

Because $E(e) \in Z(N) \subseteq (N' \cap M)_E$, it follows that $E(e)e$ belongs to the centralizers of both $(\varphi\circ E)_e$ and $(\psi\circ E^{-1})_e$. Therefore Lemma 1.1 shows

$$\frac{\mathrm{d}\varphi^e\circ E_e}{\mathrm{d}\psi^e} = (E(e)^{-1/2}e)\frac{\mathrm{d}(\varphi\circ E)_e}{\mathrm{d}\psi^e}(E(e)^{-1/2}e) =$$

$$= (E(e)^{-1/2}e)\frac{\mathrm{d}\varphi^e}{\mathrm{d}(\psi\circ E^{-1})_e}(E(e)^{-1/2}e) = \frac{\mathrm{d}\varphi^e}{\mathrm{d}\psi^e\circ T},$$

so that $E_e^{-1} = T$. The last assertion is immediate. ∎

PROPOSITION 1.5. *Let $E \in \mathscr{E}(M, N)$.*

(1) *If e is a nonzero projection in N with $z_M(e) = 1$ and if $\hat{E}_e = E \mid M_e$, then $\hat{E}_e \in \mathscr{E}(M_e, N_e)$ and $\hat{E}_e^{-1} \in P(N'e, M'e)$ is given by*

$$\hat{E}_e^{-1}(xe) = E^{-1}(xz_N(e))e, \quad x \in N'_+.$$

(2) *If e is a nonzero projection in M' with $z_M(e) = 1$ and if $\hat{E}^e(xe) = E(x)e$ for $x \in M$, then $\hat{E}^e \in \mathscr{E}(Me, Ne)$ and $(\hat{E}^e)^{-1} \in P(N'_e, M'_e)$ is $E^{-1} \mid N'_e$.*

Proof. (1) Clearly $\hat{E}_e \in \mathscr{E}(M_e, N_e)$. We can take $\varphi \in P(N)$ with $e \in N_\varphi$. Because $\sigma_t^{\varphi\circ E} \mid N = \sigma_t^\varphi$, we get $e \in M_{\varphi\circ E}$ as well. Also $\varphi_e = \varphi \mid N_e \in P(N_e)$ and $\varphi_e\circ\hat{E}_e = \varphi\circ E \mid M_e$. For $\psi \in P(M')$, define $\psi^e \in P(M'e)$ by $\psi^e(xe) = \psi(x)$, $x \in M'_+$, and $(\psi\circ E^{-1})^e \in P(N'e)$ by $(\psi\circ E^{-1})^e(xe) = \psi\circ E^{-1}(xz_N(e))$, $x \in N'_+$. Moreover, defining $T(xe) = E^{-1}(xz_N(e))e$ for $x \in N'_+$, we easily check that $T \in P(N'e, M'e)$ and $(\psi\circ E^{-1})^e = \psi^e\circ T$. Hence Lemma 1.3 shows

$$\frac{\mathrm{d}\varphi_e\circ\hat{E}_e}{\mathrm{d}\psi^e} = \frac{\mathrm{d}\varphi\circ E}{\mathrm{d}\psi}\Big|_{e\mathscr{H}} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi\circ E^{-1}}\Big|_{e\mathscr{H}} = \frac{\mathrm{d}\varphi_e}{\mathrm{d}\psi^e\circ T},$$

so that $\hat{E}_e^{-1} = T$.

(2) It is easily seen that $\hat{E}^e \in \mathcal{E}(Me, Ne)$ and $E^{-1} \mid N'_e \in P(N'_e, M'_e)$. Note that $z_M(e) = 1$ implies $z_N(e) = 1$. For $\varphi \in P(N)$, define $\varphi^e \in P(Ne)$ by $\varphi^e(xe) = \varphi(x)$, $x \in N_+$, and $(\varphi\circ E)^e \in P(Me)$ by $(\varphi\circ E)^e(xe) = \varphi\circ E(x)$, $x \in M_+$. Since

$$\varphi^e(\hat{E}^e(x)) = \varphi(E(x)) = (\varphi\circ E)^e(xe), \quad x \in M_+,$$

we have $\varphi^e\circ\hat{E}^e = (\varphi\circ E)^e$. Next take $\psi \in P(M')$ with $e \in (M')_\psi$. Because $\sigma_t^{\psi\circ E^{-1}} \mid M' = \sigma_t^\psi$ by [8, Theorem 4.7], we also get $e \in M_{\psi\circ E^{-1}}$. Hence Lemma 1.3 shows

$$\frac{d\varphi^e\circ\hat{E}^e}{d\psi_e} = \frac{d\varphi\circ E}{d\psi}\bigg|_{e\mathscr{H}} = \frac{d\varphi}{d\psi\circ E^{-1}}\bigg|_{e\mathscr{H}} = \frac{d\varphi}{d\psi_e\circ(E^{-1} \mid N'_e)},$$

since $\psi\circ E^{-1} \mid N'_e = \psi_e\circ(E^{-1} \mid N'_e)$. Therefore $(\hat{E}^e)^{-1} = E^{-1} \mid N'_e$. ▨

LEMMA 1.6. *Let $M_i \supseteq N_i$ be von Neumann algebras on a Hilbert space $\mathscr{H}_i$ for $i = 1, 2$. If $\varphi_i \in P(M_i)$ and $\psi_i \in P(M'_i)$, then*

$$\frac{d(\varphi_1 \otimes \varphi_2)}{d(\psi_1 \otimes \psi_2)} = \frac{d\varphi_1}{d\psi_1} \otimes \frac{d\varphi_2}{d\psi_2}.$$

Proof. In the following proof, we use the usual notations in the spatial theory (see [5, 24]). Let $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, $\varphi = \varphi_1 \otimes \varphi_2$ and $\psi = \psi_1 \otimes \psi_2$. Let q (resp. q_i) be the positive quadratic form on $\mathscr{H}$ (resp. $\mathscr{H}_i$) corresponding to φ and ψ (resp. φ_i and ψ_i). Define $A = (d\varphi_1/d\psi_1) \otimes (d\varphi_2/d\psi_2)$ and $q_A = \|A^{1/2}\xi\|^2$ for $\xi \in D(A^{1/2})$. Since $A^{1/2} = (d\varphi_1/d\psi_1)^{1/2} \otimes (d\varphi_1/d\psi_2)^{1/2}$ and $D(q_i) = D(\mathscr{H}_i, \psi_i) \cap D((d\varphi_i/d\psi_i)^{1/2})$ is a core of $(d\varphi_i/d\psi_i)^{1/2}$, it follows that the algebraic tensor product $D(q_1) \odot D(q_2)$ is a core of $A^{1/2}$. If $\xi_i \in D(\mathscr{H}_i, \psi_i)$, then $\xi_1 \otimes \xi_2 \in D(\mathscr{H}, \psi)$ and $R^\psi(\xi_1, \otimes \xi_2) = R^{\psi_1}(\xi_1) \otimes R^{\psi_2}(\xi_2)$. Hence, for every $\xi = \sum_{j=1}^n \xi_{1j} \otimes \xi_{2j}$ in $D(q_1) \odot D(q_2)$, we have

$$q(\xi) = \sum_{j,k=1}^n \varphi(R^\psi(\xi_{1j} \otimes \xi_{2j})R^\psi(\xi_{1k} \otimes \xi_{2k})^*) =$$

$$= \sum_{j,k=1}^n \varphi_1(R^{\psi_1}(\xi_{1j})R^{\psi_1}(\xi_{1k})^*)\varphi_2(R^{\psi_2}(\xi_{2j})R^{\psi_2}(\xi_{2k})^*) =$$

$$= \sum_{j,k=1}^n \left\langle \left(\frac{d\varphi_1}{d\psi_1}\right)^{1/2} \xi_{1j}, \left(\frac{d\varphi_1}{d\psi_1}\right)^{1/2} \xi_{1k} \right\rangle \left\langle \left(\frac{d\varphi_2}{d\psi_2}\right)^{1/2} \xi_{2j}, \left(\frac{d\varphi_2}{d\psi_2}\right)^{1/2} \xi_{2k} \right\rangle = q_A(\xi).$$

Therefore $D(\bar{q}) \supseteq D(A^{1/2})$ and $\bar{q}(\xi) = q_A(\xi)$ for all $\xi \in D(A^{1/2})$, where $\bar{q}$ is the closure of q. This shows that $d\varphi/d\psi \leqslant A$. Interchanging the roles of φ (resp. φ_i) and ψ (resp. ψ_i), we also get $d\psi/d\varphi \leqslant (d\psi_1/d\varphi_1) \otimes (d\psi_2/d\varphi_2) = A^{-1}$. Thus $d\varphi/d\psi = A$. ∎

PROPOSITION 1.7. *If* $E_i \in \mathscr{E}(M_i, N_i)$ *for* $i = 1, 2$, *then* $(E_1 \otimes E_2)^{-1} = E_1^{-1} \otimes E_2^{-1}$.

Proof. For $\varphi_i \in P(N_i)$ and $\psi_i \in P(M_i')$, by Lemma 1.6 we have

$$\frac{d(\varphi_1 \otimes \varphi_2)\circ(E_1 \otimes E_2)}{d(\psi_1 \otimes \psi_2)} = \frac{d((\varphi_1\circ E_1) \otimes (\varphi_2\circ E_2))}{d(\psi_1 \otimes \psi_2)} = \frac{d\varphi_1\circ E_1}{d\psi_1} \otimes \frac{d\varphi_2\circ E_2}{d\psi_2} =$$

$$= \frac{d\varphi_1}{d\psi_1\circ E_1^{-1}} \otimes \frac{d\varphi_2}{d\psi_2\circ E_2^{-1}} = \frac{d(\varphi_1 \otimes \varphi_2)}{d(\psi_1 \otimes \psi_2)\circ(E_1^{-1} \otimes E_2^{-1})},$$

as desired. ∎

When $M_i \supseteq N_i$, $i = 1, 2$, are pairs of a factor and one of its subfactors, the above proposition shows that

$$\mathrm{Index}(E_1 \otimes E_2) = (\mathrm{Index}\, E_1)(\mathrm{Index}\, E_2), \quad E_i \in \mathscr{E}(M_i, N_i).$$

LEMMA 1.8. *Let* $\varphi \in P(M)$ *and* $\psi \in P(N')$. *If* $\varphi|N$ *and* $\psi|M'$ *are semifinite, then*

$$\frac{d\varphi}{(d\psi|M')} \leqslant \frac{d(\varphi|N)}{d\psi}.$$

Proof. Let $\psi_1 = \psi|M'$. Then $\mathscr{H}_{\varphi_1} \subseteq \mathscr{H}_{\varphi}$ and $D(\mathscr{H}, \psi_1) \supseteq D(\mathscr{H}, \psi)$. For every $\xi \in D(\mathscr{H}, \psi)$, since $R^{\psi_1}(\xi) = R^{\psi}(\xi)|\mathscr{H}_{\psi_1}$, we have

$$\varphi(R^{\psi_1}(\xi)R^{\psi_1}(\xi)^*) \leqslant (\varphi|N)(R^{\psi}(\xi)R^{\psi}(\xi)^*).$$

Hence $D(q_1) \supseteq D(q)$ and $q_1(\xi) \leqslant q(\xi)$ for all $\xi \in D(q)$, where q_1 (resp. q) is the closure of positive quadratic form corresponding to φ and ψ_1 (resp. $\varphi|N$ and ψ). This shows the desired inequaltiy. ∎

PROPOSITION 1.9. *Suppose* N' *is* σ*-finite (this is the case when* $\mathscr{H}$ *is separable). If* $E \in \mathscr{E}(M, N)$ *and* $E^{-1}(1)$ *is a positive selfadjoint operator, then:*

(1) $E^{-1}(x) \geqslant x$ *for all* $x \in N'_+$,

(2) $E(x) \geqslant E^{-1}(1)^{-1}x$ *for all* $x \in M_+$.

Proof. (1) If $\varphi \in \mathscr{E}(N)$ and $\psi \in \mathscr{E}(N')$, then we have by Lemma 1.8

$$\frac{d\varphi\circ E}{d(\psi|M')} \leqslant \frac{d(\varphi\circ E|N)}{d\psi} = \frac{d\varphi}{d\psi},$$

implying $(\psi \mid M')\circ E^{-1} \geqslant \psi$ by [5, Proposition 8]. By assumptions, we can choose $\psi_0 \in \mathscr{E}(N')$ such that $\psi_0(E^{-1}(1)) < \infty$. For every $x \in N'_+$ and $\omega \in (N')^+_*$, we get

$$(\omega + n^{-1}\psi_0)(E^{-1}(x)) \geqslant (\omega + n^{-1}\psi_0)(x), \quad n \geqslant 1,$$

so that $\omega(E^{-1}(x)) \geqslant \omega(x)$. This shows $E^{-1}(x) \geqslant x$.

(2) By (1), $E^{-1}(1) \geqslant 1$ and so $E^{-1}(1)^{-1} \in Z(M)$. For $\varphi \in \mathscr{E}(M)$ and $\psi \in \mathscr{E}(M')$, since

$$\psi(E^{-1}(x)) = \psi(x^{1/2}E^{-1}(1)x^{1/2}) =$$

$$= \psi(E^{-1}(1)^{1/2}xE^{-1}(1)^{1/2}), \quad x \in M'_+,$$

we have by Lemmas 1.8 and 1.1

$$\frac{\mathrm{d}(\varphi \mid N)}{\mathrm{d}\psi\circ E^{-1}} \geqslant \frac{\mathrm{d}\varphi}{\mathrm{d}(\psi\circ E^{-1} \mid M')} = \frac{\mathrm{d}\varphi}{\mathrm{d}E^{-1}(1)^{1/2}\psi E^{-1}(1)^{1/2}} =$$

$$= \frac{\mathrm{d}E^{-1}(1)^{-1/2}\varphi E^{-1}(1)^{-1/2}}{\mathrm{d}\psi},$$

mplying $(\varphi \mid N)\circ E \geqslant E^{-1}(1)^{-1/2}\varphi E^{-1}(1)^{-1/2}$. Hence, if $x \in M_+$, then $\varphi(E(x)) \geqslant \varphi(E^{-1}(1)^{-1}x)$ for all $\varphi \in \mathscr{E}(M)$, so that $E(x) \geqslant E^{-1}(1)^{-1}x$. ▨

PROPOSITION 1.10. *If $M \supseteq L \supseteq N$ are factors, then* $\mathrm{Index}(E \mid L) \leqslant \mathrm{Index}\,E$ *for every $E \in \mathscr{E}(M, N)$.*

Proof. Suppose $\mathrm{Index}\,E < \infty$. Taking $\varphi \in \mathscr{E}(N)$ and $\psi \in \mathscr{E}(L')$ under the standard representation of M, we have by Lemma 1.8

$$\frac{\mathrm{d}\varphi\circ E}{\mathrm{d}(\psi \mid M')} \leqslant \frac{\mathrm{d}(\varphi\circ E \mid L)}{\mathrm{d}\psi} = \frac{\mathrm{d}\varphi\circ(E \mid L)}{\mathrm{d}\psi}.$$

Hence $(\psi \mid M')\circ E^{-1} \geqslant \psi\circ(E \mid L)^{-1}$, so that $\mathrm{Index}\,E \geqslant \mathrm{Index}(E \mid L)$. ▨

2. PROPERTIES OF MINIMUN INDEX

Throughout this section, let $M \supseteq N$ be a pair of a factor and one of its subfactors. As noted in [9], if $\mathrm{Index}\,E < \infty$ for some $E \in \mathscr{E}(M, N)$, then $N' \cap M$ is nite dimensional and $\mathrm{Index}\,F < \infty$ for all $F \in \mathscr{E}(M, N)$. Concerning minimizing Index E for $E \in \mathscr{E}(M, N)$, the next theorem was established in [9].

THEOREM 2.1. *Suppose* $\operatorname{Index} E < \infty$ *for some* $E \in \mathcal{E}(M, N)$.

(1) *There exists a unique* $E_0 \in \mathcal{E}(M, N)$ *such that*

$$\operatorname{Index} E_0 = \min\{\operatorname{Index} E : E \in \mathcal{E}(M, N)\}.$$

(2) *The following conditions for* $E \in \mathcal{E}(M, N)$ *are equivalent:*

(i) $E = E_0$;

(ii) $E|_{N' \cap M}$ *and* $E^{-1}|_{N' \cap M}$ *are traces and*

$$E^{-1}|_{N' \cap M} = (\operatorname{Index} E)E|_{N' \cap M};$$

(iii) $E^{-1}|_{N' \cap M} = (\operatorname{Index} E)E|_{N' \cap M}$;

(iv) $E|_{N' \cap M}$ *is a trace and for minimal central projections* $e_1, \ldots, e_n$ *in* $N' \cap M$ *with* $\sum e_i = 1$,

$$E(e_i) = \left(\frac{\operatorname{Index} E_{e_i}}{\operatorname{Index} E}\right)^{1/2}, \quad 1 \leqslant i \leqslant n.$$

The characterization by condition (iv) is readily seen from the proof in [9], while it is not explicitly stated there. This characterizaion has been also given by Longo [18]. If $N' \cap M \neq \mathbf{C}$ in Theorem 2.1, then $\operatorname{Index} E$ of $E \in \mathcal{E}(M, N)$ admits all the values in $[\operatorname{Index} E_0, \infty)$ (see [9]).

Now let us define the *minimum index* $[M : N]_0$ for a pair $M \supseteq N$ as follows:

DEFINITION 2.2. $[M: N]_0 = \min\{\operatorname{Index} E : E \in \mathcal{E}(M, N)\}$ if $\operatorname{Index} E < \infty$ for some $E \in \mathcal{E}(M, N)$ and $[M : N]_0 = \infty$ otherwise (i.e. $\mathcal{E}(M, N) = \emptyset$ or $\operatorname{Index} E = \infty$ for all $E \in \mathcal{E}(M, N)$).

In the sequel of this section, we present general properties of the minimum index.

PROPOSITION 2.3. $[M: N]_0 = [N' : M']_0$.

Proof. Suppose $[M: N]_0 = \operatorname{Index} E_0 < \infty$ with $E_0 \in \mathcal{E}(M, N)$, and let $E_0' = (\operatorname{Index} E_0)^{-1} E_0^{-1}$. Then $E_0' \in \mathcal{E}(N', M')$ and $E_0'^{-1} = (\operatorname{Index} E_0)E_0$. Hence Theorem 2.1(2) shows

$$E_0'^{-1}|_{N' \cap M} = E_0^{-1}|_{N' \cap M} = (\operatorname{Index} E_0)E_0'|_{N' \cap M},$$

so that $[N' : M']_0 = \operatorname{Index} E_0' = \operatorname{Index} E_0$. The desired equality follows from symmetry between $M \supseteq N$ and $N' \supseteq M'$. ∎

PROPOSITION 2.4. *If* $\tilde{M}$ *and* $\tilde{N}$ *are factors with* $M \supseteq \tilde{M} \supseteq \tilde{N} \supseteq N$, *then* $[\tilde{M} : \tilde{N}]_0 \leqslant [M: N]_0$.

Proof. It suffices to show $[L: N]_0 \leqslant [M: N]_0$ and $[M: L]_0 \leqslant [M: N]_0$ for any factor L with $M \supseteq L \supseteq N$. Proposition 1.10 implies the first inequailty. The second follows from the first and Proposition 2.3. ▨

For factors $M \supseteq L \supseteq N$, it is clear that $[M: N]_0 \leqslant [M: L]_0[L: N]_0$. The following two propositions are concerned with the case when the equality holds.

PROPOSITION 2.5. *If L is a factor with $M \supseteq L \supseteq N$ and $[M: N]_0 = \operatorname{Index} E_0 < \infty$ where $E_0 \in \mathscr{E}(M, N)$, then the following conditions are equivalent:*

(i) $[M: N]_0 = [M: L]_0\, [L: N]_0$;

(ii) *there exists $F \in \mathscr{E}(M, L)$ such that $E_0 \circ F = E_0$;*

(iii) $\sigma_t^{\varphi \circ E_0}(L) = L$, $t \in \mathbf{R}$, *for $\varphi \in \mathscr{E}(N)$ (independently of the choice of φ).*

Proof. (i)$\Rightarrow$(ii). By Proposition 2.4, $[M: L]_0 = \operatorname{Index} F < \infty$ with $F \in \mathscr{E}(M, L)$ and $[L: N]_0 = \operatorname{Index} G < \infty$ with $G \in \mathscr{E}(L, N)$. Since $G \circ F \in \mathscr{E}(M, N)$ and $\operatorname{Index}(G \circ F) = (\operatorname{Index} F)(\operatorname{Index} G) = [M: N]_0$, we get $G \circ F = E_0$ by Theorem 2.1(1). Hence $G = E_0 \mid L$ and so $E_0 \circ F = E_0$.

(ii) $\Rightarrow$ (i). Letting $G = E_0 \mid L$, we have

$$[M: N]_0 = \operatorname{Index}(G \circ F) = (\operatorname{Index} F)(\operatorname{Index} G) \geqslant [M: L]_0\; [L: N]_0.$$

(ii) $\Leftrightarrow$ (iii) follows immediately from [25]. ▨

PROPOSITION 2.6. *If L is a factor with $M \supseteq L \supseteq N$ and $N' \cap M = (N' \cap L) \vee \vee (L' \cap M)$, then $[M: N]_0 = [M: L]_0[L: N]_0$.*

Proof. By Proposition 2.4, we may assume that $[M: L]_0 = \operatorname{Index} F < \infty$ and $[L: N]_0 = \operatorname{Index} G < \infty$ where $F \in \mathscr{E}(M, L)$ and $G \in \mathscr{E}(L, N)$. If $x \in L' \cap M$ and $y \in N' \cap L$, then we have by Theorem 2.1 (2).

$$(G \circ F)^{-1}(xy) = F^{-1}(x) G^{-1}(y) = (\operatorname{Index} F)(\operatorname{Index} G) G \circ F(xy).$$

Hence

$$(G \circ F)^{-1} \mid N' \cap M = (\operatorname{Index} F)(\operatorname{Index} G) G \circ F \mid N' \cap M,$$

implying the desired equality. ▨

The next proposition has been independently obtained in [18].

PROPOSITION 3.7. *If $M_i \supseteq N_i$, $i = 1, 2$, are pairs of a factors and one of its subfactors, then*

$$[M_1 \otimes M_2: N_1 \otimes N_2]_0 = [M_1: N_1]_0[M_2: N_2]_0.$$

Proof. Because the assumption of Proposition 2.6 is satisfied for $M_1 \otimes M_2 \supseteq \supseteq N_1 \otimes M_2 \supseteq N_1 \otimes N_2$, Proposition 2.6 implies

$$[M_1 \otimes M_2: N_1 \otimes N_2]_0 = [M_1 \otimes M_2: N_1 \otimes M_2]_0[N_1 \otimes M_2: N_1 \otimes N_2]_0.$$

It is known [3, Lemme 2.3] that each $E \in \mathcal{E}(M_1 \otimes M_2, N_1 \otimes M_2)$ is written as $E = E_1 \otimes \mathrm{id}_{M_2}$ with $E_1 \in \mathcal{E}(M_1, N_1)$. Hence $[M_1 \otimes M_2 : N_1 \otimes M_2]_0 = [M_1 : N_1]_0$ and analogously $[N_1 \otimes M_2 : N_1 \otimes N_2]_0 = [M_2 : N_2]_0$, as desired. ∎

In the next theorem, we consider the minimum index for crossed products. Let G be a locally compact abelian group and α be an action of G on M such that $\alpha_g(N) = N$ for all $g \in G$. Let $M \times_\alpha G$ and $N \times_\alpha G$ denote the crossed products of M by α and of N by $\alpha \mid N$, respectively. Then $M \times_\alpha G \supseteq N \times_\alpha G$ canonically. Also let M^α and N^α be the fixed point subalgebras of α and $\alpha \mid N$, respectively. Here we suppose the second axiom of countability for G, which guarantees that the von Neumann algebras appearing in the following are σ-finite.

THEOREM 2.8. *Let G and α be as above.*

(1) *$M \times_\alpha G$ and $N \times_\alpha G$ are factors, then $[M \times_\alpha G : N \times_\alpha G]_0 = [M : N]_0$.*

(2) *Suppose $\alpha \mid N$ is dominant. If M^α and N^α are factors (equivalently so are $M \times_\alpha G$ and $M \times_\alpha G$), then $[M^\alpha : N^\alpha]_0 = [M : N]_0$.*

Proof. (1) Letting $\widetilde{M} = M \times_\alpha G$ and $\widetilde{N} = N \times_\alpha G$, we have

$$\widetilde{M} = (M \otimes \mathbf{B}(L^2(G)))^{\alpha \otimes \mathrm{Ad}(\rho)}, \quad \widetilde{N} = (N \otimes \mathbf{B}(L^2(G)))^{\alpha \otimes \mathrm{Ad}(\rho)},$$

where ρ (resp. λ) is the right (resp. left) regular representation of G on $L^2(G)$. Suppose $[M : N]_0 = \mathrm{Index}\, E < \infty$ with $E \in \mathcal{E}(M, N)$. For each $g \in G$, since $\alpha_g \circ E \circ \alpha_g^{-1} \in \mathcal{E}(M, N)$ and $\mathrm{Index}(\alpha_g \circ E \circ \alpha_g^{-1}) = \mathrm{Index}\, E$, we get $\alpha_g \circ E \circ \alpha_g^{-1} = E$ by Theorem 2.1 (1). Hence E commutes with α, so that $E \otimes \mathrm{id}$ commutes with $\alpha \otimes \mathrm{Ad}(\rho)$ where $\mathrm{id} = \mathrm{id}_{B(L^2(G))}$. Thus $\widetilde{E} \in \mathcal{E}(\widetilde{M}, \widetilde{N})$ can be defined by $\widetilde{E} = E \otimes \mathrm{id} \mid \widetilde{M}$. Then $\widetilde{E}(\pi_\alpha(x)) = \pi_\alpha(E(x))$ for all $x \in M$. By [29, 2.5.3] based on [16, Corollary 3.4], we can choose a finite basis $\{a_1, \dots, a_n\}$ in M for E, i.e. $\sum_{i=1}^{n} a_i E(a_i^* x) = x$ for all $x \in M$. Because

$$\sum_{i=1}^{n} \pi_\alpha(a_i) \widetilde{E}(\pi_\alpha(a_i^*) \pi_\alpha(x)(1 \otimes \lambda(g))) =$$

$$= \pi_\alpha\left(\sum_{i=1}^{n} a_i E(a_i^* x)\right)(1 \otimes \lambda(g)) =$$

$$= \pi_\alpha(x)(1 \otimes \lambda(g)), \quad x \in M,\ g \in G,$$

it follows that $\{\pi_\alpha(a_1), \dots, \pi_\alpha(a_n)\}$ is a basis for $\widetilde{E}$. Hence we obtain (see [4], [29])

$$\mathrm{Index}\, \widetilde{E} = \pi_\alpha\left(\sum_{i=1}^{n} a_i a_i^*\right) = \mathrm{Index}\, E,$$

implying $[\widetilde{M} : \widetilde{N}]_0 \leq [M : N]_0$.

Next let the action $\hat{\alpha}$ of $\hat{G}$ on $\tilde{M}$ be the dual action of α. Then $\hat{\alpha} \mid \tilde{N}$ is the dual co-action of $\alpha \mid N$. Letting $\tilde{\tilde{M}} = \tilde{M} \times_{\hat{\alpha}} \hat{G}$ and $\tilde{\tilde{N}} = \tilde{N} \times_{\hat{\alpha}} \hat{G}$, we have

$$\tilde{\tilde{M}} \cong M \otimes \mathbf{B}(L^2(G)), \quad \tilde{\tilde{N}} \cong N \otimes \mathbf{B}(L^2(G)).$$

By the above argument applied to $\tilde{M} \supseteq \tilde{N}$ and $\hat{\alpha}$ we obtain $[\tilde{\tilde{M}}: \tilde{\tilde{N}}]_0 \leqslant [\tilde{M}: \tilde{N}]_0$. Furthermore

$$[\tilde{\tilde{M}}: \tilde{\tilde{N}}]_0 = [M \otimes \mathbf{B}(L^2(G)): N \otimes \mathbf{B}(L^2(G))]_0 = [M:N]_0$$

by Proposition 2.7. Thus $[\tilde{M}: \tilde{N}]_0 = [M:N]_0$.

(2) Since $\alpha \mid N$ is dominant, we see by [24, Proposition 20.12] that so is α and

$$M \times_\alpha G \cong M^\alpha \otimes \mathbf{B}(L^2(G)), \quad N \times_\alpha G \cong N^\alpha \otimes \mathbf{B}(L^2(G)).$$

Hence the desired equality follows from (1). ▨

3. DEFINITION OF ENTROPY

Let M be a finite von Neumann algebra with a faithful normal trace τ, $\tau(1) = 1$, and N be a von Neumann subalgebra of M. The entropy $H(M \mid N)$ of M relative to N developed by Pimsner and Popa [22] is defined as follows:

$$(*) \qquad H(M \mid N) = \sup_{(x_i)} \sum_i \{\tau(\eta E_N(x_i)) - \tau(\eta x_i)\},$$

where $\eta(t) = -t \log t$ on $[0, \infty)$, E_N is the conditional expectation $M \to N$ with respect to τ [27], and the supremum is taken over all finite families $(x_1, \ldots, x_n)$ of $x_i \in M_+$ with $\sum x_i = 1$. This entropy was previously used by Connes and Størmer [7] in the case of M being finite dimensional. When M is a type II_1 factor and N is a subfactor of it, Pimsner and Popa [22, Theorem 4.4] exactly estimated $H(M \mid N)$ in terms of Jones' index as follows: If $N' \cap M$ has a nonatomic part, then $H(M \mid N) = \infty$. If $N' \cap M$ is atomic and $\{f_k\}$ is a set of atoms in $N' \cap M$ with $\sum f_k = 1$, then

$$(**) \qquad H(M \mid N) = \sum_k \tau(f_k) \log \frac{[M_{f_k} : N_{f_k}]}{\tau(f_k)^2}.$$

As a consequence of this estimate, they obtained several characterizations for the equality $H(M|N) = \log[M:N]$. Here $H(M|N) \leqslant \log[M:N]$ holds in general. As stated in [9], for a pair $M \supseteq N$, $H(M|N) = \log [M:N]$ holds if and only if E_N has the minimum index, that is, Jones' index $[M:N]$ is equal to the minimum index $[M:N]_0$.

Now, in particular, assume $[M:N] < \infty$ and choose minimal projections $f_1, \ldots, f_m$ in $N' \cap M$ with $\sum f_k = 1$. Then, since

$$[M_{f_k}: N_{f_k}] = \mathrm{Index}(E_N)_{f_k} = \tau(f_k)E_N^{-1}(f_k)$$

by [16, Proposition 4.2], equation (**) is written as

$$H(M|N) = -\sum_{k=1}^{m} \tau(f_k)\log\frac{\tau(f_k)}{E_N^{-1}(f_k)}.$$

Hence, in this case, $-H(M|N)$ is nothing but the relative entropy of $(\tau(f_1), \ldots, \tau(f_m))$ and $(E_N^{-1}(f_1), \ldots, E_N^{-1}(f_m))$.

From now on, let M be a von Neumann algebra with a fixed $\varphi \in \mathscr{W}(M)$, and N be a von Neumann subalgebra of M such that $E \in \mathscr{E}(M, N)$ with respect to φ exists (i.e. $\sigma_t^\varphi(N) = N$, $t \in \mathbf{R}$, [25]). Taking account of the above remarks, we introduce the *entropy* $K_\varphi(M|N)$ of M relative to φ and N as follows:

DEFINITION 3.1. Let $\omega = \varphi|N' \cap M$ and $\hat{\omega} = \varphi \cdot (E^{-1}|N' \cap M)$. Here $\hat{\omega}$ is a faithful normal weight on $N' \cap M$ thanks to $E^{-1}((N' \cap M)_+) \subseteq Z(M)_+^\frown$, but not necessarily bounded (possibly not semifinite). So we define the relative entropy $S(\hat{\omega}, \omega)$ of ω and $\hat{\omega}$ by

$$S(\hat{\omega}, \omega) = \inf\{S(\omega', \omega): \omega' \in (N' \cap M)_*^+,\quad \omega' \leqslant \hat{\omega}\}$$

with Araki's relative entropy $S(\omega', \omega)$ [1], [2]. (Note that the notation $S(\omega, \omega')$ is sometimes used instead of $S(\omega', \omega)$.) Define $K_\varphi(M|N) = -S(\hat{\omega}, \omega)$.

Note [8, Proposition 2.3 and Lemma 2.6] that $\hat{\omega}$ is semifinite if and only if so is $E^{-1}|N' \cap M$. Also $E^{-1}|N' \cap M$ is semifinite whenever so is $T|N' \cap M$ for some $T \in P(N', M')$ (see [8, Theorem 6.6]). Moreover, given a von Neumann subalgebra $\mathscr{A}$ of $N' \cap M$, we denote by $S_{\mathscr{A}}(\hat{\omega}, \omega)$ the relative entropy of $\omega|\mathscr{A}$ and $\hat{\omega}|\mathscr{A}$ in the above sense.

Although E^{-1} depends on the representing Hilbert space $\mathscr{H}$ for M, we have the following:

PROPOSITION 3.2. *$K_\varphi(M|N)$ is determined independently of the choice of $\mathscr{H}$.*

Proof. Let us show that $K_\varphi(M|N) = K_{\varphi\circ\alpha^{-1}}(\alpha(M)\,|\,\alpha(N))$ for any isomorphism α of M. To do this, it suffices to prove

$$(\alpha\circ E\circ\alpha^{-1})^{-1}\,|\,\alpha(N'\cap M) = \alpha\circ E^{-1}\circ\alpha^{-1}\,|\,\alpha(N'\cap M).$$

In fact, if the above holds, then

$$K_\varphi(M\,|\,N) = -S(\hat{\omega},\,\omega) = -S(\hat{\omega}\circ\alpha^{-1},\,\omega\circ\alpha^{-1}) =$$

$$= K_{\varphi\circ\alpha^{-1}}(\alpha(M)\,|\,\alpha(N)),$$

where ω and $\hat{\omega}$ are as in Definition 3.1. We may separately consider an amplification, an induction and a spatial isomorphism. First let α be an amplification, i.e. $\alpha\colon x\in M\mapsto x\otimes 1\in M\otimes 1$ where 1 is the identity operator on a Hilbert space $\mathscr{K}$. Then $\alpha\circ E\circ\alpha^{-1} = E\otimes \mathrm{id}_{\mathbf{C}}$, so that $(\alpha\circ E\circ\alpha^{-1})^{-1} = E^{-1}\otimes \mathrm{id}_{\mathbf{B}(\mathscr{K})}$ by Proposition 1.7. Hence

$$(\alpha\circ E\circ\alpha^{-1})^{-1}(x\otimes 1) = E^{-1}(x)\otimes 1 = (\alpha\circ E^{-1}\circ\alpha^{-1})(x\otimes 1)$$

for every $x\in(N'\cap M)_+$. Next let α be an induction, i.e. $\alpha\colon x\in M\mapsto xe\in Me$ where e is a projection in M' with $z_M(e) = z_N(e) = 1$. Since $(\alpha\circ E\circ\alpha^{-1})(xe) = E(x)e$ for $x\in M$, we get $(\alpha\circ E\circ\alpha^{-1})^{-1} = E^{-1}\,|\,N'_e$ by Proposition 1.5 (2), so that

$$(\alpha\circ E\circ\alpha^{-1})^{-1}(xe) = E^{-1}(xe) = (\alpha\circ E^{-1}\circ\alpha^{-1})(xe)$$

for every $x\in(N'\cap M)_+$. Finally, when α is a spatial isomorphism, it is easy to check $(\alpha\circ E\circ\alpha^{-1})^{-1} = \alpha\circ E^{-1}\circ\alpha^{-1}$. ∎

In the sequel of this section, we establish some basic properties of the entropy $K_\varphi(M\,|\,N)$. The next theorem gives a convenient expression for the computation of $K_\varphi(M\,|\,N)$.

THEOREM 3.3. (1) *If $E^{-1}\,|\,N'\cap M$ is not semifinite, then $K_\varphi(M\,|\,N) = \infty$.*

(2) *If $E^{-1}\,|\,N'\cap M$ is semifinite, then there exists a unique positive selfadjoint operator h affiliated with $(N'\cap M)_E$ such that $\hat{\omega} = h^{1/2}\omega h^{1/2}$. In this case, $h\geqslant 1$ (i.e. $\hat{\omega}\geqslant\omega$) and*

$$K_\varphi(M\,|\,N) = \varphi(\log h).$$

Proof. (2) Suppose $E^{-1}|N' \cap M$ is semifinite. By virtue of [8, Theorems 6.6 and 6.13], we have

$$\sigma_t^{\omega} = \sigma_t^{E} = \sigma_{-t}^{E^{-1}} = \sigma_{-t}^{\hat{\omega}}, \quad t \in \mathbf{R}.$$

Hence the unique existence of h stated follows from [19, Theorem 5.12]. Now let e be the spectral projection of h corresponding to $(0,1)$. Suppose $e \neq 0$ and take $E_e \in \mathscr{E}(M_e, N_e)$. Then

$$\hat{\omega}(e) = \omega(E^{-1}(e)) \geqslant \omega(E^{-1}(E(e)e)e) = \omega(E_e^{-1}(e))$$

by Proposition 1.4. Because [16, Lemma 3.1] holds without the assumption of factorness, we get $E_e^{-1}(e) \geqslant e$ and hence $\hat{\omega}(e) \geqslant \omega(e)$. This is a contradiction since $\hat{\omega}(e) = \omega(he) < \omega(e)$. Therefore $h \geqslant 1$. Let $h = \int_1^{\infty} \lambda de_\lambda$ be the spectral decomposition of h, and $\mathscr{A}$ be the abelian von Neumann subalgebra of $N' \cap M$ generated by h. For each $\omega' \in (N' \cap M)_*^+$ with $\omega' \leqslant \hat{\omega}$, using the monotonicity of relative entropy (see [15], [26]) and the expression of relative entropy in the abelian case (see [11]), we have

$$S(\omega', \omega) \geqslant S_{\mathscr{A}}(\omega', \omega) = \int \log \frac{d\omega(e_\lambda)}{d\omega'(e_\lambda)} d\omega(e_\lambda) \geqslant$$

$$\geqslant \int \log \frac{d\omega(e_\lambda)}{d\hat{\omega}(e_\lambda)} d\omega(e_\lambda) = \int \log \lambda^{-1} d\omega(e_\lambda) = -\omega(\log h).$$

Furthermore, letting $\omega_n = h_n^{1/2}\omega h_n^{1/2}$ with $h_n = \int_1^n \lambda de_\lambda + n(1 - e_n)$ for $n \geqslant 1$, we get $\omega_n \in (N' \cap M)_*^+$ and $\omega_n \leqslant \hat{\omega}$. Because the abelian von Neumann subalgebra $\mathscr{A}_n$ $(\subseteq (N' \cap M)_\omega)$ generated by h_n is sufficient for $\{\omega, \omega_n\}$ in the sense of [10], we have by [10, Theorem 4.1] (see also [21, Theoerm 4])

$$S(\omega_n, \omega) = S_{\mathscr{A}_n}(\omega_n, \omega) = -\omega(\log h_n)$$

as the above computation. Since $\omega(\log h_n) \to \omega(\log h)$ as $n \to \infty$, we obtain

$$K_\varphi(M \mid N) = \omega(\log h) = \varphi(\log h).$$

(1) When $\hat{\omega}$ is not semifinite, there is a projection e in $N' \cap M$ with $e \neq 1$ such that ω is semifinite on $(N' \cap M)_e$ and $\hat{\omega}(x) = \infty$ for every nonzero positive element x in $(N' \cap M)_{1-e}$. Here $e \in (N' \cap M)_E$ is readily verified because $\hat{\omega}\circ\sigma_t^E = \hat{\omega}$, $t \in \mathbf{R}$. Note $N_e' \cap M_e = (N' \cap M)_e$, and let $E_e \in \mathscr{E}(M_e, N_e)$, $\omega_e = \varphi \mid (N' \cap M)_e$ and $\hat{\omega}_e = \varphi\circ(E_e^{-1} \mid (N' \cap M)_e)$. Then Proposition 1.4 shows $\hat{\omega}_e \leqslant \hat{\omega} \mid (N' \cap M)_e$, so that $\hat{\omega}_e$ is semifinite. Hence, applying (2) proved above to E_e, we have $\omega_e \leqslant \hat{\omega}_e$, so that $\omega \leqslant \hat{\omega}$ on $(N' \cap M)_e$. Now let $\omega_n = e\omega e + n(1 - e)\omega(1 - e)$ for $n \geqslant 1$. Then.

$$\omega_n = \omega + (n - 1)(1 - e)\omega(1 - e) \leqslant \hat{\omega}$$

thanks to $e \in (N' \cap M)_\omega$. Therefore, using [2, Theorem 3.6], we obtain

$$S(\hat{\omega}, \omega) \leqslant S(\omega_n, \omega) = S(\omega_n, e\omega e) + S(\omega_n, (1 - e)\omega(1 - e)) \leqslant$$

$$\leqslant S(n(1 - e)\omega(1 - e), (1 - e)\omega(1 - e)) = -\omega(1 - e)\log n.$$

This implies $S(\hat{\omega}, \omega) = -\infty$ and so $K_\varphi(M|N) = \infty$. ▨

COROLLARY 3.4. *$K_\varphi(M \mid N) \geqslant 0$, and $K_\varphi(M \mid N) = 0$ if and only if $M = N$.*

Proof. Theorem 3.3 implies $K_\varphi(M \mid N) \geqslant 0$. Suppose $K_\varphi(M \mid N) = 0$. Then $h = 1$ in Theorem 3.2 (2), so that $\varphi(E^{-1}(1)) = 1$. Let e_N be the projection in N' defined in [16]. Then $\varphi(E^{-1}(e_N)) = 1$ by [16, Lemma 3.1]. Therefore $e_N = 1$, implying $M = N$. The converse is obvious. ▨

PROPOSITION 3.5. *If $e_1, \ldots, e_n$ are projections in $N' \cap M$ with $\sum e_i = 1$, then*

$$K_\varphi(M \mid N) \leqslant \sum_{i=1}^{n} \varphi(e_i)\log \frac{\varphi(E^{-1}(e_i))}{\varphi(e_i)} \leqslant \log\varphi(E^{-1}(1))$$

with convention $\log\infty = \infty$.

Proof. Let $\mathscr{A}$ be the subalgebra of $N' \cap M$ generated by $e_1, \ldots, e_n$. For each $\omega' \in (N' \cap M)_*^+$ with $\omega' \leqslant \hat{\omega}$, we have

$$S(\omega', \omega) \geqslant S_{\mathscr{A}}(\omega', \omega) = -\sum_i \omega(e_i)\log \frac{\omega'(e_i)}{\omega(e_i)} \geqslant$$

$$\geqslant -\sum_i \omega(e_i)\log \frac{\hat{\omega}(e_i)}{\omega(e_i)},$$

showing the first inequality. The second is immediate from the concavity of $\log t$.

PROPOSITION 3.6. *For $i = 1, 2$, let M_i be a von Neumann algebra with $\varphi_i \in \mathcal{S}(M_i)$, and N_i be a von Neumann subalgebra of M_i such that $E_i \in \mathcal{E}(M_i, N_i)$ with respect to φ_i exists. If $E_i^{-1} | N_i' \cap M_i$, $i = 1, 2$, are semifinite, then*

$$K_{\varphi_1 \otimes \varphi_2}(M_1 \otimes M_2 | N_1 \otimes N_2) = K_{\varphi_1}(M_1 | N_1) + K_{\varphi_2}(M_2 | N_2).$$

Proof. Let $M = M_1 \otimes M_2$, $N = N_1 \otimes N_2$, $\varphi = \varphi_1 \otimes \varphi_2$, and $E = E_1 \otimes E_2$. Note that E is the conditional expectation $M \to N$ with respect to φ. Moreover

$$N' \cap M = (N_1' \cap M_1) \otimes (N_2' \cap M_2),$$

$$(N' \cap M)_E \supseteq (N_1' \cap M_1)_{E_1} \otimes (N_2' \cap M_2)_{E_2},$$

since $\sigma_t^E = \sigma_t^{E_1} \otimes \sigma_t^{E_2}$, $t \in \mathbf{R}$. As in Definition 3.1, we define ω_i and $\hat{\omega}_i$ corresponding to φ_i and E_i for $i = 1, 2$, as well as ω and $\hat{\omega}$. From the semifiniteness of $E_i | N_i' \cap M_i$, using Proposition 1.7, we see that $E^{-1} | N' \cap M$ is also semifinite and is the tensor product of $E_1^{-1} | N_1' \cap M_1$ and $E_2^{-1} | N_2' \cap M_2$. Therefore $\hat{\omega} = \hat{\omega}_1 \otimes \hat{\omega}_2$. Let h_i be the positive selfadjoint operator affiliated with $(N_i' \cap M_i)_{E_i}$ such that $\hat{\omega}_i = h_i^{1/2} \omega_i h_i^{1/2}$. Because $h_1 \otimes h_2$ is affiliated with $(N' \cap M)_E$ and $\hat{\omega} = (h_1 \otimes h_2)^{1/2} \cdot \omega (h_1 \otimes h_2)^{1/2}$, Theorem 3.3 (2) implies

$$K_\varphi(M | N) = \varphi(\log(h_1 \otimes h_2)) = \varphi_1(\log h_1) + \varphi_2(\log h_2) =$$

$$= K_{\varphi_1}(M_1 | N_1) + K_{\varphi_2}(M_2 | N_2). \qquad \blacksquare$$

4. DECOMPOSITION THEOREMS FOR ENTROPY

In this section, let $M \supseteq N$ and $\varphi \in \mathcal{S}(M)$ be given as in Section 3 so that $E \in \mathcal{E}(M, N)$ with respect to φ exists. Let $\omega = \varphi | N' \cap M$ and $\hat{\omega} = \varphi \circ (E^{-1} | N' \cap M)$. We establish some decomposition theorems for the entropy $K_\varphi(M | N)$. Similar results for Pimsner and Popa's entropy (*) were given by Kawakami and Yoshida [14]. Furthecmore we estimate $K_\varphi(M | N)$, as equation (**) in Section 3, in terms of Kosaki's index.

First note that if L is an atomic von Neumann algebra and $\omega \in \mathcal{S}(L)$, then L_ω is atomic. In fact, because an atomic L is a direct sum of type I factors, it suffices to show the case $L = \mathbf{B}(\mathcal{K})$. In this case, let $a = \mathrm{d}\omega/\mathrm{dtr}$, a trace class operator, where tr is the usual trace on $\mathbf{B}(\mathcal{K})$. Then, since L_ω coincides with $\{x \in \mathbf{B}(\mathcal{K}) : a^{it} x a^{-it} =$

$= x, t \in \mathbf{R}\}$, the atomicness of L_ω is easily seen. Also L_ω is clearly finite. Thus we know that $(N' \cap M)_E$ is finite and atomic whenever $N' \cap M$ is atomic.

THEOREM 4.1. *Suppose $Z(M)$ is atomic.*

(1) *If $\{p_i\}$ is the set of all atoms in $Z(M)$, then*

$$K_\varphi(M \mid N) = \sum_i \varphi(\eta E(p_i)) + \sum_i \varphi(p_i) K_{\varphi_i}(Mp_i \mid Np_i),$$

where $\varphi_i = \varphi(p_i)^{-1}\varphi \mid Mp_i$. In particular if N is a factor, then $\varphi(\eta E(p_i)) = \eta\varphi(p_i)$ in the above.

(2) *If $E^{-1} \mid N' \cap M$ is semifinite (particularly if $K_\varphi(M \mid N) < \infty$), then $N' \cap M$ is atomic and $(N' \cap M)_E$ is finite atomic.*

Proof. (1) For each i, let $E_i = E_{p_i} \in \mathcal{E}(Mp_i, Np_i)$, $\omega_i = \varphi_i \mid (N' \cap M)p_i$ and $\hat{\omega}_i = \varphi_i \circ (E_i^{-1} \mid (N' \cap M)p_i)$. By Proposition 1.4, we get

$$\hat{\omega}_i(x) = \varphi_i(E^{-1}(E(p_i)x)) = \varphi(p_i)^{-1}\hat{\omega}(E(p_i)x)$$

for every $x \in ((N' \cap M)p_i)_+$. Hence it is readily verified that $\hat{\omega}$ is semifinite if and only if so is each $\hat{\omega}_i$. So we may assume by Theorem 3.3 (1) that $\hat{\omega}$ is semifinite. Let $h_i = hE(p_i)p_i$ where h is as in Theorem 3.3 (2). Because $((N' \cap M)p_i)_{E_i} = (N' \cap M)_E p_i$ thanks to $E(p_i) \in Z(N)$, h_i is affiliated with $((N' \cap M)p_i)_{E_i}$. Furthermore

$$\hat{\omega}_i(x) = \varphi(p_i)^{-1}\omega(h^{1/2}E(p_i)xh^{1/2}) =$$

$$= \omega_i(h^{1/2}E(p_i)xh^{1/2}) = \omega_i(h_i^{1/2}xh_i^{1/2})$$

for every $x \in ((N' \cap M)p_i)_+$. Hence Theorem 3.3 (2) implies

$$K_{\varphi_i}(Mp_i \mid Np_i) = \varphi_i(\log h_i) = \varphi(p_i)^{-1}\varphi(p_i \log(hE(p_i))) =$$

$$= \varphi(p_i)^{-1}\{\varphi(p_i \log h) + \varphi(p_i \log E(p_i))\} =$$

$$= \varphi(p_i)^{-1}\{\varphi(p_i \log h) - \varphi(\eta E(p_i))\},$$

so that

$$\varphi(p_i \log h) = \varphi(\eta E(p_i)) + \varphi(p_i) K_{\varphi_i}(Mp_i \mid Np_i).$$

This sums up to the desired equation. When N is a factor, $E(p_i)$ is equal to the scalar $\varphi(p_i)$, so that $\varphi(\eta E(p_i)) = \eta\varphi(p_i)$.

(2) Because $N' \cap M$ is atomic if and only if so is each $(N' \cap M)p_i$, we may assume from the proof of (1) that M is a factor. In this case, suppose $\hat{\omega} = E^{-1} | N' \cap M$ is semifinite. For $n \geqslant 1$, let e_n be the spectral projection of h corresponding to $[1, n]$. If $f_1, \dots, f_m$ are nonzero projections in $(N' \cap M)_E$ with $\sum f_k = e_n$, then we get by Proposition 1.4

$$f_k \leqslant E_{f_k}^{-1}(f_k) = E^{-1}(E(f_k)f_k)f_k \leqslant E^{-1}(f_k)f_k,$$

so that $1 \leqslant E^{-1}(f_k)$, $1 \leqslant k \leqslant m$, and hence $m \leqslant E^{-1}(e_n) \leqslant n$. This shows that $e_n(N' \cap M)_E e_n$ is finite dimensional. Since $e_n \uparrow 1$, we deduce that $(N' \cap M)_E$ is atomic. Therefore $Z(N)$ included in $Z((N' \cap M)_E)$ is atomic. Now let q be any atom in $Z(N)$ and $q_n = e_n q$. Then M_{q_n} and N_{q_n} are factors and $\operatorname{Index} E_{q_n} \leqslant E^{-1}(e_n) < \infty$, so that $N'_{q_n} \cap M_{q_n} = (N' \cap M)_{q_n}$ is finite dimensional. Since $q_n \uparrow q$, we deduce that $(N' \cap M)q$ is atomic. Thus $N' \cap M$ is atomic. ∎

THEOREM 4.2. *Suppose M is a factor.*

(1) *If $N' \cap M$ has a nonatomic part, then $K_\varphi(M \mid N) = \infty$.*

(2) *If $N' \cap M$ is atomic and $\{f_k\}$ is a set of atoms in $(N' \cap M)_E$ with $\sum f_k = 1$, then $M_{f_k} \supseteq N_{f_k}$ are factors and*

$$K_\varphi(M \mid N) = \sum_k \varphi(f_k)\log\frac{E^{-1}(f_k)}{\varphi(f_k)} = \sum_k \varphi(f_k)\log\frac{\operatorname{Index} E_{f_k}}{\varphi(E(f_k)^2)}.$$

The same holds when $\{f_k\}$ is the set of all atoms in $Z((N' \cap M)_E)$.

(3) *If $Z(N)$ is atomic and $\{q_j\}$ is the set of all atoms in $Z(N)$, then*

$$K_\varphi(M \mid N) = \sum_j \eta\varphi(q_j) + \sum_j \varphi(q_j)K_{\varphi_j}(M_{q_j} \mid N_{q_j}),$$

where $\varphi_j = \varphi(q_j)^{-1}\varphi \mid M_{q_j}$.

Proof. (1) is a corollary of Theorem 4.1 (2).

(2) Suppose $N' \cap M$ is atomic and let $\{f_k\}$ be a set of atoms in $(N' \cap M)_E$ with $\sum f_k = 1$. The factorness of N_{f_k} follows from $Z(N) \subseteq Z((N' \cap M)_E)$. Hence $\operatorname{Index} E_{f_k}$ is defined. If $E^{-1} | N' \cap M$ is not semifinite, then we get, for some k, $E^{-1}(f_k) = \infty$ and also $\operatorname{Index} E_{f_k} = \infty$ by Proposition 1.4. These together with Theorem 3.3 (1) imply the desired equation. Now assume that $E^{-1} | N' \cap M$ is semifinite. Taking a faithful normal semifinite trace $\hat{\tau}$ on $N' \cap M$, by [8, Theorem 6.6] we

obtain $T \in P(N', M')$ such that $T \mid N' \cap M = \hat{\tau}$. Then $T^{-1} \in P(M, N)$ and $T^{-1} \mid N' \cap M$ is semifinite by [8, Theorem 6.6]. Let $\tau = \varphi \circ (T^{-1} \mid N' \cap M)$. For every unitary u in $N' \cap M$, we have by [5, Proposition 8 and Theorem 9]

$$\frac{d\varphi_0 \circ uT^{-1}u^*}{d\psi} = u \frac{d\varphi_0 \circ T^{-1}}{d\psi} u^* =$$

$$= \left(u \cdot \frac{d\psi \circ T}{d\varphi_0} u^*\right)^{-1} = \frac{d\varphi_0}{d\psi \circ uTu^*},$$

where $\varphi \in P(N)$, $\psi \in P(M')$ and $uTu^* = T(u^* \cdot u)$. Hence $uT^{-1}u^* = (uTu^*)^{-1}$. But, since $uTu^* \mid N' \cap M = T \mid N' \cap M$, we get $uTu^* = T$ by [8, Theorem 6.6]. Hence $uT^{-1}u^* = T^{-1}$, so that $u\tau u^* = \tau$. This shows that τ is a faithful normal semifinite trace on $N' \cap M$. So let $a = d\hat{\omega}/d\hat{\tau}$ where $\hat{\omega} = E^{-1} \mid N' \cap M$, and $b = d\hat{\tau}/d\tau$. Then a is affiliated with $(N' \cap M)_E$ thanks to $(N' \cap M)_{\hat{\omega}} = (N' \cap M)_E$, and b is affiliated with $Z(N' \cap M)$. Since $E^{-1} \mid N' \cap M = a^{1/2}Ta^{1/2} \mid N' \cap M$, we get $E^{-1} = a^{1/2}Ta^{1/2}$. Hence $E = a^{-1/2}T^{-1}a^{-1/2}$ by Proposition 1.2, so that $\omega = a^{-1/2}\tau a^{-1/2}$. Therefore

$$\hat{\omega} = (a^{1/2}b^{1/2})\tau(a^{1/2}b^{1/2}) = (ab^{1/2})\omega(ab^{1/2}).$$

This shows that h in Theorem 3.3 (2) is equal to a^2b.

Let $a = \sum \alpha_i e_i$ be the spectral decomposition of a with distinct numbers α_i and projections $e_i \in (N' \cap M)_E$. Since

$$\sigma_t^E = \sigma_{-t}^{\hat{\omega}} = a^{-it} \cdot a^{it}, \quad t \in \mathbf{R},$$

we have $(N' \cap M)_E = (N' \cap M) \cap \{e_i\}'$, so that $\{e_i\} \subseteq Z((N' \cap M)_E)$. Because each atom in $Z((N' \cap M)_E)$ is a sum of elements of $\{f_k\}$, we deduce that a is affiliated with $\{f_k\}''$. Since $Z(N' \cap M) \subseteq Z((N' \cap M)_E)$, b is also affiliated with $\{f_k\}''$. Thus $h = a^2b$ can be written as $h = \sum \beta_k f_k$ with positive numbers β_k. Since

$$E^{-1}(f_k) = \omega(h^{1/2}f_kh^{1/2}) = \beta_k\varphi(f_k),$$

we get $\beta_k = E^{-1}(f_k)/\varphi(f_k)$. Furthermore it follows from Proposition 1.4 that

$$\operatorname{Index} E_{f_k} = E^{-1}(E(f_k)f_k) = \omega(h^{1/2}E(f_k)f_kh^{1/2}) =$$

$$= \beta_k\varphi(E(f_k)f_k) = \beta_k\varphi(E(f_k)^2).$$

Therefore

$$K_\varphi(M|N) = \varphi(\log h) = \sum_k \varphi(f_k)\log \beta_k =$$

$$= \sum_k \varphi(f_k)\log \frac{E^{-1}(f_k)}{\varphi(f_k)} = \sum_k \varphi(f_k)\log \frac{\operatorname{Index} E_{f_k}}{\varphi(E(f_k)^2)}.$$

The second assertions is also shown from the above proof.

(3) Because $N' \cap M$ is atomic if and only if so is each $(N' \cap M)q_j$, we may assume in view of (1) that $N' \cap M$ is atomic. For each j, we take a partition $q_j = \sum_k f_{jk}$ of q_j into atoms f_{jk} in $(N' \cap M)_E$. Applying (2) to $E_j = E_{q_j}$ and φ_j thanks to $(N'_{q_j} \cap M_{q_j})_{E_j} = (N' \cap M)_E q_j$, we have

$$K_{\varphi_j}(M_{q_j}|N_{q_j}) = \sum_k \varphi_j(f_{jk})\log \frac{E_j^{-1}(f_{jk})}{\varphi_j(f_{jk})}.$$

Moreover

$$E_j^{-1}(f_{jk}) = E^{-1}(E(q_j)f_{jk}) = E^{-1}(f_{jk})$$

as scalars. Therefore

$$\sum_k \varphi(f_{jk})\log \frac{E^{-1}(f_{jk})}{\varphi(f_{jk})} = \eta\varphi(q_j) + \varphi(q_j)K_{\varphi_j}(M_{q_j}|N_{q_j}).$$

The application of (2) to a partition $\sum f_{jk} = 1$ shows that the left-hand side of the above sums up to $K_\varphi(M|N)$. Thus we obtain the desired equation. ∎

THEOREM 4.3. *Suppose N is a factor.*

(1) *If $N' \cap M$ has a nonatomic part, then $K_\varphi(M|N) = \infty$.*

(2) *If $N' \cap M$ is atomic and $\{f_k\}$ is a set of atoms in $(N' \cap M)_E$ with $\sum f_k = 1$, then $M_{f_k} \supseteq N_{f_k}$ are factors and*

$$K_\varphi(M|N) = \sum_k \varphi(f_k)\log \frac{\varphi(E^{-1}(f_k))}{\varphi(f_k)},$$

$$\operatorname{Index} E_{f_k} = \varphi(E^{-1}(f_k)f_k).$$

The same holds when $\{f_k\}$ is the set of all atoms in $Z((N' \cap M)_E)$.

Proof. (1) It suffices by Theorem 4.1 (2) to show that if $K_\varphi(M|N) < \infty$, then $Z(M)$ is atomic. For each nonzero projection p in $Z(M)$, as in the proof of Theorem 4.1 (1), we have

$$\varphi(p\log h) = \eta\varphi(p) + \varphi(p)K_{\varphi_p}(Mp|Np) \geqslant \eta\varphi(p),$$

where $\varphi_p = \varphi(p)^{-1}\varphi \mid Mp$. If $Z(M)$ has a nonatomic part with support $p_0 \neq 0$, then for each $n \geqslant 1$ there are projections $p_1, \ldots, p_n$ in $Z(M)$ with $\varphi(p_i) = \varphi(p_0)/n$, so that

$$K_\varphi(M \mid N) \geqslant \sum_{i=1}^{n} \varphi(p_i \log h) \geqslant$$

$$\geqslant n\eta\left(\frac{\varphi(p_0)}{n}\right) \geqslant \varphi(p_0)\log n \to \infty,$$

as desired.

(2) The proof is almost the same as that of Theorem 4.2 (2). So we only give a sketch. The factorness of M_{f_k} follows from $Z(M) \subseteq Z((N' \cap M)_E)$. We take $T \in P(M, N)$ such that $\tau = T \mid N' \cap M$ is tracial. If $E^{-1} \mid N' \cap M$ is not semifinite, then the equation for $K_\varphi(M \mid N)$ is seen as before. Assuming the semifiniteness of $E^{-1} \mid N' \cap M$, we see that $\hat{\tau} = \varphi \circ (T^{-1} \mid N' \cap M)$ is tracial. Let $a = \mathrm{d}\hat{\omega}/\mathrm{d}\hat{\tau}$ and $b = \mathrm{d}\hat{\tau}/\mathrm{d}\tau$. Since $\varphi \circ E^{-1} = \varphi \circ a^{1/2}Ta^{1/2}$, we get $E^{-1} = a^{1/2}Ta^{1/2}$ by [8, Lemma 4.8], so that $E = a^{-1/2}Ta^{-1/2}$, implying $\omega = a^{-1/2}\tau a^{-1/2}$. Therefore $h = a^2 b$. Arguing as before, we can write $h = \sum \beta_k f_k$. Then $\varphi(E^{-1}(f_k)) = \beta_k \varphi(f_k)$ and so the equation for $K_\varphi(M \mid N)$ is obtained. Furthermore we have

$$(\mathrm{Index}\, E_{f_k})f_k = E_{f_k}^{-1}(f_k)f_k = \varphi(f_k)E^{-1}(f_k)f_k,$$

implying the equation for $\mathrm{Index}\, E_{f_k}$.

COROLLARY 4.4. *If $K_\varphi(M \mid N) < \infty$, then the following conditions are equivalent:*

(i) *$Z(M)$ is atomic;*

(ii) *$Z(N)$ is atomic;*

(iii) *$N' \cap M$ is atomic;*

(iv) *$(N' \cap M)_E$ is atomic.*

Proof. (i) $\Rightarrow$ (iii) is Theorem 4.1 (2).

(ii) $\Rightarrow$ (iii). Suppose $Z(N)$ is atomic and let $\{q_j\}$ be the set of all atoms in $Z(N)$. For each j, let $\omega_j = \varphi_j \mid (N' \cap M)q_j$ and $\hat{\omega}_j = \varphi_j \circ (E_j^{-1} \mid (N' \cap M)q_j)$ where $\varphi_j = \varphi(q_j)^{-1}\varphi \mid M_{q_j}$. Using Proposition 1.4, we have for every $x \in ((N' \cap M)q_j)_+$

$$\hat{\omega}_j(x) = \varphi_j(E^{-1}(E(q_j)x)q_j) \leqslant$$

$$\leqslant \varphi(q_j)^{-1}\hat{\omega}(x) = \varphi(q_j)^{-1}\omega(h^{1/2}xh^{1/2}).$$

This implies $h_j \leqslant \varphi(q_j)^{-1}hq_j$ where h_j is taken by Theorem 3.3 (2) for ω_j and $\hat{\omega}_j$. Hence

$$K_{\varphi_j}(M_{q_j} \mid N_{q_j}) = \varphi_j(\log h_j) \leqslant$$

$$\leqslant \varphi(q_j)^{-1}\, \varphi(q_j \log(\varphi(q_j)^{-1}h)) \leqslant$$

$$\leqslant \varphi(q_j)^{-1}\{\eta\varphi(q_j) + \varphi(\log h)\} < \infty,$$

so that $(N' \cap M)q_j$ is atomic by Theorem 4.3 (1). Thus $N' \cap M$ is atomic.

(iii) $\Rightarrow$ (iv) is already mentioned before Theorem 4.1. (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are immediate from $Z(M)$, $Z(N) \subseteq Z((N' \cap M)_E)$. ∎

When $Z(M)$ is atomic, Theorems 4.1 and 4.2 conclude that the calculation of $K_\varphi(M \mid N)$ is reduced to the case of $M \supseteq N$ being factors. Here let M be a finite von Neumann algebra with a faithful normal trace τ, $\tau(1) = 1$, and $H(M \mid N)$ be Pimsner and Popa's entropy (∗) in Section 3. According to [14, Theorems 1.2 and 1.3], the same reduction theorem as Theorem 4.1 (1) holds for $H(M \mid N)$ when N is a factor, and also (1) and (3) of Theorem 4.2 holds for $H(M \mid N)$ when M is a factor. Furthermore, in particular when M is a type II_1 factor and N is its subfactor, the estimate of $K_\tau(M \mid N)$ given in (1) and (2) of Theorem 4.2 coincides with that of $H(M \mid N)$ in [22, Theorem 4.4]. Thus we deduce the following:

COROLLARY 4.5. *Let M be a type II_1 von Neumann algebra with a faithful normal normalized trace τ, and N be a von Neumann subalgebra of M. If either M or N is a factor, then $K_\tau(M \mid N)$ is equal to Pimsner and Popa's entropy $H(M \mid N)$.*

The following example shows that our entropy $K_\varphi(M \mid N)$ contains von Neumann's entropy with multiple 2 as a special case.

EXAMPLE 4.6. Let $\varphi \in \mathscr{S}(\mathbf{B}(\mathscr{H}))$ and $a = \mathrm{d}\varphi/\mathrm{dtr}$. Then $\varphi^{-1} \in P(\mathbf{B}(\mathscr{H}))$ and let $\hat{a} = \mathrm{d}\varphi^{-1}/\mathrm{dtr}$. With the canonical $1 \in \mathscr{S}(\mathbf{C})$, we get

$$\hat{a} = \frac{\mathrm{d}\varphi^{-1}}{\mathrm{d}1} = \left(\frac{\mathrm{d}1}{\mathrm{d}\varphi}\right)^{-1} = a^{-1}.$$

If we write $a = \sum_k \alpha_k f_k$ with positive numbers α_k and rank one projections f_k, $\sum_k f_k = 1$, then by Theorem 4.2 (2).

$$K_\varphi(\mathbf{B}(\mathscr{H}) \mid \mathbf{C}) = \sum_k \varphi(f_k) \log \frac{\varphi^{-1}(f_k)}{\varphi(f_k)} =$$
$$= 2 \sum_k \eta\alpha_k = 2\mathrm{tr}(\eta a).$$

In particular, let M be the $n \times n$ matrix algebra with the normalized trace τ. Then $K_\tau(M \mid \mathbf{C}) = 2 \log n$. On the other hand, for the entropy (∗), $H(M \mid \mathbf{C}) = \log n$ (see [7]). Thus Corollary 4.5 does not hold when M is finite dimensional. In fact, [22, Theorem 4.4] does not hold in the finite dimensional case, while the theorems in this section remain true.

5. FURTHER PROPERTIES OF ENTROPY

When M is a finite von Neumann algebra, we can readily check, as stated in [22], the following properties of Pimsner and Popa's entropy from its definition (∗): For each von Neumann subalgebras $N \subseteq L \subseteq M$,

(1) $H(M \mid N) \leqslant H(M \mid L) + H(L \mid N)$,

(2) $H(M \mid N) \geqslant H(M \mid L)$,

(3) $H(M \mid N) \geqslant H(L \mid N)$.

In this section, let $M \supseteq L \supseteq N$ be von Neumann algebras with $\varphi \in \mathscr{E}(M)$, and assume that there exist $E \in \mathscr{E}(M, N)$ and $F \in \mathscr{E}(M, L)$ with respect to φ. In this case, $G = E \mid L$ is the conditional expectation $L \to N$ with respect to $\varphi \mid L$, and $E = G \circ F$. Then the entropies $K_\varphi(M \mid N)$, $K_\varphi(M \mid L)$ and $K_\varphi(L \mid N)$ $(= K_{\varphi \mid L}(L \mid N))$ are defined. So it is natural to ask whether the same properties as (1) – (3) hold for these entropies. But it is not at all easy to show them. In the following, we prove the property (1) in full generality and (2) under some assumptions. However the property (3) remains open and seems difficult to prove (even when $M \supseteq L \supseteq N$ are factors).

Besides ω and $\hat{\omega}$, let $\omega_1 = \varphi \mid L' \cap M$, $\hat{\omega}_1 = \varphi \circ (F^{-1} \mid L' \cap M)$, $\omega_2 = \varphi \mid N' \cap L$, and $\hat{\omega}_2 = \varphi \circ (G^{-1} \mid N' \cap L)$. Moreover, when $\hat{\omega}$ (resp. $\hat{\omega}_1$, $\hat{\omega}_2$) is semifinite, let h (resp. h_1, h_2) be the positive selfadjoint operator affiliated with $(N' \cap M)_E$ (resp. $(L' \cap M)_F$, $(N' \cap L)_G$) such that $\hat{\omega} = h^{1/2}\omega h^{1/2}$ (resp. $\hat{\omega}_1 = h_1^{1/2}\omega_1 h_1^{1/2}$, $\hat{\omega}_2 = h_2^{1/2}\omega_2 h_2^{1/2}$) (see Theorem 3.3).

THEOREM 5.1. (1) $K_\varphi(M \mid N) \leqslant K_\varphi(M \mid L) + K_\varphi(L \mid N)$.

(2) *If $K_\varphi(M \mid L)$ and $K_\varphi(L \mid N)$ are finite, then the following conditions are equivalent:*

(i) $K_\varphi(M \mid N) = K_\varphi(M \mid L) + K_\varphi(L \mid N)$;

(ii) *h is affiliated with $(N' \cap L) \vee (L' \cap M)$;*

(iii) $h = h_1 h_2$.

Proof. (1) We may assume that $K_\varphi(M \mid L)$ and $K_\varphi(L \mid N)$ are finite. Then $K_\varphi(M \mid L) = \varphi(\log h_1)$ and $K_\varphi(L \mid N) = \varphi(\log h_2)$ by Theorem 3.3 (2). Let $\mathscr{A} = (N' \cap L) \vee (L' \cap M)$ $(\subseteq N' \cap M)$. Since $F^{-1} \mid L' \cap M$ and $G^{-1} \mid N' \cap L$ are semifinite, we can choose nets $x_i \uparrow 1$ in $(L' \cap M)_+$ and $y_j \uparrow 1$ in $(N' \cap L)_+$ such that $F^{-1}(x_i)$ and $G^{-1}(y_j)$ are bounded. Then $x_i y_j \in \mathscr{A}_+$, $x_i y_j \uparrow 1$, and

$$E^{-1}(x_i y_j) = F^{-1}(G^{-1}(x_i y_j)) = F^{-1}(x_i^{1/2} G^{-1}(y_j) x_i^{1/2}) \leqslant$$

$$\leqslant \|G^{-1}(y_j)\| \|F^{-1}(x_i)\|,$$

so that $E^{-1} \mid \mathscr{A}$ is semifinite and hence so is $E^{-1} \mid N' \cap M$. Letting $\omega_0 = \varphi \mid \mathscr{A}$ $(= \omega \mid \mathscr{A})$ and $\hat{\omega}_0 = \varphi \circ (E^{-1} \mid \mathscr{A})$ $(= \hat{\omega} \mid \mathscr{A})$, we get $S(\hat{\omega}, \omega) \geqslant S(\hat{\omega}_0, \omega_0)$ by Definition 3.1 and the monotonicity of relative entropy. For every $t \in \mathbf{R}$, because $\sigma_t^\varphi(L) = L$, $\sigma_t^\varphi(N) = N$ and $\sigma_t^\omega = \sigma_t^\varphi \mid N' \cap M$, we get $\sigma_t^\omega(L' \cap M) = L' \cap M$ and $\sigma_t^\omega(N' \cap L) = N' \cap L$. Therefore $\sigma_t^\omega(\mathscr{A}) = \mathscr{A}$ and also $\sigma_t^{\hat{\omega}}(\mathscr{A}) = \mathscr{A}$ since $\sigma_t^{\hat{\omega}} = \sigma_{-t}^\omega$, so that we have

$$\sigma_t^{\hat{\omega}_0} = \sigma_t^{\hat{\omega}} \mid \mathscr{A} = \sigma_{-t}^\omega \mid \mathscr{A} = \sigma_{-t}^{\omega_0}, \quad t \in \mathbf{R}.$$

Hence there is a unique positive selfadjoint operator h_0 affiliated with $\mathscr{A}_{\omega_0}$ such that $\hat{\omega}_0 = h_0^{1/2}\omega_0 h_0^{1/2}$. Here $h_0 \geqslant 1$ because $\hat{\omega}_0 \geqslant \omega_0$ follows from $\hat{\omega} \geqslant \omega$. So we can obtain $S(\hat{\omega}_0, \omega_0) = -\varphi(\log h_0)$ as in the proof of Theorem 3.3 (2).

Next let us prove $h_0 = h_1 h_2$. If $x \in (L' \cap M)_F$, then

$$E(xy) = G(F(xy)) = G(F(yx)) = E(yx), \quad y \in M,$$

and if $x \in (N' \cap L)_G$, then

$$E(xy) = G(xF(y)) = G(F(y)x) = E(yx), \quad y \in M.$$

Thanks to $\mathscr{A}_{\omega_0} = (N' \cap M)_E \cap \mathscr{A}$, these show that $(L' \cap M)_F \subseteq \mathscr{A}_{\omega_0}$ and $(N' \cap L)_G \subseteq \mathscr{A}_{\omega_0}$. Hence $h_1 h_2$ is affiliated with $\mathscr{A}_{\omega_0}$. For each $a \in (L' \cap M)_+$ and each $b \in (N' \cap L)_+$, since $c = G^{-1}(h_2^{-1/2} b h_2^{-1/2})$ is a positive selfadjoint operator affiliated with $Z(L)$ and since $F(a) \in Z(L)$, we have

$$\hat{\omega}_0(h_1^{-1/2}h_2^{-1/2}abh_2^{-1/2}h_1^{-1/2}) = \varphi(F^{-1}(G^{-1}(h_1^{-1/2}h_2^{-1/2}abh_2^{-1/2}h_1^{-1/2}))) =$$

$$= \varphi(F^{-1}(h_1^{-1/2}a^{1/2}ca^{1/2}h_1^{-1/2})) = \omega_1(a^{1/2}ca^{1/2}) = \varphi(c^{1/2}ac^{1/2}) =$$

$$= \varphi(c^{1/2}F(a)c^{1/2}) = \varphi(F(a)^{1/2}cF(a)^{1/2}) = \varphi(G^{-1}(h_2^{-1/2}F(a)bh_2^{-1/2})) = \omega_2(F(a)b) = \omega_0(ab).$$

This shows $h_0 = h_1 h_2$. Therefore

$$K_\varphi(M \mid N) = -S(\hat{\omega}, \omega) \leqslant -S(\hat{\omega}_0, \omega_0) =$$

$$= \varphi(\log h_0) = \varphi(\log h_1 + \log h_2) = K_\varphi(M \mid L) + K_\varphi(L \mid N).$$

(2) The semifiniteness of $\hat{\omega}$ follows from assumption and the proof of (1). Since $\sigma_t^\omega(\mathscr{A}) = \mathscr{A}$, $t \in \mathbf{R}$, as shown above, there exists the conditional expectation $E_{\mathscr{A}}$ with respect to ω. First let us prove $E_{\mathscr{A}}(h) = h_0$ $(= h_1 h_2)$. If $x \in (N' \cap M)_\omega$ and $y \in \mathscr{A}$, then

$$\omega(E_{\mathscr{A}}(x)y) = \omega(xy) = \omega(yx) = \omega(yE_{\mathscr{A}}(x)).$$

This shows $E_{\mathscr{A}}((N' \cap M)_\omega) \subseteq \mathscr{A}_{\omega_0}$, so that $E_{\mathscr{A}}(h)$ is affiliated with $\mathscr{A}_{\omega_0}$. Taking $h_n \in ((N' \cap M)_\omega)_+$ with $h_n \uparrow h$, we have for every $y \in \mathscr{A}_+$

$$\omega_0(E_{\mathscr{A}}(h)^{1/2}yE_{\mathscr{A}}(h)^{1/2}) = \lim_{n\to\infty} \omega_0(y^{1/2}E_{\mathscr{A}}(h_n)y^{1/2}) =$$

$$= \lim_{n\to\infty} \omega(y^{1/2}h_n y^{1/2}) = \omega(h^{1/2}yh^{1/2}) = \hat{\omega}_0(y) = \omega_0(h_0^{1/2}yh_0^{1/2}),$$

so that $E_{\mathscr{A}}(h) = h_0$. Hence (ii) $\Leftrightarrow$ (iii) is shown. Also (iii) $\Rightarrow$ (i) is immediate from the proof of (1).

(i) $\Rightarrow$ (ii). Condition (i) implies

$$\omega(E_{\mathscr{A}}(\log h)) = \omega(\log h) = \omega(\log h_0) = \omega(\log E_{\mathscr{A}}(h)) < \infty.$$

Because $E_{\mathscr{A}}(\log h) \leqslant \log E_{\mathscr{A}}(h)$ follows from the operator concavity of $\log t$, we get $E_{\mathscr{A}}(\log h) = \log E_{\mathscr{A}}(h)$. Since

$$\log h = \int_0^\infty \{(1+t)^{-1} - (h+t)^{-1}\} \mathrm{d}t,$$

we have

$$E_{\mathscr{A}}(\log h) = \int_0^\infty \{(1+t)^{-1} - E_{\mathscr{A}}((h+t)^{-1})\} \mathrm{d}t,$$

and also

$$\log E_{\mathscr{A}}(h) = \int_0^\infty \{(1+t)^{-1} - E_{\mathscr{A}}(h+t)^{-1}\} \mathrm{d}t.$$

Furthermore the operator convexity of t^{-1} $(t > 0)$ implies that $E_{\mathscr{A}}((h+t)^{-1}) \geqslant E_{\mathscr{A}}(h+t)^{-1}$ for every $t > 0$. Noting the strong operator continuity of $E_{\mathscr{A}}((h+t)^{-1})$ and $E_{\mathscr{A}}(h+t)^{-1}$ in $t > 0$, we obtain $E_{\mathscr{A}}((h+t)^{-1}) = E_{\mathscr{A}}(h+t)^{-1}$ for all $t > 0$. For each $0 < \delta < t$,

$$E_{\mathscr{A}}((h+t-\delta)^{-1}(h+t)^{-1}) = E_{\mathscr{A}}\left(\frac{(h+t-\delta)^{-1} - (h+t)^{-1}}{\delta}\right) =$$

$$= \frac{E_{\mathscr{A}}(h+t-\delta)^{-1} - E_{\mathscr{A}}(h+t)^{-1}}{\delta} =$$

$$= E_{\mathscr{A}}(h+t-\delta)^{-1}\left(\frac{E_{\mathscr{A}}(h+t) - E_{\mathscr{A}}(h+t-\delta)}{\delta}\right) E_{\mathscr{A}}(h+t)^{-1} =$$

$$= E_{\mathscr{A}}(h+t-\delta)^{-1} E_{\mathscr{A}}(h+t)^{-1}.$$

Letting $\delta \downarrow 0$ in the above, we deduce that $E_{\mathscr{A}}((h+t)^{-2}) = E_{\mathscr{A}}(h+t)^{-2}$ for all $t > 0$. Now let $a = (h+1)^{-1}$. Then $E_{\mathscr{A}}(a^2) = E_{\mathscr{A}}(a^{-1})^{-2} = E_{\mathscr{A}}(a)^2$, so that

$$E_{\mathscr{A}}((E_{\mathscr{A}}(a) - a)^2) = E_{\mathscr{A}}(a^2) - E_{\mathscr{A}}(a)^2 = 0,$$

implying $E_{\mathscr{A}}(a) = a$. This shows (ii). ▨

THEOREM 5.2. *$K_\varphi(M \mid N) \geqslant K_\varphi(M \mid L)$ holds in each of the following cases:*
(a) *either $Z(M)$ or $Z(N)$ is atomic, and $E^{-1}(1)$ is bounded,*
(b) *M is a factor and $G^{-1} \mid N' \cap L$ is semifinite.*

Proof. Case (a). Since $E^{-1}(1)$ is bounded, $H_\varphi(M \mid N) < \infty$. Hence it follows from Corollary 4.4 that $Z(M)$, $Z(N)$ and $N' \cap M$ are all atomic. If $\{p_i\}$ is the set of all atoms in $Z(M)$, then by Theorem 4.1 (1)

$$K_\varphi(M \mid N) = \sum_i \varphi(\eta E(p_i)) + \sum_i \varphi(p_i) K_{\varphi_i}(Mp_i \mid Np_i),$$

$$K_\varphi(M \mid L) = \sum_i \varphi(\eta F(p_i)) + \sum_i \varphi(p_i) K_{\varphi_i}(Mp_i \mid Lp_i),$$

where $\varphi_i = \varphi(p_i)^{-1}\varphi \mid Mp_i$. By the operator concavity of η, we have

$$\eta E(p_i) = \eta(G(F(p_i))) \geqslant G(\eta F(p_i)),$$

so that $\varphi(\eta E(p_i)) \geqslant \varphi(\eta F(p_i))$. Hence it suffices to show that $K_\varphi(Mp_i \mid Np_i) \geqslant$ $\geqslant K_\varphi(Mp_i \mid Lp_i)$ for all i. Here $E_{p_i}^{-1}(p_i)$ is bounded by Proposition 1.4. Thus we can assume that M is a factor.

Because $Z(L) \subset (L' \cap M)_F \subseteq (N' \cap M)_E$ (see the proof of Theorem 5.1 (1)), it follows that $Z(L)$ is atomic. If $\{q_j\}$ is the set of all atoms in $Z(L)$, then by Theorem 4.2 (3).

$$K_\varphi(M \mid L) = \sum_j \eta\varphi(q_j) + \sum_j \varphi(q_j) K_{\varphi_j}(M_{q_j} \mid L_{q_j}),$$

where $\varphi_j = \varphi(q_j)^{-1}\varphi \mid M_{q_j}$. For each j, we take a partition $q_j = \sum_k f_{jk}$ of q_j into atoms f_{jk} in $(N' \cap M)_E$. Since $E_{q_j}^{-1}(f_{jk}) \leqslant E^{-1}(f_{jk})$ by Proposition 1.4, we have using Theorem 4.2 (2) twice

$$K_\varphi(M \mid N) = \sum_j \sum_k \varphi(f_{jk}) \log \frac{E^{-1}(f_{jk})}{\varphi(f_{jk})} \geqslant$$

$$\geqslant \sum_j \sum_k \varphi(f_{jk}) \log \frac{E_{q_j}^{-1}(f_{jk})}{\varphi(f_{jk})} =$$

$$= \sum_j \eta\varphi(q_j) + \sum_j \varphi(q_j) \sum_k \varphi_j(f_{jk}) \log \frac{E_{q_j}^{-1}(f_{jk})}{\varphi_j(f_{jk})} =$$

$$= \sum_j \eta\varphi(q_j) + \sum_j \varphi(q_j) K_{\varphi_j}(M_{q_j} \mid N_{q_j}).$$

Hence it suffices to show that $K_{\varphi_j}(M_{q_j} \mid N_{q_j}) \geqslant K_{\varphi_j}(M_{q_j} \mid L_{q_j})$ for all j. Here $E_{q_j}^{-1}(q_j)$ is bounded. Thus we can assume that L, as well as M, is a factor.

Now suppose $M \supseteq L$ are factors and $E^{-1}(1) = F^{-1}(1)G^{-1}(1) < \infty$ where $E^{-1}(1)$, $F^{-1}(1)$ and $G^{-1}(1)$ are all scalars. Let $\tilde{\omega} = E^{-1}(1)^{-1}\hat{\omega}$ and $\tilde{\omega}_1 = F^{-1}(1)^{-1}\hat{\omega}_1$. Since

$$\tilde{\omega}(x) = E^{-1}(1)^{-1}F^{-1}(G^{-1}(x)) =$$

$$= E^{-1}(1)^{-1}G^{-1}(1)F^{-1}(x) = \tilde{\omega}_1(x), \quad x \in L' \cap M,$$

$\tilde{\omega}_1 = \tilde{\omega} \mid L' \cap M$ as well as $\omega_1 = \omega \mid L' \cap M$. By [2, Theorem 3.6], we have

$$K_\varphi(M \mid N) = -S(E^{-1}(1)\tilde{\omega}, \omega) = \log E^{-1}(1) - S(\tilde{\omega}, \omega),$$

$$K_\varphi(M \mid L) = -S(F^{-1}(1)\tilde{\omega}_1, \omega_1) = \log F^{-1}(1) - S(\tilde{\omega}_1, \omega_1).$$

Let $\tilde{E} = G^{-1}(1)^{-1}G^{-1} \mid N' \cap M$. Then $\tilde{E} \in \mathcal{E}(N' \cap M, L' \cap M)$. Since

$$\tilde{\omega}_1(\tilde{E}(x)) = F^{-1}(1)^{-1}G^{-1}(1)^{-1}F^{-1}(G^{-1}(x)) =$$

$$= E^{-1}(1)^{-1}E^{-1}(x) = \tilde{\omega}(x), \quad x \in N' \cap M,$$

we get $\tilde{\omega} = \tilde{\omega}_1 \circ \tilde{E}$ and hence $\tilde{\omega} \circ \tilde{E} = \tilde{\omega}$, that is, $\tilde{E}$ is the conditional expectation with respect to $\tilde{\omega}$. Define $\omega' = \omega \circ \tilde{E}$. Applying Proposition 1.9 (1) to G under the standard representation of M, we get $\tilde{E}(x) \geqslant G^{-1}(1)^{-1}x$ for $x \in (N' \cap M)_+$, so that $\omega' \geqslant G^{-1}(1)^{-1}\omega$. Therefore

$$S(\omega', \omega) \leqslant S(G^{-1}(1)\omega, \omega) = \log G^{-1}(1)$$

by [2, Theorem 3.6]. Furthermore it follows from [20, Theorem 5] (see also [10, Theorem 3.2]) that

$$S(\tilde{\omega}, \omega) = S(\tilde{\omega}_1, \omega_1) + S(\omega', \omega).$$

Combining the above equalities and inequality altogether, we obtain

$$K_\varphi(M \mid N) = \log E^{-1}(1) - S(\tilde{\omega}_1, \omega_1) - S(\omega', \omega) \geqslant$$

$$\geqslant \log E^{-1}(1) - \log G^{-1}(1) - S(\tilde{\omega}_1, \omega_1) = K_\varphi(M \mid L).$$

Case (b). We may suppose $K_\varphi(M \mid N) < \infty$. Then $Z(N)$ and $N' \cap M$ are atomic by Corollary 4.4, and hence $Z(L)$ is atomic. Arguing as in Case (a) and noting the semifiniteness of $G_q^{-1} \mid (N' \cap L)q$ for any atom q in $Z(L)$, we can assume further that L is a factor. Here let $\tilde{p}$ be the support of the semifinite part of $F^{-1}|L'\cap M$. Since $E^{-1}|N'\cap M$ is semifinite, there is a net $x_i\uparrow 1$ in $(N'\cap M)_+$ with $E^{-1}(x_i)=F^{-1}(G^{-1}(x_i))<\infty$. Then the support $\tilde{p}_i$ of $G^{-1}(x_i)\in(L'\cap M)_+^{\wedge}$ is contained in $\tilde{p}$ and $\tilde{p}_i\uparrow 1$. Hence $F^{-1}|L'\cap M$ is semifinite together with $G^{-1}|N'\cap L$. Thus we can choose, in view of Theorem 3.3 (2), sequences $\{p_n\}$ and $\{q_n\}$ of projections in $(L'\cap M)_E$ and $(N'\cap L)_G$, respectively, such that $p_n\uparrow 1$, $q_n\uparrow 1$, $F^{-1}(p_n)<\infty$ and $G^{-1}(q_n)<\infty$. Letting $e_n=p_nq_n$, we get $e_n\in(N'\cap M)_E$ and $e_n\uparrow 1$ (here $e_1\neq 0$ may be assumed). For each n, there exists $F_n\in\mathcal{E}(M_{e_n},L_{e_n})$, as well as $E_n=E_{e_n}$, with respect to $\varphi_n=\varphi(e_n)^{-1}\varphi|M_{e_n}$, and

$$E_n^{-1}(e_n)\leqslant E^{-1}(e_n)=F^{-1}(p_n)G^{-1}(q_n)<\infty.$$

Thus it follows from the case (a) that

$$K_{\varphi_n}(M_{e_n}|N_{e_n})\geqslant K_{\varphi_n}(M_{e_n}|L_{e_n}),\quad n\geqslant 1.$$

Partitioning each e_n-e_{n-1} into atoms in $(N'\cap M)_E$, we choose a set $\{f_k\}$ of atoms in $(N'\cap M)_E$ with $\sum f_k=1$ such that $e_n=\sum_{k\in I_n}f_k$ for all n where $I_1\subseteq\subseteq I_2\subseteq\ldots$. Then Theorem 4.2 (2) implies

$$K_{\varphi_n}(M_{e_n}|N_{e_n})=\sum_{k\in I_n}\varphi_n(f_k)\log\frac{E_n^{-1}(f_k)}{\varphi_n(f_k)}=$$

$$=\log\varphi(e_n)+\frac{1}{\varphi(e_n)}\sum_{k\in I_n}\varphi(f_k)\log\frac{E_n^{-1}(f_k)}{\varphi(f_k)}.$$

Here $\varphi(e_n)\uparrow 1$. For each k, when $n\to\infty$ with $k\in I_n$, we have

$$\varphi(f_k)\leqslant\varphi_n(f_k)\leqslant E_n^{-1}(f_k)=E^{-1}(E(e_n)f_k)\uparrow E^{-1}(f_k).$$

Therefore

$$K_\varphi(M|N)=\sum_k\varphi(f_k)\log\frac{E^{-1}(f_k)}{\varphi(f_k)}=\lim_{n\to\infty}K_{\varphi_n}(M_{e_n}|N_{e_n}).$$

Next let us obtain the complete expression of $F_n^{-1}\in P((L_{e_n})',M'_{e_n})$. It is readily verified that $M_{e_n}=(M_{p_n})_{e_n}$, $L_{e_n}=(L_{p_n})_{e_n}$ and $F_n=F_{p_n}|M_{e_n}$. Hence, using Propositions 1.5 (1) and 1.4, we have for every $x\in L'_+$

$$F_n^{-1}(e_nxe_n)=F_n^{-1}((p_nxp_n)e_n)=F_{p_n}^{-1}(p_nxp_n)e_n=$$

$$=F^{-1}(F(p_n)p_nxp_n)e_n=\varphi(p_n)F^{-1}(p_nxp_n)e_n.$$

Moreover

$$((L_{e_n})' \cap M_{e_n})_{F_n} = (L'_{p_n} \cap N_{p_n})_{F_{p_n}} e_n =$$

$$= p_n(L' \cap M)_F p_n e_n = e_n(L' \cap M)_F e_n.$$

Partitioning each $p_n - p_{n-1}$ into atoms in $(L' \cap M)_F$, we choose a set $\{g_k\}$ of atoms in $(L' \cap M)_F$ with $\sum g_k = 1$ such that $p_n = \sum_{k \in J_n} g_k$ for all n where $J_1 \subseteq \subseteq J_2 \subset \dots$. Then $\{g_k q_n : k \in J_n\}$ is a set of atoms in $e_n(L' \cap M)_F e_n$ with $\sum_{k \in J_n} g_k q_n = e_n$. Hence Theorem 4.2 (2) implies

$$K_{\varphi_n}(M_{e_n} \mid L_{e_n}) = \sum_{k \in J_n} \varphi_n(g_k q_n) \log \frac{F_n^{-1}(g_k q_n)}{\varphi_n(g_k q_n)} =$$

$$= \log \varphi(e_n) + \frac{1}{\varphi(e_n)} \sum_{k \in J_n} \varphi(g_k q_n) \log \frac{F_n^{-1}(g_k q_n)}{\varphi(g_k q_n)}.$$

For each k, when $n \to \infty$ with $k \in J_n$, we have $\varphi(g_k q_n) \uparrow \varphi(g_k)$ and from the above expression of F_n^{-1}

$$\varphi(g_k q_n) \leqslant F_n^{-1}(g_k q_n) = F_n^{-1}(e_n g_k e_n) =$$

$$= \varphi(p_n) F^{-1}(g_k) \uparrow F^{-1}(g_k)$$

as scalars. Thus we obtain

$$K_\varphi(M \mid L) = \sum_k \varphi(g_k) \log \frac{F^{-1}(g_k)}{\varphi(g_k)} \leqslant$$

$$\leqslant \liminf_{n \to \infty} K_{\varphi_n}(M_{e_n} \mid L_{e_n}) \leqslant$$

$$\leqslant \lim_{n \to \infty} K_{\varphi_n}(M_{e_n} \mid N_{e_n}) = K_\varphi(M \mid N).$$ ▨

When $E^{-1} \mid N' \cap M$ is semifinite (particularly when $K_\varphi(M \mid N) < \infty$), $F^{-1}|L' \cap \cap M$ is semifinite as shown in Case (b), but it is not known whether so is $G^{-1} \mid N' \cap L$.

6. MINIMUM INDEX AND ENTROPY

Throughout this section, let $M \supseteq N$ be a pair of a factor and one of its subfactors such that $[M: N]_0 = \operatorname{Index} E_0 < \infty$ where $E_0 \in \mathscr{E}(M, N)$. For each $E \in \mathscr{E}(M, N)$, because $E \mid N' \cap M$ and $E^{-1} \mid N' \cap M$ are scalar-valued, the entropy

$K_\varphi(M|N)$ is independent of the choice of $\varphi \in \mathcal{S}(M)$ with $\varphi\circ E = \varphi$. So we use the notation $K_E(M|N)$ instead of $K_\varphi(M|N)$. In fact, $K_E(M|N) = -S(\hat{\omega}, \omega)$ where $\omega = E|N' \cap M$ and $\hat{\omega} = E^{-1}|N' \cap M$.

Proposition 3.5 shows that $K_E(M|N) \leqslant \log \operatorname{Index} E$ for every $E \in \mathcal{E}(M, N)$. Furthermore we have the following:

Proposition 6.1. *$K_E(M|N) \leqslant \log[M:N]_0$ for every $E \in \mathcal{E}(M, N)$.*

Proof. Let $\tau = E_0|N' \cap M$. By Theorem 2.1 (2), τ is a trace and $E_0^{-1}|N' \cap M = [M:N]_0\tau$. For each $E \in \mathcal{E}(M, N)$, let $\omega = E|N' \cap M$, $\hat{\omega} = E^{-1}|N' \cap M$ and $a = \mathrm{d}\omega/\mathrm{d}\tau$. Since $E|N' \cap M = a^{1/2}E_0a^{1/2}|N' \cap M$, we get $E = a^{1/2}E_0a^{1/2}$, so that $E^{-1} = a^{-1/2}E_0^{-1}a^{-1/2}$ by Proposition 1.2. Hence

$$\hat{\omega} = [M:N]_0a^{-1/2}\tau a^{-1/2} = [M:N]_0a^{-1}\omega a^{-1}.$$

Therefore we have by Theorem 3.3 (2)

$$K_E(M|N) = \omega(\log([M:N]_0a^{-2})) = \log[M:N]_0 - 2\omega(\log a) =$$

$$= \log[M:N]_0 + 2\tau(\eta a) \leqslant \log[M:N]_0,$$

because $\tau(\eta a) \leqslant \eta\tau(a) = 0$. ∎

Proposition 6.2. *If $E \in \mathcal{E}(M, N)$, then*

$$K_E(M|N) + S(\tilde{\omega}, \omega) = \log \operatorname{Index} E,$$

$$K_E(M|N) + \frac{1}{2}\|\tilde{\omega} - \omega\|^2 \leqslant \log \operatorname{Index} E,$$

where $\omega = E|N' \cap M$ and $\tilde{\omega} = (\operatorname{Index} E)^{-1}E^{-1}|N' \cap M$.

Proof. By definition,

$$K_E(M|N) = -S((\operatorname{Index} E)\tilde{\omega}, \omega) = \log \operatorname{Index} E - S(\tilde{\omega}, \omega).$$

Moreover it is known [10, Theorem 3.1] that

$$\|\tilde{\omega} - \omega\|^2 \leqslant 2S(\tilde{\omega}, \omega).$$ ∎

The next theorem gives new characterizations, besides those in Theorem 2.1 (2), for $E \in \mathcal{E}(M, N)$ having the minimum index.

Theorem 6.3. *The following conditions for $E \in \mathcal{E}(M, N)$ are equivalent:*

(i) $\operatorname{Index} E = [M:N]_0$, i.e. $E = E_0$;

(ii) $K_E(M|N) = \log[M:N]_0$;

(iii) $K_E(M \mid N) = \log \operatorname{Index} E$;

(iv) *for every nonzero projection e in* $N' \cap M$,

$$\operatorname{Index} E_e = E(e)^2 \operatorname{Index} E;$$

(v) *for every nonzero projections* $e_1, \ldots, e_n$ *in* $N' \cap M$ *with* $\sum e_i = 1$,

$$\sum_{i=1}^{n} E(e_i) \log \frac{\operatorname{Index} E_{e_i}}{E(e_i)^2} = \log \operatorname{Index} E.$$

Proof. In the proof of Proposition 6.1, $E = E_0$ if and only if $\omega = \tau$ (i.e. $a = 1$). This is equivalent to $\tau(\eta a) = 0$. Hence (i) $\Leftrightarrow$ (ii). Also (i) $\Leftrightarrow$ (iii) is immediate from Proposition 6.2. If $E = E_0$ and e is a nonzero projection in $N' \cap M$, then

$$\operatorname{Index} E_e = E(e) E^{-1}(e) = E(e)^2 \operatorname{Index} E$$

by [16, Proposition 4.2] and Theorem 2.1 (2). Thus (i) $\Rightarrow$ (iv). (iv) $\Rightarrow$ (v) is obvious. Finally (v) $\Rightarrow$ (iii) follows from Theorem 4.2 (2). ▨

THEOREM 6.4. *If* $N' \cap M \neq \mathbf{C}$, *then*

$$\{K_E(M \mid N) : E \in \mathscr{E}(M, N)\} = (\log \alpha,\ \log[M : N]_0],$$

where

$$\alpha = [M : N]_0 \min\{E_0(e) : e \textit{ is a nonzero projection in } N' \cap M\} =$$
$$= \min\{[M_e : N_e]_0 : e \textit{ is a nonzero projection in } N' \cap M\}.$$

Proof. For $E \in \mathscr{E}(M, N)$, let $a = d\omega/d\tau$ where $\omega = E \mid N' \cap M$ and $\tau = E_0 \mid N' \cap M$. Then

$$K_E(M \mid N) = \log[M : N]_0 + 2\tau(\eta a)$$

by the proof of Proposition 6.1. Taking the spectral decomposition $a = \sum \alpha_i e_i$ of a, since $\sum \alpha_i \tau(e_i) = \tau(a) = 1$, we have

$$0 \geqslant \tau(\eta a) = \sum_i \eta(\alpha_i) \tau(e_i) =$$
$$= \sum_i \alpha_i \tau(e_i) \log \tau(e_i) + \sum_i \eta(\alpha_i \tau(e_i)) \geqslant$$
$$\geqslant \sum_i \alpha_i \tau(e_i) \log \tau(e_i) \geqslant \log \beta,$$

where

$$\beta = \min\{\tau(e) : e \text{ is a nonzero projection in } N' \cap M\}.$$

Here a varies over all strictly positive elements in $N' \cap M$ with $\tau(a) = 1$. Since $N' \cap M \neq \mathbf{C}$, it is easy to see that the equality $\tau(\eta a) = \log \beta$ does not occur in the above. Hence it follows that $\tau(\eta a)$ admits all values in $(\log \beta, 0]$. Therefore we obtain

$$\{K_E(M \mid N) : E \in \mathscr{E}(M, N)\} = (\log \alpha,\ \log[M : N]_0]$$

with $\alpha = [M: N]_0 / \beta^2$. Moreover

$$\alpha = [M: N]_0 \min_e E_0(e)^2 = \min_e \operatorname{Index}(E_0)_e = \min_e [M_e: N_e]_0,$$

where the minimums are taken over nonzero projections e in $N' \cap M$. ∎

COROLLARY 6.5 *The following conditions are equivalent:*

(i) $\inf\{K_E(M \mid N): E \in \mathcal{E}(M, N)\} = 0$;

(ii) *there exists a projection* e *in* $N' \cap M$ *such that* $M_e = N_e$.

Proof. When $N' \cap M = \mathbf{C}$, each of the conditions (i) and (ii) is equivalent to $M = N$. When $N' \cap M \neq \mathbf{C}$, (i) $\Leftrightarrow$ (ii) follows from Theorem 6.4. ∎

In fact, because of the restriction of index values, the above conditions hold if the infimum in (i) is less than log 2.

THEOREM 6.6. *Let* L *be a factor with* $M \supseteq L \supseteq N$. *If* $[M: L]_0 = \operatorname{Index} F < \infty$ *and* $[L: N]_0 = \operatorname{Index} G < \infty$ *where* $F \in \mathcal{E}(M, L)$ *and* $G \in \mathcal{E}(L, N)$, *then the following conditions are equivalent:*

(i) $[M: N]_0 = [M: L]_0[L: N]_0$, i.e. $E_0 = G \circ F$;

(ii) $K_{G \circ F}(M \mid N) = K_F(M \mid L) + K_G(L \mid N)$.

Proof. Let $E = G \circ F$ and h, h_1, h_2 be as in Theorem 5.1. Then $h_1 = [M: L]_0 1$ and $h_2 = [L: N]_0 1$ by assumption.

(i) $\Rightarrow$ (ii). Since $\operatorname{Index} E = (\operatorname{Index} F)(\operatorname{Index} G) = [M: N]_0$, we have $h = [M: N]_0 1$ by Theorem 2.1 (2). Hence (ii) holds by Theorem 5.1 (2).

(ii) $\Rightarrow$ (i). Theorem 5.1 (2) implies $h = h_1 h_2 = [M: L]_0[L: N]_0 1$, so that (i) holds by Theorem 2.1 (2). ∎

Under the assumption of Theorem 6.6, it seems that the equivalent equalities (i) and (ii) do not hold in general, while we have no explicit example. But we remark the following: Let $N \subseteq M_0 = M \subseteq M_1 \subseteq M_2 \subseteq \dots$ be the tower of factors with $E_n \in \mathcal{E}(M_n, M_{n-1})$, $n \geq 1$, obtained by repeating the basic construction [16] from E_0. Then it is immediate that $[M_n: M_{n-1}]_0 = \operatorname{Index} E_n = [M: N]_0$ for all $n \geq 1$. In particular, when M is a type II_1 factor and Jones' index $[M: N]$ is equal to $[M: N]_0$, it is known [23, Theorem 3.1] that $H(M_n \mid N) = (n+1)H(M \mid N)$, equivalently $[M_n: N]_0 = [M: N]_0^{n+1}$ for all $n \geq 1$. According to [17], these equalities of minimum indices can be extended to the general case of basic constructions.

We end the paper with an example illustrating the relation between the minimum index and the entropy.

EXAMPLE 6.7. Let R be the hyperfinite type II_1 factor with the normalized trace τ and R_λ be Jones' subfactor [12] of R with $\lambda = [R: R_\lambda]^{-1} < 1/4$. Take $t > 0$ with $t(1-t) = \lambda$. According to [22], there exist a projection f in $R_\lambda' \cap R$ with $\tau(f) = t$ and an isomorphism $\theta: R_f \to R_{1-f}$ such that $R_\lambda = \{x \oplus \theta(x): x \in R_f\}$ and $R_\lambda' \cap R = \mathbf{C}f + \mathbf{C}(1-f)$. Let τ' be the normalized trace on R_λ'. Since

$$1 = [R_f: (R_\lambda)_{1-f}] = [R: R_\lambda]\tau(f)\tau'(f),$$

we get $\tau'(f) = 1 - t$. Hence [9, Theorem 2 (1)] shows

$$[R: R_\lambda]_0 = [R: R_\lambda]\{(\tau(f)\tau'(f))^{1/2} + (\tau(1 - f)\tau'(1 - f))^{1/2}\}^2 =$$

$$= 4 < [R: R_\lambda],$$

and $E_0 \in \mathscr{E}(R, R_\lambda)$ having the minimum index 4 is given by $E_0 = a^{1/2}E_{R_\lambda}a^{1/2}$ where E_{R_λ} is the conditional expectation with respect to τ and $a = (2\lambda)^{-1}((1 - t)f + t(1 - f))$. We have

$$\{\mathrm{Index}\, E: E \in \mathscr{E}(R, R_\lambda)\} = [4, \infty),$$

$$\{K_E(R \mid R_\lambda): E \in \mathscr{E}(R, R_\lambda)\} = (0, \log 4].$$

Also note [22, Corollary 5.3] that Pimsner and Popa's entropy $H(R \mid R_\lambda)$ is equel to $2(\eta t + \eta(1 - t)) < \log 4$.

Notes added in proof. (1) For general von Neumann algebras $M \supseteq N$ and $\varphi \in \mathscr{S}(M)$, Connes [6] introduced the entropy $H_\varphi(M \mid N)$ of M relative to N and φ, which extends the entropy (*) in Section 3. The relation between Connes' entropy and the minimum index is discussed in [30].

(2) It is proved by Longo [31] that the chain rule of the minimum index (i.e. he equality (i) of Theorem 6.6) holds in general.

This research was supported in part by Grant-in-Aid fot Scientific Research.

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Received May 4, 1989.