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SUMS OF HOMOMORPHISMS ON BANACH LATTICES

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In this paper, we consider the problem of characterizing operators between Banach lattices which may be expressed as sums of lattice homomorphisms and the solid vector space generated by such operators. We will actually characterize a somewhat larger class of operators having a "local" representation as a sum of homomorphisms. In addition, we will examine questions of absolute continuity and compactness with respect to these operators.

Operator T from Banach lattice E to Banach lattice F is a *local homomorphism* if there is an increasing net of orthomorphisms $\{M_\alpha\}$ on F whose supremum is the identity, such that each $M_\alpha \circ T$ is a finite sum of lattice homomorphisms. If E and F may be represented as (extended real valued) functions on compact spaces X and Y respectively, then it will be shown in Section 2 that T is a local homomorphism if and only if T can be expressed as an integral $Tf(y) = \int k(y, x)f(x)\mathrm{d}\sigma$, where σ is the counting measure, $k(y, x)$ is non-zero for at most a finite number of x, and y is in a dense open subset of Y. In Section 3, we show that if operator S is absolutely continuous with respect to local homomorphism T in the sense of Luxemburg, then S is also a local homomorphism. Section 4 investigates questions related to compactness of local homomorphisms. For example, if T is a compact local homomorphism from $C(X)$ to $C(Y)$ (X and Y compact) and $0 \leqslant S \leqslant T$, then S is also a compact local homomorphism.

1. PRELIMINARIES

For Banach lattices E and F, $L(E, F)$ will denote the class of bounded linear operators from E to F. $T \in L(E, F)$ is a *lattice homomorphism* if $Tf \wedge Tg = 0$, for all f and g in E satisfying $f \wedge g = 0$. The set of all homomorphisms will be written $\mathrm{Hom}(E, F)$. An operator $M \in L(E, E)$ which is band preserving, i.e. $M(B) \subset B$ for every band (order-closed ideal) B of E is called an *orthomorphism*, and the set of all such operators will be denoted Orth(E). When an appropriate representation for E as a function space is possible, Orth(E) can be identified with the collection of multiplication operators on E (see [8, Theorem 141.1]). If an orthomorphism is representable as a multiplication by g, we will use the symbol g to denote this operator.

Let $C(X)$ and $C(Y)$ be the spaces of real valued continuous functions on compact Hausdorff spaces X and Y. If $T \in L(C(X), C(Y))$ is a lattice homomorphism then there is a positive $r \in C(Y)$ and a function $\varphi: Y \to X$ which is continuous when $r(y) \neq 0$ such that $Tf(y) = r(y)f(\varphi(y))$ (see [2, Theorem 7.22]). This type of representation may be generalized to larger classes of Banach lattices ([4], [1]).

$C'(X)$ may be identified with the space of bounded Radon measures on X. Each $x \in X$ can be identified with a corresponding point measure $\hat{x} \in C'(X)$ in a natural way. A linear operator $T \in L(C(X), C(Y))$ is determined by the values of its adjoint $T^*: C'(Y) \to C'(X)$, restricted to these point measures in Y, and in fact $L(C(X), C(Y))$ may be identified with the set of functions from Y to $C'(X)$ continuous in the $\sigma(C'(X), C(X))$ topology on $C'(X)$. Lattice homomorphisms are then precisely those operators for which $T^*y = r(y)\varphi\hat{\ }(y)$, with $\varphi: Y \to X$ continuous when $r(y) \neq 0$. For additional background information on positive operators see [2], [6], [7], [8].

If $T: C(X) \to C(Y)$ is a sum of lattice homomorphisms, then for each $y \in Y$, there is a collection $\{x_1, x_2, \ldots, x_n\} \subset X$ and positive real numbers $a_1, \ldots, a_n$ so that $T^*y = \sum_{i=1}^{n} a_i\hat{x}_i$ (i.e. T^*y is a sum of elements of Hom($C(X)$, **R**)). However, the condition $T^*y = \sum_{i=1}^{n} a_i\hat{x}_i$ for all $y \in Y$ is not sufficient to guarantee that T is a sum of homomorphisms, as demonstrated in the following example.

EXAMPLE 1. Let $Y = X = [0, 1]$ and let $\varphi_1, \varphi_2: [0, 1] \to [0, 1]$ be any continuous functions satisfying $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(y) \neq \varphi_2(y)$ when $y \neq 0$. Define h from $[0, 1]$ to $C'([0, 1])$ by

$$h(y) = \begin{cases} \dfrac{1+\sin(1/y)}{2}\hat{\varphi}_1(y) + \dfrac{1-\sin(1/y)}{2}\hat{\varphi}_2(y), & y \neq 0 \\ \hat{\varphi}_1(0,) & y = 0 \end{cases}$$

and let $T: C([0, 1]) \to C([0, 1])$ be given by $Tf(y) = \langle h(y), f\rangle$.

Suppose $T = S_1 + \ldots + S_n$ where each S_i is a lattice homomorphism ($S_i^*(\cdot) = r_i(\cdot)\hat{\psi}_i(\cdot)$, $r_i: [0, 1] \to \mathbf{R}$ is continuous and $\psi_i: [0, 1] \to [0, 1]$ is continuous when $r_i \neq 0$). If $r_i(y_0) \neq 0$, then $\psi_i(y) = \varphi_j(y)$, with $j = 1$ or 2, in a neighborhood of y_0. If $\psi_i(y)$ equals both φ_1 and φ_2 infinitely often near $y = 0$, then $r_i(y) = 0$ infinitely often near 0 and $\lim_{y \to 0} r_i(y) = 0$. If $\psi_i(y) = \varphi_1(y)$ (or $\psi_i(y) = \varphi_2(y)$) in a neighborhood of $y = 0$, then $r_i(y) \leqslant (1 + \sin(1/y))/2$ $(r_i(y) \leqslant (1 - \sin(1/y))/2)$ near $y = 0$ and thus $\lim_{y \to 0} r_i(y) \leqslant 0$. We thus obtain a contradiction, since it is impossible that $T^*(0)$ have the appropriate representation.

In the preceding example, $T^*y = a_1\hat{x}_1 + a_2\hat{x}_2$ for each $y \in Y$. Although T cannot be expressed as a sum of lattice homomorphisms, T could be written as such a sum if we can "avoid" $y = 0$. More explicitly, if $g \in C([0, 1])$ satisfies $g(0) = 0$, then the operator defined by T followed by multiplication by g can be written as a sum of two lattice homomorphisms. This suggests the following definition.

DEFINITION. Let E and F be Banach lattices. $T \in L(E, F)$ is a *local homomorphism* if there is a net $\{M_\alpha\} \subset \mathrm{Orth}(E)$ with $M_\alpha \uparrow I_E$, the identity operator on E, and $M_\alpha \circ T$ is a finite sum of lattice homomorphisms for each α.

2. CHARACTERIZATION OF LOCAL HOMOMORPHISMS

We begin by considering local homomorphisms from $C(X)$ to $C(Y)$, where X and Y are compact Hausdorff spaces. $1_X \in C(X)$ will denote the constant one function. Note that the orthomorphism $\bar{1}_X$ is the identity on $C(X)$.

THEOREM 1. *Let $T: C(X) \to C(Y)$. The following are equivalent.*

i) *T is a local homomorphism.*

ii) *There is a dense open set $G \subset Y$ such that for each $y_0 \in G$ there exist a neighborhood $B(y_0)$ of y_0 and non-negative functions $f_1, \ldots, f_n \in C(X)$ satisfying $\sum_{i=1}^{n} f_i = 1_X$ and*

$$(T \circ \bar{f}_i)^* y \in \mathrm{Hom}(C(X), \mathbf{R})$$

for all $y \in B(y_0)$.

iii) *There is a dense open set $H \subset Y$ such that for each $y \in H$ there are real numbers $a_1, \ldots, a_n$ and $x_1, \ldots, x_n \in X$ satisfying*

$$T^*y = \sum_1^n a_i \hat{x}_i .$$

iv) *There is a dense open set $H \subset Y$ and a real valued function k on $Y \times X$ such that, for each $y \in H$, $k(y, x) = 0$ except for a finite number of $x \in X$, and*

$$Tf(y) = \int_X k(y, x) f(x) \mathrm{d}\sigma$$

for all $f \in C(X)$, where σ is the counting measure.

Proof. We note first that the relation $Tf(y) = \langle T^*y, f\rangle$ implies that (iii) and (iv) are equivalent. To show that (i) implies (iii), suppose that $\{g_\alpha\} \subset C(Y)$, $g_\alpha \uparrow 1_Y$ and that $\bar{g}_\alpha \circ T = \sum_{i=1}^{n} S_i$ where each S_i is a lattice homomorphism, i.e. $S_i f(y) = r_i(y) f(\varphi_i(y))$ for an appropriate r_i and φ_i. Let A be the set $\{y \in Y;\, g_\alpha(y) = 0$ for every $\alpha\}$. Since $g_\alpha \uparrow 1_Y$, A has empty interior. Let $H = A^c$. For $y_0 \in H$, find g_{α_0} with $g_{\alpha_0}(y_0) > 0$. Thus

$$(g_{\alpha_0} \circ T)^* y_0 = g_{\alpha_0}(y_0) T^*(y_0) = \sum_{i=1}^{n} r_i(y_0)\hat{\varphi}_i(y_0).$$

We may let $a_i = r_i(y_0)/g_{\alpha_0}(y_0)$ and $x_i = \varphi_i(y_0)$ to obtain (iii).

For (iii) $\Rightarrow$ (ii), we will need the following fact: *If* $T^*y_0 = \sum_{i=1}^{n} a_i\hat{x}_i$, $a_i > 0$, *and the* x_i *are distinct elements of* X, *then there is a neighborhood* U_0 *of* y_0 *such that for each* $y \in U_0$ *we have* $T^*y = \sum_{i=1}^{m} b_i\hat{z}_i$, *where* $b_i > 0$ *depends on* y, $z_i \in X$ *are distinct, and* $m \geqslant n$.

To verify this, find continuous functions s_i, $i = 1, \ldots, n$, such that $s_i(x_i) = 1$ and $s_i \wedge s_j = 0$ when $i \neq j$. Suppose that $y_\alpha \to y_0$ and $T^*y_\alpha = \sum_{i=1}^{m_\alpha} b_i(\alpha)\hat{z}_i(\alpha)$, $m_\alpha < n$. This would imply that $\max_i |\langle T^*y_\alpha - T^*y_0,\, s_i\rangle| \geqslant \min\{a_1, \ldots, a_n\}$, a contradiction.

Assume that (iii) is satisfied. We will call $y_0 \in H$ a *pathological point* if $T^*y_0 = \sum_{i=1}^{n} a_i\hat{x}_i$ and every neighborhood of y_0 contains a y such that $T^*y = \sum_{i=1}^{m} b_i\hat{z}_i$ for distinct z_i, $b_i > 0$, and $m > n$. We let P denote the set of all pathological points and note that P is closed. Let $P_n = \left\{ y \in P;\; T^*y = \sum_{i=1}^{m} b_i\hat{z}_i,\, m \leqslant n \right\}$. P_n is closed by the fact cited above. P_n has empty interior, since any non-empty open set contained in P would necessarily contain points y where $T^*y = \sum_{i=1}^{m} b_i\hat{z}_i$ with m arbitrarily large. Since $P = \bigcup_n P_n$, the Baire category theorem implies that P has empty interior.

Let $G = H \cap P^c$ and $y_0 \in G$. Suppose that $T^*y_0 = \sum_{i=1}^{n} a_i\hat{x}_i$ for distinct x_i and $a_i > 0$. From the note at the beginning of the proof of (iii) implies (ii) and the fact that $y_0 \notin P$, there is a neighborhood U_0 of y_0 such that $T^*y = \sum_{i=1}^{n} a_i(y)\hat{x}_i(y)$

(n fixed) for all $y \in U_0$ ($a_i(y_0) = a_i$). We may find open sets $U_1, \ldots, U_n \subset X$ with $\bar{U}_1, \ldots, \bar{U}_n$ disjoint and $x_i \in U_i$. The sets U_i may be used to determine a $\sigma(C'(X), C(X))$ neighborhood V of T^*y_0 by letting V be the union of all sets of the form $\{v \in C'(X) \,;\, |\langle v - T^*y_0, g_i\rangle| < a_i,\ i = 1, \ldots, n\}$ where $g_i \in C(X)$, $g_i(x_i) = 1$, and support$(g_i) \subset U_i$. We may then let $B(y_0) = U_0' \cap U_0$ where $U_0' = (T^*)^{-1}(V)$ in Y. For $y \in B(y_0)$, $y \in U_0$ implies $T^*y = \sum_{i=1}^{n} a_i(y)\hat{x}_i(y)$. $T^*y \in V$ implies that one of the $x_j(y)$ must be in each U_i (we may assume, without loss of generality, that $x_i(y) \in U_i$). We complete the proof by finding $f_i \in C(X)$ with $f_i(x) = 1$ on U_i, $f_i \geqslant 0$, and $\sum_{i=1}^{n} f_i = 1_X$.

We now consider (ii) implies (i). Find $B(y_0)$ and $f_1, \ldots, f_n$ as in (ii). In addition, let $V(y_0)$ be a neighborhood of y_0 so that $\bar{V}(y_0) \subset B(y_0)$. Let $h_0 \in C(Y)$ be a function satisfying $0 \leqslant h_0(y) \leqslant 1$, $h_0(y) = 1$ on $\bar{V}(y_0)$ and $h_0(y) = 0$ on $B(y_0)^c$.

Suppose $W \subset G$ is open, and we have found $h_1', \ldots, h_m' \in C(Y)$ and $f_1', \ldots, f_m' \in C(X)$ such that $T^*y = \sum_{i=1}^{m} h_i'(y)(T\circ\bar{f}_i')^*y$ where $h_i'(y)(T\circ\bar{f}_i')^*y \in \mathrm{Hom}(C(X), \mathbf{R})$ for all $y \in W$. Form the sum

$$h_0(y) \sum_{i=1}^{n} (T\circ\bar{f}_i)^*y + (1 - h_0(y)) \sum_{i=1}^{m} h_i'(y)\,(T\circ\bar{f}_i')^*y,$$

where $y \in W \cup V(y_0)$. We may rearrange and combine the terms of the sum above to obtain a representation for T^*y with $y \in V(y_0) \cup W$ containing $m + n$ terms, similar to the representation for $y \in W$.

If $K \subset G$ is compact, the collection $\{V(y) \,;\, y \in K\}$ is a cover for K. Apply the previous argument successively to the sets of a finite subcover of K to obtain $W \supset K$, $\{f_i\} \subset C(X)$, and $\{h_i\} \subset C(Y)$ with $T^*(y) = \sum_{i=1}^{n} h_i(y)\,(T\circ\bar{f}_i)^*y$ for $y \in W$, where $h_i(y)\,(T\circ\bar{f}_i)^*y \in \mathrm{Hom}(C(X), \mathbf{R})$.

If $g_k \in C(Y)$ with the support of g_K contained in K, then $\bar{g}_K\circ T$ may be written as a sum of homomorphisms, i.e.

$$(\bar{g}_K\circ T)^*y = \sum_{i=1}^{n} r_i(y)\hat{\varphi}_i(y)$$

where

$$\varphi_i(y) = \frac{(T\circ\bar{f}_i)^*y}{\|(T\circ\bar{f}_i)^*y\|}$$

and

$$r_i(y) = \|(T\circ\bar{f}_i)^*y\|h_i(y)g_K(y).$$

COROLLARY. *$T: C(X) \to C(Y)$ is a local homomorphism if and only if there is a dense open set $G \subset Y$ such that, for each $y_0 \in G$, there is a $g_0 \in C(Y)$ with $g_0(y_0) > 0$ and non-negative $f_1, \ldots, f_n \in C(X)$, $\sum_{i=1}^{n} f_i = 1_X$, such that each $g_0 \cdot T f_i$ is a lattice homomorphism.*

Proof. Suppose that T is a local homomorphism. For $y_0 \in G$, find $B(y_0)$ and $f_1, \ldots, f_n$ as in Theorem 1, condition (ii). Let g_0 satisfy $g_0(y_0) = 1$ with the support of g_0 contained in $B(y_0)$. Conversely, given g_0 and $f_1, \ldots, f_n$, let $B(y_0) = \{y : g_0(y) > 1/2\}$ to verify condition (ii) in Theorem 1.

We will now consider Banach lattices that can be represented (in the sense of Schaefer) as continuous extended real valued functions. Given a Banach lattice E with quasi-interior point e, we will identify E with continuous extended real-valued functions on a compact topological space X, each function finite on a dense subset (e.g. see [7, Theorem 4.5]). The ideal generated by the quasi-interior point e is dense in E and corresponds to $C(X)$. The operators on E correspond to the point-wise defined operations for the functions on sets of finiteness. We will identify the elements in E with functions on X.

Let E and F be Banach lattices with quasi-interior points. Assume T is a lattice homomorphism from E to F, E and F are represented as continuous extended real-valued functions on X and Y respectively, and $T(e)$ is finite on Y. Now T from E to F can be realized as a weighted composition operator (see [4, Theorem 1]) in the sense that $Tf(y) = r(y)f(\varphi(y))$ for all y in $\{y\,; r(y) > 0\}$ and $Tf(y) = 0$ for all y in the interior of $r^{-1}(0)$, where φ from Y to X and r from Y to $\mathbf{R}$ are continuous on $\{y : r(y) > 0\}$. For brevity, we write $Tf(y) = r(y)f(\varphi(y))$ for all y in a dense subset of Y (r and φ continuous only on $\{y : r(y) > 0\}$). If so described, this determines a unique continuous extended real-valued function on Y.

THEOREM 2. *Let T map a Banach lattice E with a quasi-interior point e into a Banach lattice F. The following are equivalent.*

(i) *T is a local homomorphism.*

(ii) *Let X be a representation space for E (with $e = 1_X$) and Y a representation space for F with Te finite on Y. There is a dense open set $H \subset Y$ such that for each y in H, there are real numbers $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$ in X satisfying*

$$(Tf)(y) = \sum_{i=1}^{n} a_i f(x_i)$$

for all $f \geqslant 0$ in E.

(iii) *For X and Y as in* (ii), *and σ the counting measure,*

$$Tf = \int k(\cdot, x) f(x) d\sigma$$

where k is a real-valued function on $Y \times X$ and for each y in H, $k(y, x) = 0$ except for a finite number of x in X. It is understood by this equality that for each f in E,

$$(Tf)(y) = \int k(y, x)f(x)\mathrm{d}\sigma$$

for each y in a dense set and Tf is uniquely determined by continuity.

Proof. We first show that (i) implies (ii). Let E and F be Banach lattices with representation spaces X (with $e = 1_X$) and Y (with Te finite) for E and F respectively. Now T restricted to $C(X)$ (the ideal generated by e) maps $C(X)$ into $C(Y)$. From Theorem 1, we know that T (on $C(X)$) satisfies $T^* y = \sum a_i \hat{x}_i$ for each y in a dense open subset H. Furthermore, as in the proof of Theorem 1, $g_\alpha T = \sum_{i=1}^{m} S_i$ for homomorphisms S_i. The lattice homomorphism S_i can be realized as a type of weighted composition operator (see [4, Theorem 1]) so that $(S_i f) = a_i f(x_i)$ for all y in a dense open set H_i. Now $H_\alpha = \bigcap_{i=1}^{m} H_i \cap (g_\alpha^{-1}(0))^c$ is dense and open in $(g_\alpha^{-1}(0))^c$. Thus for $H = \bigcup H_\alpha$, condition (ii) is satisfied.

Condition (iii) is just a restatement of condition (ii). We now prove that (ii) implies (i). Restricted to $C(X)$, condition (iii) of Theorem 1 is satisfied so that $\overline{g}_\alpha \circ T = \sum S_i$ for functions in $C(X)$. Now each S_i as a map from $C(X)$ into F is dominated by $g_\alpha \circ T$ and can thus be extended to a positive operator on all of E ($S_i(f \wedge ne)$ is Cauchy in F). In turn, since $S_i((f \wedge g) \wedge ne) = S_i[(f \wedge ne) \wedge (g \wedge ne)] = 0$ for f and g positive and disjoint, it follows that S_i is a lattice homomorphism on E and (i) is satisfied.

We observe that condition (iii) implies that if $T: E \to F$ is a local homomorphism and $0 \leqslant S \leqslant T$, then S is a local homomorphism. Since the set of local homomorphisms is clearly closed under finite summation, we have the following corollary.

COROLLARY. *The collection of all differences of local homomorphisms is a solid vector subspace of $L(E, F)$.*

The set of local homomorphisms is larger than the solid subspace of $L(E, F)$ generated by finite sums of lattice homomorphisms. For a given local homomorphism T, there is no uniform bound on the number of terms in the representation of $T^* y$ as a sum of elements of Hom($C(X)$, **R**). Even if such a uniform bound were imposed, however, T still would not necessarily be dominated by a sum of homomorphisms, as demonstrated by the following example.

EXAMPLE 2. Let $Y = [0, 1]$ and $X = \alpha(\mathbf{N}) = \{1, 2, \ldots, \omega\}$, the one point compactification of the natural numbers. Let $\{r_i\} \subset C([0, 1])$ be a sequence of non-zero functions with the following properties: $(r_i^{-1}(0))^c$ is connected, $r_1(0) = 1$, $r_i \wedge r_j = 0$ if $|i - j| > 1$, and $\sum_{i=1}^{\infty} r_i(y) = 1$ for $y \in [0, 1]$. Note that these properties imply that for each $y \in [0, 1)$, $r_i(y) > 0$ for at most two r_i. Define $h: [0, 1] \to C'(\alpha(\mathbf{N}))$ as follows

$$h(y) = \begin{cases} r_1(y)\hat{1} + r_2(y)\hat{2} + r_3(y)\hat{3} + \ldots, & y \in [0, 1) \\ \hat{\omega}, & y = 1. \end{cases}$$

Note that h is continuous for the $\sigma(C', C)$ topology on $C'(\alpha\mathbf{N})$. Define $T: C(\alpha\mathbf{N}) \to C([0,1])$ by $Tf(y) = \langle h(y), f \rangle$. Let $\{y_m\} \subset [0, 1]$ satisfy $r_m(y_m) = 1$ (which implies $y_m \to 1$). Note that $\varphi: [0, 1] \to \alpha\mathbf{N}$ is continuous if and only if φ is constant. Consider $\sum_{i=1}^{n} S_i$, where each S_i is a lattice homomorphism and $S_i^* y = s_i(y)\hat{\varphi}_i(y)$ for appropriate s_i and φ_i. If $s_i(1) > 0$, then $\varphi_i(y)$ is constant in a neighborhood of $y = 1$, and $\varphi_i(y_m) = \hat{m}$ for at most a finite number of m. We may conclude that there is an M_i so that $\varphi_i(y_m) \neq \hat{m}$ when $m > M_i$. Alternatively, if $s_i(1) = 0$, there is a number M_i so that $s_i(y_m) < 1/2n$ when $m > M_i$. Thus, if $m > \max\{M_1, \ldots, M_n\}$, then $\sum_{i=1}^{n} s_i(y_m)\hat{\varphi}_i(y_m)$ cannot be greater than $(1/2)\hat{m}$, and we conclude that it is impossible for T to be dominated by $\sum_{i=1}^{n} S_i$.

3. ABSOLUTE CONTINUITY FOR LOCAL HOMOMORPHISMS

Recall (Luxemburg, [5]) that $S: E \to F$ is absolutely continuous with respect to T if Sf is in the band generated by Tf for all $f \geqslant 0$ in E. If S is absolutely continuous with respect to T, we write $S \ll T$. In this section, we show that if T is a local homomorphism and positive $S \ll T$, then S is also a local homomorphism.

LEMMA 1. *Let S and T be positive linear operators from E to F with T a lattice homomorphism and $S \ll T$. Then S is a lattice homomorphism. Further, let E and F be represented as continuous extended real-valued functions on X and Y respectively with $S(1_X)$ and $T(1_X)$ finite on Y. Given that $Tf(y) = r_1(y)f(\varphi(y))$ for y in a dense subset, then $Sf(y) = r_2(y)f(\varphi(y))$ for a dense subset, and $r_2(y) = 0$ if $r_1(y) = 0$.*

Proof. Suppose $g \wedge f = 0$ (implying $Tf \wedge Tg = 0$). Since $Sf = \bigvee_n (Sf \wedge nTf)$ and $Sg = \bigvee_m (Sg \wedge mTg)$, we have

$$Sf \wedge Tg = (\bigvee_n (Sf \wedge nTf)) \wedge (\bigvee_m (Sg \wedge mTg)) =$$

$$= \bigvee_n ((Sf \wedge nTf) \wedge (\bigvee_m (Sg \wedge mTg))) =$$

$$= \bigvee_n \bigvee_m (Sf \wedge nTf \wedge Sg \wedge mTg) = 0.$$

For a proof of the last assertion, assume that $Sf(y) = r_2(y)f(\varphi_2(y))$.

Let $A_1 = \{y : r_1(y) > 0\}$ and $A_2 = \{y : r_2(y) > 0\}$, and let Z_1^0 denote the interior of $r_1^{-1}(0)$. Now $Z_1^0 \cup A_1$ is dense in Y. For y in $Z_1^0 \cup A_2$, there exists a $g > 0$ in $C(X)$ such that $Sg(y) = 1$ and Tg is zero on a neighborhood of y. Since $Sg \wedge nTg \uparrow \uparrow Sg$, this is impossible, so that $Z_1^0 \cap A_2 = \emptyset$. Now $A_1 \cap A_2$ is dense in A_2. For y in $A_1 \cap A_2$, assume that $\varphi(y) \neq \varphi_2(y)$. Choose f_1 and f_2 in $C(X)$ so that $f_1 \wedge f_2 = 0$ and $f_1(\varphi(y)) > 0$ and $f_2(\varphi_2(y)) > 0$. This contradicts $Sf_1 \wedge nTf_1 \uparrow Sf_1$ since Tf_1 is zero on a neighborhood of y but $Sf_1(y) > 0$. Thus $\varphi = \varphi_2$ on $A_1 \cap A_2$ (a dense subset of A_2). We may now write $Sf(y) = r_2(y)f(\varphi(y))$ for all y in a dense subset, namely $(A_1 \cap A_2) \cup Z_2^0$, where Z_2^0 is the interior of $r_2^{-1}(0)$. Here, if $r_1(y) = 0$ then $r_2(y) = 0$.

THEOREM 3. *Let E and F be Banach lattices with compact representation spaces X and Y respectively. Let S and T be positive linear operators from E to F with $S(1_X)$ and $T(1_X)$ positive. If T is a local homomorphism, then the following are equivalent.*

(i) *$S \ll T$ (in the sense of Luxemburg).*

(ii) *If $Tf(y) = \sum a_i f(x_i)$ for all y in a dense subset of y, then $Sf(y) = \sum b_i f(x_i)$ and $a_i = 0$ implies that $b_i = 0$.*

In this case, S is also a local homomorphism.

Proof. Assume that (ii) is satisfied. Then for $f \geq 0$ in E, $Sf(y) \wedge nTf(y) \uparrow \uparrow Sf(y)$ for all y in a dense subset and hence $Sf \wedge nTf \uparrow Sf$, establishing (i). Now assume that (i) is satisfied. As in the corollary to Theorem 1, $\bar{g}_\alpha \circ T = \sum_i \bar{g}_\alpha \circ T \circ \bar{f}_i$ for T restricted to $C(X)$ and each $\bar{g}_\alpha \circ T \circ \bar{f}_i$ is a lattice homomorphism. For $f \geq 0$ in E, $f \wedge ne$ converges to f in norm and $\bar{g}_\alpha \circ T(f \wedge ne) \to \bar{g}_\alpha \circ Tf$ and $\sum_i \bar{g}_\alpha \circ T \circ \bar{f}_i(f \wedge ne) \to \sum_i (\bar{g}_\alpha \circ T \circ \bar{f}_i)f$ so that $\bar{g}_\alpha \circ T = \sum_i \bar{g}_\alpha \circ T \circ \bar{f}_i$ on E. Since $\bar{g}_\alpha \circ S \ll \bar{g}_\alpha \circ T$ and $\bar{g}_\alpha \circ S \circ \bar{f}_i \ll \bar{g}_\alpha \circ T \circ \bar{f}_i$, Lemma 1 implies that $\bar{g}_\alpha \circ S \circ \bar{f}_i$ is a lattice homomorphism. In addition, assume that $Tf(y) = \sum_i a_i f(x_i)$ for all y in a dense subset. Now $Tf(y) = \sum_i (\bar{g}_\alpha \circ T \circ \bar{f}_i)f(y)$ for y in a dense subset. Now Lemma 1 applies to $\bar{g}_\alpha \circ S \circ \bar{f}_i \ll \bar{g}_\alpha \circ T \circ \bar{f}_i$ so that $\sum_i (\bar{g}_\alpha \circ S \circ \bar{f}_i)f(y) = Sf(y) = \sum_i b_i f(x_i)$ and if $a_i = 0$, then $b_i = 0$.

4. COMPACTNESS PROPERTIES FOR LOCAL HOMOMORPHISMS

In this section, we consider compact local homomorphisms, deriving a number of results, including a characterization of the norm closure of the set of compact local homomorphisms. We restrict our attention to operators from $C(X)$ to $C(Y)$ where X and Y are compact and Hausdorff.

PROPOSITION. *Let T be a positive compact operator from $C(X)$ to $C(Y)$. Suppose that $T = \bigvee_{i=1}^{n} T_i$, where each T_i is a lattice homomorphism. Then each T_i is compact.*

Proof. An operator S from $C(X)$ to $C(Y)$ is compact if and only if $S^*: Y \to C'(X)$ is *norm* continuous (e.g. see [3, VI. 7.1]) Since T is a sum of homomorphisms, for each y we have $T^*y = \sum_{i=1}^{k} b_i\hat{x}_i$, where $k \leqslant n$ and the $x_i \in X$ are distinct. Thus $T_j^*y = a\,\hat{x}_{i_0}$ for some i_0. If we let $\{y_\alpha\}$ be a net with $y_\alpha \to y$, we may complete the proof by showing $\|T_j^*y_\alpha - T_j^*y\| \to 0$. Since $T_j^*y_\alpha \to T_j^*y$ $(\sigma(C', C))$, there is an α_0 so that $T_j^*y_\alpha \wedge \hat{x}_i = 0$ when $i \neq i_0$ and $\alpha \geqslant \alpha_0$. If $T_j^*y_\alpha = a_\alpha\hat{x}_\alpha$ and $x_\alpha \neq x_{i_0}$, then $\|T^*y_\alpha - T^*y\| \geqslant \|T_j^*y_\alpha\|$ where $\alpha \geqslant \alpha_0$.

If we assume $T_j^*y \neq 0$, the previous inequality, along with the fact that $\|T^*y_\alpha - T^*y\| \to 0$, imply that there is an α_1 so that $T_j^*y_\alpha = a_\alpha x_{i_0}$ for $\alpha \geqslant \alpha_1$. Thus

$$\|T_j^*y_\alpha - T_j^*y\| = \|a_\alpha\hat{x}_{i_0} - a\hat{x}_{i_0}\| = |a_\alpha - a|.$$

Since

$$|a_\alpha - a| = |\langle (a_\alpha - a)\hat{x}_{i_0}, 1_X\rangle| \to 0,$$

we conclude

$$\|T_j^*y_\alpha - T_j^*y\| \to 0.$$

Finally, if $\|T_j^*y\| = 0$, then

$$\|T_j^*y_\alpha - T_j^*y\| = \|T_j^*y_\alpha\| = \langle T_j^*y_\alpha, 1\rangle \to 0.$$

Thus $T_j^*: Y \to C'(X)$ is norm continuous, and T_j is compact.

We denote by C'_a the band of $C'(X)$ generated by $\{\hat{x}\,;\, x \in X\}$. This band will be called the *atomic* part of C'.

LEMMA 2. *Let $T: C(X) \to C(Y)$ be a compact local homomorphism. Then T^* maps Y to C'_a.*

Proof. Suppose $z \in Y$ and $\{z_\alpha\} \subset H$ is a net converging to z with H as in Theorem 1, (iii). Since $T^*z_\alpha \in C'_a$, $T^*z_\alpha\}$ converges to T^*z in norm, and bands in $C'(X)$ are norm closed, we have $T^*z \in C'_a$.

THEOREM 4. *Let $T\colon C(X) \to C(Y)$ be a compact local homomorphism and suppose $S\colon C(X) \to C(Y)$ with $0 \leqslant S \leqslant T$. Then S is also a compact local homomorphism.*

Proof. That S is a local homomorphism follows from Theorem 1. To show that S is compact, let $y_\alpha \to y$ in Y. Since T^*y is in C'_a, we may write $T^*y = \sum_{i=1}^{\infty} a_i\hat{x}_i$ for some sequence $\{x_i\} \subset X$. $T^*y_\alpha \to T^*y$ in norm implies that $T^*y_\alpha = \sum_{i=1}^{\infty} a_{i,\alpha}\hat{x}_i + v_\alpha$ where $\|v_\alpha\| \to 0$. Hence $S^*y = \sum_{i=1}^{\infty} b_i\hat{x}_i$ and $S^*y_\alpha = \sum_{i=1}^{\infty} b_{i,\alpha}\hat{x}_i + \mu_\alpha$ with $0 \leqslant b_{i,\alpha} \leqslant a_{i,\alpha}$, $0 \leqslant b_i \leqslant a_i$, and $0 \leqslant \mu_\alpha \leqslant v_\alpha$. Also note that $\sum_{i=1}^{\infty} b_{i,\alpha}\hat{x}_i \to \sum_{i=1}^{\infty} b_i\hat{x}_i$ $(\sigma(C', C))$ and $\left\| \sum_{i=1}^{\infty} b_{i,\alpha}\hat{x}_i - \sum_{i=1}^{\infty} b_i\hat{x}_i \right\| = \sum_{i=1}^{\infty} |b_{i,\alpha} - b_i|$. Given positive integer, k, we have

$$\sum_{i=k}^{\infty} |b_{i,\alpha} - b_i| \leqslant \sum_{i=k}^{\infty} |a_{i,\alpha}| + |a_i| \leqslant$$

$$\leqslant 2\sum_{i=k}^{\infty} |a_i| + \|T^*y_\alpha - T^*y\|.$$

Given $\varepsilon > 0$, if we choose K sufficiently large so that $\sum_{i=K}^{\infty} |a_i| < \varepsilon/5$ and choose α_0 so that $\sum_{i=1}^{K-1} |b_{i,\alpha} - b_i|$, $\|T^*y_\alpha - T^*y\|$, and $\|\mu_\alpha\|$ are all less than $\varepsilon/5$ when $\alpha \geqslant \alpha_0$, then we may conclude

$$\|S^*y_\alpha - S^*y\| = \left\| \sum_{i=1}^{\infty} (b_{i,\alpha} - b_i)\hat{x}_i + \mu_\alpha \right\| \leqslant$$

$$\leqslant \sum_{i=1}^{K-1} |b_{i,\alpha} - b_i| + \sum_{i=K}^{\infty} |b_{i,\alpha} - b_i| + \|\mu_\alpha\| <$$

$$< \frac{\varepsilon}{5} + \left(2\left(\frac{\varepsilon}{5}\right) + \frac{\varepsilon}{5}\right) + \frac{\varepsilon}{5} = \varepsilon$$

when $\alpha \geqslant \alpha_0$, and hence S is compact.

THEOREM 5. *Let T: $C(X) \to C(Y)$ be compact and positive. T is in the norm closure of the collection of compact local homomorphisms (and, in fact, the norm closure of the collection of finite sums of homomorphisms) if and only if T^* maps Y to C'_a.*

Proof. We begin by assuming that for compact T, there is a sequence $\{S_n\}$ of compact local homomorphisms so that $\|S_n - T\| \to 0$. Since $S_n^* : Y \to C'_a$ and C'_a is norm closed, it follows that $T^* : Y \to C'_a$.

To prove the reverse implication, let T be compact and $T^* : Y \to C'_a$. Given a collection $\{x_1, \ldots, x_N\} \subseteq X$, write $T^*y = \sum_{i=1}^{N} a_i(y)\hat{x}_i + \sum_{i=1}^{\infty} b_i\hat{z}_i$ where $\{b_i\}$ and $\{z_i\}$ depend on y and the x_i and z_i are distinct. For $y_1, y_2 \in Y$, we have $\|T^*y_1 - T^*y_2\| \geqslant \sum_{i=1}^{N} |a_i(y_1) - a_i(y_2)|$. Hence, each $a_i(y)$ is continuous. Given $y_0 \in Y$, we may write $T^*y_0 = \sum_{i=1}^{\infty} a_i\hat{x}_i$. Given $\varepsilon > 0$, there is an N so that $\left\|T^*y_0 - \sum_{i=1}^{N} a_i\hat{x}_i\right\| < \varepsilon/3$. For each y, express T^*y as $\sum_{i=1}^{N} a_i(y)\hat{x}_i + \sum_{i=1}^{\infty} b_i\hat{z}_i$ $(a_i(y_0) = a_i)$. There is a neighborhood V of y_0 so that

$$\sum_{i=1}^{N} |a_i - a_i(y)| < \varepsilon/3 \quad \text{and} \quad \|T^*y - T^*y_0\| < \varepsilon/3 \quad \text{for } y \in V.$$

Thus, for $y \in V$ we conclude

$$\left\|T^*y - \sum_{i=1}^{N} a_i(y)\hat{x}_i\right\| \leqslant \|T^*y - T^*y_0\| + \left\|T^*y_0 - \sum_{i=1}^{N} a_i\hat{x}_i\right\| +$$

$$+ \left\|\sum_{i=1}^{N} (a_i - a_i(y))\hat{x}_i\right\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Repeat the above for each $y \in Y$ to find $V(y)$, a neighborhood of y, on which an inequality similar to the above holds. Obtain a finite collection of these, $V_1, \ldots, V_n$, so that on V_k we have

$$\left\|Ty^* - \sum_{i=1}^{N_k} a_{i,k}(y)\hat{x}_{i,k}\right\| < \varepsilon.$$

Combine these into a single collection $\{c_1(y), \ldots, c_m(y)\}$ and $\{z_1, \ldots, z_m\}$ and let $\varphi_i(y) = z_i$. Note that

$$\left\| T^* y - \sum_{i=1}^{m} c_i(y)\, \hat{\varphi}_i(y) \right\| \leqslant \left\| T^* y - \sum_{i=1}^{N_k} a_{i,k}(y) \hat{x}_{i,k} \right\|$$

for each k to complete the proof of the theorem.

As a final comment, we note that if $C(Y)$ is Dedekind complete and $T\colon C(X) \to C(Y)$ satisfies $T^* y = \sum_{i=1}^{n} a_i \hat{x}_i$ (n fixed) for each $y \in Y$, then T is the sum of n lattice homomorphisms. A proof of this will be given in a future paper dealing with sums of homomorphisms on Dedekind complete spaces.

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